Nuclear reactions in astrophysics

Pierre Descouvemont

Université Libre de Bruxelles
Brussels, Belgium

Content of the lectures

1. **Introduction**
    1. Reaction networks
    2. Needs for astrophysics
    3. Specificities of nuclear astrophysics

2. **Low-energy cross sections**
    1. Definitions
    2. General properties
    3. S-factor

3. **Reaction rates**
    1. Definitions
    2. Gamow peak
    3. Resonant and non-resonant rates

4. **General scattering theory (simple case: spins 0, no charge, single channel)**
    1. Different models
    2. Optical model
    3. Scattering amplitude and cross sections
    4. Phase-shift method
    5. Resonances
    6. Generalizations: Coulomb interaction, absorption, non-zero spins

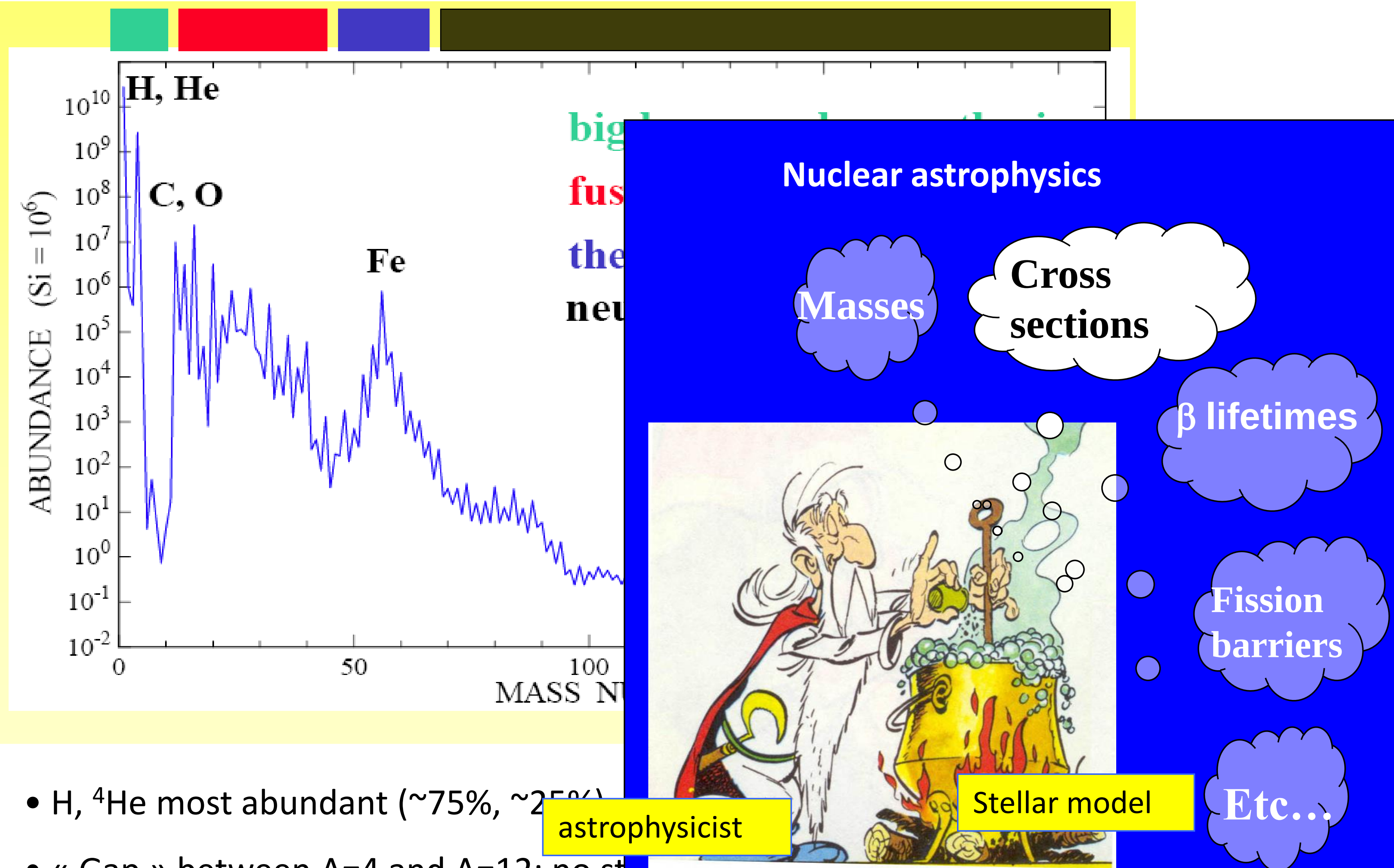
5. Models used in nuclear astrophysics

1. Brief overview
2. The potential model : Radiative-capture reactions
3. The R-matrix method
4. Microscopic models

1. Introduction

1. Introduction

Goal of nuclear astrophysics: understand the abundances of the elements



- H, ^{4}He most abundant (~75%, ~2
- « Gap » between A=4 and A=12: no st
- Iron peak (very stable)

1. Introduction

- Years ~ 1940-50: Hoyle, Gamow

  Role of nuclear reactions in stars

  - Energy production
  - Nucleosynthesis (Hoyle state in ^{12}C)

- 1957: B_2FH: Burbidge, Burbidge, Fowler, Hoyle (Rev. Mod. Phys. 29 (1957) 547)

  Wikipedia site: http://en.wikipedia.org/wiki/B%C2%B2FH

  Cycles: pp chain: converts 4p → ^4He

  CNO cycle: converts 4p → ^4He (via ^{12}C)

  s (slow) process: (n,γ) capture followed by β decay

  r (rapid) process: several (n,γ) captures

  p (proton) process: (p,γ) capture

- Nucleosynthesis:

  Primordial (Bigbang): 3 first minutes of the Universe

  Stellar: star evolution, energy production

- Essentially two (experimental) problems in nuclear astrophysics

  Low energies → very low cross sections (Coulomb barrier)

  Need for radioactive beams

  → in most cases a theoretical support is necessary (data extrapolation)

1. Introduction

Reaction networks: set of equations with abundances of nucleus m: Y_m

$$\frac{dY_m}{dt} = -\lambda_m Y_m$$ → Destruction of m by β decay: $\lambda_m = 1/\tau_m$

$$+\sum_k \lambda_k^{(m)} Y_k$$ → Production of m by β decay from elements k

$$-\sum_k Y_m Y_k [mk]^{(m+k)}$$ → Destruction of m by reaction with k

$$+\sum_{k,l} Y_k Y_l [kl]^{(m)}$$ → Production of m by reaction k+l→m

with $[kl]^{(m)} \sim$ <σv>, <σv>=reaction rate (strongly depends on temperature)

In practice:

- Many reactions are involved (no systematics)
- σ must be known at very low energies → very low cross sections
- Reactions with radioactive elements are needed
- At high temperatures: high level densities → properties of many resonances needed

1. Introduction

Specificities of nuclear astrophysics

- low energies (far below the Coulomb barrier)
  → small cross sections
  (in general not accessible in laboratories at stellar energies)
  → low angular momenta (selection of resonances)

- radioactive nuclei
  → need for radioactive beams (^{7}Be, ^{13}N, ^{18}F, …)

- different types of reactions:
  - transfer (α,n), (α,p), (p,α), etc…
  - radiative-capture: (p,γ), (α,γ), (n,γ), etc…
  - weak processes: p+p → d+ e^+ + ν
  - fusion: $^{12}C+^{12}C$, $^{16}O+^{16}O$, etc.

- different situations
  - capture, transfer
  - resonant, non resonant
  - low level density (light nuclei), high level density (heavy nuclei)
  - peripheral, internal processes

→ **different approaches, for theory and for experiment**

1. Introduction

Some key reactions

- $d(\alpha,\gamma)^6Li$, $^3He(\alpha,\gamma)^7Be$: Big-Bang
- Triple α, $^{12}C(\alpha,\gamma)^{16}O$: He burning
- $^7Be(p,\gamma)^8B$: solar neutrino problems
- $^{18}F(p,\alpha)^{15}O$: nova nucleosynthesis
- Etc…

2. Low-energy cross sections

- Definitions
- General properties
- S-factor

2. Low-energy cross sections

Types of reactions: general definitions valid for all models

Type	Example	Origin
Transfer	$^3He(^3He,2p)\alpha$	Strong
Radiative capture	$^2H(p,\gamma)^3He$	Electromagnetic
Weak capture	$p+p \rightarrow d+ e^+ + \nu$	Weak

cross section decreases

2. Low-energy cross sections

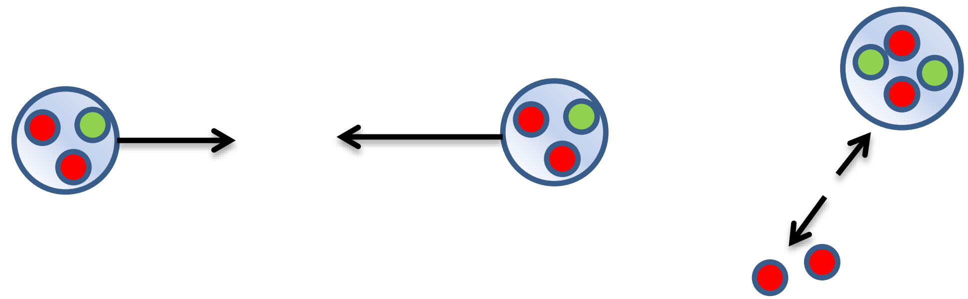
- Transfer: A+B → C+D (σ_t, strong interaction, example: ^{3}He(d,p)^{4}He)

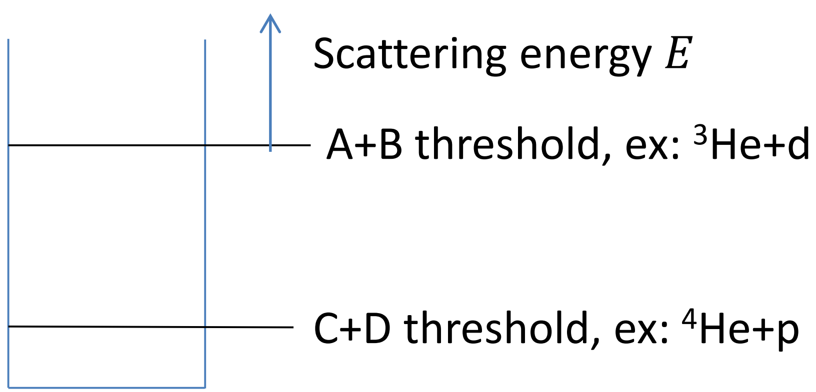
$$\sigma_{t,c\to c'}(E) = \frac{\pi}{k^2}\sum_{J\pi}\frac{2J+1}{(2I_1+1)(2I_2+1)}\left|U^{J\pi}_{cc'}(E)\right|^2$$

$U^{J\pi}_{cc'}(E)$ =collision (scattering) matrix (obtained from scattering theory → various models)

c, c' =entrance and exit channels

Transfer reaction:
Nucleons are transfered





Compound nucleus, ex: ^{5}Li

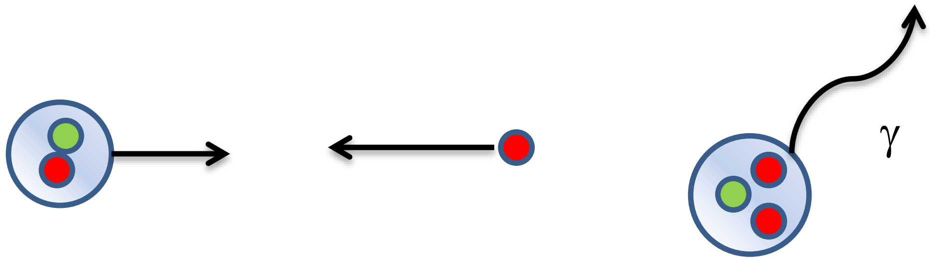
2. Low-energy cross sections

- Radiative capture : A+B → C+γ (σ_C, electromagnetic interaction, example: $^{12}C(p,\gamma)^{13}N$)

$$\sigma_C^{J_f\pi_f}(E) \sim \sum_{\lambda}\sum_{J_i\pi_i} k_\gamma^{2\lambda+1} \left| < \Psi^{J_f\pi_f} \| \mathcal{M}_\lambda \| \Psi^{J_i\pi_i}(E) > \right|^2$$

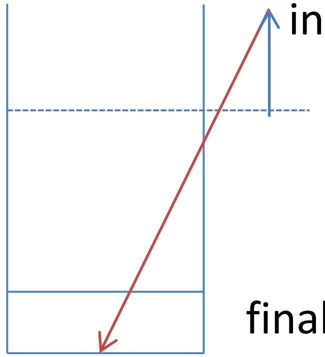
$J_f\pi_f$=final state of the compound nucleus C
$\Psi^{J_i\pi_i}(E)$=initial scattering state of the system (A+B)
$\mathcal{M}_{\lambda\mu}$=electromagnetic operator (electric or magnetic): $\mathcal{M}_{\lambda\mu} \sim e\, r^\lambda Y_\lambda^\mu(\Omega_r)$

Capture reaction:
A photon is emitted



Long wavelength approximation:
Wave number $k_\gamma = E_\gamma/\hbar c$, wavelength: $\lambda_\gamma = 2\pi/k_\gamma$
Typical value: $E_\gamma = 1$ MeV, $\lambda_\gamma \approx 1200$ fm >> typical dimensions of the system (R)
→ $k_\gamma R \ll 1$= Long wavelength approximation

2. Low-energy cross sections



initial state E>0 , contains all $J_i\pi_i$,

A+B threshold, ex: ^{12}C+p

final states E_f <0, specific $J_f\pi_f$

$$\sigma_C^{J_f\pi_f}(E) \sim \sum_{J_i\pi_i}\sum_{\lambda} k_\gamma^{2\lambda+1}\left|< \Psi^{J_f\pi_f}\|\mathcal{M}_\lambda\|\Psi^{J_i\pi_i}(E) >\right|^2$$

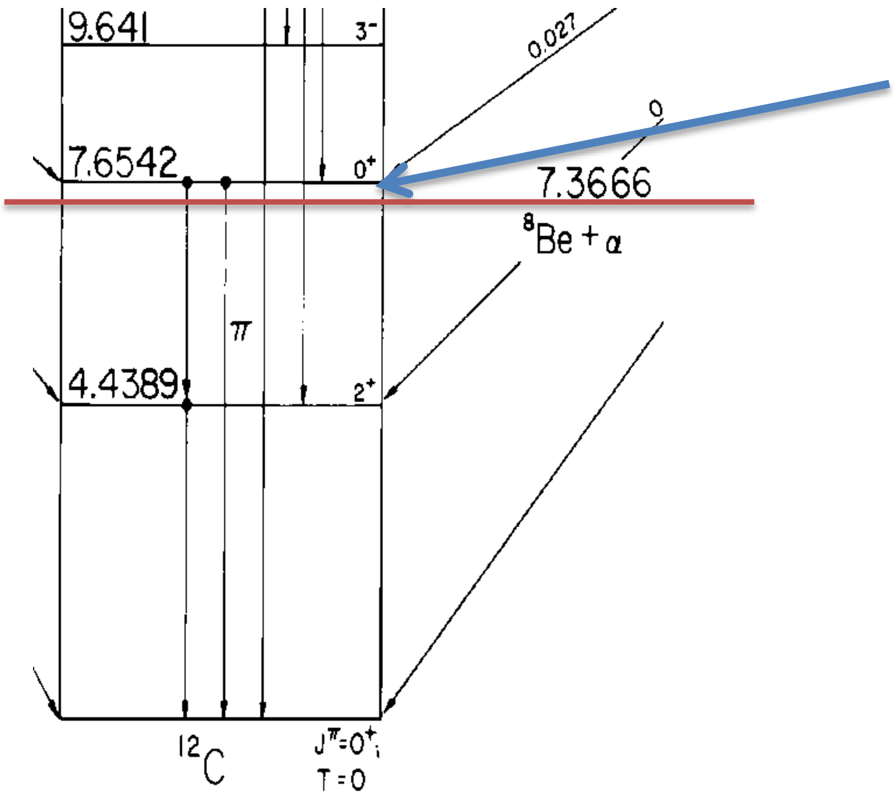
- $k_\gamma = (E - E_f)/\hbar c$ = photon wave number
- In practice
  - Summation over λ limited to 1 term (often E1, or E2/M1 if E1 is forbidden)

    $$\frac{E2}{E1} \sim (k_\gamma R)^2 \ll 1$$ (from the long wavelength approximation)
  - Summation over $J_i\pi_i$ limited by selection rules

    $$|J_i - J_f| \leq \lambda \leq J_i + J_f$$

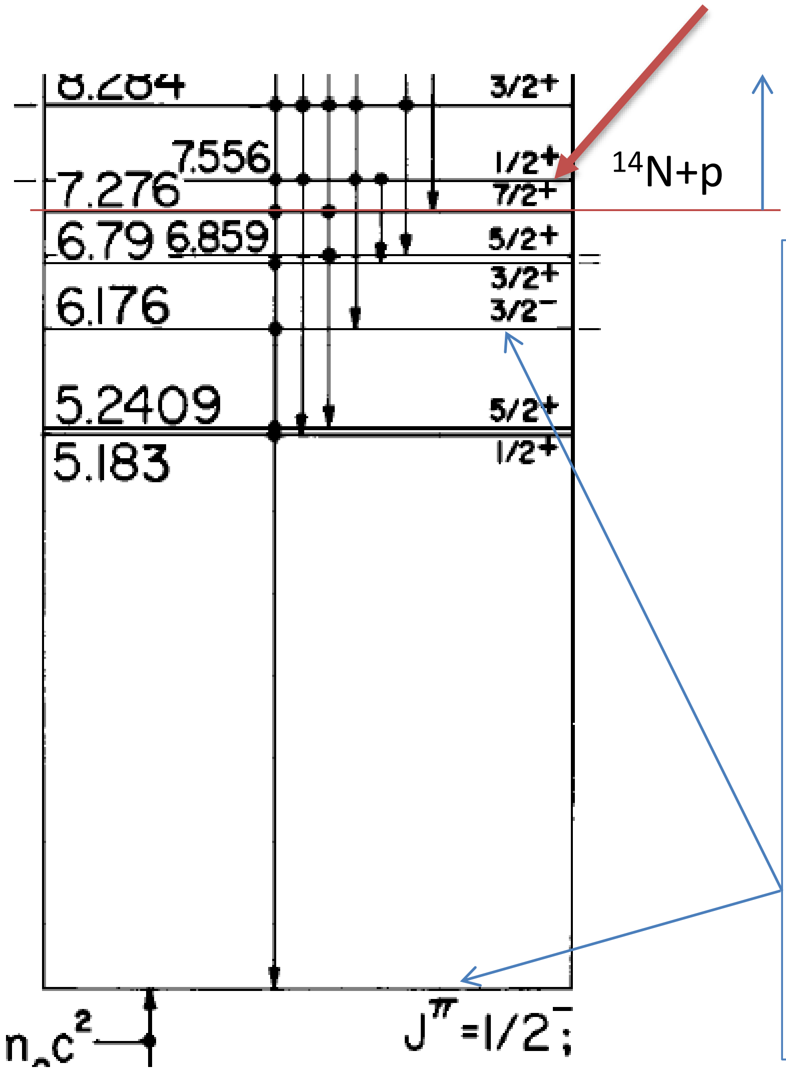
    $\pi_i\pi_f = (-1)^\lambda$ for electric, $\pi_i\pi_f = (-1)^{\lambda+1}$ for magnetic

Example 1: $^{8}Be(\alpha,\gamma)^{12}C$



- Initial partial wave $J_i = 0^+$ (includes the Hoyle state.
- E2 dominant (E1 forbidden in N=Z)
- → essentially the $J_f = 2^+$ state is populated.

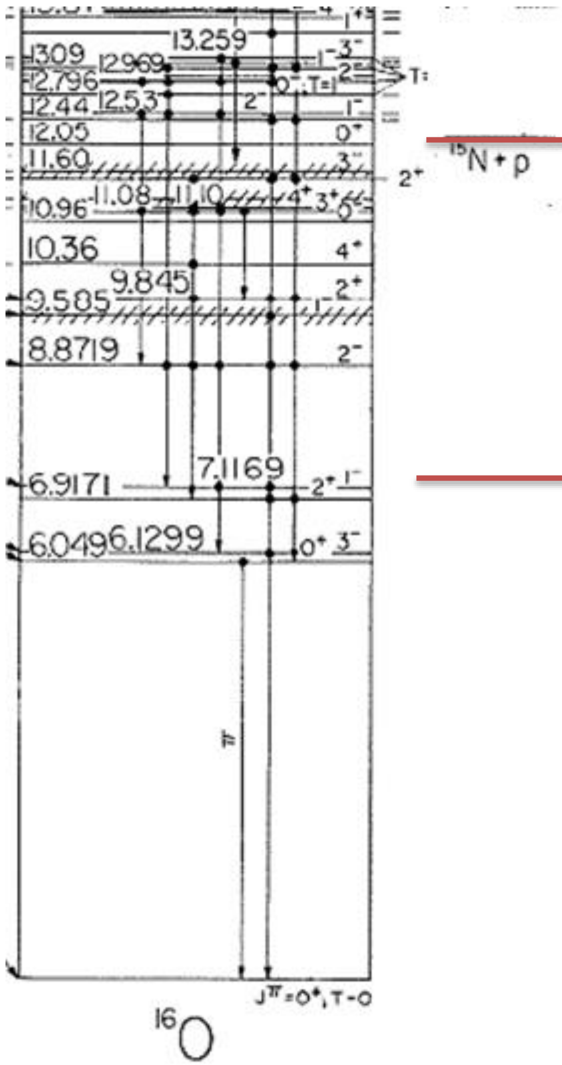
Example 2: $^{14}N(p,\gamma)^{15}O$



E (all J_i values)

- Spin of ^{14}N: I_1=1$^+$, proton I_2=1/2$^+$
- Channel spin I:
  $$|I_1 - I_2| \leq I \leq I_1 + I_2$$
  → $I = 1/2, 3/2$
- Orbital momentum ℓ
  $$|I - \ell| \leq J_i \leq I + \ell$$
- At low energies, $\ell = 0$ is dominant → $J_i = 1/2^+, 3/2^+$
- multipolarity E1 → transitions to $J_f = 1/2^-, 3/2^-, 5/2^-$
- Resonance $1/2^+$ determines the cross section

Example 3: $^{12}C(\alpha,\gamma)^{16}O$, $^{15}N(p,\gamma)^{16}O$, $^{15}N(p,\alpha)^{12}C$



^{15}N+p threshold
$^{15}N(p,\gamma)^{16}O$ and $^{15}N(p,\alpha)^{12}C$ are open
→ $^{15}N(p,\gamma)^{16}O$ negligible

$^{12}C+\alpha$ threshold
only possibility: $^{12}C(\alpha,\gamma)^{16}O$
→ $^{12}C(\alpha,\gamma)^{16}O$ (very) important

2. Low-energy cross sections

- Weak capture ($p+p \rightarrow d+\nu+e^-$): tiny cross section
  → no measurement (only calculations)

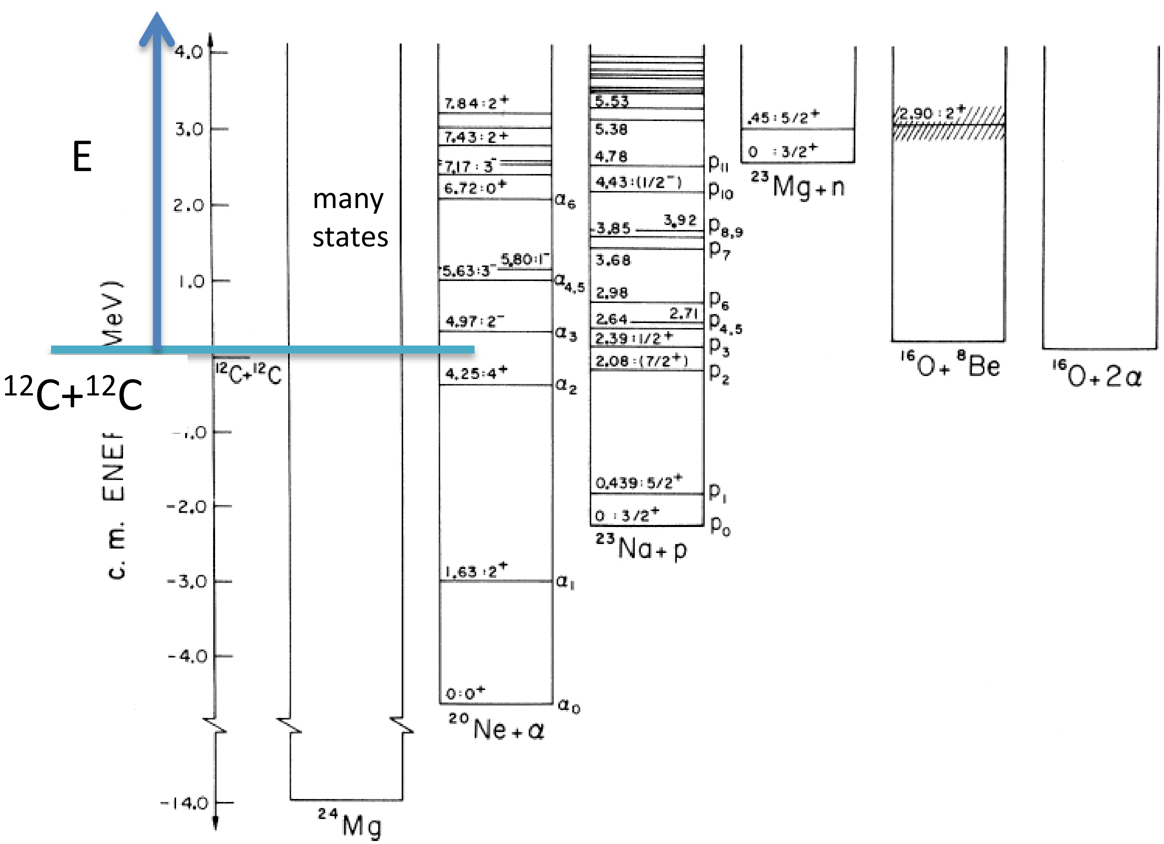
$$\sigma_W^{J_f\pi_f}(E) \sim \sum_{J_i\pi_i} \left| < \Psi^{J_f\pi_f} \| O_\beta \| \Psi^{J_i\pi_i}(E) > \right|^2$$

  - Calculations similar to radiative capture
  - O_β= Fermi ($\sum_i t_{i\pm}$) and Gamow-Teller ($\sum_i t_{i\pm}\sigma_i$) operators

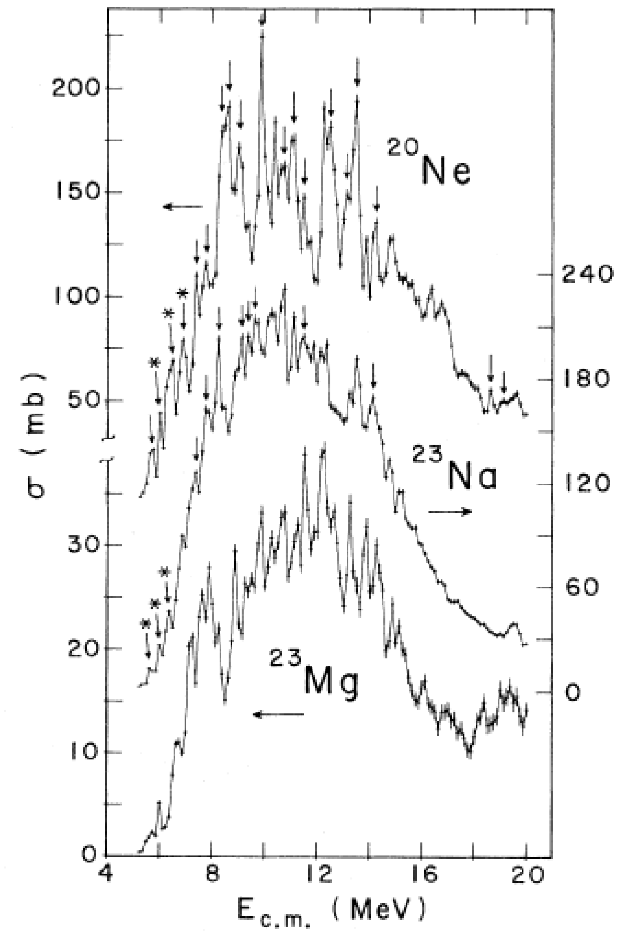
  - $^3He+p \rightarrow {}^4He+\nu+e^-$: produces high-energy neutrinos (more than tiny!)

