PROCEEDINGS OF SCIENCE

An Introduction to Entanglement Entropy

Kulaxizi Manuela[∗†]
Physique Théorique et Mathématique, Université Libre de Bruxelles and International Solvay Institutes
E-mail: manuela.kulaxizi@gmail.com

The purpose of these lecture notes is to give an introduction to entanglement entropy with an emphasis on holography. Apart from the standard definition and properties of entanglement entropy, we explicitly derive the relation between the coefficients of the conformal anomaly and the universal term in the entropy of a d–dimensional CFT. We also discuss the prescription for calculating the entanglement entropy holographically in theories dual to Einstein-Hilbert and Lovelock gravity and present several examples in detail.

Ninth Modave Summer School in Mathematical Physics
September 1-7, 2013
Modave, Belgium

[∗]Speaker.
[†]A footnote may follow.

PoS(Modave 2013)003

1. Introduction and Outline.

Entanglement is one of the most fascinating aspects of quantum physics but remains mysterious in many ways; the outcome of a local measurement is under certain conditions correlated to the outcome of another local measurement far away. This phenomenon has been the starting point for the development of several branches of physics, such as quantum information and communication.

A widely used measure of entanglement, is entanglement entropy. This quantity is none other but the von Neumann entropy S_A associated to a subsystem A of the total system produced by tracing out the degrees of freedom in the complement A^c of A. For continuous systems, such as quantum field theories (QFTs), one usually defines a subsystem by specifying a submanifold A within the d–dimensional spacetime the QFT lives. One then intuitevely understands the entanglement entropy as the entropy of an observer in A who cannot be accessed by A^c since information is lost by "tracing over" A^c. This is somehow reminscent of black hole physics and is one of the reasons entanglement entropy has been an active area of study for several years. Nevertheless, the most important advances in our understanding occured fairly recently, after the "marriage" of entanglement entropy with holography (see for instance [1, 2]).

In these lecture notes, we will present an introduction to entanglement entropy and its holographic interpretation. In section 2, we start with the definition and properties of entanglement entropy for general QFTs. In section 3 we discuss the replica trick as a standard approach for calculating entanglement entropy. In section 4 we give the prescription for calculating entanglement entropy in conformal field theories (CFTs) with a dual description in terms of Einstein-Hilbert gravity and explicitly discuss the cases where the entangling surface is an infinitely long strip or a disk. We conclude this section with a study of entanglement entropy at finite temperature. In section 5 we relate the coefficient of the logarithmic term in entanglement entropy with the coefficients of the conformal anomaly. This is done, after a short exposition of the methods for computing integrals of polynomials of the curvature tensor on manifolds with conical singularities. The presentation in 5.1 contains all the relevant information for readers interested in understanding these techniques with other objectives, such as applications in gravitational theories. We conclude section 5 with a computation of the coefficient of the universal term in entanglement entropy in a four dimensional CFT for two distinct entangling surfaces, that of a disk (in 5.2) and of a cylinder (in 5.3). Next we present a small introduction to Lovelock gravity and in section 7 we give the prescription for computing entanglement entropy in Lovelock theories of gravity. We then explicity compute the coefficient of the universal term in entanglement entropy of the dual CFT for entangling sufaces of spherical and cylindrical shapes. The holographic results in sections 7.1 and 7.2 are in complete agreement with those of sections 5.2 and 5.3 respectively.

2. Basic definition and properties of entanglement entropy.

2.1 Definition of entanglement entropy.

Consider a quantum system in its ground state $|\Psi\rangle$ and assume that the ground state has no degeneracy. The total density matrix of the system is then $\rho_{tot.} = \langle\Psi|\Psi\rangle$ and the total entropy is vanishing, $S_{tot.} = -\text{Tr}\rho_{tot.} \ln \rho_{tot} = 0$. Let us now consider dividing the system in two subsystems A and its complement A^c. We will further assume that the total Hilbert space can be written as a direct product of the Hilbert spaces associated to the two sybsystems, $\mathscr{H} = \mathscr{H}_A \bigotimes \mathscr{H}_{A^c}$. Imagine an observer who can only access one subsystem, e.g. the subsystem A. For him/her the total system will be described by the *reduced* density matrix ρ_A, obtained from the total density matrix by tracing over all states which belong in the Hilbert space of A^c,

$$\rho_A = \mathrm{Tr}_{A^c} \rho_{tot.} . \tag{2.1}$$

The same observer will then measure the von Neumman entropy

$$S_A = -\mathrm{Tr}_A (\rho_A \ln \rho_A) , \tag{2.2}$$

which is defined as the *entanglement entropy* of the subsystem A.

The entanglement entropy measures how much entangled or quantum a state of a given system is. In general, one can consider time-varying states and then it is necessary to specify the time $t = t_0$ for which measurements take pace. Here we will focus on static systems and will not discuss issues pertaining to time–dependence.

Let us illustrate the notion of entanglement entropy with a specific example. Consider two non-interacting, binary systems A, B such that $\mathscr{H}_{tot.} = \mathscr{H}_A \bigotimes \mathscr{H}_B$, and denote the respective basis of the Hilbert spaces as $\{|0\rangle, |1\rangle\}_A$ and $\{|0\rangle, |1\rangle\}_B$. A generic state of the total system is $|\Psi\rangle = \sum_{i,j} c_{ij} |i\rangle_A \bigotimes |j\rangle_B$.

Suppose that the total system is in a product state, e.g., $|\Psi_1\rangle = |0\rangle_A \bigotimes |0\rangle_B$ and compute the entanglement entropy S_A. The first step is to evaluate the reduced denisty matrix ρ_A

$$\rho_A = \sum_B \rho_{tot} = \sum_{i=0,1} \langle i| \left(|0\rangle_A \bigotimes |0\rangle_B \right) \left({}_A\langle 0| \bigotimes_A \langle 0| \right) |i\rangle = |0\rangle_A \bigotimes_A \langle 0| , \tag{2.3}$$

which can be written

$$\rho_A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} . \tag{2.4}$$

According to the definition (2.2)

$$S_A = -\mathrm{Tr}_A \rho_A \ln \rho_A = \sum_i \rho_{ii} \ln \rho_{ii} = 1 \ln 1 + 0 \ln 0 = 0 . \tag{2.5}$$

Let us now repeat the exercise for the system in the state $|\Psi_2\rangle = \frac{|0\rangle_A \otimes |1\rangle_B - |1\rangle_A \otimes |0\rangle_B}{\sqrt{2}}$. In this case, the reduced denisty matrix is

$$\begin{aligned} \rho_A &= \frac{1}{2} \sum_{i=0,1} \langle i| \left(|0\rangle_A \bigotimes |1\rangle_B - |1\rangle_A \bigotimes |0\rangle_B \right) \left({}_A\langle 0| \bigotimes {}_B\langle 1| - {}_A\langle 1| \bigotimes {}_B\langle 0| \right) |i\rangle \Rightarrow \\ &= \frac{|0\rangle_A {}_A\langle 0| + |1\rangle_A {}_A\langle 1|}{2} , \end{aligned} \tag{2.6}$$

and can be alternatively expressed as

$$\rho_A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} . \tag{2.7}$$

The entanglement entropy is then non-vanishing and equal to

$$S_A = -\mathrm{Tr}_A \rho_A \ln \rho_A = \ln 2 . \tag{2.8}$$

We thus see that S_A is zero whenever the system is in a product state.

2.2 Properties of entanglement entropy.

- When the total system is in a pure state, the entanglement entropy associated to a subregion A is equal to the entanglement entropy associated to the complement of the subregion, $S_A = S_{A^c}$. This is not true when the system is in a mixed state, at finite temperature for example.

- For two generic subsystems A, B the *strong subadditivity* inqeuality holds

$$S_A + S_B \geq S_{A\cup B} + S_{A\cap B} \tag{2.9}$$

  It follows that if A, B, C are three *non-intersecting* regions

$$S_B + S_{A\cup B\cup C} \leq S_{A\cup B} + S_{B\cup C}, \qquad S_A + S_C \leq S_{A\cup B} + S_{B\cup C} \tag{2.10}$$

- There exists a class of systems whose entanglement entropy additionaly satisfy the *monogamy* property

$$I_{A,B,C} = S_{A\cup B\cup C} - S_{A\cup B} - S_{A\cup C} - S_{B\cup C} + S_A + S_B + S_C \leq 0 \tag{2.11}$$

  Intuitively when the entanglement entropy for a system in a specific state is monogamous, it always decreases as the partitioning of the system increases. Curiously, monogamy is a property of quantum field theories with holographic duals.

- Entanglement entropy is a divergent quantity in continuous systems. Intuitevely, entanglement is stronger close to the boundary separating the subsystem from the whole. In generic higher dimensional quantum field theories, the leading divergenece of the entanglement entropy follows the "area" law

$$S_A \simeq \frac{Area(\partial A)}{\varepsilon^{d-2}} + \cdots, \tag{2.12}$$

  where ∂A denotes the boundary seperating the two regions and ε the ultraviolet cutoff (the lattice spacing in discrete systems).

For a conformal field theory in d-spacetime dimensions one typically has the following expansion

$$S_A = \frac{g_{d-2}(\partial A)}{\varepsilon^{d-2}} + \cdots + \frac{g_1(\partial A)}{\varepsilon} + g_0(\partial A)\ln\varepsilon + s_A \tag{2.13}$$

where the functions $g_i(\partial A)$ depend only on the details of the boundary of the region A and s_A represents the finite term in the entanglement entropy. When A has a single characteristic length scale R, e.g. A is a ball of radius R, then $g_i(\partial A)$ are homogeneous functions pf degree i in R, *i.e.*, $g_i(\partial A) \propto R^i$. In general the functions g_i are cutoff dependent and as such can be termed non-physical. Except for g_0 which is cutoff independent and universal. It is a characteristic function of the CFT related to coefficients of the conformal anomaly. In a two–dimensional CFT for instance, S_A is found to be [3, 4]

$$S_A = \frac{c}{3}\ln\frac{\ell}{\varepsilon} \tag{2.14}$$

where c is the central charge of the two-dimensional CFT. Notice that the “area law” is not valid in two spacetime dimensions.

