PROCEEDINGS OF SCIENCE

Zakharov-Shabat Systems and Conformal Immersions Induced by Dirac Spinors

Floyd L. Williams
Department of Mathematics and Statistics, University of Massachusetts, 710 N. Pleasant Street, Amherst, MA, USA 01003
E-mail: williams@math.umass.edu

We compute the conformal immersion of a surface S in $\mathbf{R}^3$ induced by a deformed Dirac spinor associated with a Zakharov-Shabat system. As an application we consider classical configurations of strings in $\mathbf{R}^3$ given in the case when the mean curvature of S is constant.

Third International Satellite Conference on Mathematical Methods in Physics
21 - 26 October, 2013
Londrina - PR (Brazil)

PoS(ICMP 2013)015

1. Introduction

In the well-known 1972 paper [1], V. Zakharov and A. Shabat applied the inverse scattering method with its associated linear Zakharov-Shabat (Z-S) system. These play a prominent and ground-breaking role in the analysis of various nonlinear evolution equations $\psi_t = f(\psi, \psi_x, \psi_{xx}, \psi_{xxx}, \ldots)$ such as sine-Gordon, modified Korteweg-de Vries (mKdV), Novikov-Veselov, or the nonlinear Schrodinger equation

$$i\psi_t + \psi_{xx} + \frac{2}{1-\alpha^2}|\psi|^2\psi = 0, \quad i^2 = -1, \quad \alpha \neq \pm 1, \tag{1.1}$$

for example, for which a Lax pair was constructed. The method was applied earlier, and initially, for the KdV equation by C. Gardner, J. Greene, M. Kruskal, and R. Miura [2].

The Z-S system of interest here is given by

$$\begin{aligned} r'(x) + i\lambda r(x) &= 2p(x)s(x) \\ s'(x) - i\lambda s(x) &= -2p(x)r(x) \end{aligned} \tag{1.2}$$

where generally $r(x), s(x)$ are complex-valued functions of a real variable x, the potential function $p(x)$ is *real-valued*, and the spectral parameter λ is complex. We also have interest in *deformed* solutions $r(x;t)$, $s(x;t)$ of a deformed version of 1.2 for a real parameter t. For the particular choices $\lambda = i/2$, $p(x) = -U(x)$, $s(x) = r_1(x)$, $r(x) = r_2(x)$ one obtains from 1.2 the system

$$\begin{aligned} r_1'(x) + \frac{r_1(x)}{2} &= 2U(x)r_2(x) \\ r_2'(x) - \frac{r_2(x)}{2} &= -2U(x)r_1(x) \end{aligned} \tag{1.3}$$

which for

$$L := \frac{d}{dx} + \begin{bmatrix} 1/2 & -2U \\ 2U & -1/2 \end{bmatrix}, \quad r = (r_1, r_2)^\top \quad \text{(the transpose of } (r_1, r_2)\text{)} \tag{1.4}$$

can be written as

$$Lr = 0. \tag{1.5}$$

Similarly for the Cauchy-Riemann operators ∂_z, $\bar{\partial}_z$ and the Dirac operator $\mathcal{D}$ given for $z = x + iy$ by

$$\begin{aligned} \partial_z &= \tfrac{\partial}{\partial z} := \tfrac{1}{2}\left(\tfrac{\partial}{\partial x} - i\tfrac{\partial}{\partial y}\right) \\ \bar{\partial}_z &= \tfrac{\partial}{\partial \bar{z}} := \tfrac{1}{2}\left(\tfrac{\partial}{\partial x} + i\tfrac{\partial}{\partial y}\right) \\ \mathcal{D} &:= \begin{bmatrix} \partial_z & 0 \\ 0 & \partial_{\bar{z}} \end{bmatrix} + \begin{bmatrix} 0 & -U \\ U & 0 \end{bmatrix}. \end{aligned} \tag{1.6}$$

we shall also consider *Dirac spinors* $\psi = (\psi_1, \psi_2)^\top$ given by

$$\mathcal{D}\psi = 0\colon \frac{\partial \psi_1}{\partial z} = U\psi_2, \quad \frac{\partial \psi_2}{\partial \bar{z}} = -U\psi_1, \quad \bar{U} = U. \tag{1.7}$$

The bar $^{-}$ denotes complex conjugation as usual.