2. Low-energy cross sections

- Fusion: similar to transfer, but with many output channels
  → statistical treatment
  → optical potentials

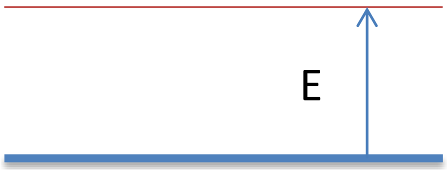


experimental cross section
Satkowiak et al. PRC 26 (1982) 2027



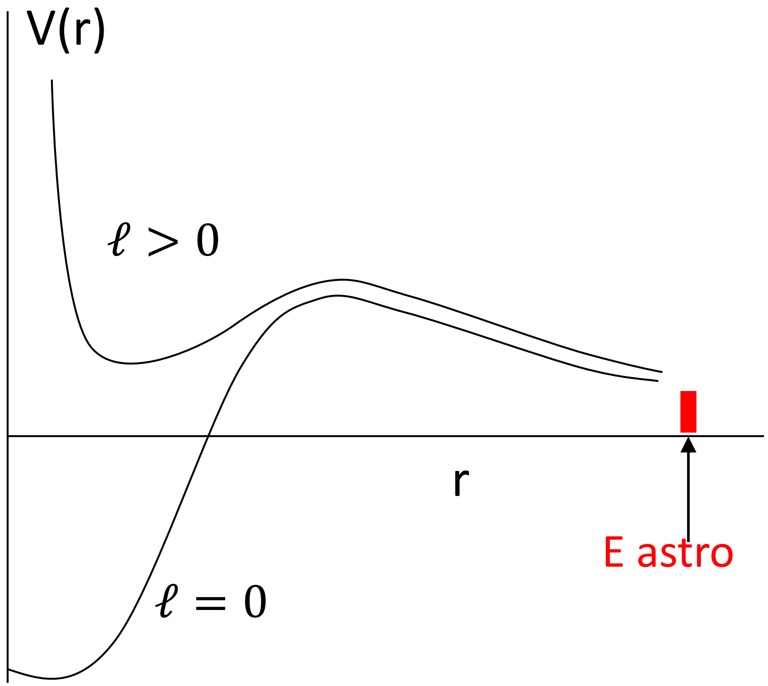
2. Low-energy cross sections

General properties (common to all reactions)



Scattering energy E: wave function $\Psi_i(E)$ common to all processes

Reaction threshold



- Cross sections dominated by Coulomb effects
  Sommerfeld parameter $\eta = Z_1 Z_2 e^2/\hbar v$

- Coulomb functions at low energies
  $$F_\ell(\eta, x) \to \exp(-\pi\eta)\,\mathcal{F}_\ell(x),$$
  $$G_\ell(\eta, x) \to \exp(\pi\eta)\,\mathcal{G}_\ell(x),$$

- Coulomb effect: strong E dependence : $\exp(2\pi\eta)$
  neutrons: $\sigma(E) \sim 1/v$

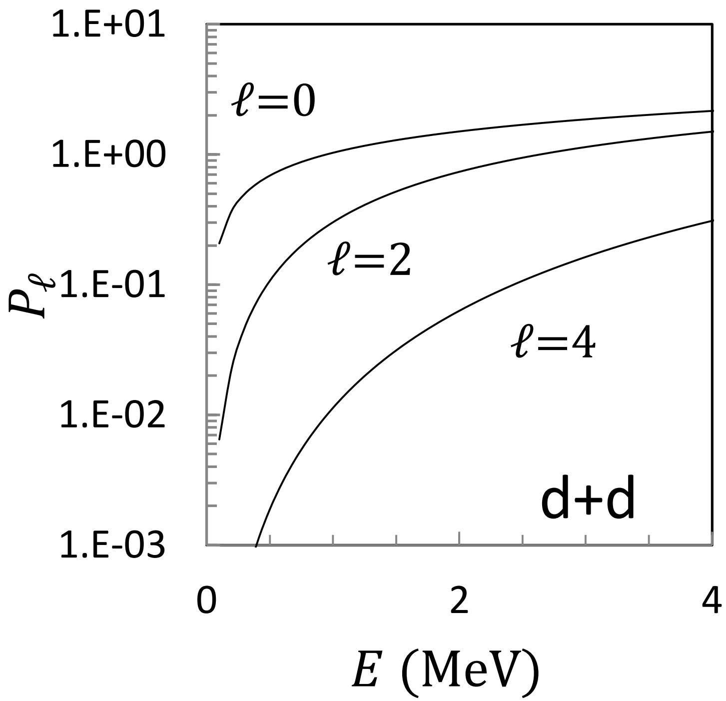
- Strong ℓ dependence
  Centrifugal term: $\sim \frac{\hbar^2}{2\mu}\frac{\ell(\ell+1)}{r^2}$
  → stronger for nucleons ($\mu \approx 1$) than for α ($\mu \approx 4$)

2. Low-energy cross sections

General properties: specificities of the entrance channel → common to all reactions

- All cross sections (capture, transfer) involve a summation over ℓ: $\sigma(E) = \sum_\ell \sigma_\ell(E)$
- The partial cross sections $\sigma_\ell(E)$ are proportional to the penetration factor

$$P_\ell(E) = \frac{ka}{F_\ell(ka)^2 + G_\ell(ka)^2}$$ (a =typical radius)



Consequences

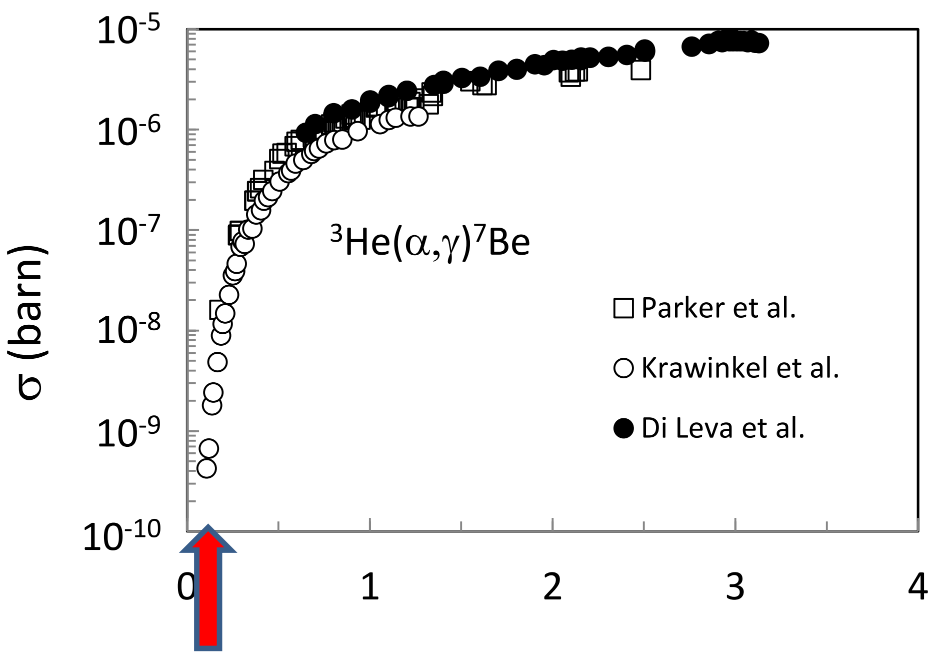
- $\ell > 0$ are often negligible at low energies
- $\ell = \ell_{min}$ is dominant (often $\ell_{min} = 0$)
- For $\ell = 0$, $P_0(E) \sim \exp(-2\pi\eta)$

Astrophysical S factor: $S(E) = \sigma(E) E \exp(2\pi\eta)$ (Units: E*L^2: MeV-barn)

- removes the coulomb dependence → only nuclear effects
- weakly depends on energy →$\sigma(E) \approx S_0 \exp(-2\pi\eta)/E$ (any reaction at low E)

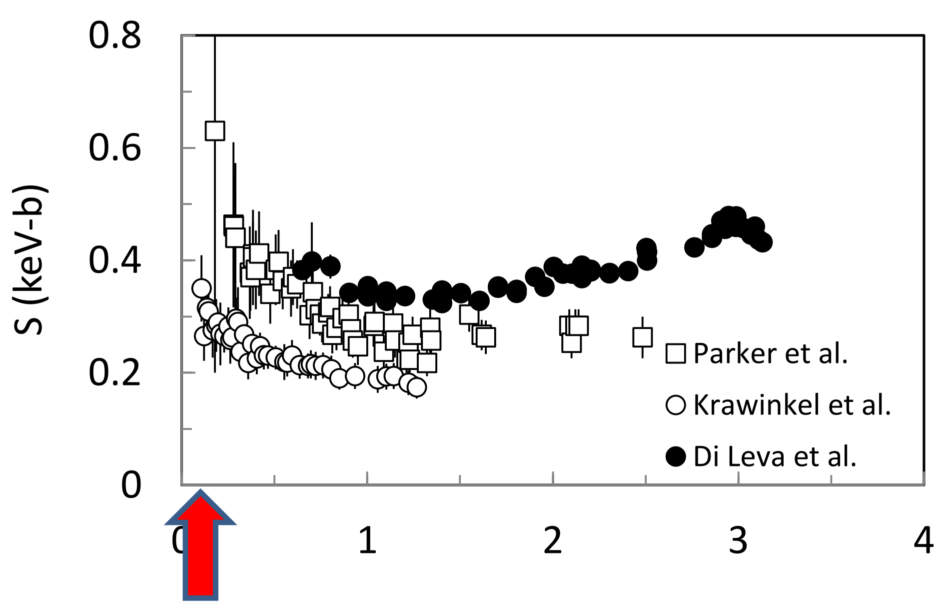
2. Low-energy cross sections

non resonant: $S(E) = \sigma(E) E \exp(2\pi\eta)$



Example: $^{3}He(\alpha,\gamma)^{7}Be$ reaction

- Cross section σ(E) Strongly depends on energy
- Logarithmic scale



S factor

- Coulomb effects removed
- Weak energy dependence
- Linear scale

2. Low-energy cross sections

Resonant cross sections: Breit-Wigner form

$$\sigma_R(E) \approx \frac{\pi}{k^2}\frac{(2J_R+1)}{(2I_1+1)(2I_2+1)}\frac{\Gamma_1(E)\Gamma_2(E)}{(E_R-E)^2+\Gamma^2/4}$$

- J_R, E_R=spin, energy of the resonance
- Valid for any process (capture, transfer)
- Valid for a single resonance → several resonances need to be added (if necessary)

- Γ_1=Partial width in the entrance channel (strongly depends on E, ℓ)

  $\Gamma_1(E) = 2\gamma_1^2 P_\ell(E)$ with γ_1^2=reduced width (does not depend on E)

  $P_\ell(E) \sim \exp(-2\pi\eta)$

  A resonance at low energies is always narrow (role of $P_\ell(E)$)

- Γ_2=Partial width in the exit channel (weakly depends on E, ℓ)
  - Transfer: $\Gamma_2(E) = 2\gamma_2^2 P_{\ell_f}(E+Q)$ (in general $Q \gg E$ → $P_{\ell_f}(E+Q)$ almost constant)
  - Capture: $\Gamma_2(E) \sim \left(E-E_f\right)^{2\lambda+1} B(E\lambda)$ → weak energy dependence

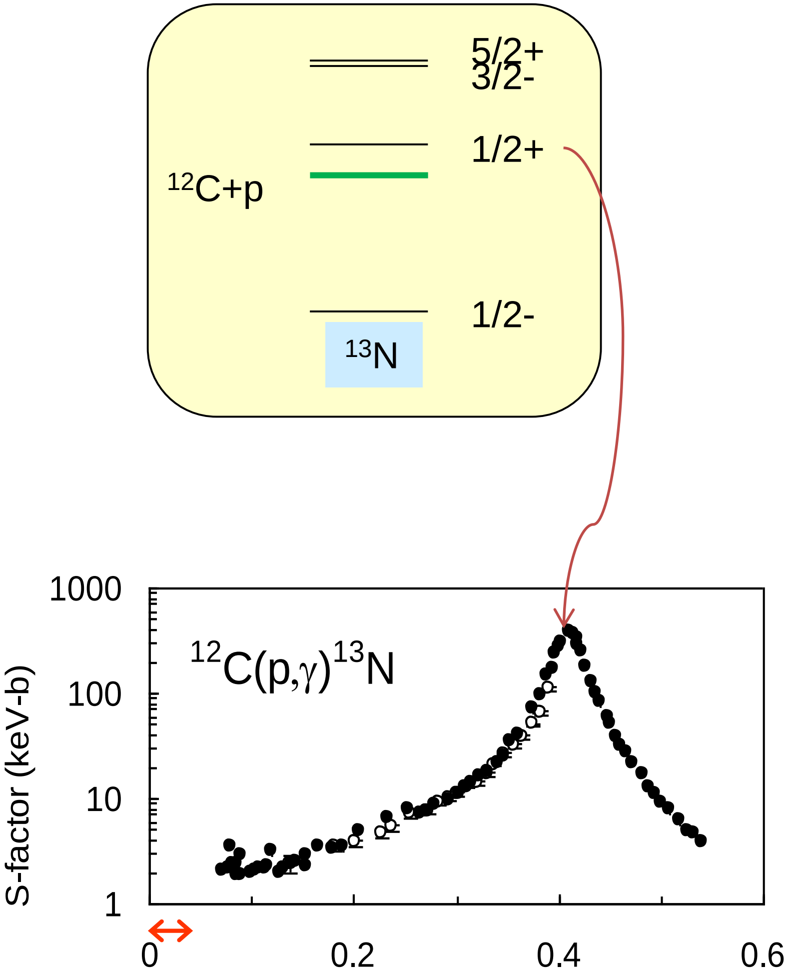
- S factor near a resonance $S(E) = \sigma(E)E\exp(2\pi\eta)$

$$S_R(E) \sim \frac{\gamma_1^2\Gamma_2}{(E_R-E)^2+\Gamma^2/4} P_\ell(E)\exp(2\pi\eta)$$

Almost constant

→ Simple estimate at low E (at the Breit-Wigner approximation)

2. Low-energy cross sections



$$S_R(E) \sim \frac{\gamma_1^2 \Gamma_2}{(E_R - E)^2 + \Gamma^2/4} P_\ell(E) \exp(2\pi\eta)$$

$$\sim \frac{\gamma_1^2 \Gamma_2}{(E_R - E)^2 + \Gamma^2/4}$$

- For $\ell = 0$: $P_0(E) \exp(2\pi\eta) \sim$ constant
- For $\ell > 0$, $P_\ell(E) \ll P_0(E)$
  → $\ell > 0$ resonances are suppressed

In ^{12}C(p,γ)^{13}N:

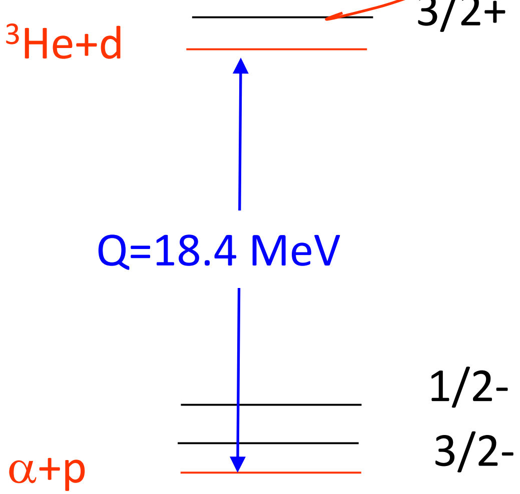
- Resonance $1/2^+$: $\ell = 0$
- Resonances $3/2^-$, $5/2^+$ $\ell = 1,2$ → negligible

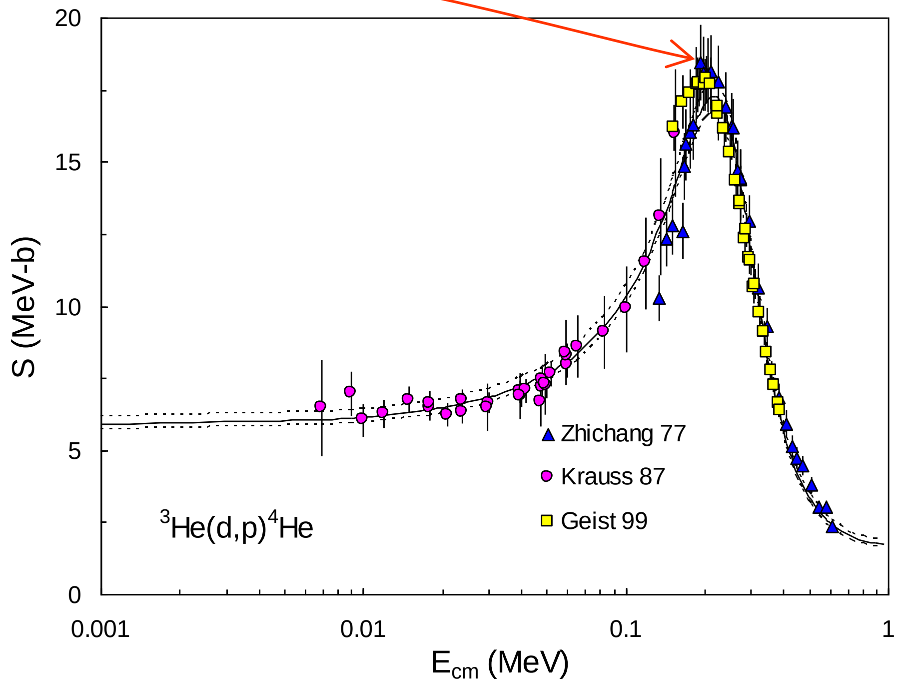
Note: BW is an approximation

- Neglects background, external capture
- Assumes an isolated resonance
- Is more accurate near the resonance energy

2. Low-energy cross sections

$^3He(d,p)^4He$: isolated resonance in a transfer reaction



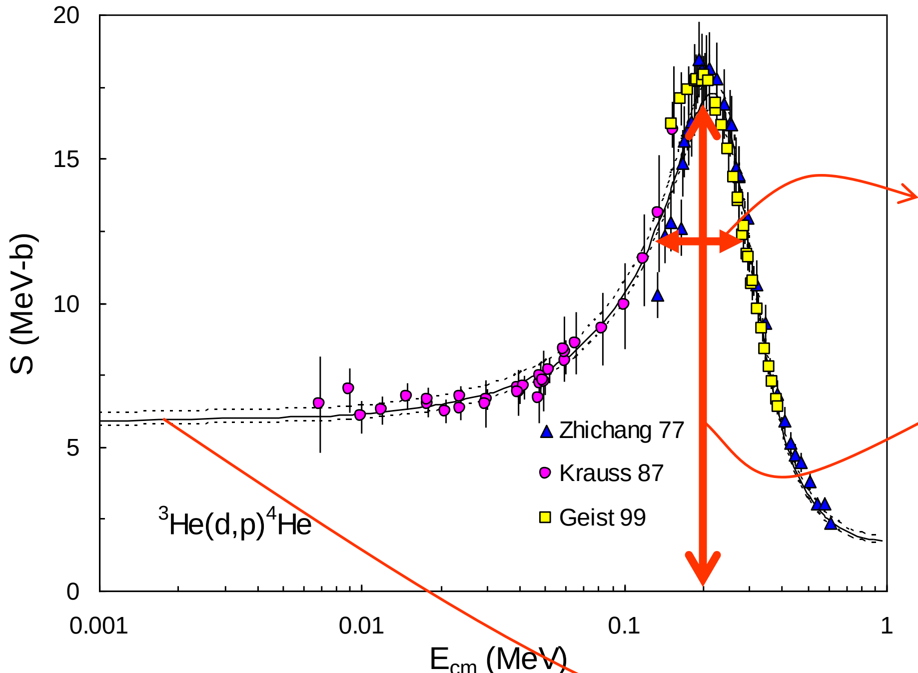


3/2+ resonance:

- Entrance channel: spin S=1/2,3/2, parity + → $\ell = 0,2$
- Exit channel: spin S=1/2, parity + → $\ell = 1$

Breit Wigner approximation

$$\sigma_{dp}(E) \approx \frac{\pi}{k^2} \frac{(2J_R+1)}{(2I_1+1)(2I_2+1)} \frac{\Gamma_d(E)\Gamma_p(E)}{(E_R-E)^2+\Gamma^2/4}$$



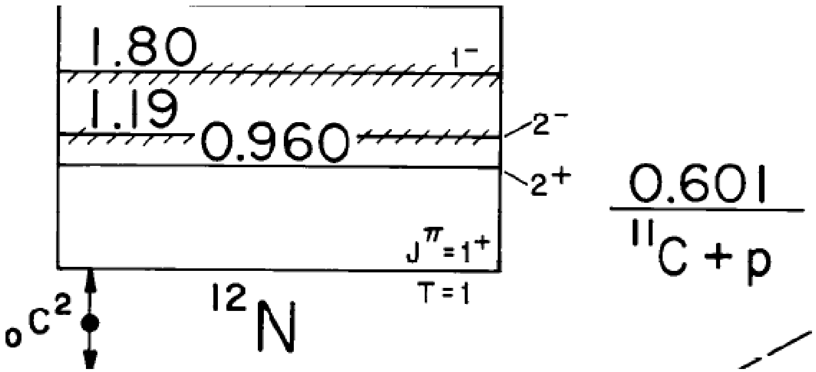
Width at half maximum=total width Γ

Amplitude: $\sim \Gamma_d\ \Gamma_p/\Gamma^2$

S(E) constant for $\ell = 0$

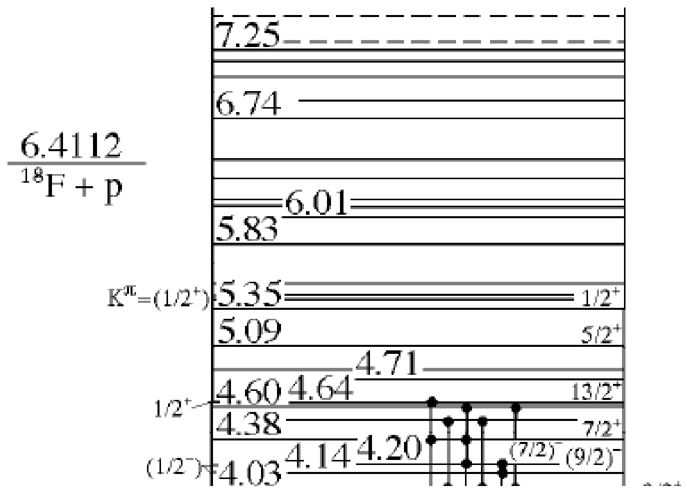
Two comments: 1. Selection of the main resonances
2. Going beyond the Breit-Wigner approximation

1. Selection of the main resonances



$^{11}C(p,\gamma)^{12}N$ (spin $^{11}C=3/2^-$)

- Resonance 2^-: $\ell = 0$, E1
- Resonance 2^+: $\ell = 1$, E2/M1
  → negligible



$^{18}F(p,\alpha)^{15}O$ (spin $^{18}F=1^+$)

- Many resonances
- Only $\ell = 0$ resonances are important
  → $J = 1/2^+, 3/2^+$ only

➔ In general a small number of resonances play a role

2. Going beyond the Breit-Wigner approximation

- How to go beyond the BW approximation?
- Problem of vocabulary
  - Direct capture
  - External capture
  - Non-resonant capture = « direct » capture

→ confusion!