3. The replica trick.

Computing the entanglement entropy is a notoriously difficult task. A first principles, analytic calculation of entanglement entropy has been performed only for special theories and shapes/regions[1]. Aside from techniques pertaining to certain classes of theories, e.g. two–dimensional CFTs or free theories, the generic approach to computing entanglement entropy is the "replica trick". One first expresses (2.2) in terms of $\mathrm{Tr}\rho_A{}^n$ for integer n, then analytically continues the expression to generic n and finally takes the limit $n \to 1$.

To find an appropriate expression for entanglement entrpy in terms of ρ_A^n let us first rewrite S_A in terms of λ, the eigenvalues of the reduced density matrix ρ_A,

$$S_A = -\mathrm{Tr}_A \rho_A \ln \rho_A = -\sum_\lambda \lambda \ln \lambda \,. \tag{3.1}$$

Assuming that it is possible to define ρ_A^n for integer n and then analytically continue to all n, we can easily show that the following expressions are equivalent and equal to eq. (3.1)

$$S_A = -\lim_{n\to 1} n \frac{\partial}{\partial n} \mathrm{Tr}_A \rho_A^n \tag{3.2}$$

$$S_A = -\lim_{n\to 1} n \frac{\partial}{\partial n} \ln \mathrm{Tr}_A \rho_A^n \tag{3.3}$$

$$S_A = -\lim_{n\to 1} \frac{\mathrm{Tr}_A \rho_A^n}{n-1} \tag{3.4}$$

$$S_A = -\lim_{n\to 1} \left(\frac{\partial}{\partial n} - 1 \right) \ln \mathrm{Tr}_A \rho_A^n \,. \tag{3.5}$$

Before explaining how these identities can be used to compute entanglement entropy, let us see why they are true.

It is easy to understand why eq. (3.2) is valid. Simply express $\mathrm{Tr}\rho_A^n$ in terms of its eigenvalues $\mathrm{Tr}\rho_A^n = \sum \lambda^n$ to find that

$$-\lim_{n\to 1} n \frac{\partial}{\partial n} \mathrm{Tr}_A \rho_A^n = -\lim_{n\to 1} \frac{\partial}{\partial n} \sum \lambda^n = -\lim_{n\to 1} n \sum \lambda^n \ln \lambda = -\sum \lambda \ln \lambda \,. \tag{3.6}$$

Eq.(3.3) can then be proven, with the help of eq.(3.2) and the identity $\mathrm{Tr}\rho_A = 1$. Starting from (3.3) and the identity $\mathrm{Tr}\rho_A = 1$ leads to (3.4), *i.e.*,

$$\begin{aligned} -\lim_{n\to 1} \frac{\partial}{\partial n} \left(\ln \mathrm{Tr}_A \rho_A^n - n \ln \mathrm{Tr}_A \rho_A \right) &= -\lim_{n\to 1} \left(\frac{\partial}{\partial n} \ln \mathrm{Tr}_A \rho_A^n - \ln \mathrm{Tr}_A \rho_A \right) = \\ &= -\lim_{n\to 1} \left(\frac{\partial}{\partial n} - 1 \right) \ln \mathrm{Tr}_A \rho_A^n \,. \end{aligned} \tag{3.7}$$

Finally, we can show eq.(3.5), starting from eq.(3.3) and using the definition of the derivative operator

$$-\lim_{n\to 1} n \frac{\partial}{\partial n} \ln \mathrm{Tr}_A \rho_A^n = -\lim_{n\to 1} \lim_{\alpha-1\to 0} \frac{\ln \mathrm{Tr}\rho_A^{n+\alpha-1} - \ln \mathrm{Tr}\rho_A^n}{\alpha-1} = -\lim_{\alpha\to 1} \frac{\ln \mathrm{Tr}\rho^\alpha}{\alpha-1} \,. \tag{3.8}$$

[1]For results in two–dimensional CFTs the reader is advised to consult [5]. For a review in free field theories see [6] and for spherical entangling surfaces in particular [7, 8, 9].

As previously mentioned, the replica trick instructs us to compute $\mathrm{Tr}\rho_A^n$ and then use either of the identities above to evaluate S_A. We will show how to do this in the path integral formalism. For reasons of convenience, we will consider a theory living in two spacetime dimensions and a region A being the interval $x \in [u,v]$ at $\tau = 0$ in flat Euclidean space with coordinates $(x,\tau) \in \mathbb{R}^2$.

Before writting down the path integral expression for ρ_A and ρ_A^n it is useful to recall how to write the thermal density matrix for the same system at temperature $T \equiv \beta^{-1}$, namely,

$$\begin{aligned}[\rho]_{\phi\phi'} &= \rho\left(\{\phi(x)\}|\{\phi'(x')\}\right) = \\ &= Z^{-1}(\beta)\int [\mathscr{D}\phi(y,\tau)]\, e^{-S}\Pi_{x'}\delta\left(\phi(y,0)-\phi'(x')\right)\ \delta\left(\phi(y,\tau)-\phi(x)\right),\end{aligned} \tag{3.9}$$

where the Euclidean action is $S = \int_0^\beta \mathscr{L}\, d\tau\, dx$ with $\mathscr{L}$ the Lagrangian of the system. The normalization factor Z is the partition function $Z(\beta) = \mathrm{Tr}e^{-\beta H}$ which ensures that $\mathrm{Tr}\rho = 1$. Here H is the Hamiltonian of the system. The rows and columns of the thermal density matrix are labelled by the values of the fields at $\tau = 0$ and $\tau = \beta$ which are $\phi(y,0)$ and $\phi(y,\beta)$. The δ function terms in (3.9) impose precisely these "boundary conditions", effectively removing the integration over $\{\phi(x),\phi'(x')\}$. The partition function Z can be obtained from the same path integral by setting $\{\phi(x)\} = \{\phi'(x')\}$ and integrating over $\phi(x)$. In the path integral, this has the effect of sewing the edges of the space along $\tau = 0$ and $\tau = \beta$ to form a cylinder.

Consider now the *reduced* density matrix ρ_A. It is straighforward to write down a path integral expression for ρ_A starting from (3.9). One takes the limit $\beta^{-1} \to 0$ and integrates over all $\{\phi(x)\} = \{\phi'(x')\}$ but only for those points $x \in A^c$. This process has the effect of producing a cut along the interval $A = (u,v)$ at $\tau = 0$. The boundary conditions we need to impose are now at the points $\tau^+ = 0^+$ and $\tau^- = 0^-$.

$$\begin{aligned}[\rho_A]_{\phi\phi'} &= \left(\{\phi(x)\}|\{\phi'(x')\}\right) = \\ &= Z_1^{-1}\int_{-\infty}^{+\infty} [\mathscr{D}\phi(y,\tau)]\, e^{-S}\,\Pi_{x\in A}\,\delta\left(\phi(y,0^+)-\phi_0(x)\right)\ \delta\left(\phi(y,0^-)-\phi_0(x')\right),\end{aligned} \tag{3.10}$$

where Z_1 denotes the vaccuum partition function on $\mathbb{R}^2$ and ensures that $\mathrm{Tr}_A\rho_A = 1$ (see Fig.1a).

To construct ρ_A^n we make n-copies of (3.10)

$$(\rho_A)_{\phi_1\phi_1'}(\rho_A)_{\phi_2\phi_2'}\cdots(\rho_A)_{\phi_n\phi_n'}, \tag{3.11}$$

where we identify ϕ_{i-1}' with ϕ_i and subsequently integrate over ϕ_i. Next we need to evaluate $\mathrm{Tr}_A\rho_A^n$. Taking the trace amounts to identifying ϕ_n' with ϕ_1 in (3.11) and integrating over all $\phi_1(x)$. In this way, $\mathrm{Tr}\rho_A^n$ is computed from a path integral on an n-sheeted Riemann surface $\mathscr{R}_n$. This is described pictorially in Fig.1b.

$$\mathrm{Tr}\rho_A^n = Z_1^{-n}\int_{\mathscr{R}_n}[\mathscr{D}\phi]\, e^{-S} \equiv \frac{Z_n}{(Z_1)^n} \tag{3.12}$$

We can now compute the entanglement entropy using for example, (3.5),

$$S_A = \lim_{n\to 1}\left(\frac{\partial}{\partial n}-1\right)\ln\frac{Z_n}{(Z_1)^n} \tag{3.13}$$

In practice, to evaluate the path integral on $\mathscr{R}_n$ one usually defines *twist* operators ($\mathscr{T}_n, \mathscr{T}_{-n} = \mathscr{T}_n^\dagger$), with the help of which $\mathrm{Tr}\rho_A^n$ can be written as a two-point function $\mathrm{Tr}\rho_A^n = \langle\mathscr{T}_n\mathscr{T}_{-n}\rangle$ (see [5] and

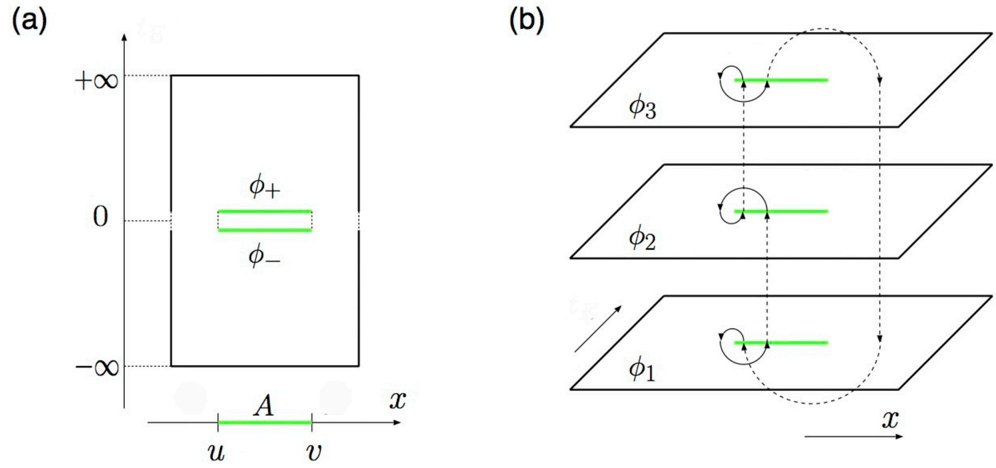

Figure 1: (a) depicts the reduced density matrix ρ_A in the path integral representation. (b) is a pictorial representation of $\mathscr{R}_n$. This picture is reproduced here from [12].

references therein for more details on twist operators). We would like to finish this section with a disclaimer; although the replica trick is assumed to be valid in arbitrary dimensions, it can only be rigorously proven in two spacetime dimensions. Nevertheless, there is strong evidence that it is true in higher dimensions (particularly from holography) and we will assume so in the following.