Given a Dirac spinor ψ in 1.7, B. Konopelchenko [3] has shown in fact that one can construct a conformal immersion $X = (X^1, X^2, X^3)\colon \mathbf{R}^2 \to \mathbf{R}^3$ of a surface S in Euclidean 3-space $\mathbf{R}^3$—a construction by which the classical Weierstrass-Enneper formula/representation of minimal surfaces (the case with $U = 0$) is neatly generalized. The Konopelchenko representation, which is equivalent to the K. Kenmotsu representation [4, 5], appears in earlier work also—of people like L. Eisenhart [6] and A. Bobenko [7], for example. Here the coordinate functions $X^j\colon \mathbf{R}^2 \to \mathbf{R}$, $j = 1,2,3$, the first fundamental form ds^2, the Gaussian curvature K, and the mean curvature H of S are given by the concrete formulas

$$\frac{\partial X^1}{\partial z} = i\left(\bar{\psi}_1^2 + \psi_2^2\right), \tag{1.8a}$$

$$\frac{\partial X^2}{\partial z} = \bar{\psi}_1^2 - \psi_2^2, \tag{1.8b}$$

$$\frac{\partial X^3}{\partial z} = -2\bar{\psi}_1\psi_2, \tag{1.8c}$$

$$ds^2 = 4u^2(dx^2+dy^2), \qquad u := |\psi_1|^2 + |\psi_2|^2, \tag{1.9}$$

$$K = -\frac{1}{u^2}\frac{\partial^2 \log u}{\partial_z \bar{\partial}_z} = -\frac{1}{4u^2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\log u, \qquad H = \frac{U}{u}.$$

Equations 1.8a, 1.8b, 1.8c are the differential version of the generalized Weierstrass-Enneper representation (the integration version) in (2) of [8] for example.

The Konopelchenko-Weierstrass-Enneper representation is a powerful tool for the analysis of Polyakov string theory problems [8]. It also has a connection to Liouville-Beltrami gravity [9], and it allows for a new approach in relating integrable systems and conformally immersed surfaces in $\mathbf{R}^3$ and $\mathbf{R}^4$ [3, 5, 10, 11]. In [9] it is also shown that for certain constant mean density surfaces S (i.e. for $Hu^2 = a$ constant) the Polyakov extrinsic action is preserved under the modified Novikov-Veselov hierarchy of flows.

V. Varlamov in [12] considers the important particular case of a surface of revolution, for which explicit fundamental solutions (Jost functions) of the Z-S system 1.2 are obtained, by way of Bargmann potentials. Integrable deformations of the corresponding explicit spinor field ψ (in the one-soliton case) are expressed by the mKdV equation. There are some trivial errors in [12] that we correct.

Given the deformed Varlamov spinor field ψ just mentioned (see equation 2.10 in Section 2 below) we compute the corresponding deformed surface S. Thus we solve the system in 1.8 (in fact, we solve a deformed version) and we compute the first fundamental form ds^2 and the Gaussian and mean curvatures K and H of S. In particular we show that H is *constant* when the spectral parameter λ in 1.2 is pure imaginary. As an application of our results, we consider in the final section (Section 4) classical configurations of strings in $\mathbf{R}^3$.

2. Solutions of a Deformed Z-S System

We describe the fundamental solutions (Jost functions) $\phi_1^+(x,\lambda)$, $\phi_2^+(x,\lambda)$ of the Z-S system 1.2 constructed by Varlamov in Section 4 of [12], by way Bargmann potentials. Replacing $p(x)$ by

PoS(ICMP 2013)015

a suitable function $p(x;t)$ (for a deformation variable t), we also consider his deformed solutions $\phi_1^+(x,\lambda,t)$, $\phi_2^+(x,\lambda,t)$ of a corresponding deformed version of 1.2. As indicated in the introduction there are a few trivial errors in [12]. In particular the expressions for $\phi_1^+(x,\lambda)$ and $\phi_1^+(x,\lambda,t)$ have a missing factor that we correct for the record.

On page 10 of [12], w and $b(x)$ should read $w = 2\alpha e^{\phi}\cosh(\mu x - \phi)$, $b(x) = \pm 2\mu\,\mathrm{sech}(\mu x - \phi)$ (rather than $w = 2e^{\phi}\cosh(\mu - \phi)$, $b(x) = \pm 2\mu\,\mathrm{sech}(\mu - \phi)$, which in fact are constants). The factor $4\mu^2$ for $a(x)$ there should read 2μ instead, and the expressions for $\phi_1^+(x,\lambda)$, $\phi_1^+(x,\lambda,t)$ in equations (15), (19) should have the denominator $\mu[2i\lambda - \mu]$, and not just $[2i\lambda - \mu]$ alone.