- External capture $\sigma(E) = |M_{int} + M_{ext}|^2$

  With $\sigma_{BW}(E) = |M_{int}|^2$

  $M_{ext} \sim C$, with C=Asymptotic Normalization Constant (ANC) is needed

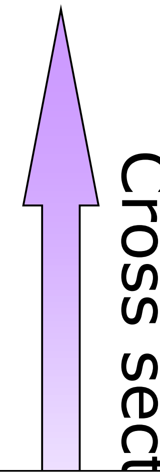
- Non resonant capture : $\sigma(E) = \sum_\ell \sigma_\ell(E) = \sigma_R(E) + \sum_{\ell \neq \ell_R} \sigma_\ell(E)$

  → scanning the resonance is necessary

2. Low-energy cross sections

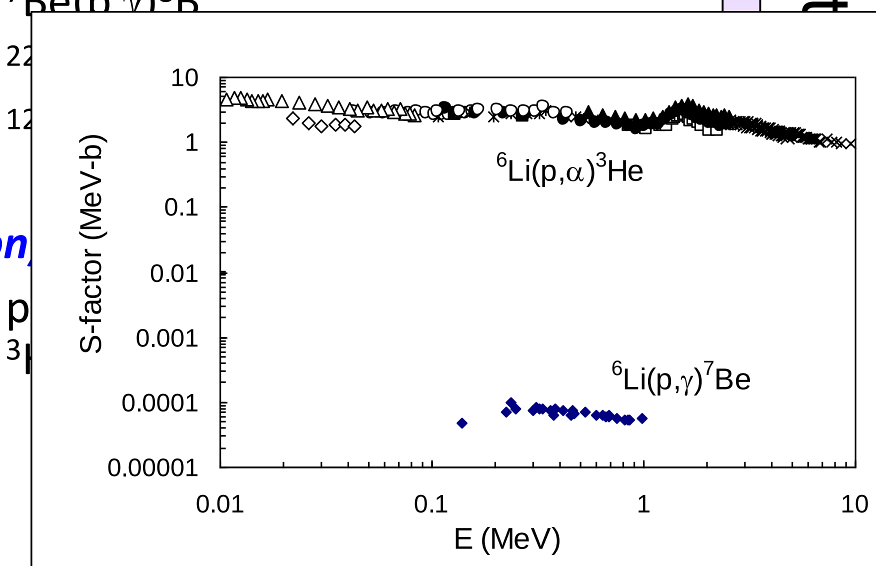
Many different situations

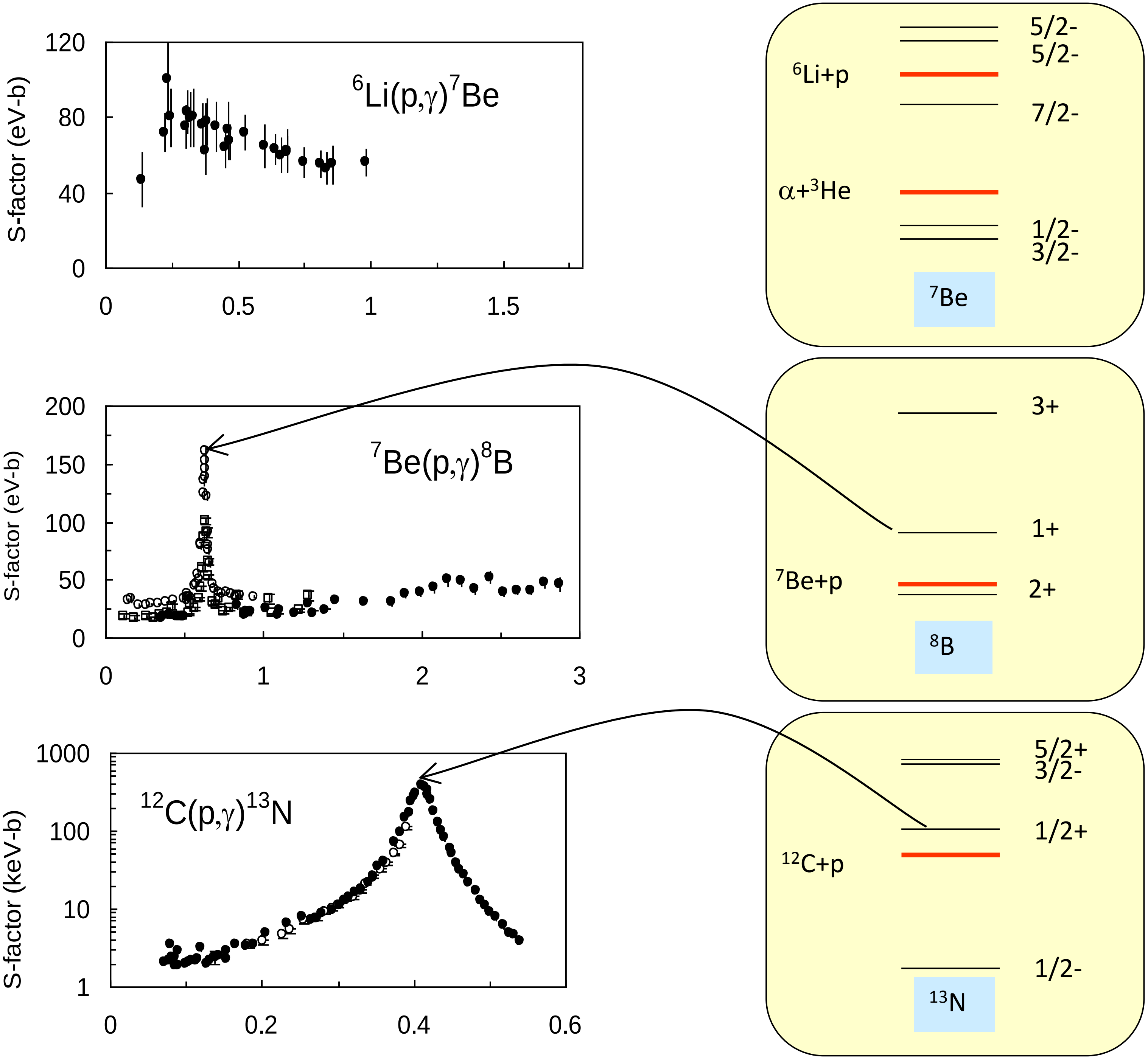
- *Transfer cross sections (**strong interaction**)*
  - Non resonant: $^{6}Li(p,\alpha)^{3}He$
  - Resonant, with $\ell_R=\ell_{min}$: $^{3}He(d,p)\alpha$
  - Resonant, with $\ell_R>\ell_{min}$: $^{11}B(p,\alpha)^{8}Be$
  - Multiresonance: $^{22}Ne(\alpha,n)^{25}Mg$



- *Capture cross sections (**electromagnetic interaction**)*
  - Non resonant: $^{6}Li(p,\gamma)^{7}Be$
  - Resonant, with $\ell_R=\ell_{min}$: $^{12}C(p,\gamma)^{13}N$
  - Resonant, with $\ell_R>\ell_{min}$: $^{7}Be(p,\gamma)^{8}B$
  - Multiresonance: 22
  - Subthreshold state: 12

- *Weak capture cross sections (**weak interaction**)*
  - Non resonant p
    3



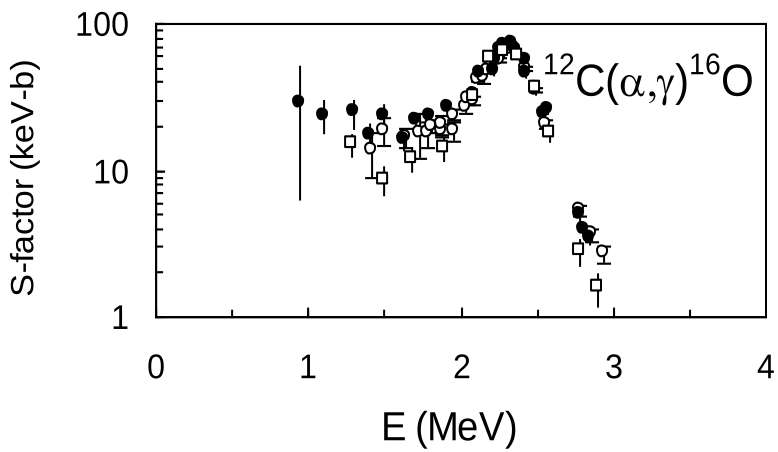


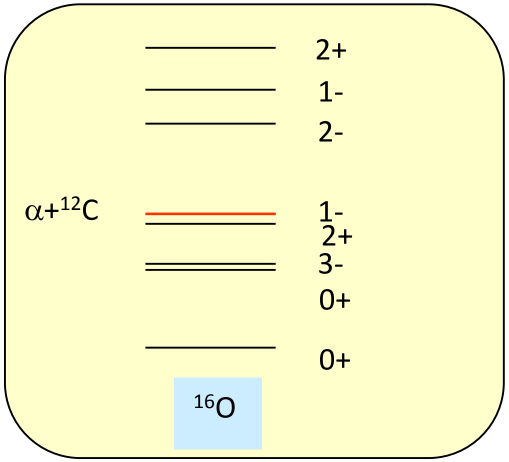
Non resonant

Resonant
$\ell_{min}=0$, E1
$\ell_R=1$, M1

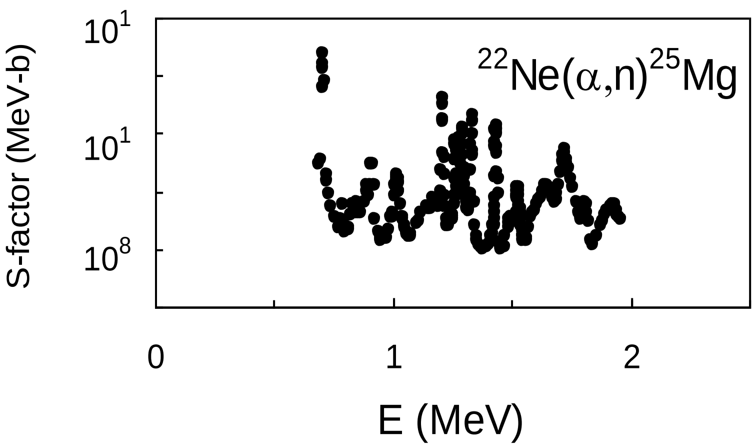
Resonant
$\ell_{min}=0$
$\ell_R=0$

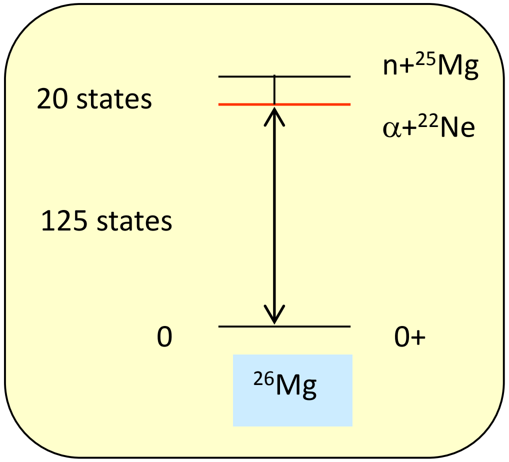
2. Low-energy cross sections





Subthreshlod states 2^+, 1^-





Multiresonant

General situation for heavy nuclei

3. Reaction rates

1. Definitions
2. Gamow peak
3. Non-resonant rates
4. Resonant rates

3. Reaction rates

1. Definition

Quantity used in astrophysics: reaction rate (integarl over the energy E)

$$N_A < \sigma v > = N_A \int \sigma(E) v N(E,T) dE$$

- Definition valid for resonant and non−resonant reactions
- N_A=Avogadro number
- T=temperature, v = velocity, k_B= Boltzmann constant ($k_B \sim \frac{1}{11.6}\,\mathrm{MeV}/10^9 K$)
- $N(E,T) = \left(\frac{8E}{\pi \mu m_N (k_B T)^3}\right)^{1/2} \exp\left(-\frac{E}{k_B T}\right)$ = Maxwell−Boltzmann distribution
- $\frac{1}{N_A<\sigma v>}$ =typical reaction time
- 2 approaches
  - numerical
  - analytical: non-resonant and resonant reactions treated separately

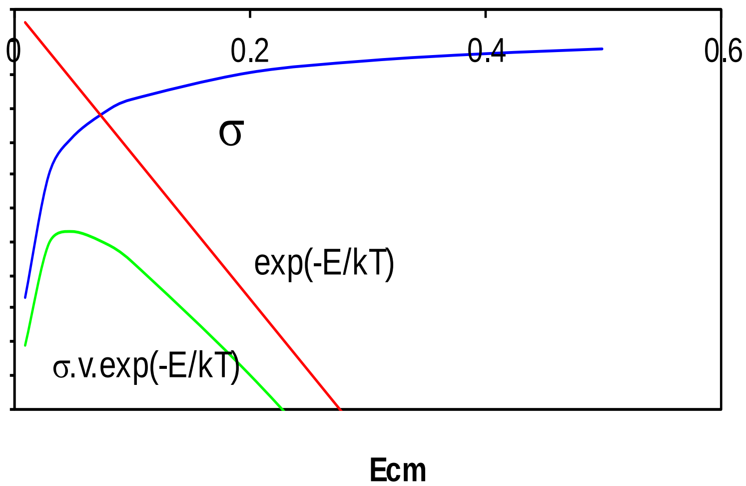
➔ essentially two energy dependences: $\exp\left(-\frac{E}{k_B T}\right)$: decreases with E

$\exp(-2\pi\eta)$: increases with E

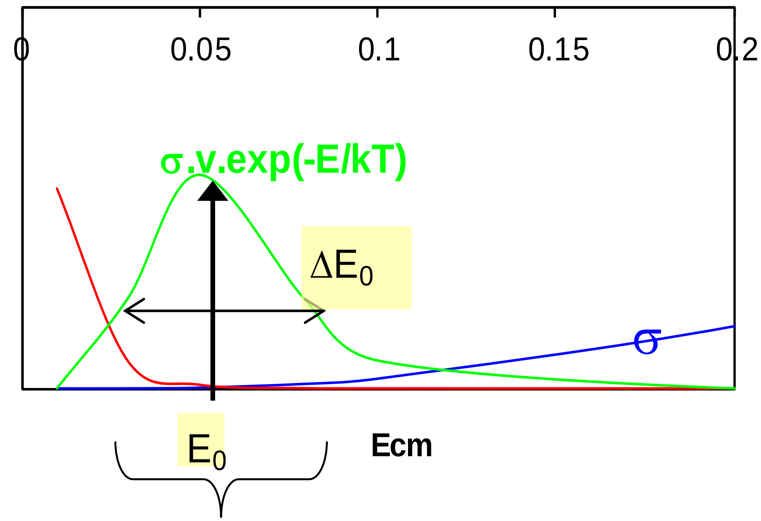
2. The Gamow peak

Defines the energy range relevant for the reaction rate (non-resonant reactions)

Logarithmic scale



Linear scale



Gamow peak (~ Gaussian, depends on T)

Gamow peak : $E_0 = 0.122\mu^{1/3}(Z_1 Z_2 T_9)^{2/3}$MeV: **lower than the Coulomb barrier**

increases with T

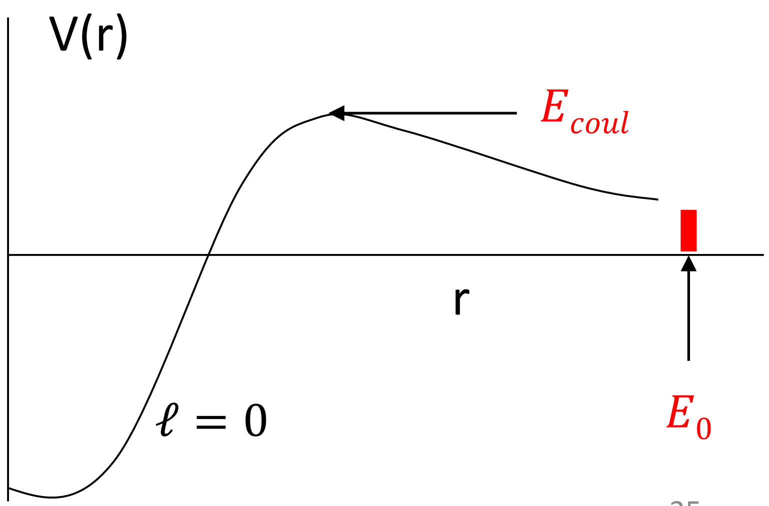
$$\Delta E_0 = 0.237\,\mu^{1/6}(Z_1 Z_2)^{1/3}\,T_9^{5/6}\ \text{MeV}$$

=Energy range where $\sigma(E)$ must be known ($T_9 = T$ in 10^9K)

Examples

Reaction	T (10^9 K)	E_0 (MeV)	ΔE_0 (MeV)	E_{coul} (MeV)	$\sigma(E_0)/\sigma(E_{coul})$
d + p	0.015	0.006	0.007	0.3	10^{-4}
$^3He + {}^3He$	0.015	0.021	0.012	1.2	10^{-13}
$\alpha + {}^{12}C$	0.2	0.3	0.17	3	10^{-11}
$^{12}C + {}^{12}C$	1	2.4	1.05	7	10^{-10}

- ❑ $E_0/E_{coul} \approx 0.3\, T_9^{2/3}$ (p and α)
- ❑ At low T_9, $E_0 \ll E_{coul}$ (coulomb barrier)
- ❑ Very low cross sections at stellar temperatures (different for neutrons: no barrier)



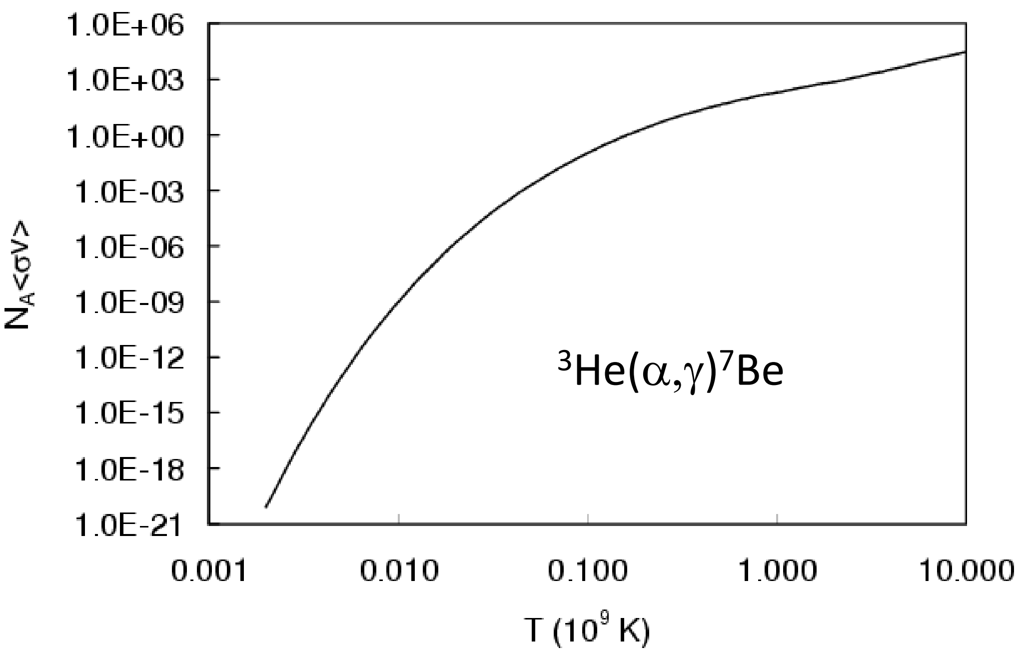
3. Non-resonant reaction rates

- Approximation: Taylor expansion about the minimum $E = E_0$: $2\pi\eta + E/k_BT \approx c_0 + \left(\frac{E-E_0}{2\Delta E_0}\right)^2$

$$\text{Then } <\sigma v> \approx \left(\frac{8}{\pi\mu m_N (k_BT)^3}\right)^{1/2} \exp\left(-3\frac{E_0}{k_BT}\right) \int S(E) \exp\left(-\left(\frac{E-E_0}{2\Delta E_0}\right)^2\right)\ dE$$

- $S(E)$ is assumed constant ($= S(E_0)$) in the Gamow peak

$$\rightarrow <\sigma v> \sim S(E_0)\exp\left(-3\frac{E_0}{k_BT}\right)/T^{2/3}, \text{with } \mathrm{E}_0 = 0.122\mu^{1/3}(Z_1Z_2T_9)^{2/3}\ \mathrm{MeV}$$



Very fast variation with the temperature

4. Resonant reaction rates

- General definition: $N_A < \sigma v > = N_A \int \sigma(E) v N(E,T) dE$

  here $\sigma(E)$ is given by the Breit-Wigner approximation

$$\sigma(E) \approx \frac{\pi}{k^2} \frac{(2J_R + 1)}{(2I_1 + 1)(2I_2 + 1)} \frac{\Gamma_1(E)\Gamma_2(E)}{(E_R - E)^2 + \Gamma^2/4}$$

- This provides

$$< \sigma v >_R = \left(\frac{2\pi}{\mu m_N k_B T} \right)^{3/2} \hbar^2 \omega\gamma \exp\left(-\frac{E_R}{k_B T} \right)$$

$$\omega\gamma = \frac{2J_R + 1}{(2I_1 + 1)(2I_2 + 1)} \frac{\Gamma_1 \Gamma_2}{\Gamma_1 + \Gamma_2}$$

  - $\omega\gamma$=resonance « strength »
  - Γ_1,Γ_2=partial widths in the entrance and exit channels
  - For a reaction (p,γ) : $\Gamma_\gamma << \Gamma_p \rightarrow \omega\gamma \sim \Gamma_\gamma$

- Valid for capture and transfer
- Rate strongly depends on the resonance enrgy

➔ In general: competition between resonant and non-resonant contributions

Tail contribution: for a given resonance

For a resonance: $< \sigma v > \sim \int S(E)\exp(-2\pi\eta - E/k_B T)\, dE$

- Non resonant: $S(E) \approx S_0$: 1 maximum at $E = E_0$
- Resonant: $S(E)$=BW : 2 maxima at $E = E_R$ does not depend on T
  $E = E_0$: depends on T

→ 2 contributions to the rate : $N_A < \sigma v > \approx N_A < \sigma v >_R + N_A < \sigma v >_T$

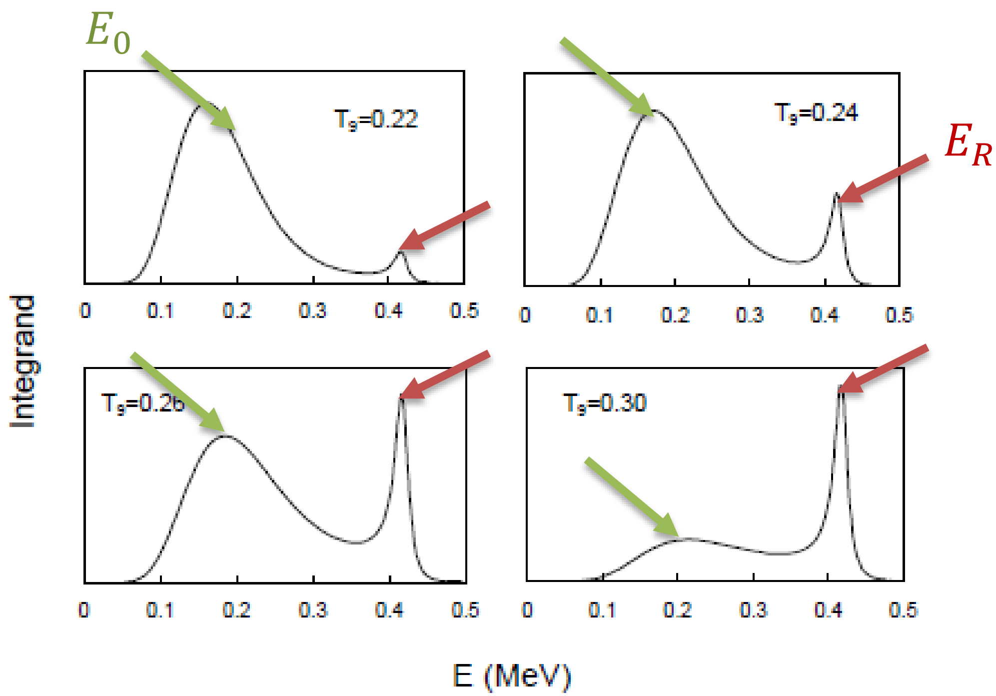
« resonant » ($N_A < \sigma v >_R$) « tail » ($N_A < \sigma v >_T$)