4. Holographic entanglement entropy.

Let us consider a d–dimensional CFT with a dual description in terms of Einstein-Hilbert gravity in AdS$_{d+1}$ spacetime (for a review on AdS/CFT see for example [10]). It is natural to expect that entanglement entropy can be computed holographically as a geometrical quantity, in a similar manner to thermal entropy. This is why the authors of [11, 12] conjectured that the entanglement entropy of a spatial[2] region A on the boundary of AdS is given by

$$S_A = \frac{1}{4G_N^{(d+1)}} \int_\Sigma \sqrt{\sigma} \tag{4.1}$$

where Σ is defined as the minimal area surface which asymptotes to the boundary of the spatial region A, (∂A).

For spherical regions in holographic CFTs, the presence of a $U(1)$ symmetry, allows one to map the entanglement entropy to the horizon entropy of hyperbolic black holes, and explicitly prove the conjecture [14]. In the general case, $U(1)$ symmetry is absent and the conformal map exploited in [14] does not hold. Recently, however, Lewkowycz and Maldacena devised a method to calculate the gravitational entropy of a region without $U(1)$ symmetry [1]. With the help of the replica trick they succeeded in proving the Ryu-Takayanagi conjecture.

Several properties of entanglement entropy are immediately obvious from Eq.(4.1). For instance, when the spatial region A extends to the whole of space, entanglement entropy coincides with statistical entropy. At finite temperature eq. (4.1) naturally reduces to the Bekenstein-Hawking

[2]The generalization to the covariant case is discussed in [13].

entropy formula whereas for vanishing temperature, the dual gravitational description contains no horizon and the entropy vanishes as it should. Other properties of the entanglement entropy like strong subadditivity [15] or the fact that A and its complement A^c have the same entropy, are also satisfied by the holographic EE formula.

4.1 The holographic entanglement entropy of a straight belt.

In this section we will compute the entanglement entropy of a d-dimensional CFT for a belt-shaped region A of infinite length H and finite width ℓ using holography. The CFT lives on the d–dimensional flat spacetime boundary of AdS_{d+1}. We start by writting the ambient AdS_{d+1} metric in a convenient parametrization, taking into account the symmetries of the problem. Here it is simply

$$ds^2 = \frac{dz^2 - dt^2 + dx^2 + \sum_i dx_i^2}{z^2}, \quad i = 1, 2, \cdots, d-2. \tag{4.2}$$

where $z = 0$ corresponds to the boundary of AdS_{d+1} and we set the radius of curvature of AdS to unity, $L_{AdS} = 1$. The belt has width ℓ along the x–direction and is infinitely extended along the rest of the boundary spatial dimensions x_i.

The induced metric $h_{\mu\nu}$ on a static surface within a bulk spacetime with metric $g_{\mu\nu}$ is given by

$$h_{\mu\nu} = \frac{\partial x^\rho}{\partial X^\mu}\frac{\partial x^\sigma}{\partial X^\nu} g_{\rho\sigma}, \tag{4.3}$$

where x^μ are the ambient spatial coordinates and X^μ the embedding ones. For the entangling surface of the belt, symmetry allows to parametrize the surface by a single function $x(z)$ and choose embedding coordinates $X^\mu \equiv (z, x_2, \cdots, x_d)$ such that

$$ds_A^2 = \frac{\sum_i dx_i^2}{z^2} + \frac{dz^2}{z^2}\left[1 + \left(\frac{\partial x_1}{\partial z}\right)^2\right]. \tag{4.4}$$

Substituting (4.4) in (4.1) leads to

$$S_A = \frac{1}{4G_N^{d+1}} \int d^{d-2}x\, dz \frac{\sqrt{1+\dot{x}_1^2}}{z^{d-1}} = \frac{H^{d-2}}{4G_N^{d+1}} \int dz \frac{\sqrt{1+\dot{x}_1^2}}{z^{d-1}}, \tag{4.5}$$

Solving the equation of motion following from (4.5) determines the entangling surface

$$\frac{\dot{x}_1}{z^{d-1}\sqrt{1+\dot{x}_1^2}} = const. \quad \Rightarrow \quad \dot{x}_1^2 = \frac{z^{2(d-1)}}{z_*^{2(d-1)} - z^{2(d-1)}} \tag{4.6}$$

where we conveniently expressed the constant of motion as $c = z_*^{-d+1}$. The entangling surface dips into the bulk all the way to z_* which can be expressed in terms of the characteristic width ℓ of the belt as follows

$$\begin{aligned} \frac{\ell}{2} &= \int_0^{\frac{l}{2}} dx_1 = \int_0^{z_*} dz \dot{x}_1 = \int_0^{z_*} dz \frac{z^{d-1}}{\sqrt{z_*^{2(d-1)} - z^{2(d-1)}}} = z_* \int_0^1 dy \frac{y^{d-1}}{\sqrt{1-y^{2(d-1)}}} \Rightarrow \\ \Rightarrow \quad \frac{\ell}{2} &= z_* \frac{\sqrt{\pi}\,\Gamma\left[\frac{d}{2(d-1)}\right]}{\Gamma\left[\frac{1}{2(d-1)}\right]} \end{aligned}. \tag{4.7}$$

The final step in computing the entanglement entropy is to evaluate (4.5) on the the entangling surface (4.6)

$$\begin{aligned} S_A &= 2\frac{H^{d-2}}{4G_N^{d+1}} z_*^{-d} \int_0^1 dy \left(1-y^{2(d-1)}\right)^{-\frac{1}{2}} y^{-(d-1)} = \\ &= 2\frac{H^{d-2}}{4G_N^{d+1}} z_*^{-d} \left\{ \int_\varepsilon^1 \frac{dy}{y^{d-1}} + \int_0^1 \frac{dy}{y^{d-1}} \left[\left(1-y^{2(d-1)}\right)^{-\frac{1}{2}} - 1\right] \right\}, \end{aligned} \tag{4.8}$$

where we isolated the area divergent term and are left to compute

$$I_0 = \int_0^1 \frac{dy}{y^{d-1}} \left[\left(1-y^{2(d-1)}\right)^{-\frac{1}{2}} - 1\right] = \frac{1}{d-2} - \frac{\sqrt{\pi}}{d-2} \frac{\Gamma\left[\frac{d}{2(d-1)}\right]}{\Gamma\left[\frac{1}{2(d-1)}\right]}. \tag{4.9}$$

Using (4.9) leads to the final result for the entanglement entropy of a straight belt

$$S_A = \frac{1}{4G_N^{d+1}} \left[\frac{2}{d-2} \left(\frac{H}{\varepsilon}\right)^{d-2} - \frac{2^{d-1}\pi^{\frac{d-1}{2}}}{d-2} \left(\frac{\Gamma\left[\frac{d}{2(d-1)}\right]}{\Gamma\left[\frac{1}{2(d-1)}\right]} \right)^{d-1} \left(\frac{H}{\ell}\right)^{d-2} \right]. \tag{4.10}$$

Restoring the units, we see that $S_A \propto \frac{L_{AdS}^{d-1}}{G_N^{d+1}}$. For the case of $\mathcal{N}=4$ SU(N) Super Yang Mills (SYM) theory we can express the overal coefficient purely in terms of gauge theory parameters. From John Estes's lecture notes in the same volume, we learn that graviton scattering determines the ten–dimensional Newton's constant to be $G_N^{(10)} \propto \alpha'^4 g_s^2$, where α' is related to the string tension and g_s denotes the string coupling constant. The five dimensional Newton's constant is then given by

$$G_N^{(5)} = \frac{G_N^{(10)}}{L_{AdS}^5 Vol(S^5)} = \frac{G_N^{(10)}}{L_{AdS}^5 Vol(S^5)}, \tag{4.11}$$

where we used the fact that the S^5 radius is equal to that of AdS_5 and we applied the fomula for the volume of hyperspheres

$$Vol(S^n) = \frac{2\pi^{\frac{n}{2}}}{\Gamma[\frac{n}{2}]}. \tag{4.12}$$

Using eq. (4.11) and the expression for the radius of AdS in terms of the gauge coupling $L_{AdS}^4 = \alpha'^2 g_s N$ we find that

$$\frac{L_{AdS}^3}{G_N^{(5)}} = \frac{L_{AdS}^8 \pi^3}{G_N^{(10)}} \propto N^2, \tag{4.13}$$

Let us finish this section by making the following comments on (4.10), which represents the entanglement entropy of a belt for a conformal gauge theory at strong 't Hooft coupling $\lambda = g_{YM} N^2$ and large N with a dual gravitational description:

- The expected area divergent term is the first term in brackets in (4.10). Observe that there is no $\ln\varepsilon$-term. This implies that the finite term which behaves like $\left(\frac{H}{\ell}\right)^{d-2}$ is physical and independent of the cutoff ε.
- For the case of $\mathcal{N}=4$ SYM theory, where a precise dictionary exists, we find that the entanglement entropy is independent of the 't Hooft coupling constant and proportional to N^2, the number of degrees of freedom.