Instead of ϕ_j^+, $j = 1,2$, we shall work with multiples $r(x) = A\phi_1^+(x,\lambda)$, $s(x) = A\phi_2^+(x,\lambda)$ of them, for some constant $A = 2\mu[2i\lambda - \mu]$. Thus we express the Varlamov solution of the Z-S system 1.2 above as follows:

$$\begin{aligned} r(x) &= e^{-i\lambda x}[2\mu\tanh(\mu x - \phi) + 4i\lambda] \\ s(x) &= \pm 2\mu e^{-i\lambda x}\,\mathrm{sech}(\mu x - \phi) \end{aligned} \tag{2.1}$$

for

$$2p(x) = \pm\mu\,\mathrm{sech}(\mu x - \phi) \tag{2.2}$$

for real constants $\mu, \phi \in \mathbf{R}$, $\mu \neq 0$.

With 2.2 as motivation now define, for t real, $p(x;t)$ by

$$2p(x;t) = \pm\mu\,\mathrm{sech}(\mu x - \phi(t)) \tag{2.3}$$

for a suitable function $\phi(t)$. By direct differentiation with respect to x and t

$$\begin{aligned} &2p_{xxx}(x;t) + 48p(x;t)^2 p_x(x,t) = \mp\mu^4\tanh(\mu x - \phi(t))\,\mathrm{sech}(\mu x - \phi(t)), \\ &2p_t(x;t) = \pm\mu\dot{\phi}(t)\tanh(\mu x - \phi(t))\,\mathrm{sech}(\mu x - \phi(t)) \end{aligned} \tag{2.4}$$

from whence it follows that $p(x;t)$ is a solution of the mKdV equation

$$p_t = p_{xxx} + 24p^2 p_x \tag{2.5}$$

if

$$\begin{aligned} &\dot{\phi}(t) = -\mu^3 : \phi(t) = -\mu^3 t + \nu, \quad \nu \in \mathbf{R} \\ &2p(x;t) := \pm\mu\,\mathrm{sech}(\mu x + \mu^3 t - \nu). \end{aligned} \tag{2.6}$$

It is now clear from 2.1, 2.2 that the functions

$$\begin{aligned} r(x;t) &:= e^{-i\lambda x}[2\mu\tanh(\mu x + \mu^3 t - \nu) + 4i\lambda] \\ s(x;t) &:= \pm 2\mu e^{-i\lambda x}\,\mathrm{sech}(\mu x + \mu^3 t - \nu) \end{aligned} \tag{2.7}$$

solve the deformed version

$$r_x(x;t) + i\lambda r(x;t) = 2p(x;t)s(x;t) \tag{2.8}$$

$$s_x(x,t) - i\lambda s(x,t) = -2p(x;t)r(x;t)$$

of 1.2 for $p(x;t)$ defined in 2.6, since the differentiation in 2.8 is with respect to x only, and not t. These functions are A times the functions $\phi_1^+(x,\lambda,t)$, $\phi_2^+(x,\lambda,t)$ in [12], respectively, for the same constant $A = 2\mu[2i\lambda - \mu]$ above. For convenience we shall denote the function of x and t in 2.7 by θ:

$$\theta(x,t) := \mu x + \mu^3 t - \nu \tag{2.9}$$

Now given 2.8 it follows [10, 11] that the functions ψ_1, ψ_2 given by

$$\psi_1(x,y;t) := e^{\lambda y}s(x;t) := \pm 2\mu e^{-\lambda i(x+iy)} \operatorname{sech}\theta(x,t) \tag{2.10}$$
$$\psi_2(x,y;t) := e^{\lambda y}r(x;t) := e^{-\lambda i(x+iy)}[2\mu \tanh\theta(x,t) + 4i\lambda]$$

(for $\theta(x,t)$ in 2.9) satisfy the system

$$\psi_{1_z}(x,y;t) = U(x;t)\psi_2(x,y;t) \tag{2.11}$$
$$\psi_{2_{\bar z}}(x,y;t) = -U(x;t)\psi_1(x,y;t)$$

for

$$U(x;t) := -p(x;t) := \mp\frac{\mu}{2}\operatorname{sech}\theta(x,t). \tag{2.12}$$

Thus, comparing 2.11 with 1.7, we see that $\psi = (\psi_1, \psi_2)^\top$ is a deformed Dirac spinor. Here $-U$ satisfies the modified Korteweg-de Vries equation 2.5.