- $< \sigma v >_R = \left(\frac{2\pi}{\mu m_N k_B T}\right)^{3/2} \hbar^2 \omega\gamma \exp\left(-\frac{E_R}{k_B T}\right)$
- $< \sigma v >_T \sim S(E_0) \exp\left(-3\frac{E_0}{k_B T}\right)/T^{2/3}$, with $S(E_0) \sim \frac{\Gamma_1(E_0)\Gamma_2(E_0)}{(E_R - E_0)^2 + \Gamma^2/4}$

- Both contributions depend on temperature: in most cases one term is dominant
- « Critical temperature »: when $E_0 = E_R$ → separation not valid

3. Reaction rates

Example $^{12}C(p,\gamma)^{13}N$: $E_R = 0.42\ MeV$
Integrant $S(E)\exp(-2\pi\eta - E/k_BT)$



$^{12}C(p,\gamma)^{13}N$

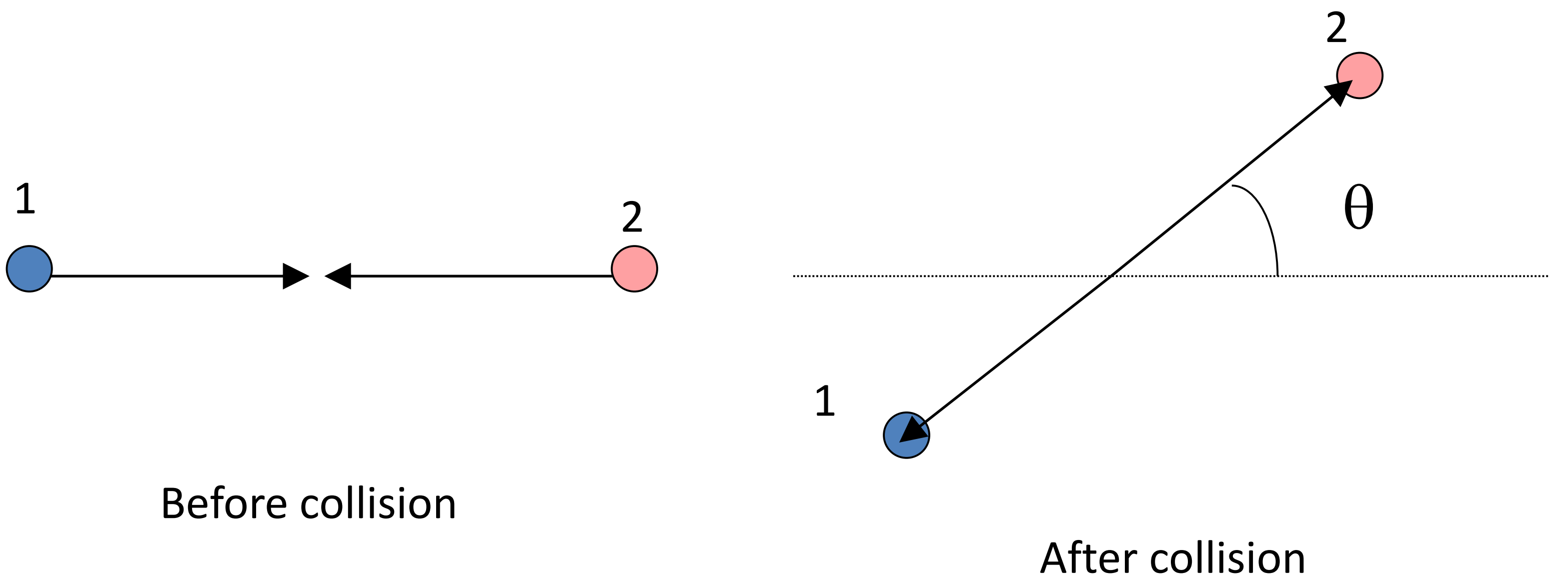
Above $T_9 \approx 0.3$: « resonant » contribution is dominant
requires $E_R, \omega\gamma$ only (no individual partial widths)
strongly depends on E_R: $\exp(-E_R/k_BT)$

Below $T_9 \approx 0.2$: $E_0 \ll E_R$: « tail » contribution is dominant
requires both widths
weakly depends on E_R: $1/((E_R - E_0)^2 + \Gamma^2/4)$

4. General scattering theory

1. Different models
2. Potential/optical model
3. Scattering amplitude and cross section (elastic scattering)

Scheme of the collision (elastic scattering)



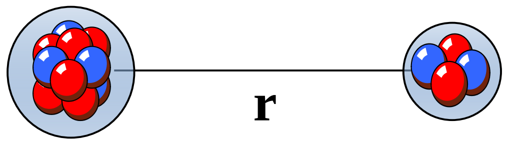
Center-of-mass system

1. Different models

Schrödinger equation: $H\Psi(\boldsymbol{r_1}, \boldsymbol{r_2}, \ldots \boldsymbol{r_A}) = E\Psi(\boldsymbol{r_1}, \boldsymbol{r_2}, \ldots \boldsymbol{r_A})$ with $E > 0$: scattering states

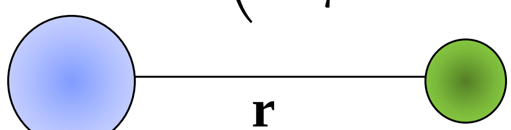
- A-body equation (microscopic models) $H = \sum_i t_i + \frac{1}{2}\sum_{i,j} v_{ij}(\boldsymbol{r_i} - \boldsymbol{r_j})$

  v_{ij}=nucleon-nucleon interaction



- Optical model: internal structure of the nuclei is neglected
  the particles interact by a nucleus-nucleus potential
  absorption simulated by the imaginary part = optical potential

$$H\Psi(\boldsymbol{r}) = \left(-\frac{\hbar^2}{2\mu}\Delta + V(\boldsymbol{r})\right)\Psi(\boldsymbol{r}) = E\Psi(\boldsymbol{r})$$



- Additional assumptions: elastic scattering
  no Coulomb interaction
  spins zero

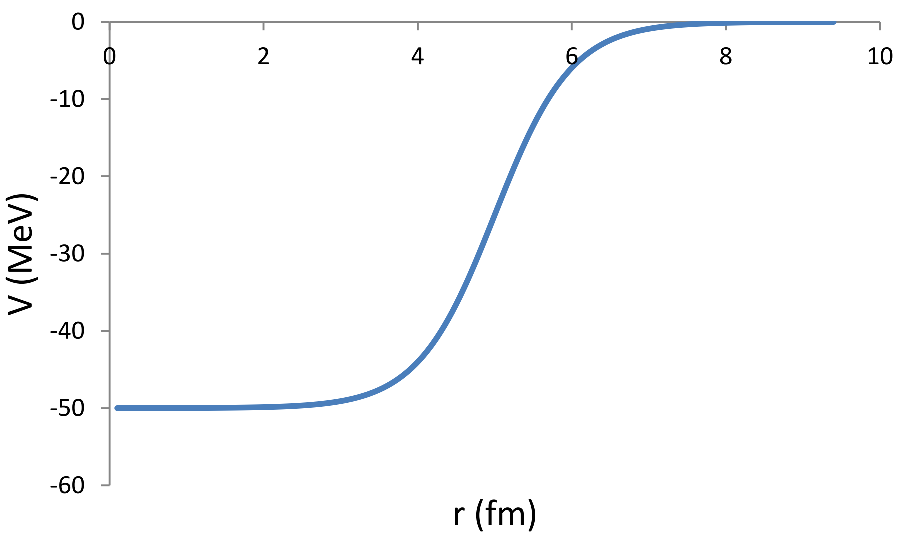
2. Potential/Optical model

Two contributions to the nucleus-nucleus potential: nuclear $V_N(r)$ and Coulomb $V_C(r)$

Typical nuclear potential: $V_N(r)$ (short range, attractive)

- examples: Gaussian $V_N(r) = -V_0 \exp(-(r/r_0)^2)$

  Woods-Saxon: $V_N(r) = -\frac{V_0}{1+\exp\left(\frac{r-r_0}{a}\right)}$
- Real at low energies
- parameters are fitted to experiment
- no analytical solution of the Schrödinger equation



Woods-Saxon potential
r_0 =range (~sum of the radii)
a= diffuseness (~0.5 fm)

Figure: V_0=50 MeV, r_0=5 fm, $a = 0.5$ fm

4. General scattering theory

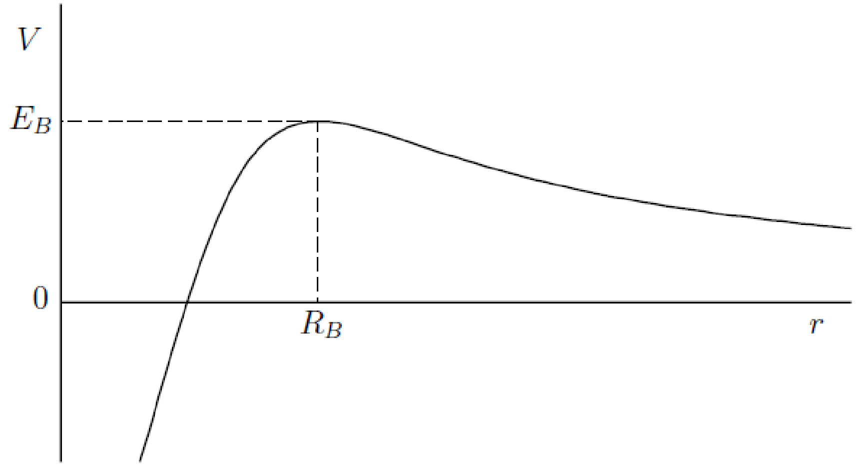
Coulomb potential: long range, repulsive

- « point-point » potential : $V_C(r) = \frac{Z_1 Z_2 e^2}{r}$
- « point-sphere » potential : (radius R_C)

$$V_C(r) = \frac{Z_1 Z_2 e^2}{r} \text{ for } r \geq R_C$$
$$V_C(r) = \frac{Z_1 Z_2 e^2}{2R_C}\left(3 - \left(\frac{r}{R_C}\right)^2\right) \text{ for } r \leq R_C$$

Total potential : $V(r) = V_N(r) + V_C(r)$: presents a maxium at the Coulomb barrier

- radius $r = R_B$
- height $V(R_B) = E_B$

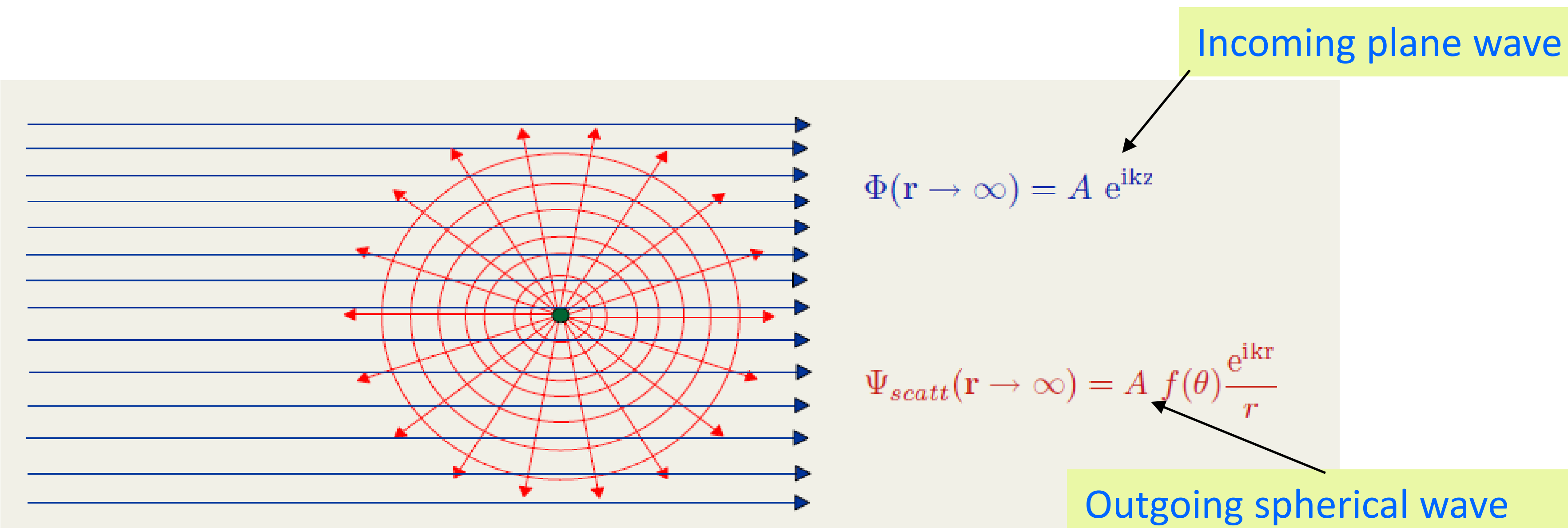


3. Scattering amplitude and cross section

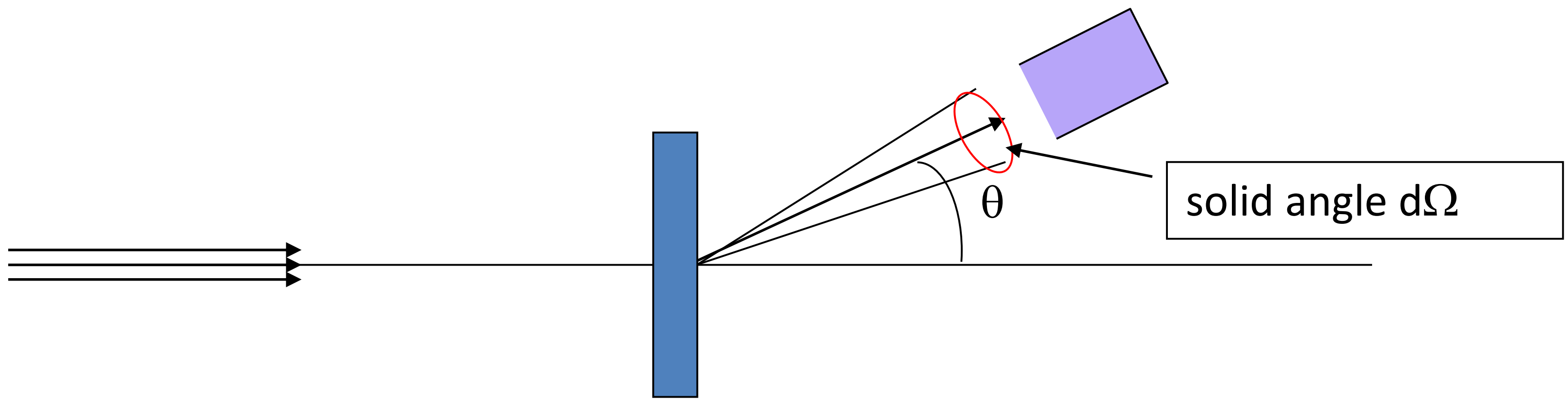
$$H\Psi(\boldsymbol{r}) = \left(-\frac{\hbar^2}{2\mu}\Delta + V(\boldsymbol{r})\right)\Psi(\boldsymbol{r}) = E\Psi(\boldsymbol{r})$$

At large distances : $\Psi(\boldsymbol{r}) \to A\left(e^{i\boldsymbol{k}\cdot\boldsymbol{r}} + f(\theta)\frac{e^{ikr}}{r}\right)$ (with *z* along the beam axis)

where: k=wave number: $k^2 = 2\mu E/\hbar^2$
A =amplitude (scattering wave function is not normalized to unity)
$f(\theta)$ =scattering amplitude (length)



4. General scattering theory



Cross section: $$\frac{d\sigma}{d\Omega} = |f(\theta)|^2, \sigma = \int \frac{d\sigma}{d\Omega} d\Omega$$

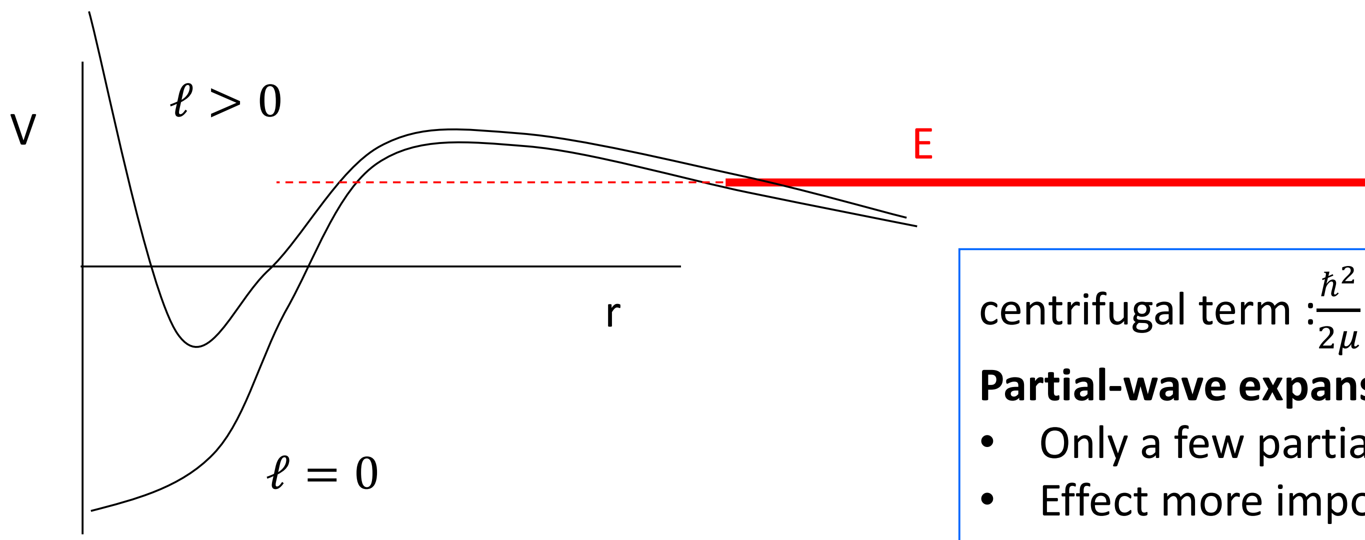
- Cross section obtained from the asymptotic part of the wave function
  General problem for scattering states: the wave function must be known up to large distances
- "Direct" problem: determine σ from the potential
- "Inverse" problem : determine the potential V from σ
- Angular distribution: E fixed, θ variable
- Excitation function: θ variable, E fixed,

4. General scattering theory

Main issue: determining the scattering amplitude $f(\theta)$ (and wave function $\Psi(\boldsymbol{r})$)

At low energies: partial wave expansion: $\Psi(\boldsymbol{r}) = \sum_{lm} \Psi_l(r) Y_l^m(\theta, \phi)$

- Scattering wave function necessary to compute cross sections
- Must be determined for each partial wave l
- Main interest: few partial waves at low energies



centrifugal term : $\frac{\hbar^2}{2\mu} \frac{\ell(\ell+1)}{r^2}$

Partial-wave expansion

- Only a few partial waves contribute
- Effect more important for nucleon-nucleus: $\mu \approx 1$
- Strongest for neutron: no barrier for $\ell = 0$.

4. Phase-shift method

- Goal: solving the Schrodinger equation

$$\left(-\frac{\hbar^2}{2\mu}\Delta + V(\boldsymbol{r})\right)\Psi(\boldsymbol{r}) = E\Psi(\boldsymbol{r})$$

with a partial-wave expansion

$$\Psi(\boldsymbol{r}) = \sum_{\ell,m} \frac{u_\ell(r)}{r} Y_\ell^m(\Omega_r)\, Y_\ell^{m*}(\Omega_k)$$

- Simplifying assumtions
  - neutral systems (no Coulomb interaction)
  - spins zero
  - single-channel calculations → elastic scattering

- The wave function is expanded as

$$\Psi(\boldsymbol{r}) = \sum_{\ell,m} \frac{u_\ell(r)}{r} Y_\ell^m(\Omega_r)\, Y_\ell^{m*}(\Omega_k)$$

- This provides the Schrödinger equation for each partial wave ($\Omega_k = 0 \ \rightarrow m = 0$)

$$-\frac{\hbar^2}{2\mu}\left(\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2}\right) u_\ell + V(r) u_\ell = E u_\ell$$

- Large distances : $r \rightarrow \infty, V(r) \rightarrow 0$

$u_\ell'' - \frac{\ell(\ell+1)}{r^2} u_\ell + k^2 u_\ell = 0$ Bessel equation $\rightarrow u_\ell(r) = r j_\ell(kr), r n_\ell(kr)$

- Remarks
  - must be solved for all ℓ values
  - at low energies: few partial waves in the expansion
  - at small r: $u_\ell(r) \rightarrow r^{\ell+1}$

4. General scattering theory

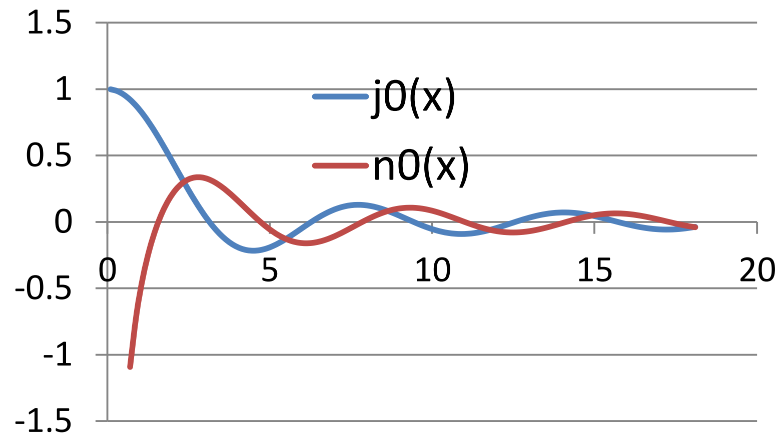
For small x: $j_l(x) \to \frac{x^l}{(2l+1)!!}$

$n_l(x) \to -\frac{(2l-1)!!}{x^{l+1}}$

For large x: $j_l(x) \to \frac{1}{x}\sin(x - l\pi/2)$

$n_l(x) \to -\frac{1}{x}\cos(x - l\pi/2)$

Examples: $j_0(x) = \frac{\sin x}{x}$, $n_0(x) = -\frac{\cos\ x}{x}$



At large distances: $u_\ell(r)$ is a linear combination of $rj_\ell(kr)$ and $rn_\ell(kr)$

$$u_\ell(r) \to C_l\, r\, \left(j_\ell(kr)\ - \tan \delta_\ell \times n_\ell(kr)\right)$$

With δ_ℓ = phase shift (provides information about the potential): If V=0 → $\delta_\ell = 0$

Derivation of the elastic cross section

- Identify the asymptotic behaviours

$$\Psi(\boldsymbol{r}) \to A\left(e^{i\boldsymbol{k}\cdot\boldsymbol{r}} + f(\theta)\frac{e^{ikr}}{r}\right)$$

$$\Psi(\boldsymbol{r}) \to \sum_\ell C_\ell\big(j_\ell(kr) \; - \tan\delta_\ell \times n_\ell(kr)\big)Y_\ell^0(\Omega_r)\sqrt{\frac{2\ell+1}{4\pi}}$$

- Provides coefficients C_ℓ and scattering amplitude $f(\theta)$ (elastic scattering)

$$f(\theta,E) = \frac{1}{2ik}\sum_{\ell=0}^{\infty}(2\ell+1)(\exp(2i\delta_\ell(E)) - 1)\, P_\ell(\cos\theta)$$

$$\frac{d\sigma(\theta,E)}{d\Omega} = |f(\theta,E)|^2$$

- Integrated cross section (neutral systems only)

$$\sigma = \frac{\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)\sin^2\delta_\ell$$

- In practice, the summation over ℓ is limited to some ℓ_{max}

4. General scattering theory

$$\frac{d\sigma(\theta,E)}{d\Omega} = |f(\theta,E)|^2 \text{ with } f(\theta,E) = \frac{1}{2ik}\sum_{\ell=0}^{\infty}(2\ell+1)(\exp(2i\delta_\ell(E))-1)\,P_\ell(\cos\theta)$$

➔ factorization of the dependences in E and θ
low energies: small number of ℓ values ($\delta_\ell \to 0$ when ℓ increases)
high energies: large number (→ alternative methods)