PoS(Modave 2013)003

4.2 The holographic entanglement entropy of a circular disk.

Here we sketch the computation of entanglement entropy for a spherical region of radius R. The appropriate parametrization of AdS_{d+1} in this case is

$$ds^2 = \frac{dz^2 - dt^2 + dr^2 + r^2 d\Omega_{d-2}^2}{z^2}\,. \tag{4.14}$$

Symmetry allows for an entangling surface with a profile of the type $z(r)$ so we choose embedding coordinates $X^\mu = (r, \theta, \phi_1, \cdots, \phi_{d-2})$ and express the induced metric on the surface as follows

$$ds_{EE}^2 = \frac{dr^2}{z^2}\left[1 + \left(\frac{\partial z}{\partial r}\right)^2\right] + \frac{r^2}{z^2} d\Omega_{d-1}^2\,. \tag{4.15}$$

Substituting (4.15) into (4.1) yields

$$S_A = \int d^{d-1}X\sqrt{h} = \int d\Omega_{d-2}\frac{dr}{r}\frac{r^{d-1}}{z^{d-1}}\,. \tag{4.16}$$

The equations which determine the entangling surface then read

$$rz\ddot{z} + (d-1)z\dot{z}^2 + (d-1)z\dot{z} + dr\dot{z}^2 + dr = 0 \quad \Rightarrow \quad z^2(r) = R^2 - r^2\,. \tag{4.17}$$

Substituting the solution (4.17) into (4.16) we evaluate the area of the entangling surface by firstly isolating the divergences and then computing the finite part of the integral. The result is

$$S_B = \frac{L_{AdS}^{d-1}}{G_N^{d+1}}\left[g_{d-2}\left(\frac{R}{\varepsilon}\right)^{d-2} + g_{d-3}\left(\frac{R}{\varepsilon}\right)^{d-3} + \cdots + \begin{cases} g_1\left(\frac{R}{\varepsilon}\right) + g_0, & d = odd \\ g_2\left(\frac{R}{\varepsilon}\right)^2 + g_1 \ln\frac{R}{\varepsilon} + g_0, & d = even \end{cases}\right]. \tag{4.18}$$

The leading divergent term in (4.18) is proportional to the area of the spherical region in both even and odd–dimensional CFTs. However, the universal, physical terms in the entanglement entropy depend on the dimensionality of the spacetime where the field theory lives. In even spacetimes, a logarithmically divergent term appears and its coefficient represents the universal contribution to entanglement entropy. As we will show in section 5.2, the coefficient g_1 is indeed physical and is proportional to the coefficient a of the euler term in the conformal anomaly in $d = 4$ dimenions. This is important since a has a special behavior along renormalization group flows [16] – its value in the UV fixed point CFT is always greater than that of the IR fixed point. In odd dimensional spacetimes, the logarithmically divergent term is absent and the universal term coincides with the finite term, g_0, in entanglement entropy. g_0 is related to the partition function of the CFT on a sphere [14] and it has been conjectured [17] and shown [18] to decrease along RG flows in $d = 3$ spacetime dimensions.

4.3 The holographic entanglement entropy of a straight belt at finite temperature.

The final example we will study in this section is the entanglement entropy of an infintely long belt for a CFT at finite temperature. Turning on temperature in the field theory side corresponds to changing the background spacetime from pure AdS$_{d+1}$ to the AdS$_{d+1}$ Schwarzschild black hole [10]. We set $d=4$ for convenience. The black hole metric is

$$ds^2 = L_{AdS}^2 \left[\frac{du^2}{h(u)\,u^2} + u^2 \left(h(u)dt^2 + \sum_{i=1}^{2} dx_i^2 \right) + dx^2 \right] \quad \text{where} \quad h(u) = 1 - \frac{u_0^4}{u^4}, \tag{4.19}$$

and the position of the horizon is related to the temperature of the dual CFT through $u_0 = \pi T$. The entangling surface can be parametrized by a single function $u(x)$ and the induced metric on the surface reads

$$ds_A^2 = L_{AdS}^2 u^2 \left[\left(1 + \frac{1}{h(u)u^4} \right) dx^2 + dx_2^2 + dx_3^2 \right]. \tag{4.20}$$

Substituting (4.20) into (4.1) results in

$$S_A = \frac{L_{AdS}^3}{4G_N^{(5)}} H^2 \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} dx\, u^3 \sqrt{1 + \frac{u'^2}{u^4 - u_0^4}}. \tag{4.21}$$

The integrand in (4.21) does not depend explicitly on x and solving for the profile $u(x)$ yields

$$u'(x) = \left[(u^4 - u_0^4) \left(\frac{u^6}{u_*^6} - 1 \right) \right]^{\frac{1}{2}}. \tag{4.22}$$

u_* is related to the width ℓ of the belt through

$$\begin{aligned} \frac{\ell}{2} &= \int_{u_*}^{\infty} \frac{du}{(u^4 - u_0^4)^{\frac{1}{2}} \left(\frac{u^6}{u_*^6} - 1 \right)^{\frac{1}{2}}} = \\ &= \frac{1}{u_*} \int_0^1 dy\, y^3 (1 - y^6)^{-\frac{1}{2}} (1 - y^4 b^4)^{-\frac{1}{2}} = \\ &= \frac{1}{u_*} I_0(b), \end{aligned} \tag{4.23}$$

where we defined a new variable $y \equiv \frac{u_*}{u}$ and parameter $b \equiv \frac{u_0}{u_*}$ and denoted by $I_0(b)$ the integral on the second line of (4.23). Physical quantities of the dual field theory will depend on the dimensionless ratio $\ell \pi T = \ell u_0 = 2bI_0(b)$.

Evaluating (4.21) on the solution (4.22) yields

$$\begin{aligned} S_A &= \frac{L_{AdS}^3}{G_N^{(5)}} 2H^2 \int_{u_*}^{\infty} du \frac{u^6}{(u^4 - u_0^4)^{\frac{1}{2}} (u^6 - u_*^6)^{\frac{1}{2}}} = \\ &= \frac{L_{AdS}^3}{G_N^{(5)}} 2H^2 u_*^2 \int_0^1 dy\, y^{-3} (1 - y^6)^{-\frac{1}{2}} (1 - b^4 y^4)^{-\frac{1}{2}} = \\ &= \frac{L_{AdS}^3}{G_N^{(5)}} 2H^2 \frac{4I_0^2(b)}{\ell^2} I(b), \end{aligned} \tag{4.24}$$

where we defined

$$I(b) \equiv \int_0^1 dy y^{-3} (1 - y^6)^{-\frac{1}{2}} (1 - b^4 y^4)^{-\frac{1}{2}}. \tag{4.25}$$

The entanglement entropy of a straight belt at finite temperature is then expressed as

$$S_s = \frac{L_{AdS}^3}{4G_N^{(5)}}\,\ell H^2\,\frac{8I_0(b)^2 I(b)}{\ell^3}, \qquad \Rightarrow \qquad S_s \propto N^2 \ell H^2 u_0^3 \left(\frac{I(b)}{b^3 I_0(b)}\right). \tag{4.26}$$

In the final expression we have rewritten the entanglement entropy in terms of the physical variables of the dual CFT. It is interesting to study the behavior of (4.26) at high temperatures, *i.e.*, when $b \simeq 1$ and $\ell \simeq H$. In the limit $b \to 1$ the integrals $I_0(b)$ and $I(b)$ exhibit the same kind of divergent behavior close to $y \simeq 1$. This observation, assuming the UV divergent behavior of $I(b)$ at $y \simeq 0$ is regulated, leads to

$$\lim_{b\to 1} \frac{I(b)}{b^3 I_0(b)} \to 1\,, \tag{4.27}$$

which implies an extensive result for entanglement entropy, namely, $S_s \sim N^2 \ell H^2 T^3$. The entanglement entropy is proportional to the volume of the belt ℓH^2. This result is in agreement with field theoretic expectations. It is clear from the definition of entanglement entropy that it includes a contribution from thermal entropy. This behavior implies that S_A will have a very differnet behavior before and after the confinement/deconfinement transition and can be used as an order parameter [19]. Finally, the fact that entangelement entropy does not only account for quantum fluctuations, is a generic feature of systems in a mixed state. In these cases, other measures of entanglement must be used (e.g. entanglement negativity [20, 21]).

5. Entanglement entropy and conformal anomaly.

Consider a $(d+1)$–dimensional CFT in flat space and a smooth, connected entangling surface A which depends on a single characteristic scale R, e.g. a spherically shaped region. The change of the entanglement entropy under a rescaling of the radius is equal to

$$R\frac{d}{dR}S_A(R) = -\lim_{n\to 1}\left(\frac{\partial}{\partial n} - 1\right) R\frac{d}{dR}\ln Z_n\,, \tag{5.1}$$

where $Z_n = \mathrm{Tr}\rho_A^n$ denotes the partition function of the n–sheeted space $\mathscr{R}_n$. Clearly, a rescaling of R is equivalent to a Weyl rescaling of the metric of R_n, at least for $n=1$. Here we will assume that it is the same for any $n \geq 1$. Under this assumption we can rewrite (5.1) as follows

$$R\frac{d}{dR}S_A(R) = -\lim_{n\to 1}\left(\frac{\partial}{\partial n} - 1\right) 2\int d^d x\, g^{\mu\nu}\frac{\delta}{\delta g^{\mu\nu}}\ln Z_n\,, \tag{5.2}$$

The variation of the finite part of the effective action under Weyl rescalings is related to the expectation value of the stress energy tensor

$$-2g^{\mu\nu}\frac{\delta}{\delta g^{\mu\nu}}\ln Z = \sqrt{g}\, g_{\mu\nu}\langle T^{\mu\nu}\rangle \tag{5.3}$$

which combined with (5.2) leads to

$$R\frac{d}{dR}S_A^{fin.}(R) = \lim_{n\to 1}\left(\frac{\partial}{\partial n} - 1\right)\int_{R_n} d^{d+1}x\sqrt{g}\, g^{\mu\nu}\langle T_{\mu\nu}\rangle\,. \tag{5.4}$$

In odd spacetime dimensions there is no conformal anomaly, *i.e.*, $\langle T^{\mu}_{\mu} \rangle = 0$, and thus the finite part of entanglement entropy is a constant, independent of the radius[3]. On even dimensional spacetimes however, the conformal anomaly does not vanish. In fact, the conformal anomaly of a CFT on a manifold $\mathscr{M}$ is equal to [22, 23]

$$\int_{\mathscr{M}} d^{d+1}x\sqrt{g} g^{\mu\nu} \langle T_{\mu\nu} \rangle = \int_{\mathscr{M}} d^{d+1}x\sqrt{g} \left(\sum C_i I_i - (-1)^{\frac{d}{2}} A E_d \right) . \tag{5.5}$$

where E_d is the d–dimensional Euler density, a topological invariant, whereas I_i are i conformal (Weyl) invariants whose total number depends on the dimensionality of the spacetime $(d+1)$. The invariants are constructed out of the curvature tensors of spaces of $\frac{d}{2}$ or $(\frac{d+1}{2} - k)$–dimensionality and $2k$ covariant derivatives. C_i and A are the anomaly coefficients[4].