3. The Deformed Surface of ψ

In the introduction it was noted that a Dirac spinor $\psi = (\psi_1, \psi_2)^\top$ gives rise to a surface S immersed in $\mathbf{R}^3$, by work of Konopelchenko and others, where the immersion $X\colon \mathbf{R}^2 \to \mathbf{R}^3$ is determined by the system of equations 1.8. The formula $ds^2 = 4u^2(dx^2 + dy^2)$ in 1.9 shows that S is indeed *conformally* immersed. The main result presented here is a computation of S in the case of the deformed spinor ψ with ψ_1, ψ_2 given by 2.10. Thus we find an explicit solution $X = (X^1, X^2, X^3)$ of the system

$$X_z^1(x,y;t) = i[\bar\psi_1{}^2(x,y;t) + \psi_2^2(x,y;t)] \tag{3.1a}$$
$$X_z^2(x,y;t) = \bar\psi_1{}^2(x,y;t) - \psi_2^2(x,y;t) \tag{3.1b}$$
$$X_z^3(x,y;t) = -2\bar\psi_1(x,y;t)\psi_2(x,y;t), \tag{3.1c}$$

which is a deformed version of 1.8a, 1.8b, 1.8c. ψ_1, ψ_2 in 2.10 depend of course on the spectral parameter λ also, which first appears in the Z-S system 1.2—which we take to be non-zero: $\lambda \in \mathbf{C} - \{0\}$ as we have no interest in the trivial case.

The first step is to explicate the system 3.1 by writing out its real and imaginary parts. We write $\lambda = \lambda_1 + i\lambda_2$ of course with $\lambda_1, \lambda_2 \in \mathbf{R}$. Using the definition of ∂_z in 1.6 and the definition of ψ_1, ψ_2 in 2.10 one eventually finds that the system 3.1 is equivalent to the following system of equations:

$$\frac{1}{2}\frac{\partial X^1}{\partial x} = 4e^F\{[4(\lambda_2^2-\lambda_1^2)-4\mu\lambda_2\tanh\theta+\mu^2-2\mu^2\operatorname{sech}^2\theta]\sin E+4[2\lambda_1\lambda_2-\mu\lambda_1\tanh\theta]\cos E\} \tag{3.2a}$$

$$-\frac{1}{2}\frac{\partial X^1}{\partial y} = 4e^F\{4[\mu\lambda_1\tanh\theta-2\lambda_1\lambda_2]\sin E+[4(\lambda_2^2+\lambda_1^2)-4\mu\lambda_2\tanh\theta+\mu^2]\cos E\} \tag{3.2b}$$

$$\frac{1}{2}\frac{\partial X^2}{\partial x} = 4e^F\{[4\mu\lambda_2\tanh\theta-4(\lambda_2^2-\lambda_1^2)-\mu^2-2\mu^2\operatorname{sech}^2\theta]\cos E+4[2\lambda_1\lambda_2-\mu\lambda_1\tanh\theta]\sin E\} \tag{3.2c}$$

$$-\frac{1}{2}\frac{\partial X^2}{\partial y} = 4e^F\{4[2\lambda_1\lambda_2-\mu\lambda_1\tanh\theta]\cos E+[4(\lambda_2^2-\lambda_1^2)-4\mu\lambda_2\tanh\theta+\mu^2]\sin E\} \tag{3.2d}$$

$$\frac{1}{2}\frac{\partial X^3}{\partial x} = e^F\operatorname{sech}\theta[\mp 8\mu^2\tanh\theta\pm 16\mu\lambda_2] \tag{3.2e}$$

$$-\frac{1}{2}\frac{\partial X^3}{\partial y} = \mp 16\mu\lambda_1 e^F\operatorname{sech}\theta, \tag{3.2f}$$

where we write

$$F := 2(\lambda_2 x+\lambda_1 y), \quad E := 2(\lambda_1 x-\lambda_2 y) \tag{3.3}$$

for convenience, and (again) θ is given by 2.9. Here equations (3.2a, 3.2b), (3.2c, 3.2d), (3.2e, 3.2f) are equivalent to equations 3.1a, 3.1b, 3.1c, respectively.