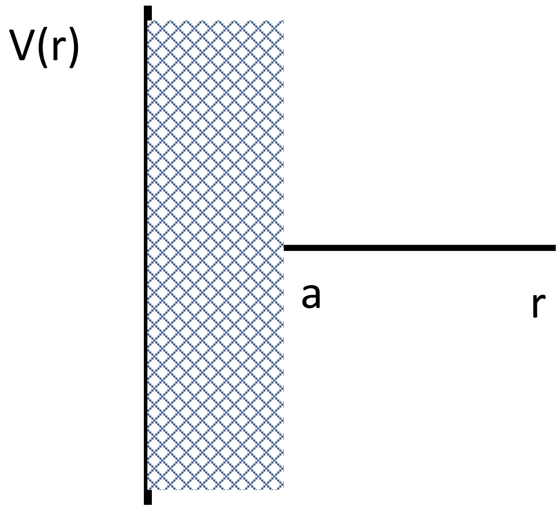
General properties of the phase shifts

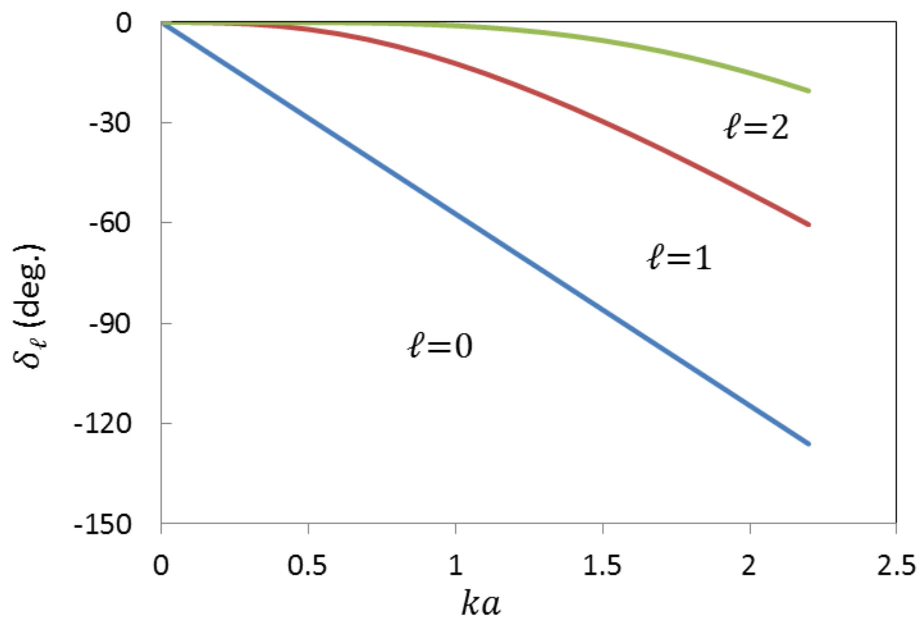
1. The phase shift (and all derivatives) are continuous functions of E
2. The phase shift is known within nπ: $\exp 2i\delta = \exp(2i(\delta + n\pi))$
3. Levinson theorem
   - $\delta_\ell(E=0)$ is arbitrary
   - $\delta_\ell(0) - \delta_\ell(\infty) = \mathrm{N}\pi$, where N is the number of bound states in partial wave ℓ
   - Example: p+n, $\ell = 0$: $\delta_0(0) - \delta_0(\infty) = \pi$ (bound deuteron)
     $\ell = 1$: $\delta_1(0) - \delta_1(\infty) = 0$ (no bound state for $\ell = 1$)

4. General scattering theory

- Example: hard sphere (radius a)
- continuity at $r = a \rightarrow j_\ell(ka) - \tan \delta_\ell \times n_\ell(ka) = 0$ $\quad \rightarrow \tan \delta_\ell = \frac{j_\ell(ka)}{n_\ell(ka)}$

$\rightarrow \delta_0 = -ka$



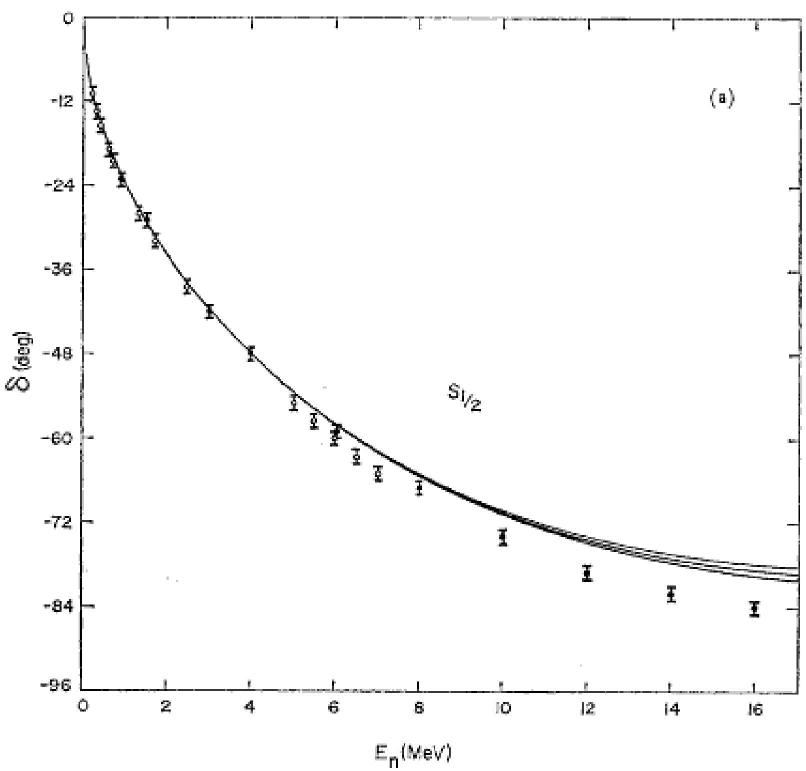


At low energies: $\quad \delta_\ell(E) \rightarrow -\frac{(ka)^{2\ell+1}}{(2\ell+1)!!(2l-1)!!}$, in general: $\delta_\ell(E) \sim k^{2\ell+1}$

➔ Strong difference between $\ell = 0$ (no barrier) et $\ell \neq 0$ (centrifugal barrier) (typical to neutron-induced reactions) example : α+n phase shift $\ell = 0$

consistent with the hard sphere ($a \sim 2.2$ fm)

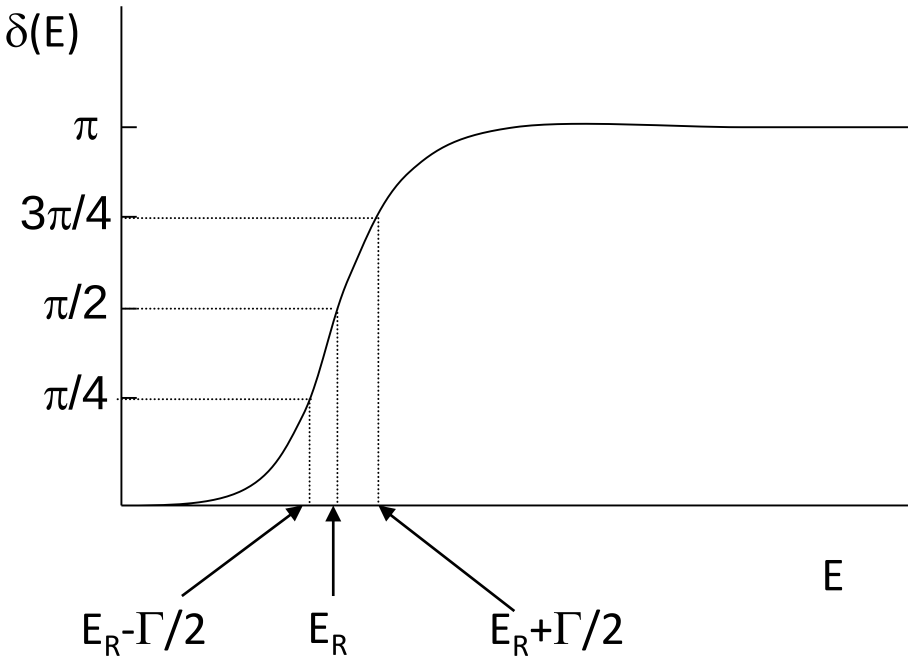
R. A. ARNDT AND L. D. ROPER



5. Resonances

Resonances: $\delta_R(E) \approx \operatorname{atan} \frac{\Gamma}{2(E_R - E)}$ = Breit-Wigner approximation

E_R=resonance energy
Γ=resonance width: related to the lifetime $\Gamma\tau = \hbar$



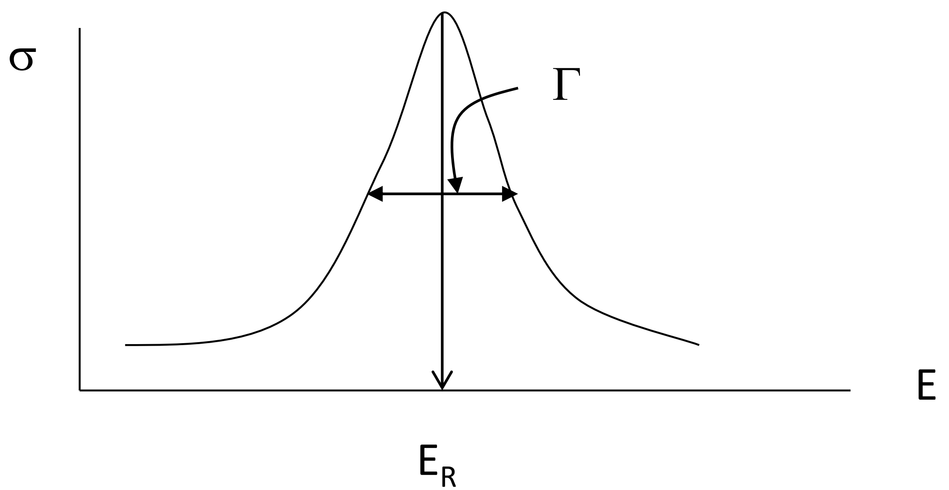
- Narrow resonance: Γ small, τ large
- Broad resonance: Γ large, τ small
- Bound states: $\Gamma = 0$, $E_R < 0$

4. General scattering theory

Cross section (for neutrons)

$\sigma(E) = \frac{\pi}{k^2}\sum_\ell (2\ell+1)|\exp(2i\delta_\ell) - 1|^2$ maximum for $\delta = \frac{\pi}{2}$

Near the resonance: $\sigma(E) \approx \frac{4\pi}{k^2}(2\ell_R + 1)\frac{\Gamma^2/4}{(E_R - E)^2 + \Gamma^2/4}$, where ℓ_R=resonant partial wave

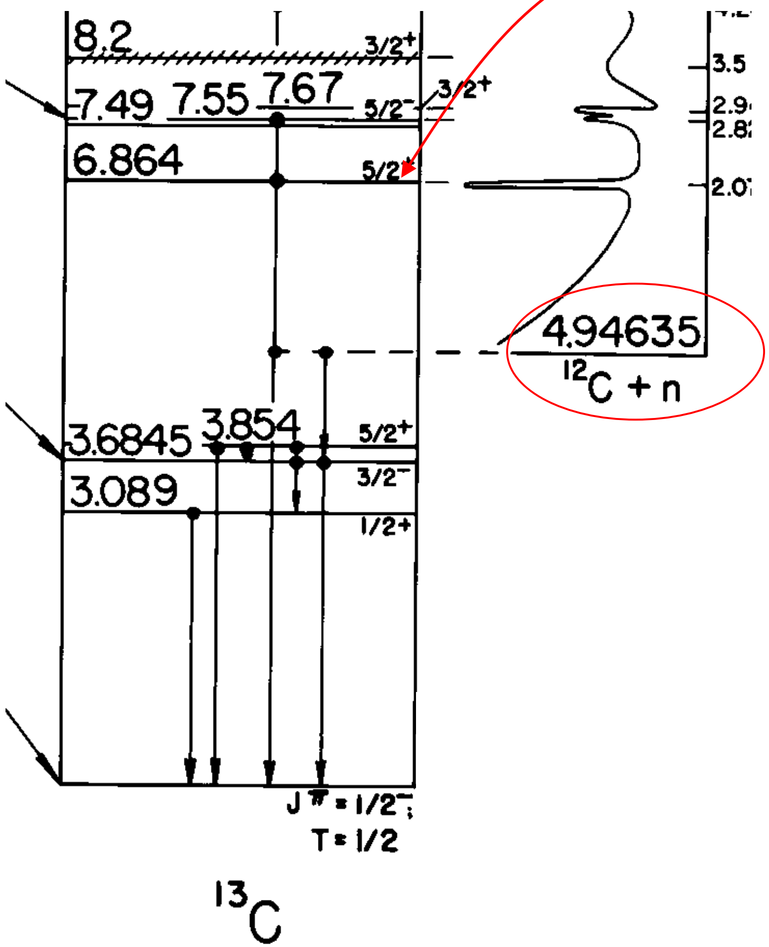


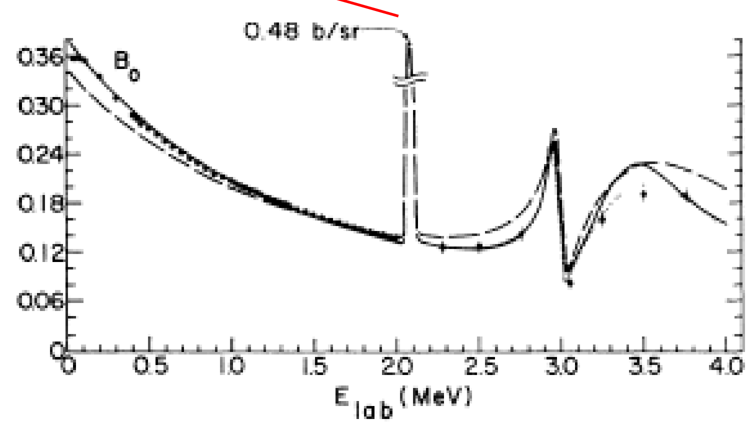
In practice:

- Peak not symmetric (Γ depends on E)
- « Background » neglected (other ℓ values)
- Differences with respect to Breit-Wigner

4. General scattering theory

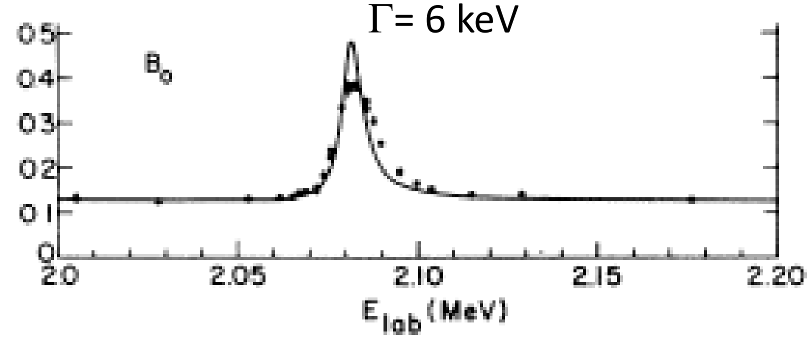
Example: n+^{12}C





E_{cm}=1.92 MeV

Γ= 6 keV



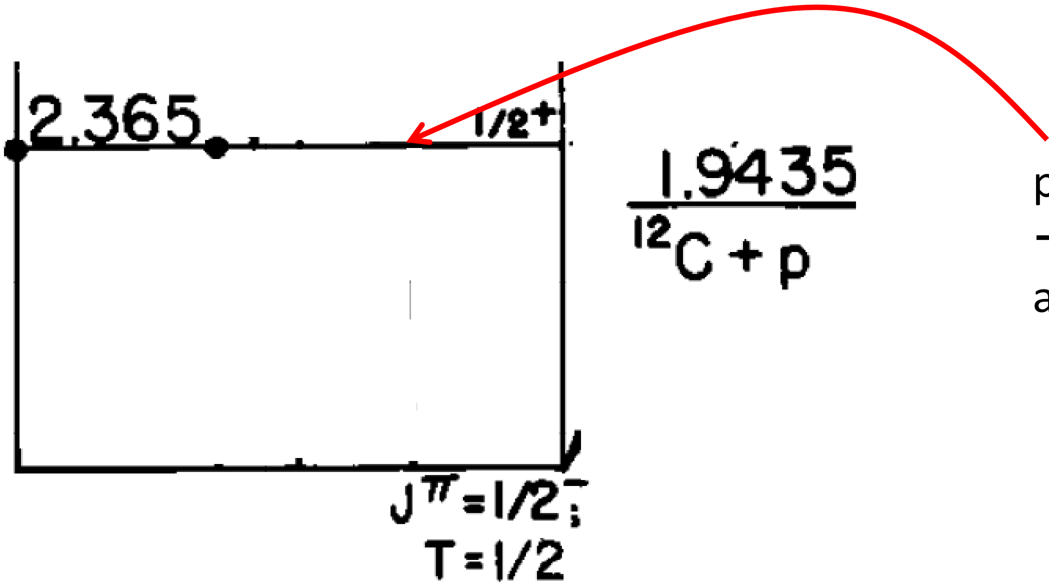
Comparison of 2 typical times:

a. Lifetime of the resonance: $\tau_R = \hbar/\Gamma \approx \frac{197}{3.10^{23}\times 6.10^{-3}} \approx 1.1 \times 10^{-19} s$

b. Interaction time without resonance: $\tau_{NR} = d/v \approx 5.2 \times 10^{-22} s$ ➔ $\tau_{NR} << \tau_R$

Narrow resonances

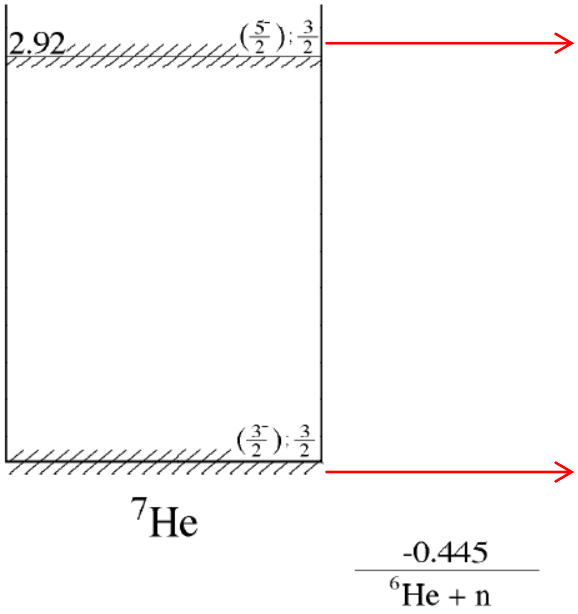
- Small particle width
- long lifetime
- can be approximetly treated by *neglecting the asymptotic behaviour of the wave function*



proton width=32 keV
→ can be described in a bound-state approximation

Broad resonances

- Large particle width
- Short lifetime
- *asymptotic behaviour of the wave function is important*
  → rigorous scattering theory
  → bound-state model complemented by other tools (complex scaling, etc.)



very broad resonance: Γ=1990 keV

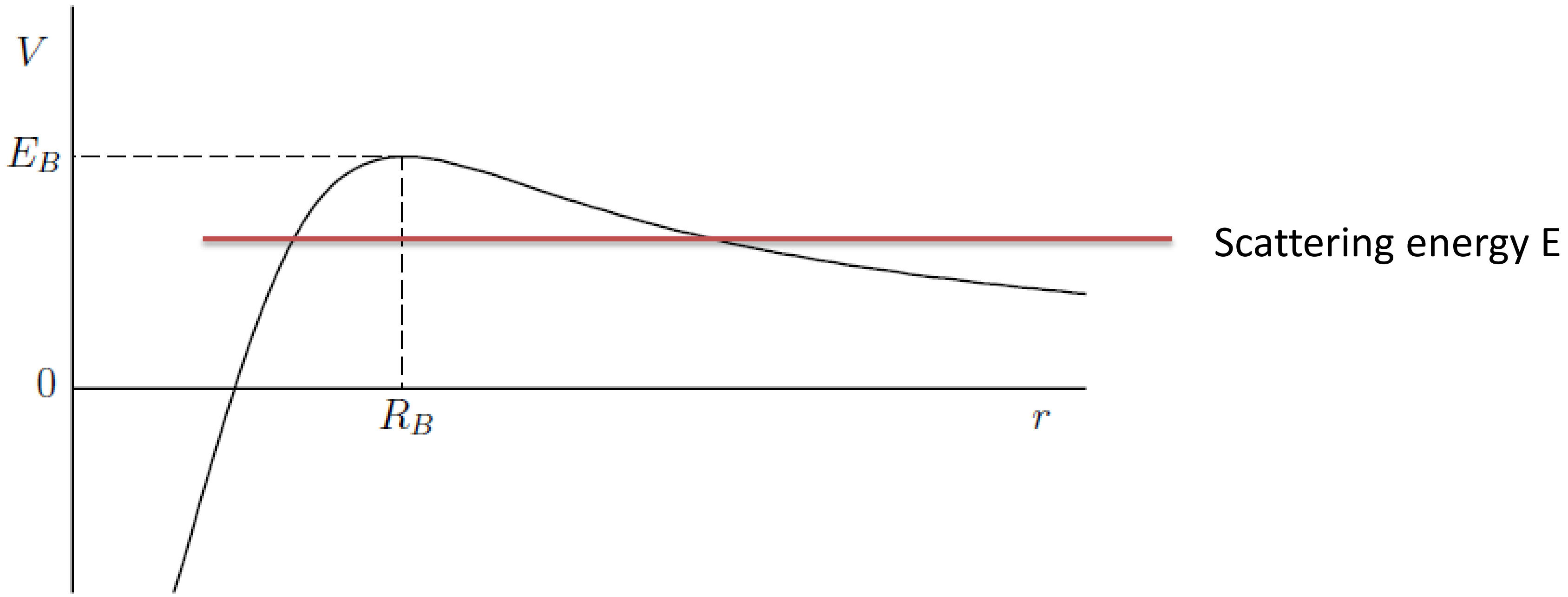
ground state unstable: Γ=120 keV

5. Generalizations

- Extension to charged systems
- Numerical calculation
- Optical model (high energies → absorption)
- Extension to multichannel problems

5. Generalizations

Generalization 1: charged systems



$E \gg E_B$: weak coulomb effects (V negligible with respect to E)
$E < E_B$: strong coulomb effects (ex: nuclear astrophysics)

4. Generalizations

A. Asymptotic behaviour

Neutral systems

$$\left(-\frac{\hbar^2}{2\mu}\Delta + V_N(r) - E\right)\Psi(\boldsymbol{r}) = 0$$

$$\Psi(\boldsymbol{r}) \to \exp(i\boldsymbol{k}\cdot\boldsymbol{r}) + f(\theta)\frac{\exp(ikr)}{r}$$

Charged systems

$$\left(-\frac{\hbar^2}{2\mu}\Delta + V_N(r) + \frac{Z_1Z_2e^2}{r} - E\right)\Psi(\boldsymbol{r}) = 0$$

$$\Psi(\boldsymbol{r}) \to \exp(i\boldsymbol{k}\cdot\boldsymbol{r} + i\eta\ln(\boldsymbol{k}\cdot\boldsymbol{r} - kr)) + f(\theta)\frac{\exp(i(kr - \eta\ln 2kr))}{r}$$

$$\eta = \frac{Z_1Z_2e^2}{\hbar v}$$

- Sommerfeld parameter
- « measurement » of coulomb effects
- Increases at low energies
- Decreases at high energies

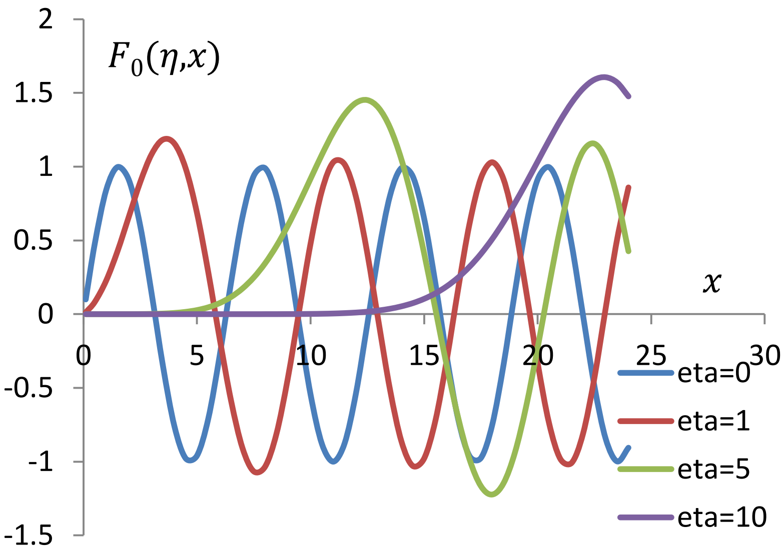
B. Phase shifts with the coulomb potential

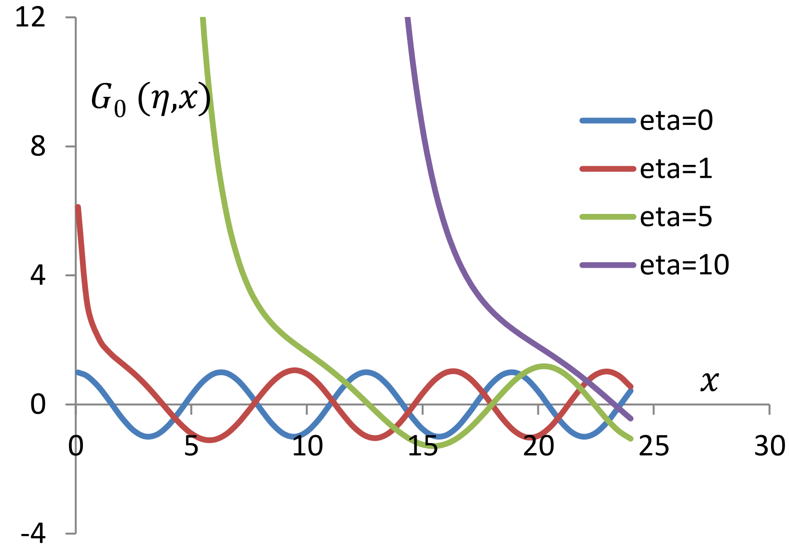
Neutral system: $\left(\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2} + k^2\right) R_\ell = 0$