The simplest example of a conformal anomaly is in $d = 2$ dimensions, where

$$g^{\mu\nu} \langle T_{\mu\nu} \rangle = \frac{c}{12} R \tag{5.6}$$

In $d = 4$ dimensions on the other hand, the conformal anomaly reads

$$\begin{aligned} g^{\mu\nu} \langle T_{\mu\nu} \rangle &= \frac{1}{64\pi^2 90} (c I_4 - a E_4) \\ I_4 &= W^2_{\mu\nu\rho\sigma} = R^2_{\mu\nu\rho\sigma} - 2R^2_{\mu\nu} + \frac{1}{3} R^2 \\ E_4 &= R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2 \end{aligned} \tag{5.7}$$

It is interesting to remark that for $\mathscr{N} = 4$ SYM, the anomaly coefficients (c, a) are equal and furthermore $c = a \propto N^2$. The same is true, namely $c = a$, for any CFT with a dual gravitational description based on Einstein gravity.

It is clear from (5.5) that in flat space where curvature tensors vanish identically, conformal invariance is not anomalous and $\langle T_{\mu\nu} \rangle = 0$. However, $\langle T_{\mu\nu} \rangle \neq 0$ for an even dimensional manifold $\mathscr{R}_n$ with conical singularities. A rescaling of the characteristic length scale R then relates the finite part of the entanglement entropy in even d–dimensions to the integrated anomaly on $\mathscr{R}_n$,

$$R \frac{d}{dR} S^{fin.}_A(R) = \lim_{n \to 1} \left(\frac{\partial}{\partial n} - 1 \right) \int_{\mathscr{R}_n} d^{d+1}x\sqrt{g} \left(\sum C_i I_i - (-1)^{\frac{d}{2}} A E_d \right) . \tag{5.8}$$

This is the main result of this subsection. A natural question arises: can we explicitly compute the integral in (5.8)? To do so, we must know how to deal with integrals on manifolds with conical singularities. In what follows we will describe some basic techniques for doing so.

5.1 Integrals of curvature tensors on manifolds with conical singularities.

Methods for dealing with integrals of curvature invariants *e.g.*, $\int R^2$, $\int R^2_{\mu\nu\rho\sigma}, \cdots$ on manifolds with conical singularities have been developed over the course of several years starting with the work of [24]. Here we will review the basic ideas; for more details the reader is advised to consult [25, 26, 27] and references therein as well as [2] for a recent development on the subject. We will start by discussing the case of two–dimensional spaces, and then proceed with the generalization to arbitrary dimensions.

[3] Obviously, the variation under a rescaling of the characteristic length R is opposite to the variation under a rescaling of the cutoff.

[4] We did not include possible regularization scheme dependent terms in 5.5.

5.1.1 Two–dimensional cones.

The metric (rather pseudometric) of a two-dimensional cone C_n in polar coordinates is simply the flat space metric with an angular variable of wrong periodicity, e.g.

$$ds^2 = dr^2 + r^2 d\tau^2, \quad \text{with} \quad \tau \in (0, 2\pi n)\,. \tag{5.9}$$

A 'smooth" cone can be produced with a change in the periodicity of the τ–cooordinate or equivalently by changing the metric component g_{rr} in (5.9) to $g_{rr} = n$, *i.e.*,

$$ds^2 = n^2 dr^2 + r^2 d\tau^2\,. \tag{5.10}$$

Let us write the generic form of a two–dimensional space M_n with the topology of a cone as

$$ds^2_{M_n} = e^{\sigma(r)}\left(dr^2 + r^2 d\tau^2\right) \equiv e^{-\sigma(r)} ds^2_{C_n}, \quad \text{with} \quad \tau \in (0, 2\pi n)\,. \tag{5.11}$$

The conformal factor behaves in the vicinity of $r = 0$ as

$$\sigma(r) \simeq_{r\to 0} \sigma_1(\tau) r^2 + \sigma_2(\tau) r^4 + \mathscr{O}(r)\,, \tag{5.12}$$

with the σ_i being arbitrary functions of the angular variable. A constant term is absent from (5.12) because it can be asborbed with a redefinition of the radial variable r. The asymptotics of the conformal factor (5.12) ensure that the conical singularity is due to the cone C_n (we will actually have an example with a different parametrization in the following, where this is not true).

We would like to compute the integral $I = \int \sqrt{g} R$ on the manifold with metric (5.11). Since the space is singular we will regularize it and compute it on the regularized space. We will then send the regularization parameter to zero and check if the result is independent of the regulator. The singularity in (5.11) resides in C_n, our task is thus to find a smooth analog for C_n. This is simple. A regulated geometry $\widetilde{C}_n$ can be produced by replacing g_{rr} of C_n in a region $r \in (0, r_0)$ close to the singularity, with an appropriate function $f_n(r, b)$ such that $\lim_{a\to 0} f_n(r, a) = 1$ and $\lim_{r\to 0} f_n(r, a) = n$. The first requirement comes from demanding that the original metric is reproduced when the regulator is send to zero and the second from demanding that the geometry is smooth (as indicated by (5.10)) in the presence of the regulator. There is of course a plethora of functions of this type. Some of the choices which appeared in the literature already (e.g. [26], [1]) are the following

$$f_n(r, a) = \frac{r^2 + b^2 n^2}{r^2 + b^2} \tag{5.13}$$

$$f_n(r, a) = 1 + (n-1) e^{-\frac{r^2}{a^2}}\,. \tag{5.14}$$

We can now pick the regulator of our choice and evaluate the Ricci scalar curvature on the regularized metric of $\widetilde{M}$ given by

$$ds^2_{\widetilde{M}} = e^{\sigma} ds^2_{\widetilde{C}} \equiv e^{\sigma}\left(f_n(r, a) dr^2 + r^2 d\tau^2\right)\,. \tag{5.15}$$

If $R_{\widetilde{C}}$ denotes the Ricci scalar on the regularized cone then

$$R_{\widetilde{M}} = e^{-\sigma} R_{\widetilde{C}} - e^{-\sigma} (\nabla^2_{\widetilde{C}} \sigma), \quad \text{with} \quad R_{\widetilde{C}} = \frac{f_n'}{r f_n^2}\,. \tag{5.16}$$

Finally, we have that

$$\begin{aligned}\int \sqrt{g} R_{\widetilde{M}} &= \int_0^\infty dr\, r \int_0^{2\pi n} d\tau\, e^{-\sigma} e^{\sigma} f_n^{\frac{1}{2}} r \frac{f_n'}{r f_n^2} - \int_0^\infty \int_0^{2\pi n} dr\, r\, d\tau \sqrt{f_n} \left(\nabla^2_{\widetilde{C}} \sigma\right) = \\ &= 2\pi n \int_0^\infty dr f_n' f_n^{-\frac{3}{2}} - \int_0^\infty \int_0^{2\pi n} dr\, r\, d\tau \sqrt{f_n} \left(\nabla^2_{\widetilde{C}} \sigma\right) = \\ &= -4\pi(n-1) - \int_0^\infty \int_0^{2\pi n} dr\, r\, d\tau \sqrt{f_n} \left(\nabla^2_{\widetilde{C}} \sigma\right) .\end{aligned} \tag{5.17}$$

Notice that the first term is independent of the regularization parameter a. The second term on the other hand is finite when $a \to 0$. Moreover, it coincides with the integral of the Ricci scalar computed on the smooth domain M_n/Σ, where Σ is the singular set at $r = 0$ (in our two–dimensional example Σ is simply the point $r = 0$). When $a \to 0$ we thus find

$$\lim_{a\to 0} \int_{\widetilde{M}_n} R = -4\pi(n-1) + \int_{M_n/\Sigma} R \,. \tag{5.18}$$

Even though (5.18) is derived in two spacetime dimensions, it is true for arbitrary dimensions. In higher dimensions however it is possible to consider arbitrary polynomials of curvature invariants and their integrals, the computation of which might be more subtle. In particular, the result is not in general independent of the regularization parameter.

5.1.2 Higher dimensional cones.

Consider a static, Euclidean spacetime M_n with conical singularities at $r = 0$ and an angular coordinate ranging from $\tau \in (0, 2\pi n)$. The singular codimension two surface is denoted by Σ. The manifold M_n near the conical singularity has locally the structure of a product space $C_n \times \Sigma$ where C_n is the two–dimensional cone. When this is true globally as well, the cone is called symmetric and is invariant under $U(1)$ rotations. Equivalently, Σ can be embedded in M_n with vanishing extrinsic curvatures $K^{(i)}$. Otherwise, M_n is called a squashed cone and the geometry near Σ is a warped product of C_n and Σ.