The system (3.2e, 3.2f) is the easiest, by far, to solve. A solution is given by

$$X^3(x,y;t) = \pm 16\mu e^F\operatorname{sech}\theta(x,t) \tag{3.4}$$

for F, θ in 3.3, 2.9. A method for solving the system (3.2c, 3.2d) is analogous to that for solving (3.2a, 3.2b) so we sketch it for the latter system. It is easier to integrate equation 3.2b with respect to y than to integrate equation 3.2a with respect to x. One checks that for F, E in 3.3

$$\int e^F\sin E\,dy = \frac{e^F[\lambda_1\sin E+\lambda_2\cos E]}{2(\lambda_1^2+\lambda_2^2)}+f_1(x) \tag{3.5}$$
$$\int e^F\cos E\,dy = \frac{e^F[\lambda_1\cos E-\lambda_2\sin E]}{2(\lambda_1^2+\lambda_2^2)}+f_2(x)$$

for functions of integration $f_1(x)$, $f_2(x)$; remember that $\lambda=\lambda_1+i\lambda_2\neq 0$ by assumption. Now since $\theta(x,t)$ is independent of y, the formulas in 3.5 determine $X_1(x,y;t)$, up to a knowledge of $f_1(x)$, $f_2(x)$. One next differentiates $X_1(x,y;t)$ with respect to x and uses 3.2a, as usual. This is a bit tedious but there are some fortunate simplifications along the way. Eventually one finds that equation 3.2a forces the condition

$$(16\mu\lambda_1\tanh\theta-32\lambda_1\lambda_2)f_1'(x)+16\mu^2\operatorname{sech}\theta[\lambda_1 f_1(x)-\lambda_2 f_2(x)] \tag{3.6}$$
$$+[16(\lambda_2^2-\lambda_1^2)-16\mu\lambda_2\tanh\theta+4\mu^2]f_2'(x)=0$$

on $f_1(x)$, $f_2(x)$—a condition that is satisfied for the choices $f_1(x) = f_2(x) = 0$, or $f_1(x) = \lambda_2$, $f_2(x) = \lambda_1$, for example. We make the former choice $f_1(x) = f_2(x) = 0$. In the end, again as the arguments are analogous for the system (3.2c, 3.2d), one finds that

$$
\begin{aligned}
X_1(x,y;t) = \frac{e^F}{\lambda_1^2+\lambda_2^2}&[(32\lambda_1\lambda_2 - 16\mu\lambda_1\tanh\theta(x,t))(\lambda_1\sin E + \lambda_2\cos E) \\
&+ (16(\lambda_1^2-\lambda_2^2) + 16\mu\lambda_2\tanh\theta(x,t) - 4\mu^2)(\lambda_1\cos E - \lambda_2\sin E)] \\
X_2(x,y;t) = \frac{e^F}{\lambda_1^2+\lambda_2^2}&[(16\mu\lambda_1\tanh\theta(x,t) - 32\lambda_1\lambda_2)(\lambda_1\cos E - \lambda_2\sin E) \\
&+ (16(\lambda_1^2-\lambda_2^2) + 16\mu\lambda_2\tanh\theta(x,t) - 4\mu^2)(\lambda_1\sin E + \lambda_2\cos E)]
\end{aligned}
\tag{3.7}
$$

for $F = 2(\lambda_2 x + \lambda_1 y)$, $E = 2(\lambda_1 x - \lambda_2 y)$, $\theta(x,t) = \mu x + \mu^3 t - \nu$.

In summary, formulas 3.4, 3.7 therefore provide for an explicit solution of the system 3.1 (or equivalently for the system 3.2) and thus for the conformal immersion $X = (X^1, X^2, X^3): \mathbf{R}^2 \to \mathbf{R}^3$ of the deformed surface S induced by the deformed Dirac spinor $\psi = (\psi_1, \psi_2)^\top$ given in definition 2.10.