Bessel equation : solutions $j_\ell(kr), n_\ell(kr)$

Charged system: $\left(\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2} - 2\frac{\eta k}{r} + k^2\right) R_\ell = 0$:

Coulomb equation: solutions $F_\ell(\eta, kr), G_\ell(\eta, kr)$





5. Generalizations

- Incoming and outgoing functions (complex)

$$I_\ell(\eta,x) = G_\ell(\eta,x) - iF_\ell(\eta,x) \to e^{-i(x-\frac{\ell\pi}{2}-\eta\ln 2x+\sigma_\ell)}$$: incoming wave

$$O_\ell(\eta,x) = G_\ell(\eta,x) + iF_\ell(\eta,x) \to e^{i(x-\frac{\ell\pi}{2}-\eta\ln 2x+\sigma_\ell)}$$: outgoing wave

- Phase-shift definition
  - neutral systems : $R_\ell(r) \to rA\big(j_\ell(kr) - \tan\delta_\ell\, n_\ell(kr)\big)$
  - charged systems: $R_\ell(r) \to A\big(F_\ell(\eta,kr) + \tan\delta_\ell\, G_\ell(\eta,kr)\big)$

$$\to B(\cos\delta_\ell\, F_\ell(\eta,kr) + \sin\delta_\ell\, G_\ell(\eta,kr))$$

$$\to C\big(I_\ell(\eta,kr) - U_\ell O_\ell(\eta,kr)\big)$$

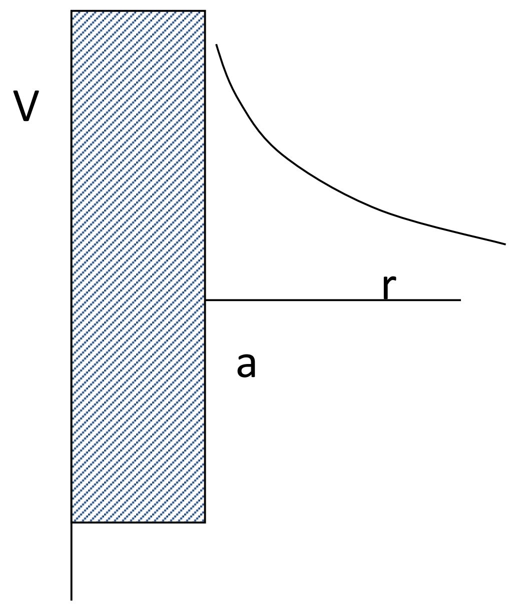
3 equivalent definitions (amplitude is different)

Collision matrix (=scattering matrix)

$$U_\ell = e^{2i\delta_\ell}$$: modulus $|U_\ell| = 1$

5. Generalizations

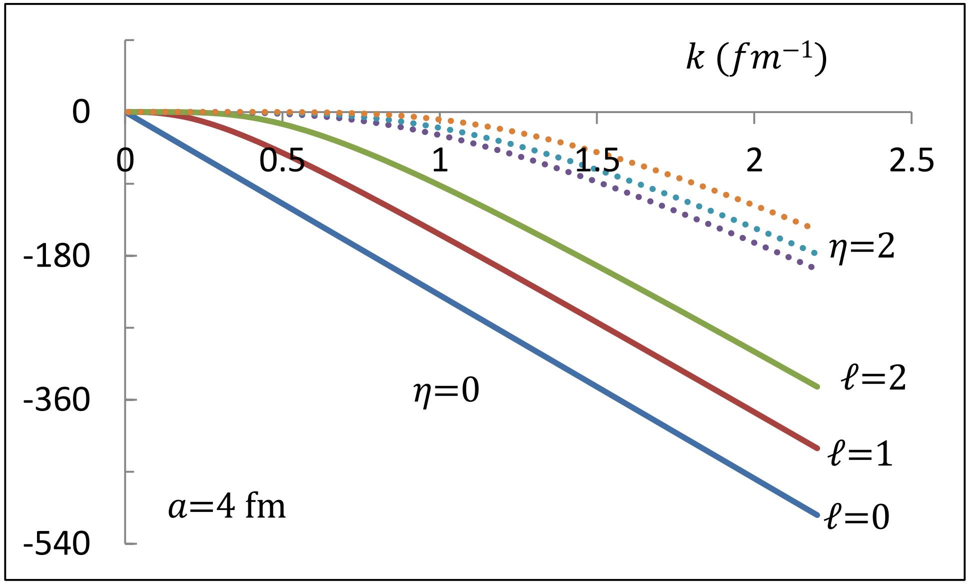
Example: hard-sphere potential



$$V(r) = \frac{Z_1 Z_2 e^2}{r} \, for \; r > a$$

$$\infty \; for \; r < a$$

phase shift: $\tan \delta_\ell = -\frac{F_\ell(\eta, ka)}{G_\ell(\eta, ka)}$



5. Generalizations

C. Rutherford cross section

For a Coulomb potential ($V_N = 0$):

- scattering amplitude : $f_c(\theta) = -\frac{\eta}{2k \sin^2 \theta/2} e^{2i(\sigma_0 - \eta \ln \sin \theta/2)}$
- Coulomb phase shift for $\ell = 0$: $\sigma_0 = \arg \Gamma(1 + i\eta)$

We get the Rutherford cross section:

$$\frac{d\sigma_C}{d\Omega} = |f_c(\theta)|^2 = \left(\frac{Z_1 Z_2 e^2}{4E \sin^2 \theta/2}\right)^2$$

- Increases at low energies
- Diverges at $\theta = 0$ → no integrated cross section

D. Cross sections with nuclear and Coulomb potentials

- The general defintions

$$f(\theta) = \frac{1}{2ik}\sum_{\ell=0}^{\infty}(2\ell+1)(\exp(2i\delta_\ell)-1)\,P_\ell(\cos\theta)$$
$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2$$

are still valid

- $Problem$: very slow convergence with ℓ
  → separation of the nuclear and coulomb phase shifts

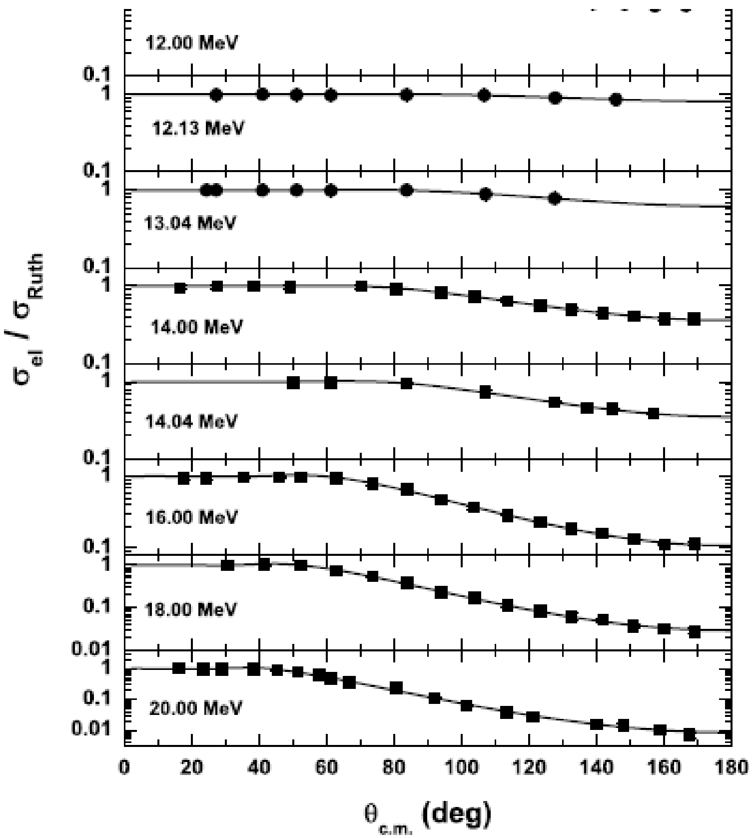
$$\delta_\ell = \delta_\ell^N + \sigma_\ell$$
$$\sigma_\ell = \arg\Gamma(1+\ell+i\eta)$$

- Scattering amplitude $f(\theta)$ written as $f(\theta) = f^C(\theta) + f^N(\theta)$
  - $f^C(\theta) = \frac{1}{2ik}\sum_{\ell=0}^{\infty}(2\ell+1)(\exp(2i\sigma_\ell)-1)\,P_\ell(\cos\theta) = -\frac{\eta}{2k\sin^2\theta/2}e^{2i(\sigma_0-\eta\ln\sin\theta/2)}$
    → analytical
  - $f^N(\theta) = \frac{1}{2ik}\sum_{\ell=0}^{\infty}(2\ell+1)\exp(2i\sigma_\ell)\;(\exp(2i\delta_\ell^N)-1)\,P_\ell(\cos\theta)$
    → converges rapidly

5. Generalizations

Total cross section: $\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = |f^C(\theta) + f^N(\theta)|^2$

- Nuclear term dominant at 180°
- Coulomb term coulombien dominant at small angles → used to normalize experiments
- Coulomb amplitude strongly depends on the angle → $\frac{d\sigma/d\Omega}{d\sigma_C/d\Omega}$
- Integrated cross section $\int \frac{d\sigma}{d\Omega} d\Omega$ is not defined



System ^{6}Li+^{58}Ni

- $E_{cm} = \frac{58}{64} E_{lab}$
- Coulomb barrier
$$E_B \sim \frac{3 * 28 * 1.44}{7} \sim 17 \text{ MeV}$$
- Below the barrier: $\sigma \sim \sigma_C$
- Above E_B : σ is different from σ_C

5. Generalizations

Generalization 2: numerical calculation

For some potentials: analytic solution of the Schrödinger equation

In general: no analytical solution → numerical approach

$$-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2}u_\ell(r) + (V(r) - E)u_\ell(r) = 0$$

with: $$V(r) = V_N(r) + \frac{Z_1 Z_2 e^2}{r} + \frac{\hbar^2}{2\mu}\frac{\ell(\ell+1)}{r^2}$$

$$u_\ell(r) \rightarrow F_\ell(kr,\eta)\cos\delta_\ell + G_\ell(kr,\eta)\sin\delta_\ell$$

Numerical solution : discretization N points, with mesh size h

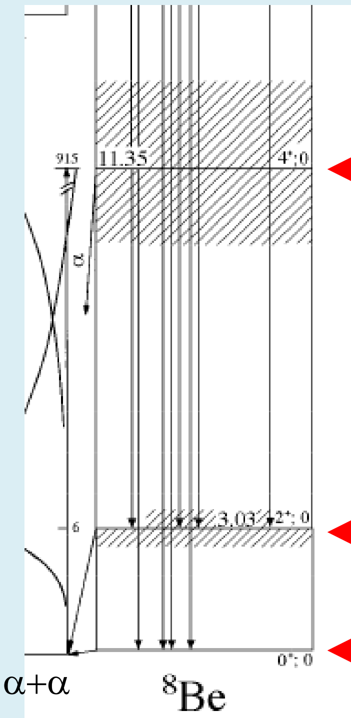
- $u_l(0) = 0$
- $u_l(h) = 1$ (or any constant)
- $u_l(2h)$ is determined numerically from $u_l(0)$ and $u_l(h)$ (Numerov algorithm)
- $u_l(3h), \dots u_l(Nh)$
- for large r: matching to the asymptotic behaviour → phase shift

Bound states: same idea (but energy is unknown)

5. Generalizations

Example: α+α

Experimental spectrum of ^{8}Be

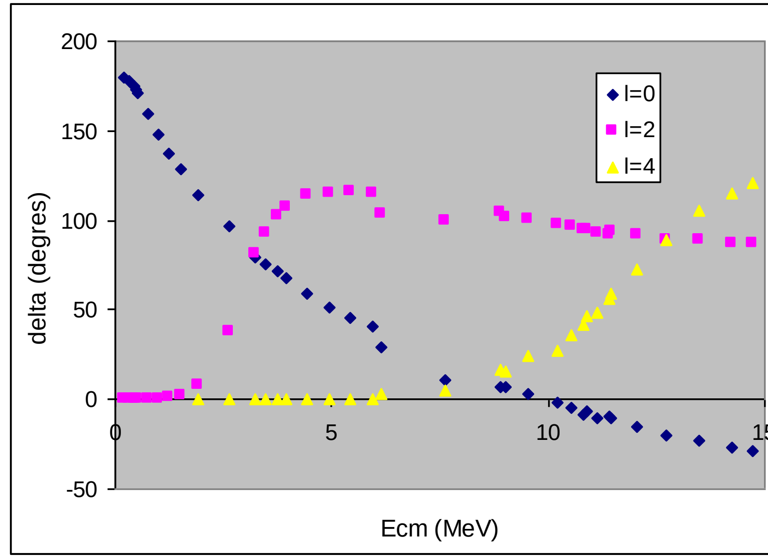


4^+
E~11 MeV
Γ~3.5 MeV

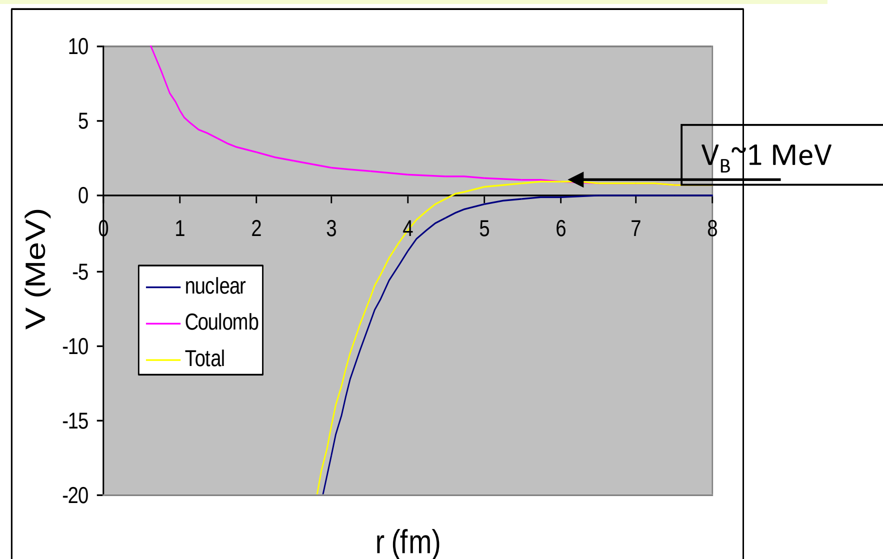
2^+
E~3 MeV
Γ~1.5 MeV

0^+
E=0.09 MeV
Γ=6 eV

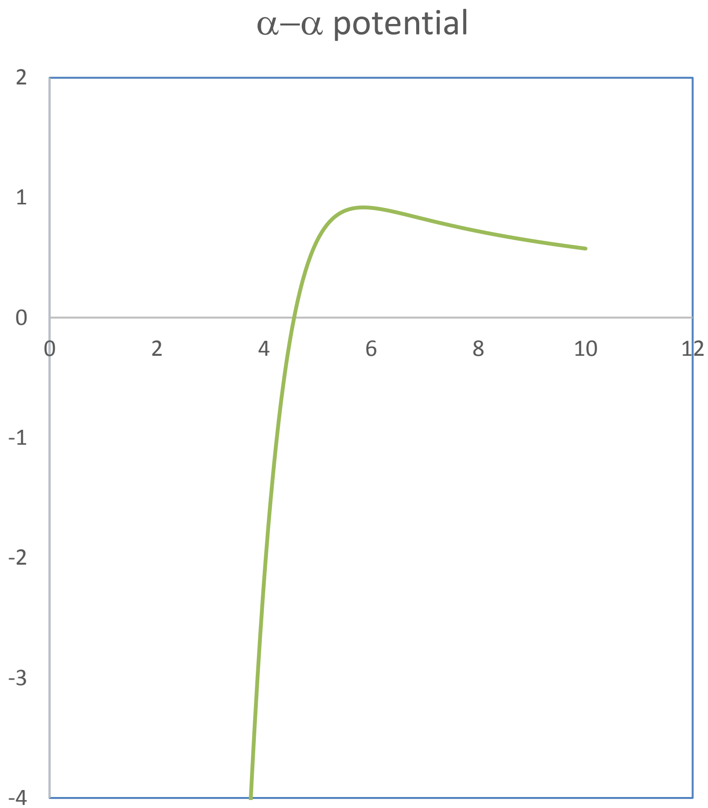
Experimental phase shifts



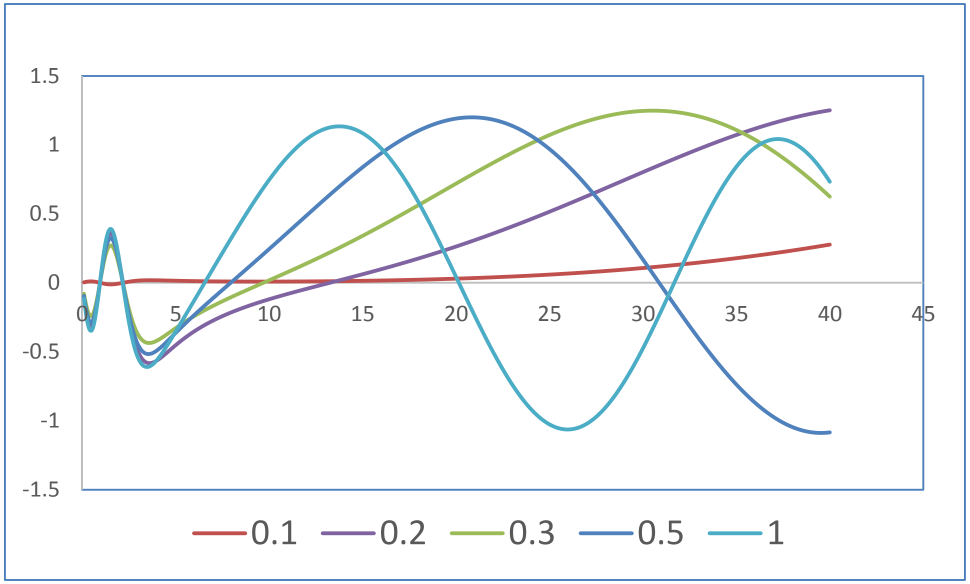
Potential: $V_N(r)=-122.3*\exp(-(r/2.13)^2)$

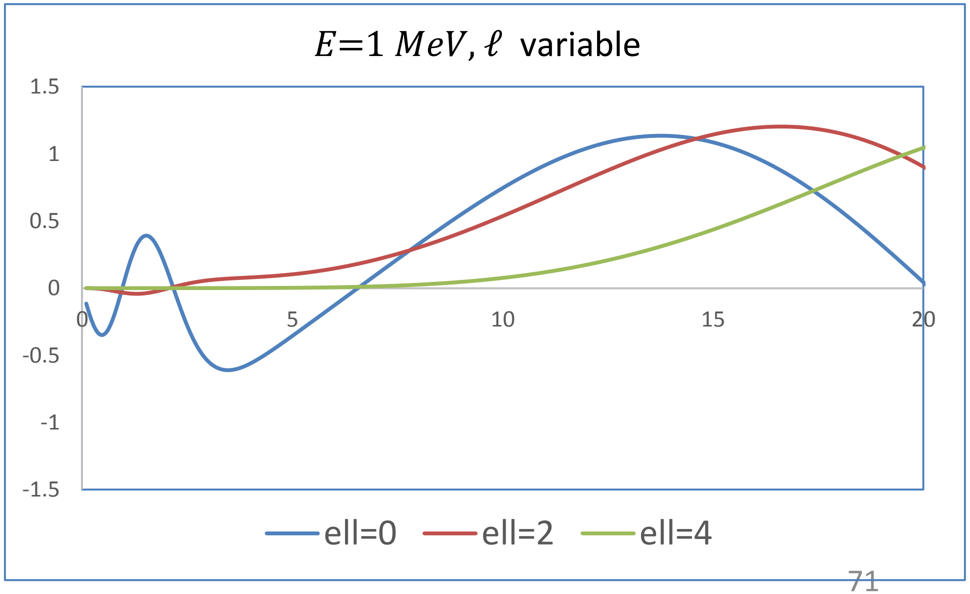


5. Generalizations



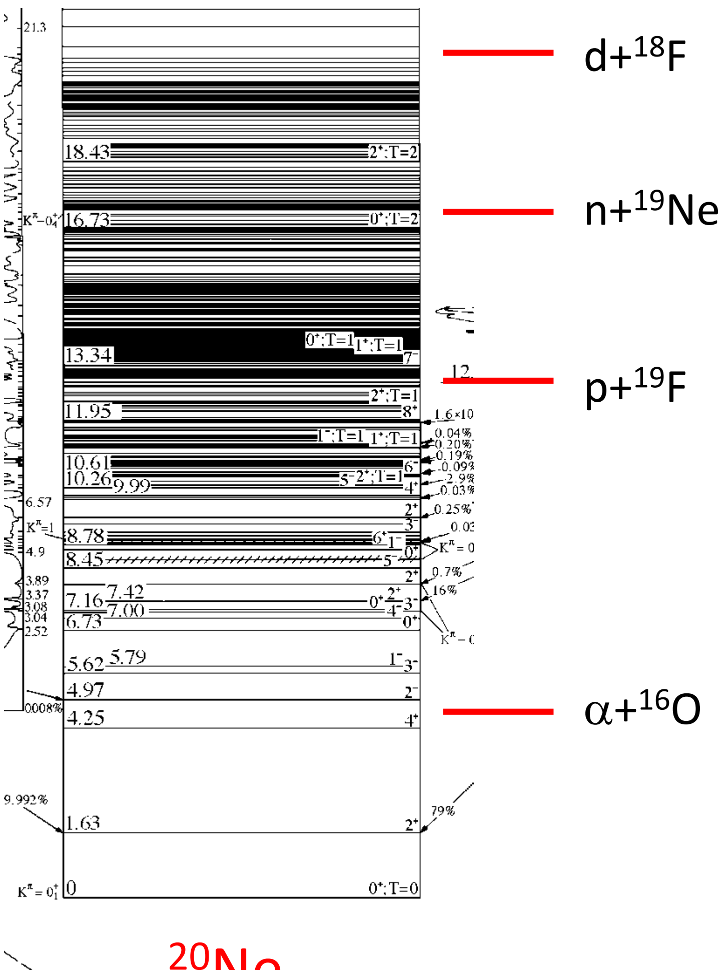
α+α wave function for $\ell = 0$





Generalization 3: complex potentials $V = V_R + iW$

Goal: to simulate absorption channels



^{20}Ne

High energies:

- many open channels
- strong absorption
- potential model extended to complex potentials (« optical »)

Phase shift is complex: $\delta = \delta_R + i\delta_I$

collision matrix: $U = \exp(2i\delta) = \eta \exp(2i\delta_R)$

where $\eta = \exp(-2\delta_I) < 1$

Elastic cross section

$$\frac{d\sigma}{d\Omega} = \frac{1}{4k^2} \left| \sum_\ell (2\ell+1)(\eta_\ell \exp(2i\delta_\ell) - 1) P_\ell(\cos\theta) \right|^2$$

Reaction cross section:

$$\sigma = \frac{\pi}{k^2} \sum_\ell (2\ell+1)(1-\eta_\ell^2)$$

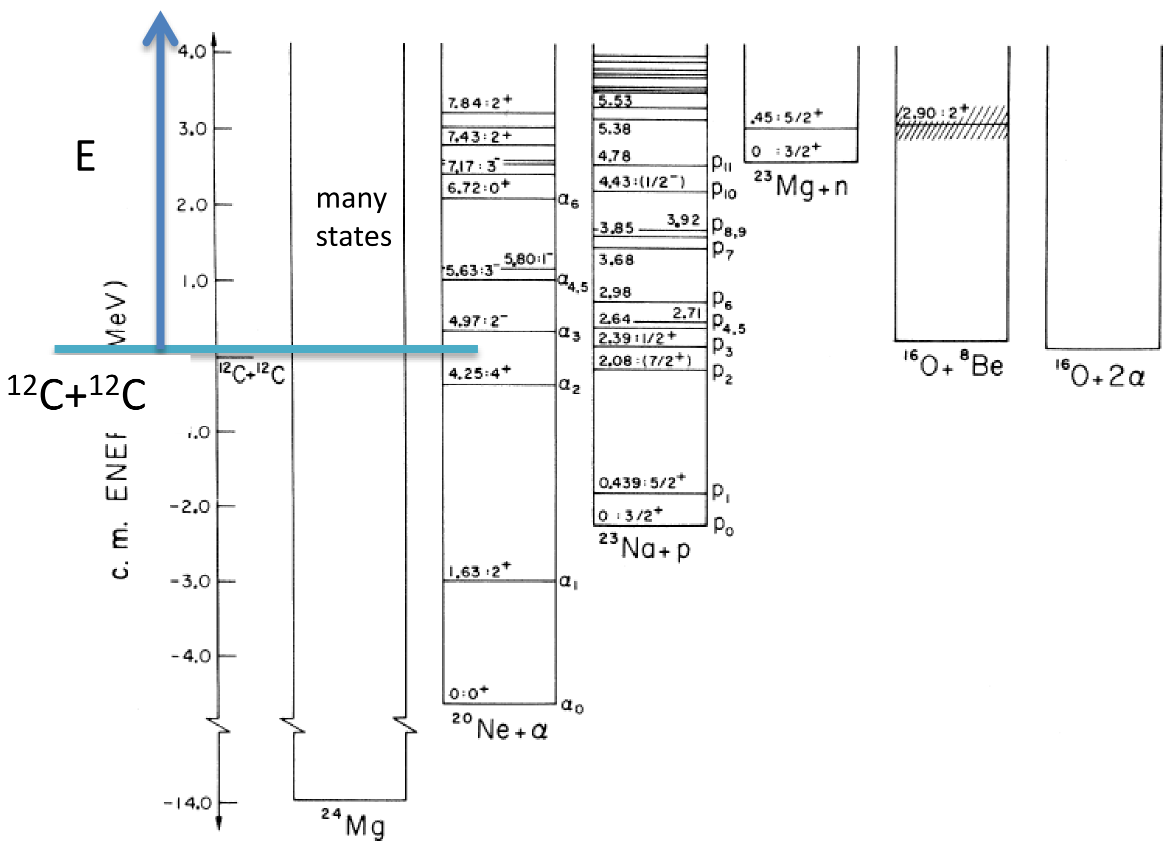
5. Generalizations

In astrophysics, optical potentials are used to compute fusion cross sections

Fusion cross section: includes many channels

Example: $^{12}C+^{12}C$: Essentially $^{20}Ne+\alpha$, $^{23}Na+p$, $^{23}Mg+n$ channels