To treat the higher dimensional case, it is useful to consider a different parametrization of the two–dimensional cone (5.9) for which

$$ds^2_{C_n} = \rho^{2(n-1)} \left(d\rho^2 + \rho^2 d\phi^2\right), \quad \text{with} \quad \phi \in (0, 2\pi) \,. \tag{5.19}$$

As shown in [2] this parametrization allows for a unified regularization scheme for both types of cones. Given this and that the symmetric cone is a special case of the squashed one, we will focus on the latter.

In the vicinity of the singularity $\rho = 0$ the metric of the squashed cone M_n takes the form

$$\begin{aligned} ds^2 &= \rho^{2(n-1)} \left(d\rho^2 + s(\rho) d\phi^2\right) + \left(\gamma_{\mu\nu}(u,\rho) + 2K^{(i)}_{\mu\nu} x^{(i)}\right) du^i du^j , \quad \phi \in (0, 2\pi) \\ s(\rho) &= \rho^2 + \mathscr{O}(\rho^4), \quad \gamma_{\mu\nu}(u,\rho) = \gamma_{\mu\nu}(u) + \mathscr{O}(\rho^2) \,. \end{aligned} \tag{5.20}$$

Here x^i with $i = 1, 2$ are defined as

$$x^1 = r \cos\tau = \frac{\rho^n}{n} \cos n\phi , \quad x^2 = r \sin\tau = \frac{\rho^n}{n} \sin n\phi \,, \tag{5.21}$$

and $K^{(i)}_{\mu\nu}$ denotes the extrinsic curvature of Σ, *i.e.*,

$$K^i_{\mu\nu} = h^\lambda_\mu h^\rho_\nu \left(\nabla_\rho n^{(i)}_\lambda\right), \qquad h^\lambda_\mu = \delta^\lambda_\mu - \sum_i (n_{(i)})_\mu (n_{(i)})^\lambda \,. \tag{5.22}$$

For the symmetric cone the extrinsic curvature term is absent and there is no linear term in the expansion of the metric on Σ close to $\rho = 0$.

The regularized geometry $\widetilde{M}_n$ can be simply written as follows

$$ds^2 = e^{2A_n(\rho,a)}\left(d\rho^2 + s(\rho)d\phi^2\right) + \left(\gamma_{\mu\nu}(u,\rho) + 2K^{(i)}_{\mu\nu}x^{(i)}\right) du^i du^j \,, \tag{5.23}$$

A convenient choice for the warp factor with regularization parameter a is

$$A = \frac{n-1}{2}\ln\left(\rho^2 + a^2\right). \tag{5.24}$$

Suppose now that we want to evaluate the integral of an invariant $\mathscr{R}$, which is an order m polynomial of components of the Riemann tensor. Just like in two dimensions, we replace (5.20) with (5.23) and compute the integral. In the limit $a \to 0$ the integral behaves like [27]

$$\int_{\widetilde{M}_n} d^dx\sqrt{g}\mathscr{R} = \frac{A_k}{a^k} + \frac{A_{k-1}}{a^{k-1}} + \cdots + A_0 + \mathscr{O}(a) \tag{5.25}$$

where the power of the leading divergence k depends on the order m of the polynomial $\mathscr{R}$. It can be shown that for static spacetimes in the limit $n \to 1$ the leading behavior is given by A_0 which is proportional to $(n-1)$ while all other terms A_k with $k > 0$ are of order $\mathscr{O}((n-1)^2)$. Hence, in the limit $n \to 1$ the result is independent of the regulator. It is this amazing fact which allows us to explicitly calculate integrals of the type (5.25) relevant for entanglement entropy, either from the CFT (e.g. [28]) or the holographic side (e.g. [27, 2]).

In particular, let us consider (5.8) for a four–dimensional CFT, *i.e.*,

$$R\frac{d}{dR}S^{fin.}_A(R) = \lim_{n\to 1}\left(\frac{\partial}{\partial n} - 1\right)\int_{M_n} d^{d+1}x\sqrt{g}\,\frac{1}{90\times 64\pi^2}\left(cI_2 - aE_4\right). \tag{5.26}$$

Using the technique illustrated above one finds that

$$\begin{aligned}\int_{\tilde{M}_n} d^4x\sqrt{g}E_4 &= n\int_{M_1} d^4x\sqrt{g}E_4 + 8\pi(1-n)\int_\Sigma d^2y\sqrt{\gamma}R_\Sigma + \mathscr{O}\left((n-1)^2\right)\\ \int_{\tilde{M}_n} d^4x\sqrt{g}I_4 &= n\int_{M_1} d^4x\sqrt{g}I_4 + 8\pi(1-n)\int_\Sigma d^2y\sqrt{\gamma}K_\Sigma + \mathscr{O}\left((n-1)^2\right),\end{aligned} \tag{5.27}$$

where R_Σ is the induced Ricci scalar on Σ, expressed in terms of the ambient spacetime Riemann and Ricci curvature tensors and the extrinsic curvature K as

$$R_\Sigma = R - 2R_{ii} - R_{ijij} + K^2 - \mathrm{Tr}\left(K^2\right), \tag{5.28}$$

and K_Σ is a conformal invariant defined as follows

$$K_\Sigma = \left(R_{ijij} - R_{ii} + \frac{1}{3}R\right) - \left(\mathrm{Tr}K^2 - \frac{1}{2}K^2\right). \tag{5.29}$$

Here R_{ijij}, R_{ii} are the normal projections of the curvature tensors

$$R_{ijij} = R_{abcd} n_i^a n_j^b n_i^c n_j^d, \qquad R_{ii} = R_{ab} n_i^a n_i^b, \tag{5.30}$$

and $\mathrm{Tr}K^2 = \sum_i K^i_{\mu\nu}(K^i)^{\mu\nu}$. Finally, substituting (5.27) into (5.26) yields

$$R\frac{d}{dR} S_A^{fin.}(R) = \frac{1}{90\times 64\pi^2}\left(8\pi a\int_\Sigma R_\Sigma - 8\pi c\int_\Sigma K_\Sigma\right) \tag{5.31}$$

We will conclude this section by explicitly computing the right hand side of (5.26) in a couple of special cases; that of a spherical region of radius R and of an infinitely long cylinder of width R.

5.2 The universal term in the entanglement entropy of a circular disk.

We consider a CFT living on $\mathbb{R}^4$ which immediately implies that $R_{ijij} = R_{ii} = R = 0$. Moreover, $K_\Sigma = 0$ for the sphere. We thus only need to compute R_Σ. We start by writting the metric of S^2 as

$$d\Omega_2^2 = R^2(d\theta^2 + \sin^2\theta d\phi^2)\,. \tag{5.32}$$

The normals to Σ in flat space and the associate extrinsic curvature tensors are then

$$\left.\begin{array}{l} n^t = 1 \Rightarrow K^t = 0 \\ n_2^r = 1 \Rightarrow K^{(2)}_{\theta\theta} = K^{(2)}_{\phi\phi} = R\sin^2\theta \end{array}\right\} \Rightarrow \qquad \mathrm{Tr}K^2 = \frac{2}{R^2}, \qquad K^2 = \frac{4}{R^2}\,. \tag{5.33}$$

Substituting into eq. (5.31) using (5.28) and (5.33) yields

$$R\frac{R}{dR} S_B^{fin.} = \lim_{n\to 1}\left(n\frac{\partial}{\partial n} - 1\right)\frac{8\pi(n-1)a}{90\times 64\pi^2}\int\sqrt{h}\left(K^2 - \mathrm{Tr}K^2\right) = \frac{a}{90}, \tag{5.34}$$

which gives an explicit result for the coefficient of the logarithmic term in the entanglement entropy of a spherical region in a four dimensional CFT.

5.3 The universal term in the entanglement entropy of an infinitely long cylinder.

We focus again on a CFT living in flat space so that $R_{ijij} = R_{ii} = R = 0$. When the metric of the cylinder is written as

$$ds_c^2 = dy^2 + R^2 d\phi^2\,, \tag{5.35}$$

the extrinsic curvatures are equal to

$$\left.\begin{array}{l} n^t = 1 \Rightarrow K^t = 0 \\ n_2^r = 1 \Rightarrow K^2_{\phi\phi} = R \end{array}\right\} \Rightarrow \qquad \mathrm{Tr}K^2 = \frac{1}{R^2}, \qquad K^2 = \frac{1}{R^2}\,. \tag{5.36}$$

Substituting into (5.31) using (5.29), (5.33) and the fact that $R_\Sigma = 0$ for the cylinder yields

$$R\frac{R}{dR} S_C^{fin.} = \lim_{n\to 1}\left(n\frac{\partial}{\partial n} - 1\right)\frac{8\pi(1-n)c}{90\times 64\pi^2}\int\sqrt{h}\left(\frac{1}{2}K^2 - \mathrm{Tr}K^2\right) = \frac{c}{720}\frac{\ell}{R}\,. \tag{5.37}$$

We thus find that the coefficient of the universal term in the entanglement entropy of an infinitely long cylinder in a four–dimensional CFT is equal to $\frac{c}{720}\frac{\ell}{R}$.

Expressions (5.34), (5.37) conclude this section. In the following sections we will compute the coefficient of the universal term in entanglement entropy holographically. We will consider higher derivative gravitational theories, whose dual CFTs generically have $a \neq c$, and perform the computation for a disk and a cylinder. We will then compare the holographic results with (5.34), (5.37). Finally, let us point out that (5.34), (5.37) effectively propose an alternative characterization of the conformal anomaly coefficients (a, c) through entanglement entropy as coefficients of the logarithmic term for the entropy of two different entangling surfaces; a sphere and a cylinder.

6. Lovelock Theories of gravity and Holography.

Among all theories of gravity which contain higher derivative terms of the Riemann tensor in their action, Lovelock gravity [29, 30, 31] is special. This class of gravitational theories stands out for its simplicity and the properties it shares with Einstein-Hilbert gravity. In particular, it is the most general theory of gravity whose equations of motion involve only second order derivatives of the metric. It is also ghost free when expanded around Minkowski spacetimes. Moreover, the Palatini and metric formulations of Lovelock gravity have been shown to be equivalent [32].