One can use the formulas in 1.9 with the definitions of (ψ_1, ψ_2), U in 2.10, 2.12 to compute the first fundamental form ds^2, Gaussian curvature K, and mean curvature H of S. The result is the following for F, θ above and for

$$
u := |\psi_1|^2 + |\psi_2|^2 = 4e^F[\mu^2 + 4(\lambda_1^2+\lambda_2^2 - 4\lambda_2\tanh\theta)]: \tag{3.8}
$$

$$
ds^2 = 64e^{2F}[\mu^2 + 4(\lambda_1^2+\lambda_2^2 - 4\lambda_2\tanh\theta)]^2(dx^2+dy^2), \tag{3.9}
$$

$$
K = \frac{-\lambda_2\mu^3 e^{-2F}[4(\lambda_1^2+\lambda_2^2)+\mu^2]\tanh\theta - [4(\lambda_1^2+\lambda_2^2)+\mu^3]\tanh^3\theta + 2\mu\lambda_2\tanh^4\theta - 2\mu\lambda_2]}{8[\mu^2+4(\lambda_1^2+\lambda_2^2) - 4\mu\lambda_2\tanh\theta]^4}
$$

$$
H = \frac{\mp\mu e^{-F}\operatorname{sech}\theta}{8[\mu^2+4(\lambda_1^2+\lambda_2^2-\mu\lambda_2\tanh\theta)]} = \frac{\mp\mu e^{-F}}{8\{[\mu^2+4(\lambda_1^2+\lambda_2^2)]\cosh\theta - 4\mu\lambda_2\sinh\theta\}}.
$$

4. Remarks on Classical Configurations of Strings

As an application of the results of Section 3, we consider classical configurations of strings given by the standard Nambu-Goto-Polyakov action [13, 14]:

$$
S_{NGP} = \mu_0 \iint \sqrt{\det g}\, dx\, dy + \frac{1}{\alpha_0}\iint H^2\sqrt{\det g}\, dx\, dy \tag{4.1}
$$

with the integration taken over a world sheet. Thus the interest is in solutions of the corresponding Euler-Lagrange equation for S_{NGP} [8]:

$$
\Delta_{LB}H + 2H(H^2-K) - 2\alpha_0\mu_0 H = 0 \tag{4.2}
$$

where Δ_{LB} is the Laplace-Beltrami operator of the metric $ds^2 = 4u^2(dx^2+dy^2)$ in 1.9:

$$
\Delta_{LB} = \frac{1}{4u^2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right). \tag{4.3}
$$

Now $\sqrt{\det g} = 4u^2$ so by 1.9, $H^2\sqrt{\det g} = 4U^2$ and $u^2 = \left(|\psi_1|^2 + |\psi_2|^2\right)^2$, which means that the action in 4.1 can be expressed as

$$S_{NGP} = 4\mu_0 \iint \left(|\psi_1|^2 + |\psi_2|\right)^2 dx\,dy + \frac{4}{\alpha_0}\iint U^2\,dx\,dy. \tag{4.4}$$

By the change of variables $\phi = 1/H$, Konopelchenko and Landolfi show in [8] that equation 4.2 is transformed to the equation

$$\phi_{z\bar{z}} + [2U^2 + (\log U^2)_{z\bar{z}}]\phi - 2\alpha_0\mu_0 U^2\phi^3 = 0. \tag{4.5}$$

Here $f_{z\bar{z}} = (f_{xx} + f_{yy})/4$ for a function $f(x,y)$, by definition 1.6. A more general version of equation 4.5 is given in equation (6) of [15].

In our case

$$\phi = \mp\frac{8e^F}{\mu}\{[\mu^2 + 4(\lambda_1^2 + \lambda_2^2)]\cosh\theta - 4\mu\lambda_2\sinh\theta\} \tag{4.6}$$

by 3.9, and one computes that

$$\phi_{xx} = \mp\frac{8e^F}{\mu}[16\mu\lambda_1^2\lambda_2\sinh\theta + \{\mu^4 + 4\mu^2(\lambda_1^2 - 2\lambda_2^2) + 16\lambda_2^2(\lambda_1^2 + \lambda_2^2)\}\cosh\theta], \tag{4.7}$$

$$\begin{aligned}
\phi_{yy} &= 4\lambda_1^2\phi,\\
\phi_{xx} + 4\lambda_1^2\phi &= \mp\frac{8e^F}{\mu}[\mu^4 + 8\mu^2(\lambda_1^2 - \lambda_2^2) + 32\lambda_1^2\lambda_2^2 + 16(\lambda_1^4 + \lambda_2^4)]\cosh\theta\\
&= \mp\frac{8e^F}{\mu}[4(\lambda_1^2 + \lambda_2^2) + \mu^2 + 4\mu\lambda_2][4(\lambda_1^2 + \lambda_2^2) + \mu^2 - 4\mu\lambda_2]\cosh\theta,\\
2U^2 + (\log U^2)_{z\bar{z}} &= 0,
\end{aligned}$$