→ absorption simulated by a complex potential $V = V_R + iW$

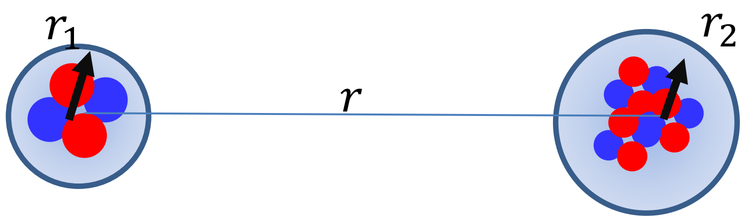


Typical potentials

A. Real part

- Woods-Saxon: $V_R(r) = -\frac{V_0}{1+\exp\left(\frac{r-r_0}{a}\right)}$ with parameters V_0, r_0, a adjusted to experiment

- Folding
  $V_R(r) = \lambda \iint dr_1 dr_2 v_{NN}(r - r_1 + r_2)\rho_1(r_1)\rho_2(r_2)$



Nucleus 1
density $\rho_1(r_1)$

Nucleus 2
density $\rho_2(r_2)$

v_{NN}=nucleon-nucleon interaction
λ =amplitude (~1), adjustable parameter
ρ_1, ρ_2=nuclear densities (in general known experimentally)

Main advantage: only one parameter λ

B. Imaginary part

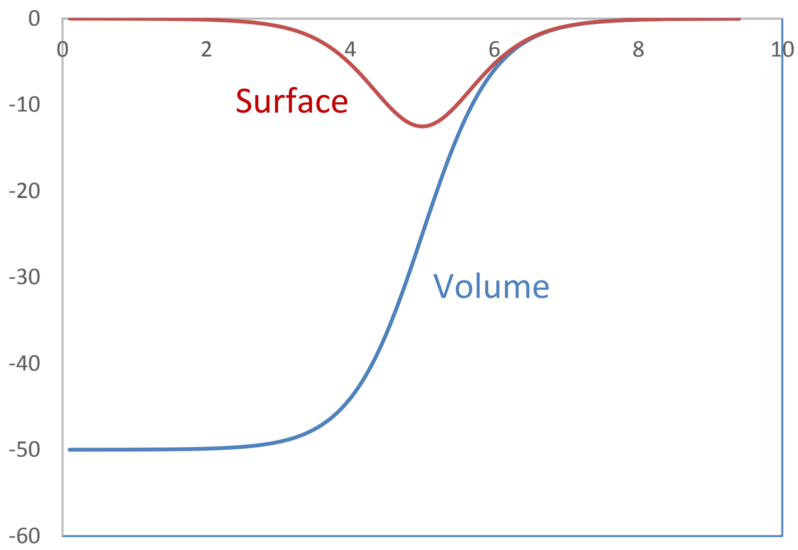
- Woods-Saxon:

  Volume: $W(r) = -W_0 f(r) = -\frac{W_0}{1+\exp\left(\frac{r-r_0}{a}\right)}$

  Surface $W(r) = -W_0 \frac{df(r)}{dr}$

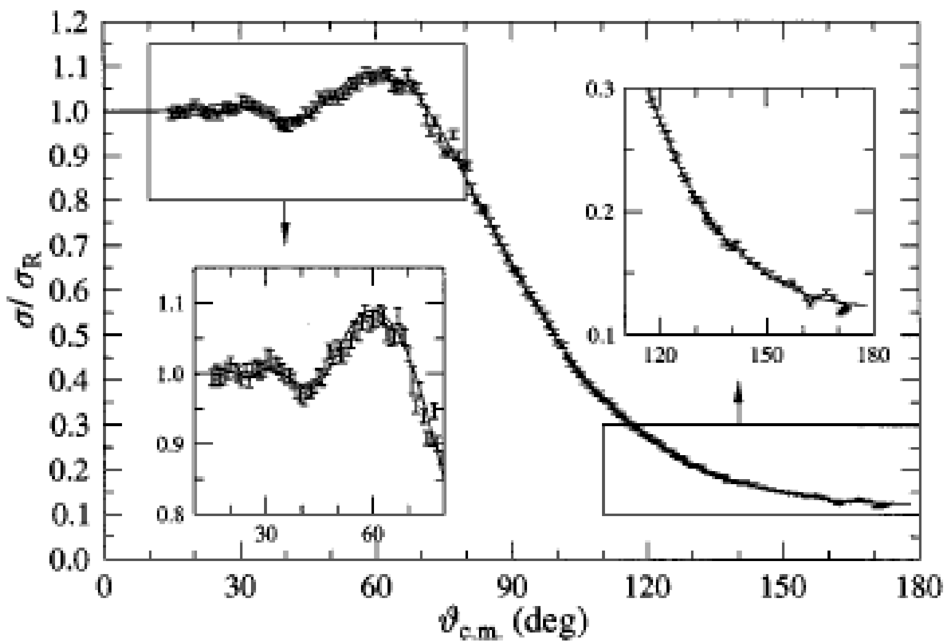
- Folding

  $W(r) = N_I V_R(r)$

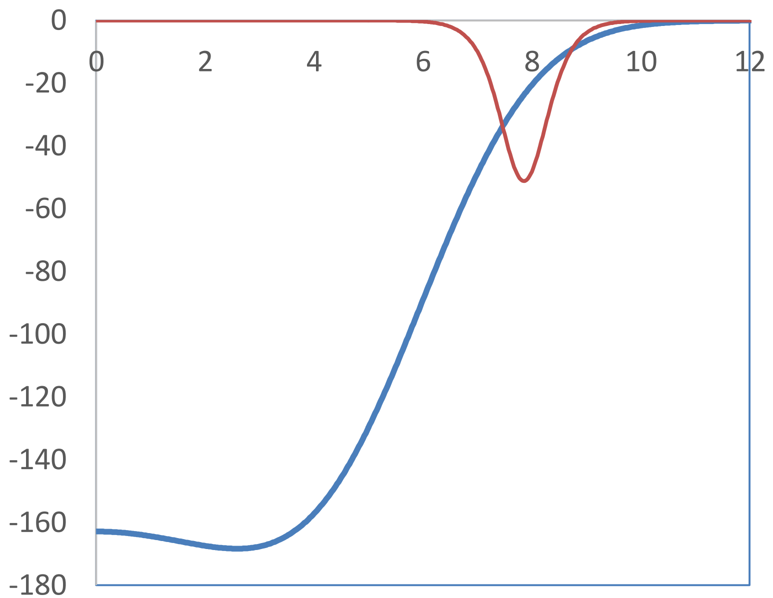


Example: α+^{144}Sm
P. Mohr et al., Phys. Rev. C55 (1997) 1523
Measurement of elastic scattering → optical potential → used for astrophysics

Elastic cross section at E_{lab}=20 MeV (E_{cm}=9.5 MeV)



α+^{144}Sm potential (folding)



6. Models used for nuclear reactions in astrophysics

6. Models used in nuclear astrophysics (for reactions)

Theoretical methods: Many different cases → no "unique" model!

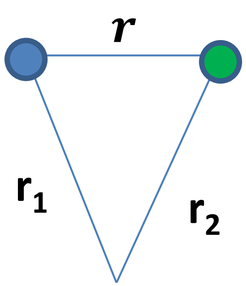
Model	Applicable to	Comments	
Potential/optical model	Capture Fusion	• Internal structure neglected • Antisymetrization approximated	Light systems Low level densities
R-matrix	Capture Transfer	• No explicit wave functions • Physics simulated by some parameters	
DWBA	Transfer	• Perturbation method • Wave functions in the entrance and exit channels	
Microscopic models	Capture Transfer	• Based on a nucleon-nucleon interaction • *A*-nucleon problems • Predictive power	
Hauser-Feshbach	Capture Transfer	• Statistical model	Heavy systems
Shell model	Capture	• Only gamma widths	

7. Radiative capture in the potential model

Potential model: two structureless particles (=optical model, without imaginary part)

- Calculations are simple
- Physics of the problem is identical in other methods
- Spins are neglected

- $\boldsymbol{R_{cm}}$=center of mass, $\boldsymbol{r}$ =relative coordinate



$$\boldsymbol{r_1} = \boldsymbol{R_{cm}} - \frac{A_2}{A}\boldsymbol{r}$$
$$\boldsymbol{r_2} = \boldsymbol{R_{cm}} + \frac{A_1}{A}\boldsymbol{r}$$

- Initial wave function: $\Psi^{\ell_i m_i}(\boldsymbol{r}) = \frac{1}{r}u_{\ell_i}(r)Y_{\ell_i}^{m_i}(\Omega)$, energy E^{ℓ_i}=scattering energy E

  Final wave function: $\Psi^{\ell_f m_f}(\boldsymbol{r}) = \frac{1}{r}u_{\ell_f}(r)Y_{\ell_f}^{m_f}(\Omega)$, energy E^{ℓ_f}

  The radial wave functions are given by:

$$-\frac{\hbar^2}{2\mu}\left(\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2}\right)u_\ell + V(r)u_\ell = E^\ell u_\ell$$

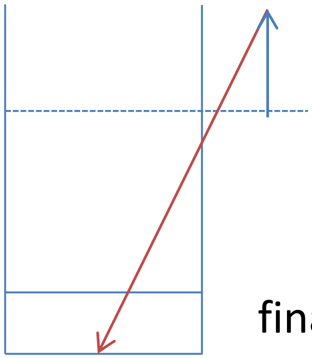
7. Radiative capture in the potential model

- Schrödinger equation: $-\frac{\hbar^2}{2\mu}\left(\frac{d^2}{dr^2}-\frac{\ell(\ell+1)}{r^2}\right)u_\ell + V(r)u_\ell = E^\ell u_\ell$
- Typical potentials:
  - coulomb =point-sphere
  - nuclear: Woods-Saxon, Gaussian
    parameters adjusted on important properties (bound-state energy, phase shifts, etc.)

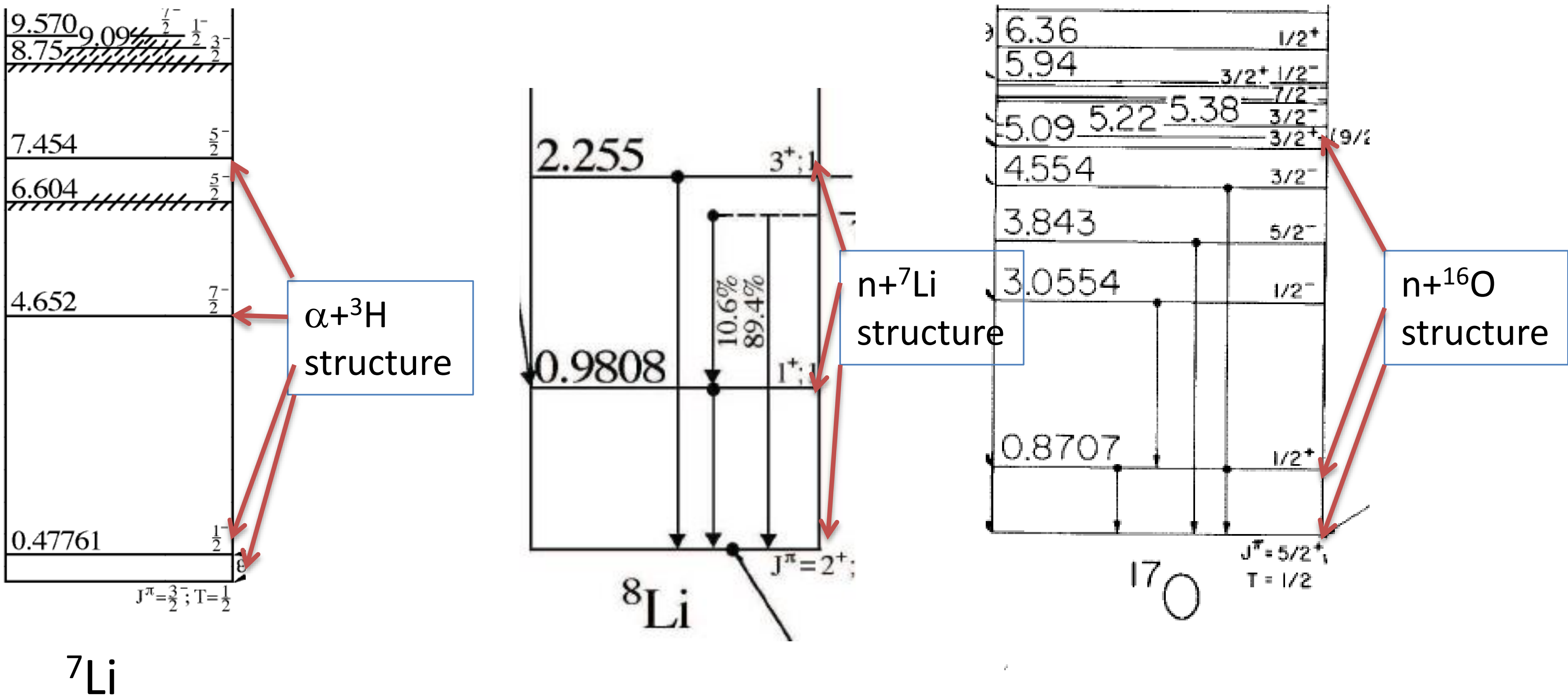
- Potentials can be different in the initial and final states

- Wave functions computed numerically (Numerov algorithm)

- Limitations
  - initial (scattering state): must reproduce resonances (if any)
  - final (bound) state: must have a A+B structure



Some typical examples



Problem more and more important when the level density increases

→ in practice: limited to low-level densities (light nuclei or nuclei close to the drip lines)

7. Radiative capture in the potential model

- Electric operator for two particles:

$$\mathcal{M}_\mu^{E\lambda} = e\left(Z_1|\boldsymbol{r_1} - \boldsymbol{R_{cm}}|^\lambda\, Y_\lambda^\mu(\Omega_{r_1 - R_{cm}}) + Z_2|\boldsymbol{r_2} - \boldsymbol{R_{cm}}|^\lambda\, Y_\lambda^\mu(\Omega_{r_2 - R_{cm}})\right)$$

which provides

$$\mathcal{M}_\mu^{E\lambda} = e\left[Z_1\left(-\frac{A_2}{A}\right)^\lambda + Z_2\left(\frac{A_1}{A}\right)^\lambda\right] r^\lambda Y_\lambda^\mu(\Omega_r) = eZ_{eff} r^\lambda Y_\lambda^\mu(\Omega_r)$$

- Matrix elements needed for electromagnetic transitions

$$< \Psi^{J_f m_f}\,|\mathcal{M}_\mu^{E\lambda}|\Psi^{J_i m_i} > = eZ_{eff} < Y_{J_f}^{m_f}|Y_\lambda^\mu|Y_{J_i}^{m_i} > \int_0^\infty u_{J_i}(r)\, u_{J_f}(r) r^\lambda dr$$

- Reduced matrix elements:

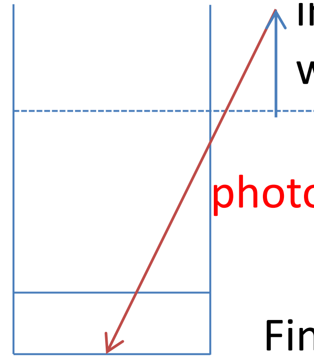
$$< \Psi^{J_f}\,\|\mathcal{M}^{E\lambda}\|\Psi^{J_i} > = eZ_{eff} < J_f 0 \lambda 0|\, J_i 0 >$$
$$\times\left(\frac{(2J_i+1)(2\lambda+1)}{4\pi(2J_f+1)}\right)^{1/2} \int_0^\infty u_{J_i}(r)\, u_{J_f}(r) r^\lambda dr$$

→ simple one-dimensional integrals

7. Radiative capture in the potential model

Assumptions:

- spins zero: $\ell_i = J_i, \ell_f = J_f$
- given values of J_i, J_f, λ



Integrated cross section

$$\sigma_\lambda(E) = \frac{8\pi}{k^2}\frac{e^2}{\hbar c} Z_{eff}^2 k_\gamma^{2\lambda+1} F(\lambda, J_i, J_f) \left| \int_0^\infty u_{J_i}(r,E)\, u_{J_f}(r) r^\lambda dr \right|^2$$

with

- $Z_{eff} = Z_1 \left(-\frac{A_2}{A}\right)^\lambda + Z_2 \left(\frac{A_1}{A}\right)^\lambda$
- $F(\lambda, J_i, J_f) =< J_i \lambda 0\, 0 | J_f 0 > (2J_i + 1) \frac{(\lambda+1)(2\lambda+1)}{\lambda(2\lambda+1)!!^2}$
- $k_\gamma = \frac{E - E_f}{\hbar c}$

Normalization

- final state (bound): normalized to unity $u_J(r) \rightarrow C W(2k_B r) \rightarrow C\exp(-k_B r)$
- initial state (continuum): $u_J(r) \rightarrow F_J(kr) \cos\delta_J + G_J(kr) \sin\delta_J$

7. Radiative capture in the potential model

Integrated vs differential cross sections

- Total (integrated) cross section:

$$\sigma(E) = \sum_{\lambda} \sigma_{\lambda}(E)$$

→ no interference between the multipolarities

- Differential cross section:

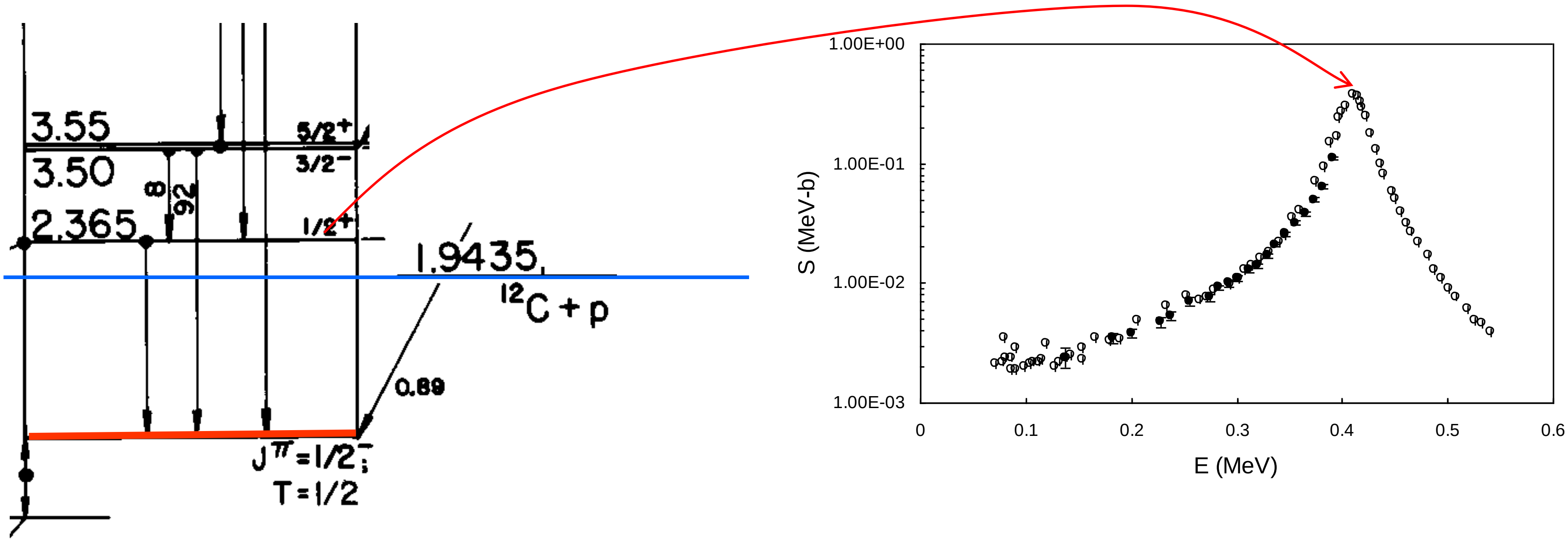
$$\frac{d\sigma}{d\theta} = \left| \sum_{\lambda} a_{\lambda}(E) P_{\lambda}(\theta) \right|^2$$

  - $P_{\lambda}(\theta)$=Legendre polynomial
  - $a_{\lambda}(E)$ are complex, $\sigma_{\lambda}(E) \sim |a_{\lambda}(E)|^2$
    → interference effects
    → angular distributions are necessary to separate the multipolarities
    →in general one multipolarity is dominant (not in $^{12}C(\alpha,\gamma)^{16}O$: E1 and E2)

7. Radiative capture in the potential model

Example: $^{12}C(p,\gamma)^{13}N$

- First reaction of the CNO cycle
- Well known experimentally
- Presents a low energy resonance ($\ell = 0 \rightarrow J = 1/2^+$)



Potential : V= -55.3*exp(-(r/2.70)^2) (final state)

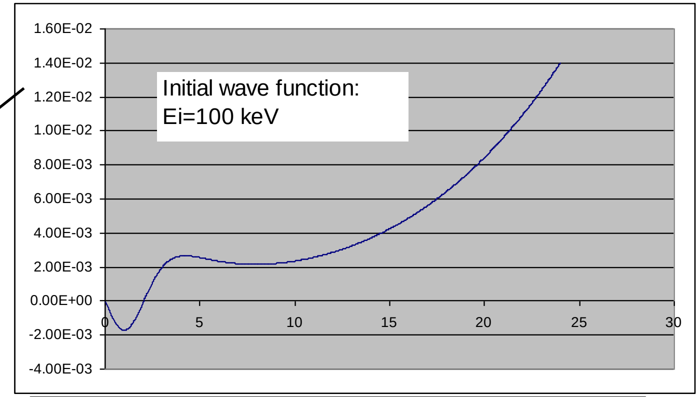
-70.5*exp(-(r/2.70)^2) (initial state)

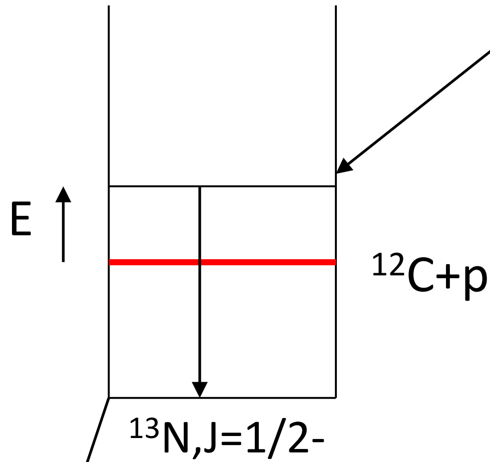
7. Radiative capture in the potential model

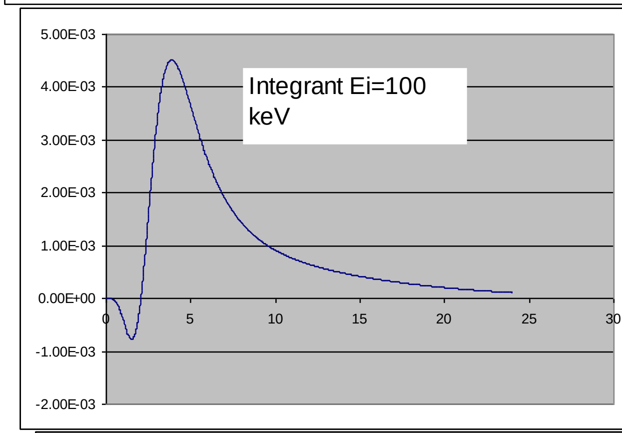
Final state: $J_f = 1/2^-$

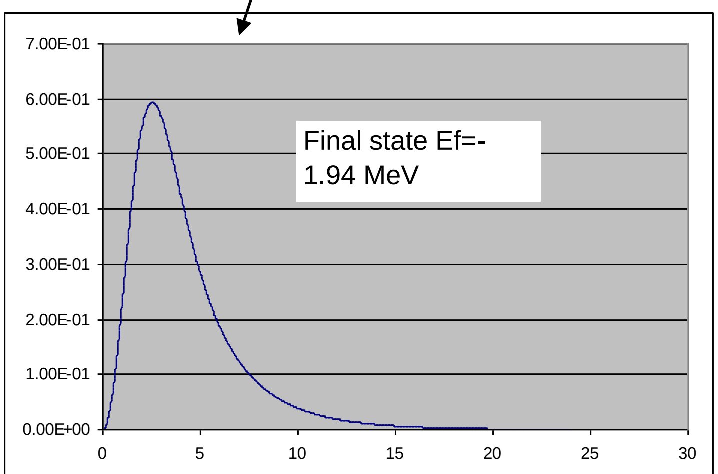
Initial state: $\ell_i = 0 \rightarrow J_i = 1/2^+$