The action for Lovelock gravity in $(d+1)$–dimensions is

$$S = \frac{1}{16\pi G_N^{d+1}} \int d^{d+1}x\sqrt{-g} \sum_{p=0}^{[\frac{d}{2}]} (-)^p \frac{(d-2p)!}{(d-2)!} \lambda_p \mathscr{L}_p \,, \tag{6.1}$$

where G_N^{d+1} is the $(d+1)$–dimensional Newton's constant, $[\frac{d}{2}]$ denotes the integral part of $\frac{d}{2}$, λ_p is the p-th order Lovelock coefficient and $\mathscr{L}_p$ is the Euler density E_{2p} of a $2p$–dimensional manifold. In $(d+1)$ dimensions all $\mathscr{L}_p$ terms with $p \geq [\frac{d}{2}]$ are either total derivatives or vanish identically.

In these lectures we focus on five dimensional gravitational theories and thus limit ourselves to the Gauss-Bonnet action. This is the simplest example of a Lovelock action, which includes just the four dimensional Euler density

$$S = \frac{1}{16\pi G_N^{(5)}} \int d^5x\sqrt{-g}\mathscr{L} \,, \quad \text{where} \quad \mathscr{L} = \left(R + \frac{12}{L^2} + \frac{\lambda L^2}{2}\mathscr{L}_2 \right) . \tag{6.2}$$

In (6.2) we introduced a cosmological constant term $\Lambda = -\frac{12}{L^2}$ and denoted the dimensionless Gauss-Bonnet parameter by λ instead of λ_2 for simplicity. The Gauss-Bonnet term $\mathscr{L}_2$ in (6.2) is equal to

$$\mathscr{L}_2 = E_4 \equiv R_{MNPQ}R^{MNPQ} - 4R_{MN}R^{MN} + R^2 \tag{6.3}$$

The equations of motion derived from (6.2) are

$$-\frac{1}{2} g_{MN}\mathscr{L} + R_{MN} + \lambda L^2 \mathscr{H}_{MN}^{(2)} = 0 \,, \tag{6.4}$$

with $\mathscr{H}_{MN}^{(2)}$ defined as follows

$$\mathscr{H}_{MN}^{(2)} = R_{MLPQ}R_N^{\ \ LPQ} - 2R_{MP}R_N^{\ \ P} - 2R_{MPNQ}R^{PQ} + RR_{MN} \,. \tag{6.5}$$

Eq. (6.4) admits AdS solutions of the form [33, 34]

$$ds^2 = \frac{L_{AdS}^2 dr^2}{r^2} + \frac{r^2}{L_{AdS}^2}\left(-dt^2 + \sum_{i=1}^{3} dx_i dx^i\right), \tag{6.6}$$

where the curvature scale of the AdS space is related to the cosmological constant via[5]

$$L_{AdS} = \frac{L}{\sqrt{f_\infty}} \quad \text{where} \quad 1 - f_\infty + \lambda_{GB} f_\infty^2 \quad \Rightarrow \quad f_\infty = \frac{2}{1+\sqrt{1-4\lambda}}. \tag{6.7}$$

Gauss-Bonnet gravity has been extensively studied in the context of the AdS/CFT correspondence. The basic aspects of the holographic dictionary established in the case of Einstein–Hilbert gravity remain the same, since the equations of motion retain their second order form. However, the additional parameter λ allows for a holographic CFT with unequal central charges (c,a) (recall that all AdS backgrounds satisfying the Einstein-Hilbert equations of motion yield $a = c$).

There are two ways to relate the gravitational parameters, the Gauss-Bonnet coupling λ, Newton's five dimensional coupling constant $G_N^{(5)}$ and the cosmological constant L, to the CFT parameters (c,a). One is via a holographic calculation of the three point function of the stress energy tensor and the other through the holographic computation of the Weyl anomaly [35, 36] . Both calculations yield the same result, which is a good consistency check. The holographic calculation of the Weyl anomaly in Gauss-Bonnet gravity was performed in [37]. Here we simply quote the results

$$\begin{aligned} c &= 45\pi \frac{L_{AdS}^3}{G_N^{(5)}} \sqrt{1-4\lambda} \\ a &= 45\pi \frac{L_{AdS}^3}{G_N^{(5)}} \left[-2+3\sqrt{1-4\lambda}\right], \end{aligned} \tag{6.8}$$

Here L_{AdS} is given in (6.7) while our conventions for the CFT central charges (c,a) are defined through the Weyl anomaly in (5.7). For the calculations of the section 7 it is convenient to express the ratio $\frac{L_{AdS}^3}{G_N^{(5)}}$ and the Gauss-Bonnet coefficient, λ, as functions of the central charges (c,a)

$$\begin{aligned} \frac{L_{AdS}^3}{G_N^{(5)}} &= \frac{1}{90\pi}(3c-a), \qquad \lambda = \frac{(a-5c)(a-c)}{4(a-3c)^2} \\ \sqrt{1-4\lambda} &= \frac{2c}{3c-a}. \end{aligned} \tag{6.9}$$

7. Entanglement Entropy in Lovelock gravity.

When the gravitational theory contains higher derivative terms the prescription of Ryu and Takayanagi must be modified. A proposal for computing entanglement entropy for CFTs holographically dual to Lovelock gravity was given in [38, 39, 40]. According to this proposal, the entanglement entropy

[5] To be specific, Gauss-Bonnet gravity admits another AdS solution with $f_\infty = \frac{2}{1-\sqrt{1-4\lambda}}$ but this solution is unstable and contains ghosts [33].

of a connected region A of the CFT dual to $(p+1)$–order Lovelock gravity can be computed by minimizing the entropy functional S_A given by

$$S_A = \frac{1}{4G_N^{(d+1)}} \sum_{p=0}^{[\frac{d}{2}]} (-)^{p+1}(p+1)\frac{(d-2p-2)!}{(d-2)!}\lambda_{p+1}\int_\Sigma \sqrt{\sigma}\mathscr{L}_p(R_\Sigma)\,. \tag{7.1}$$

Here the integral is evaluated on Σ, the co-dimension two surface which reduces to the boundary (∂A) of the entangling surface at the boundary of AdS. Σ is explicitly determined by minimizing (7.2). σ corresponds to the determinant of the induced metric on Σ whereas $\mathscr{L}_p(R_\Sigma)$ is a function of the induced curvature tensor R_Σ[6].

Eq. (7.1) coincides with the expression for the entropy of black holes in Lovelock gravity as established in [41, 25] but only when the extrinsic curvature of the horizon vanishes, *i.e.*, $K^{(i)}_{\mu\nu}=0$ [39]. In fact, it was recently proven that, contrary to ones natural intuition, Wald's entropy formula is not the correct entropy functional for computing entanglement entropy in higher curvature gravity [2]. For more details the reader is encouraged to consult [2] and references therein.

In what follows we will focus on five–dimensional Gauss-Bonnet gravity where eq. (7.1) reduces to

$$S_A = \frac{1}{4G_N^{(5)}}\int_\Sigma \sqrt{\sigma}\left(1+\lambda L^2 R_\Sigma\right)\,. \tag{7.2}$$

Our objective is to compute the entanglement entropy of a spherical and a cylindrical region and compare the result of the holographic computation with eqs. (5.34), (5.37).

7.1 The entanglement entropy of a circular disk in Lovelock gravity.

To compute the entanglement entropy of a ball of radius R, it is useful to parameterize the AdS space in the following form

$$ds^2_{AdS} = L^2_{AdS}\left[\frac{d\rho^2}{4\rho^2}+\frac{1}{\rho}\left(-dt^2+dr^2+r^2 d\Omega_2^2\right)\right]\,. \tag{7.3}$$

The first step is to identify a three dimensional surface in the bulk of AdS which reduces to a sphere of radius R at the boundary. Taking into account the symmetries of the problem we see that the surface in question is determined by a single function $r(\rho)$. With this ansatz the induced metric of the surface can be written as follows

$$ds^2_A = L^2_{AdS}\left\{\frac{1}{4\rho^2}\left[1+4\rho\left(\frac{\partial r}{\partial\rho}\right)^2\right]d\rho^2+\frac{r^2}{\rho}d\Omega_2^2\right\}\,. \tag{7.4}$$

Using (7.4) to compute the induced curvature R_Σ and substituting into (7.2) yields

$$S_A = \frac{L^3_{AdS}\Omega_2}{4G_N^{(5)}}\int d\rho\,\frac{r^2\sqrt{1+4\rho(r')^2}}{2\rho^2}\left[1+\lambda f_\infty \hat{R}\right]\,, \tag{7.5}$$

[6] To make the variational problem well-defined a boundary term should be added in (7.2). This term does not affect the solution of the embedding surface but it changes the value of the action evaluated on the solution and thus of the entanglement entropy. It turns out however that the boundary term only modifies the leading UV-divergent term in the entanglement entropy.

PoS(Modave 2013)003

where f_∞ is defined in (6.7) and $\hat{R}$ is the induced scalar curvature in units of the AdS radius

$$\hat{R} = \frac{2\left[\rho+4\rho^2(r')^2+4\rho r\left(r'+8\rho(r')^3-2\rho r''\right)-r^2\left(3+20\rho(r')^2+16\rho^2 r' r''\right)\right]}{r^2\left[1+4\rho(r')^2\right]^2}. \tag{7.6}$$

To specify the coefficient of the logarithmic term it suffices to solve the equations of motion coming from (7.5) only to the next to leading order in the neighborhood of the boundary $\rho = 0$. Doing so we find that

$$r(\rho) = R - \frac{\rho}{2R} + \cdots. \tag{7.7}$$

Substituting the solution (7.7) into (7.5) yields

$$S(B) = \frac{L^3_{AdS}\Omega_2}{4G_N^{(5)}}\int_{\varepsilon^2} d\rho\left[\frac{1-6\lambda\alpha}{2\rho^2}R^2 - \frac{1-6\lambda\alpha}{4\rho} + \mathscr{O}(\rho^0)\right]. \tag{7.8}$$

Using (6.9) and the definition of f_∞ from (6.7) we arrive at

$$S(B) = \frac{a}{90}\frac{R^2}{\varepsilon^2} + \frac{a}{90}\ln\varepsilon + \cdots, \tag{7.9}$$

which is in complete agreement with (5.34).