for U given by 2.12. Equation 4.5 therefore reduces to the equation

$$\mp\frac{e^F}{\mu}[4(\lambda_1^2 + \lambda_2^2) + \mu^2 + 4\mu\lambda_2][4(\lambda_1^2 + \lambda_2^2) + \mu^2 - 4\mu\lambda_2]\cosh\theta - \alpha_0\mu_0 U^2\phi^3 = 0. \tag{4.8}$$

Consider the simple case $\mu_0 = 0$. The first term in 4.1 (the Nambu-Goto contribution to the action) then vanishes, S_{NGP} reduces to the Willmore functional [16], and equation 4.8 holds when and only when one of the brackets there vanishes: $\mu = -2\lambda_2 \pm 2i\lambda_1$, $2\lambda_2 \pm 2i\lambda_1$. Since μ is *real*, this forces $\lambda_1 = 0$ so that $\lambda = i\lambda_2$ is pure imaginary—as in the example $\lambda = i/2$ of the introduction where Z-S system 1.3 was derived from 1.2. Thus $\mu = \pm 2\lambda_2$, and with $\lambda_1 = 0$ one has that

$$\begin{aligned}
&[\mu^2 + 4(\lambda_1^2 + \lambda_2^2)]\cosh\theta - 4\mu\lambda_2\sinh\theta \qquad (4.9)\\
&= 8\lambda_2^2 e^{\mp\theta}\\
&= 8\lambda_2^2 e^{\mp(\pm 2\lambda_2 x \pm 8\lambda_2^3 t - \nu)} \quad \text{(by 2.9)}\\
&= 8\lambda_2^2 e^{-2\lambda_2 x - 8\lambda_2^3 t \pm \nu}\\
&= 8\lambda_2^2 e^{-F} e^{-8\lambda_2^3 t \pm \nu} \quad \text{(by 3.3)}
\end{aligned}$$

which by 4.6 shows that

$$\phi(x,y;t) = \begin{cases} \mp 32\lambda_2 e^{-8\lambda_2^3 t+\nu} & \text{for } \mu = 2\lambda_2, \\ \pm 32\lambda_2 e^{-8\lambda_2^3 t-\nu} & \text{for } \mu = -2\lambda_2, \end{cases} \tag{4.10}$$

and which shows that $H = 1/\phi$ is also a *constant* independent of x, y.

In the pure imaginary case $\lambda = i\lambda_2$ the formulas 3.4, 3.7, moreover, for the immersion of S simplify:

$$\begin{aligned} X^1(x,y;t) &= -\frac{4e^{2\lambda_2 x}}{\lambda_2}[4\lambda_2^2 - 4\mu\lambda_2 \tanh\theta(x,t) + \mu^2]\sin 2\lambda_2 y \\ X^2(x,y;t) &= -\frac{4e^{2\lambda_2 x}}{\lambda_2}[4\lambda_2^2 - 4\mu\lambda_2 \tanh\theta(x,t) + \mu^2]\cos 2\lambda_2 y \\ X^3(x,y;t) &= \pm 16\mu e^{2\lambda_2 x}\operatorname{sech}\theta(x,t). \end{aligned} \tag{4.11}$$

Going back to the case $\mu = \pm 2\lambda_2$ in 4.9 we write

$$e^{2\lambda_2 x}[4\lambda_2^2 - 4\mu\lambda_2 \tanh\theta + \mu^2] = \frac{e^{2\lambda_2 x}}{\cosh\theta}[(4\lambda_2^2 + \mu^2)\cosh\theta - 4\mu\lambda_2 \sinh\theta] = \frac{8\lambda_2^2 e^{-8\lambda_2^3 t \pm \nu}}{\cosh\theta} \tag{4.12}$$

by equation 4.9. That is by 4.11

$$\begin{aligned} X^1 &= -32\lambda_2 e^{-8\lambda_2^3 t \pm \nu}\frac{\sin 2\lambda_2 y}{\cosh\theta}, \\ X^2 &= -32\lambda_2 e^{-8\lambda_2^3 t \pm \nu}\frac{\cos 2\lambda_2 y}{\cosh\theta}, \\ X^3 &= \begin{cases} \pm 32\lambda_2 \frac{e^{2\lambda_2 x}}{\cosh\theta} & \text{for } \mu = 2\lambda_2 \\ \mp 32\lambda_2 \frac{e^{2\lambda_2 x}}{\cosh\theta} & \text{for } \mu = -2\lambda_2. \end{cases} \end{aligned} \tag{4.13}$$