7.2 The entanglement entropy of an infinitely long cylinder in Lovelock gravity.

Here we study the entanglement entropy of a cylindrical surface. For the holographic computation we should find a three dimensional surface in AdS which reduces to a cylinder of radius R and length ℓ on the boundary of the AdS space. A natural parametrization of AdS space in this case is

$$ds^2_{AdS} = L^2_{AdS}\left[\frac{d\rho^2}{4\rho^2} + \frac{1}{\rho}\left(-dt^2+dz^2+dr^2+r^2d\phi^2\right)\right] \tag{7.10}$$

Symmetry considerations lead us to consider a surface described by a single function $r(\rho)$. The induced metric on the surface is

$$ds^2_A = L^2_{AdS}\left\{\frac{1}{4\rho^2}\left[1+4\rho\left(\frac{\partial r}{\partial\rho}\right)^2\right]d\rho^2 + \frac{1}{\rho}dz^2 + r^2\phi^2\right\}. \tag{7.11}$$

Substituting (7.11) into (7.2) yields

$$S_A = \frac{L^3_{AdS}}{4G_N^{(5)}}2\pi\ell\int d\rho\,\frac{r\sqrt{1+4\rho(r')^2}}{2\rho^2}\left[1+\alpha\lambda\hat{R}\right], \tag{7.12}$$

where $\hat{R}$ is again the induced curvature of the surface in units of the AdS radius

$$\hat{R} = \frac{2\left[2\rho\left(r'+8\rho(r')^3-2\rho r''\right)-r\left(3+20\rho(r')^2+16\rho^2 r' r''\right)\right]}{r\left[1+4\rho(r')^2\right]^2}. \tag{7.13}$$

The equations of motions in the vicinity of $\rho = 0$ are solved by

$$r(\rho) = R - \frac{\rho}{4R} + \cdots. \tag{7.14}$$

Evaluating (7.12) on the solution (7.14) yields

$$S_A = \frac{L^3_{AdS}}{4G^{(5)}_N} 2\pi\ell \int_{\varepsilon^2} d\rho \left[\frac{(1-6\alpha\lambda)R}{2\rho^2} - \frac{1-2\alpha\lambda}{16R\rho} + \mathscr{O}(\rho^0) \right] . \tag{7.15}$$

With the help of (6.9) and (7.13) we finally arrive at

$$S(C) = \frac{a}{90} \frac{2\pi R\ell}{4\pi\varepsilon^2} + \frac{c}{720} \frac{\ell}{R} \ln\varepsilon + \cdots . \tag{7.16}$$

which again agrees with (5.37).

References

[1] A. Lewkowycz and J. Maldacena, "Generalized gravitational entropy," JHEP **1308** (2013) 090 [arXiv:1304.4926 [hep-th]].

[2] X. Dong, "Holographic Entanglement Entropy for General Higher Derivative Gravity," JHEP **1401** (2014) 044 [arXiv:1310.5713 [hep-th], arXiv:1310.5713].

[3] C. Holzhey, F. Larsen and F. Wilczek, "Geometric and renormalized entropy in conformal field theory," Nucl. Phys. B **424** (1994) 443 [hep-th/9403108].

[4] P. Calabrese and J. L. Cardy, "Entanglement entropy and quantum field theory," J. Stat. Mech. **0406** (2004) P06002 [hep-th/0405152].

[5] P. Calabrese and J. L. Cardy, "Entanglement entropy and conformal field theory," J. Phys. A. Math. **504005**, (2009) 36 [hep-th/0905.4013].

[6] H. Casini and M. Huerta, "Entanglement entropy in free quantum field theory," J. Phys. A **42** (2009) 504007 [arXiv:0905.2562 [hep-th]].

[7] J. S. Dowker, arXiv:1009.3854 [hep-th].

[8] J. S. Dowker, arXiv:1012.1548 [hep-th].

[9] H. Casini and M. Huerta, "Entanglement entropy for the n-sphere," Phys. Lett. B **694** (2010) 167 [arXiv:1007.1813 [hep-th]].

[10] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, "Large N field theories, string theory and gravity," Phys. Rept. **323** (2000) 183 [hep-th/9905111].

[11] S. Ryu and T. Takayanagi, "Holographic derivation of entanglement entropy from AdS/CFT," Phys. Rev. Lett. **96** (2006) 181602 [hep-th/0603001].

[12] S. Ryu and T. Takayanagi, "Aspects of Holographic Entanglement Entropy," JHEP **0608** (2006) 045 [hep-th/0605073].

[13] V. E. Hubeny, M. Rangamani and T. Takayanagi, "A Covariant holographic entanglement entropy proposal," JHEP **0707** (2007) 062 [arXiv:0705.0016 [hep-th]].

[14] H. Casini, M. Huerta and R. C. Myers, "Towards a derivation of holographic entanglement entropy," JHEP **1105** (2011) 036 [arXiv:1102.0440 [hep-th]].

[15] T. Hirata and T. Takayanagi, "AdS/CFT and strong subadditivity of entanglement entropy," JHEP **0702** (2007) 042 [hep-th/0608213].

[16] Z. Komargodski and A. Schwimmer, "On Renormalization Group Flows in Four Dimensions," JHEP **1112** (2011) 099 [arXiv:1107.3987 [hep-th]].

[17] D. L. Jafferis, "The Exact Superconformal R-Symmetry Extremizes Z," JHEP **1205** (2012) 159 [arXiv:1012.3210 [hep-th]].

[18] H. Casini and M. Huerta, "On the RG running of the entanglement entropy of a circle," Phys. Rev. D **85** (2012) 125016 [arXiv:1202.5650 [hep-th]].

[19] I. R. Klebanov, D. Kutasov and A. Murugan, "Entanglement as a probe of confinement," Nucl. Phys. B **796** (2008) 274 [arXiv:0709.2140 [hep-th]].

[20] G. Vidal and R. F. Werner, "Computable measure of entanglement," Phys. Rev. A **65** (2002) 032314.

[21] P. Calabrese, J. Cardy and E. Tonni, "Entanglement negativity in quantum field theory," Phys. Rev. Lett. **109** (2012) 130502 [arXiv:1206.3092 [cond-mat.stat-mech]].

[22] L. Bonora, P. Pasti and M. Bregola, "Weyl Cocycles," Class. Quant. Grav. **3** (1986) 635.

[23] S. Deser and A. Schwimmer, "Geometric classification of conformal anomalies in arbitrary dimensions," Phys. Lett. B **309** (1993) 279 [hep-th/9302047].

[24] D. D. Sokolov and A. A. Starobinsky, "On the structure of curvature tensor on conical singularities," DAN SSSR **22** (1977) 312.

[25] M. Banados, C. Teitelboim and J. Zanelli, "Black hole entropy and the dimensional continuation of the Gauss-Bonnet theorem," Phys. Rev. Lett. **72** (1994) 957 [gr-qc/9309026].

[26] D. V. Fursaev and S. N. Solodukhin, "On the description of the Riemannian geometry in the presence of conical defects," Phys. Rev. D **52** (1995) 2133 [hep-th/9501127].

[27] D. V. Fursaev, A. Patrushev and S. N. Solodukhin, "Distributional Geometry of Squashed Cones," Phys. Rev. D **88** (2013) 044054 [arXiv:1306.4000 [hep-th]].

[28] S. N. Solodukhin, "Entanglement entropy, conformal invariance and extrinsic geometry," Phys. Lett. B **665** (2008) 305 [arXiv:0802.3117 [hep-th]].

[29] C. Lanczos, Z. Phys. **73** (1932) 147

[30] C. Lanczos, "A Remarkable property of the Riemann-Christoffel tensor in four dimensions," Annals Math. **39** (1938) 842.

[31] D. Lovelock, "The Einstein tensor and its generalizations," J. Math. Phys. **12** (1971) 498.

[32] Q. Exirifard and M. M. Sheikh-Jabbari, "Lovelock gravity at the crossroads of Palatini and metric formulations," Phys. Lett. B **661** (2008) 158 [arXiv:0705.1879 [hep-th]].

[33] D. G. Boulware and S. Deser, "String Generated Gravity Models," Phys. Rev. Lett. **55** (1985) 2656.

[34] R. -G. Cai, "Gauss-Bonnet black holes in AdS spaces," Phys. Rev. D **65** (2002) 084014 [hep-th/0109133].

[35] M. Henningson and K. Skenderis, "The Holographic Weyl anomaly," JHEP **9807** (1998) 023 [hep-th/9806087].

[36] M. Henningson and K. Skenderis, "Holography and the Weyl anomaly," Fortsch. Phys. **48** (2000) 125 [hep-th/9812032].

[37] S. 'i. Nojiri and S. D. Odintsov, "On the conformal anomaly from higher derivative gravity in AdS / CFT correspondence," Int. J. Mod. Phys. A **15** (2000) 413 [hep-th/9903033].

[38] D. V. Fursaev, "Proof of the holographic formula for entanglement entropy," JHEP **0609** (2006) 018 [hep-th/0606184].

[39] L. -Y. Hung, R. C. Myers and M. Smolkin, "On Holographic Entanglement Entropy and Higher Curvature Gravity," JHEP **1104** (2011) 025 [arXiv:1101.5813 [hep-th]].

[40] J. de Boer, M. Kulaxizi and A. Parnachev, "Holographic Entanglement Entropy in Lovelock Gravities," JHEP **1107** (2011) 109 [arXiv:1101.5781 [hep-th]].

[41] T. Jacobson and R. C. Myers, "Black hole entropy and higher curvature interactions," Phys. Rev. Lett. **70** (1993) 3684 [hep-th/9305016].