We choose $\mu = 2\lambda_2$, for example. Then $\theta := 2\lambda_2 x + 8\lambda_2^3 t - \nu \Rightarrow e^{2\lambda_2 x} = e^{\theta} e^{-8\lambda_2^3 t+\nu} \Rightarrow X^3$ can also be expressed as

$$X^3 = \pm 32\lambda_2 e^{-8\lambda_2^3 t+\nu}\frac{e^{\theta}}{\cosh\theta}. \tag{4.14}$$

One can also work directly with the Euler-Lagrange equation 4.2, using 3.9 and 4.3 to compute that

$$\Delta_{LB}H + 2H(H^2+K) = \mp\mu\left(\frac{\cosh\theta}{8}\right)^3 \frac{e^{-3F}[4(\lambda_1^2+\lambda_2^2)+\mu^2+4\mu\lambda_2][4(\lambda_1^2+\lambda_2^2)+\mu^2-4\mu\lambda_2]}{\{[\mu^2+4(\lambda_1^2+\lambda_2^2)]\cosh\theta - 4\mu\lambda_2\sinh\theta\}^4}, \tag{4.15}$$

where the product of brackets in 4.15 is the *same* product in 4.8. Thus (again) for $\mu = \pm 2\lambda_2$ with $\lambda_1 = 0$, $\mu_0 = 0$, H is a constant solution of 4.2.

References

[1] V. Zakharov and A Shabat. Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media. *Soviet Phys. JETP*, 34:62–69, 1972.

[2] C. Gardner, J. Greene, M. Kruskal, and R. Miura. Method for solving the Korteweg-de Vries equation. *Phys. Rev. Lett.*, 19:1095–1097, 1967.

[3] B. Konopelchenko. Induced surfaces and their integrable dynamics. *Studies in Appl. Math.*, 96:9–51, 1996.

[4] K. Kenmotsu. Weierstrass formula for surfaces of prescribed mean curvature. *Math. Ann.*, 245:89–99, 1979.

[5] B. Konopelchenko and I. Taimanov. Generalized Weierstrass formulae, soliton equations and Willmore surfaces, I. Tori of revolution and the mKdV equation, 1995. arXiv:dg-ga/9506011.

[6] L. Eisenhart. *A Treatise on the Differential Geometry of Curves and Surfaces*. Allyn and Bacon, 1909. also see Forgotten Books, Classical Reprint Series, 2010.

[7] A. Bobenko. Surfaces in terms of 2 by 2 matrices. Old and new integrable cases. In *Harmonic Maps and Integrable Systems*, pages 83–127. Vieweg, 1994.

[8] B. Konopelchenko and G. Landolfi. On classical configurations. *Mod. Phys. Letts. A*, 12:3161–3168, 1997.

[9] R. Carroll and B. Konopelchenko. Generalized Weierstrass-Enneper inducing, conformal immersion and gravity. *Internat. J. Mod. Physics A*, 11:1183–1216, 1996.

[10] I. Taimanov. Modified Novikov-Veselov equation and differential geometry of surfaces. *Trans. Am. Math. Soc.*, 179:133–159, 1997.

[11] I. Taimanov. Surfaces of revolution in terms of solitons. *Ann. Global Anal. Geom.*, 15:419–435, 1997.

[12] V. Varlamov. Spinor fields on the surface of revolution and their integrable deformations via the mKdV-hierarchy, 1999. arXiv:math/9906140.

[13] A. Polyakov. Quantum geometry of bosonic strings. *Phys. Letts. B*, 103:207–210, 1981.

[14] A. Polyakov. Fine structure of strings. *Nucl. Phys. B*, 268:406–412, 1986.

[15] B. Konopelchenko. On solutions of the shape equation for membranes and strings. *Phys. Letts. B*, 414:58–64, 1997.

[16] T. Willmore. *Riemannian Geometry*. Oxford Univ. Press, 1993.