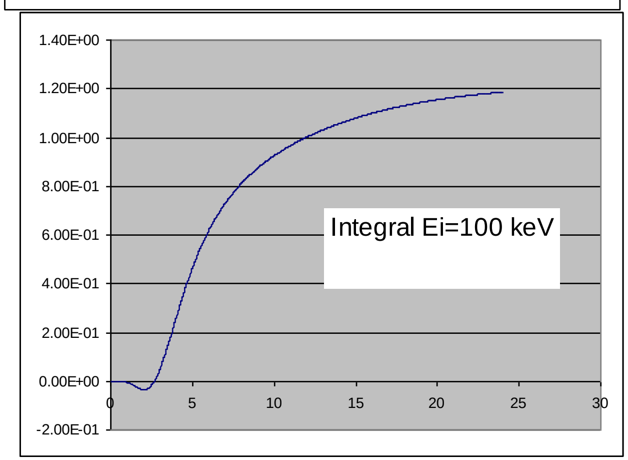
➔ E1 transition $1/2^+ \rightarrow 1/2^-$





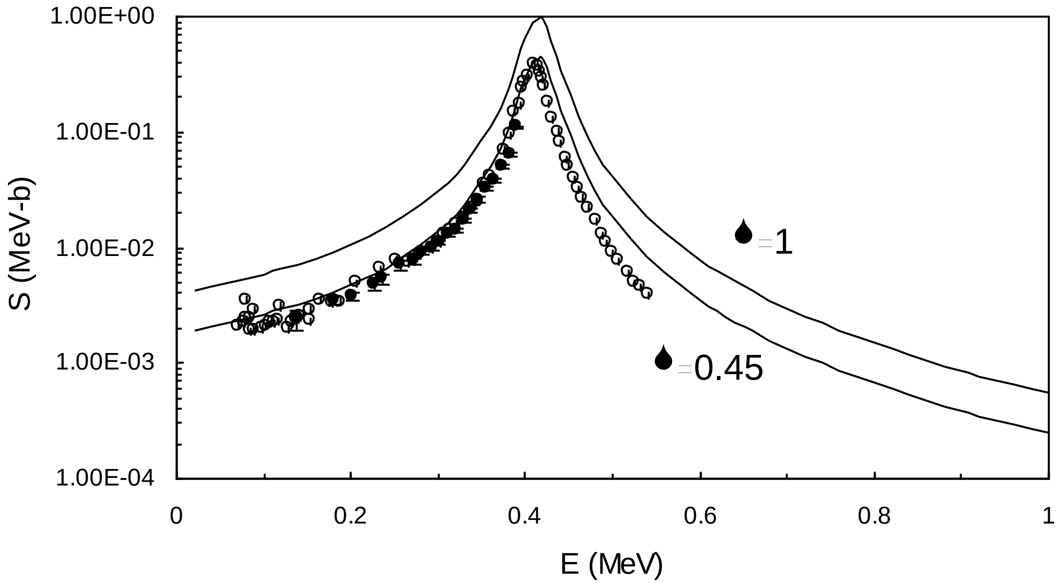






7. Radiative capture in the potential model

The calculation is repeated at all energies



Necessity of a spectroscopic factor *S*

Assumption of the potential model: $^{13}N = ^{12}C + p$
In reality $^{13}N = ^{12}C + p \oplus ^{12}C^* + p \oplus ^{9}Be + \alpha \oplus \ldots$

➔ to simulate the missing channels: $u_f(r)$ is replaced by $S^{1/2} u_f(r)$
S=spectroscopic factor

Other applications: $^{7}Be(p,\gamma)^{8}B$, $^{3}He(\alpha,\gamma)^{7}Be$, etc…

8. The R-matrix method

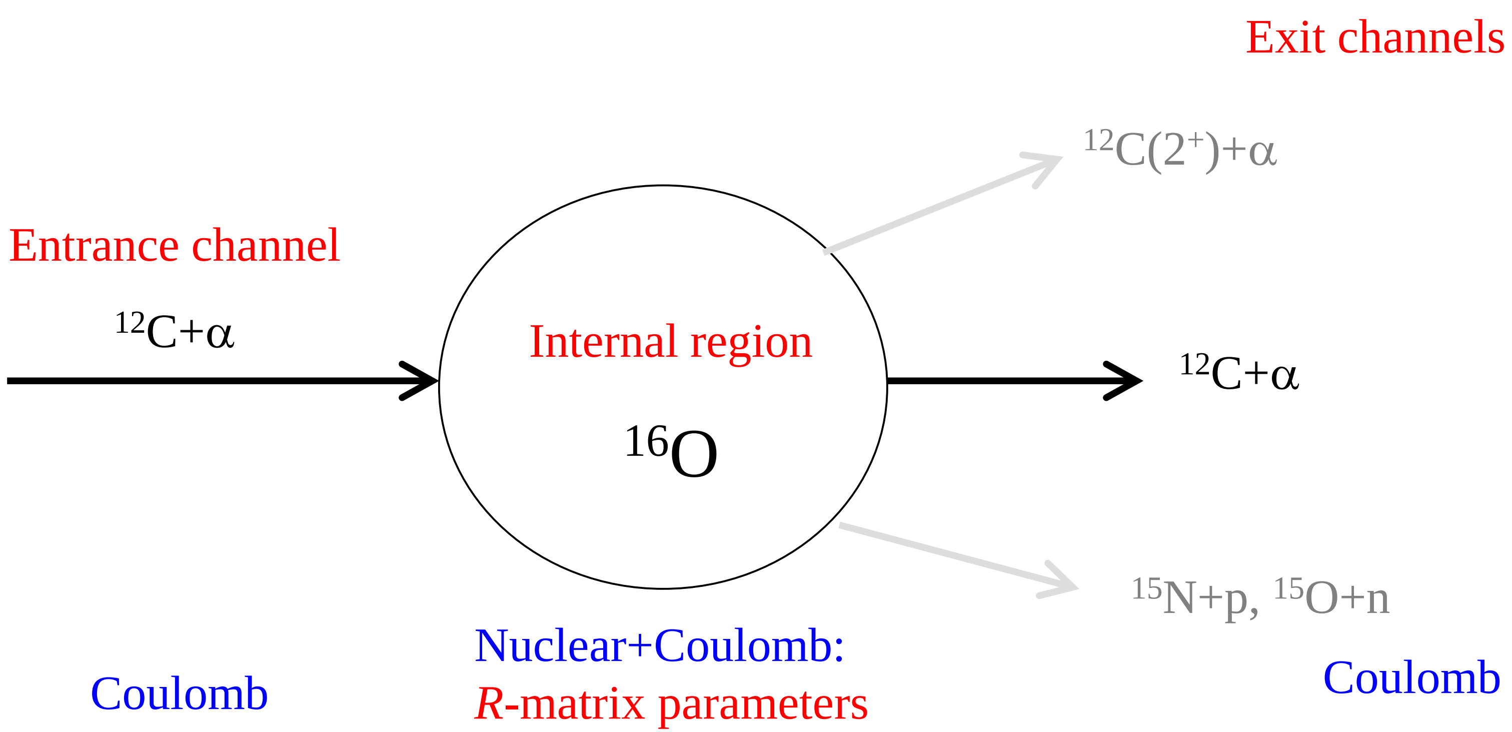
- General presentation
- Single resonance system
- Applications to elastic scattering ^{12}C+p
- Application to $^{12}C(p,\gamma)^{13}N$ and $^{12}C(\alpha,\gamma)^{16}O$

8. The R-matrix method

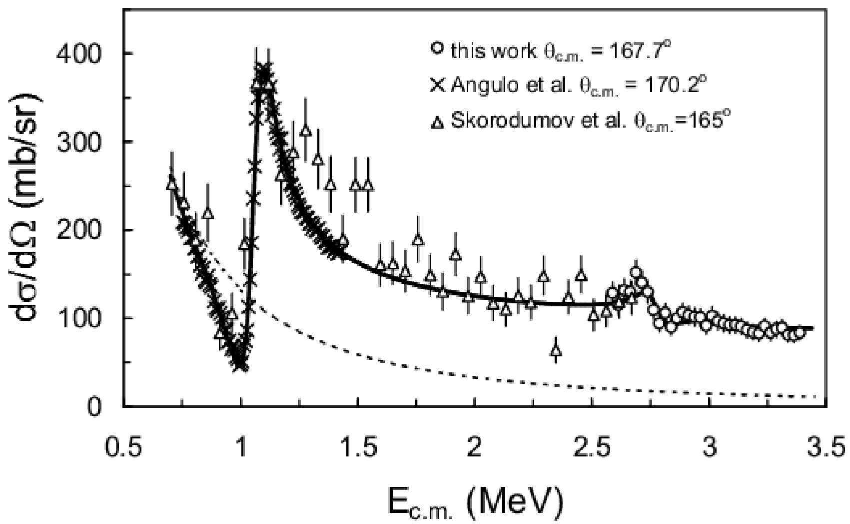
- Introduced by Wigner (1937) to parametrize resonances (nuclear physics)
  In nuclear astrophysics: used to fit data

- Provides scattering properties at all energies (not only at resonances)

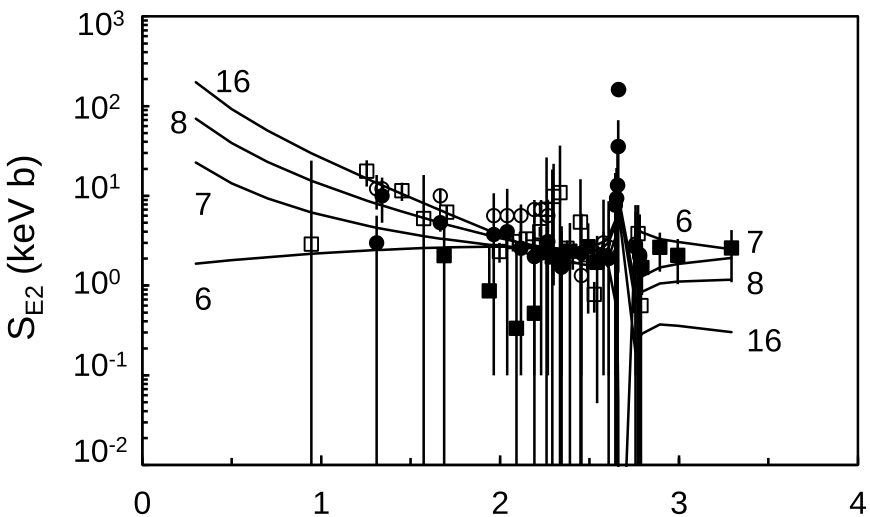
- Based on the existence of 2 regions (radius a):
  - Internal: coulomb+nuclear
  - external: coulomb



Main Goal: fit of experimental data



^{18}Ne+p elastic scattering
→ resonance properties

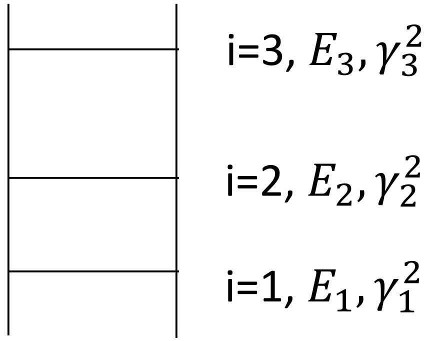


Nuclear astrophysics: $^{12}C(\alpha,\gamma)^{16}O$
→Extrapolation to low energies

8. The R-matrix method

- Internal region: The R matrix is given by a set of resonance parameters E_i, γ_i^2

$$R(E) = \sum_i \frac{\gamma_i^2}{E_i - E} = a\frac{\Psi'(a)}{\Psi(a)}$$



- External region: Coulomb behaviour of the wave function

$$\Psi(r) = I(r) - UO(r)$$

→the collision matrix *U* is deduced from the R-matrix (repeated for each spin/parity $J\pi$)

- Two types of applications:
  - **phenomenological R matrix**: γ_i^2 and E_i are fitted to the data (astrophysics)
  - **calculable R matrix**: γ_i^2 and E_i are computed from basis functions (scattering theory)

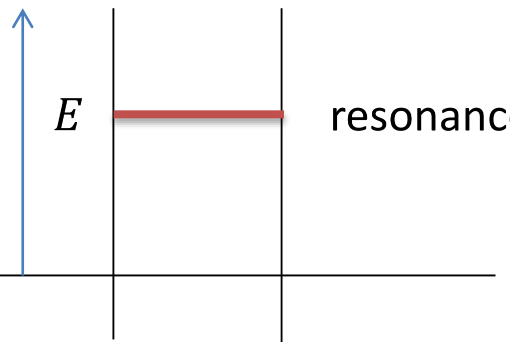
- R-matrix radius a is not a parameter: the cross sections must be insensitive to a

- Can be extended to multichannel calculations (transfer), capture, etc.

- Well adapted to nuclear astrophysics: low energies, low level densities

8. The R-matrix method

A simple case: elastic scattering with a single isolated resonance



resonance: energy E_R, width Γ (or γ^2) ="observed parameters"
"calculated" parameters: E_0, γ_0^2

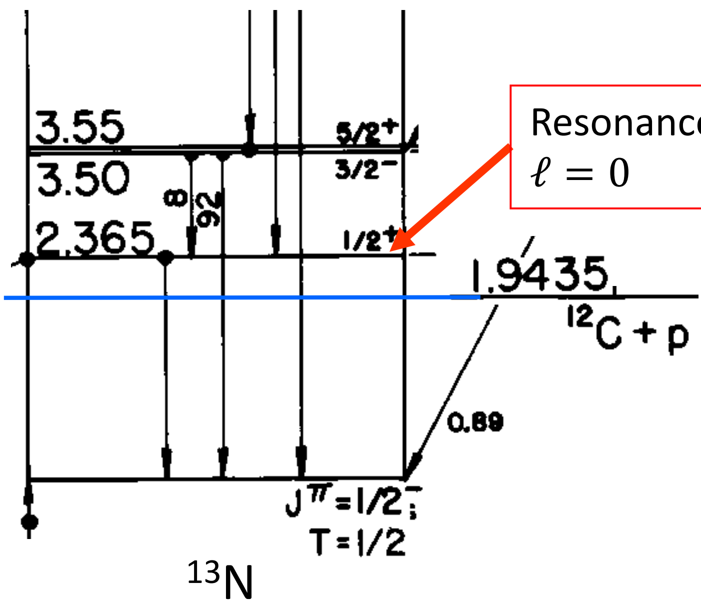
threshold

- From the total width Γ → reduced width $\Gamma = 2\gamma^2 P_l(E_R)$
  $P_l(E_R)$=penetration factor

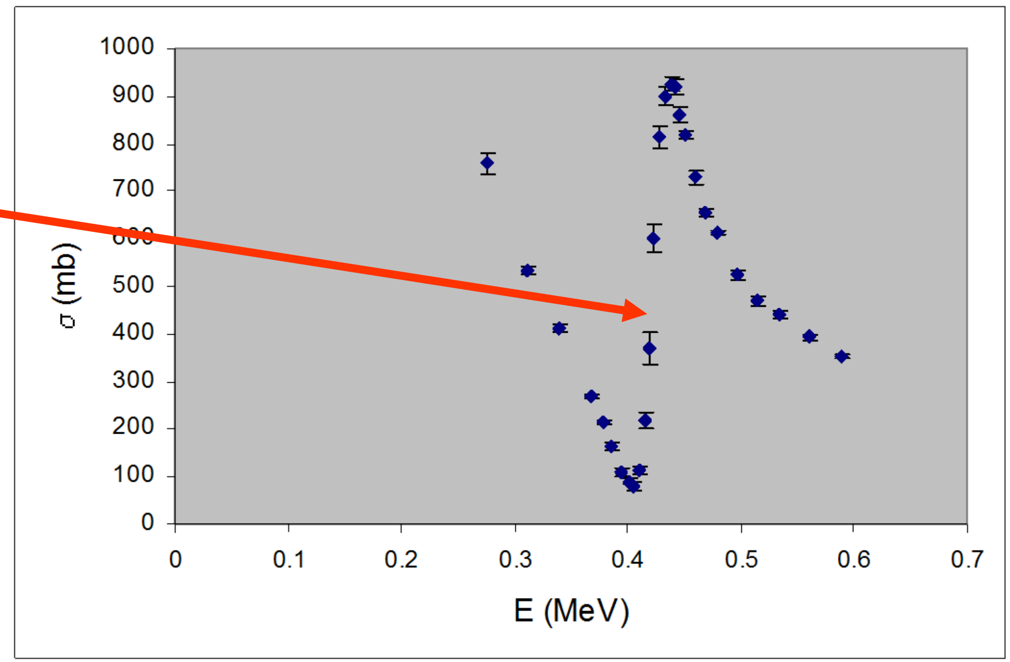
- Link between $(E_R, \gamma^2) \leftrightarrow (E_0, \gamma_0^2)$
- Calculation of the R-matrix $R(E) = \frac{\gamma_0^2}{E_0 - E}$
- Calculation of the scattering matrix: $U(E) = \frac{I(ka)}{O(ka)} \frac{1 - L^* R(E)}{1 - LR(E)}$ (must be done for each ℓ)
- Calculation of the cross section → E_0 and/or γ_0^2 can be fitted

8. The R-matrix method

Example: $^{12}C+p$: $E_R=0.42$ MeV



Resonance
$\ell = 0$



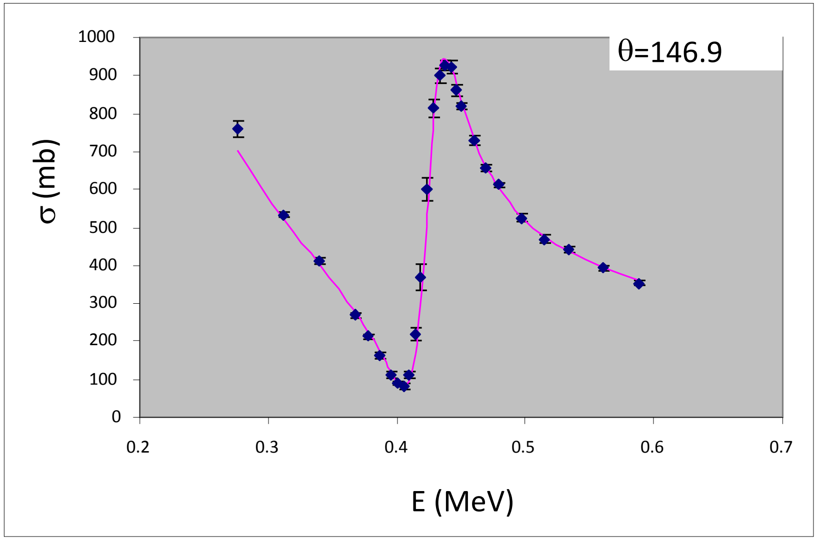
In the considered energy range: resonance J=1/2+ ($\ell = 0$)

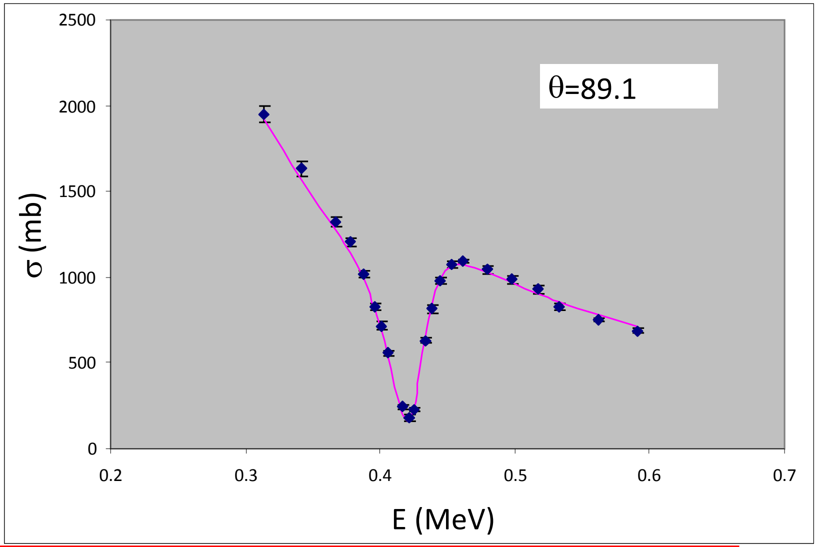
→ Phase shift for $\ell = 0$ is treated by the R matrix

→ Other phase shifts $\ell > 0$ are given by the hard-sphere approximation

First example: Elastic scattering $^{12}C+p$

Data from H.O. Meyer et al., Z. Phys. A279 (1976) 41





R matrix fits for different channel radii

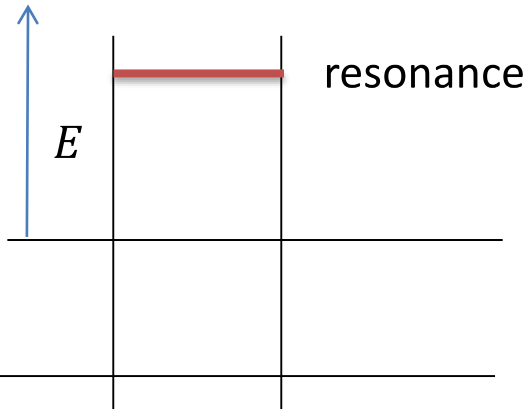
a	E_R	Γ	E_0	$\gamma_0 2$	χ^2
4.5	0.4273	0.0341	-1.108	1.334	2.338
5	0.4272	0.0340	-0.586	1.068	2.325
5.5	0.4272	0.0338	-0.279	0.882	2.321
6	0.4271	0.0336	-0.085	0.745	2.346

➔ E_R, Γ very stable with a

➔ global fit independent of a

8. The R-matrix method

Extension to transfer, example: $^{18}F(p,\alpha)^{15}O$



resonance: energy E_R, partial widths Γ_1, Γ_2 (or γ_1^2, γ_2^2) =“observed parameters”
“calculated” parameters: $E_0, \gamma_{01}^2, \gamma_{02}^2$

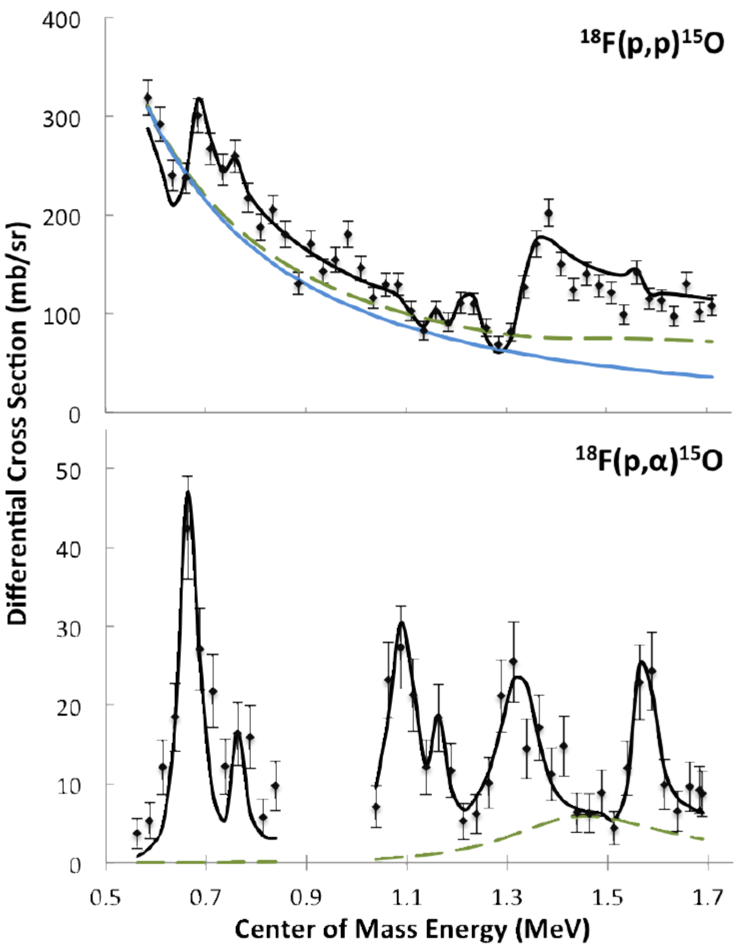
threshold 1 ($p+^{18}F$)

threshold 2 ($\alpha+^{15}O$)

- Link between $(E_R, \gamma_1^2, \gamma_2^2) \leftrightarrow E_0, \gamma_{01}^2, \gamma_{02}^2$ more complicated
- R-matrix: 2x2 matrix $R_{ii}(E) = \frac{\gamma_{01}^2}{E_0 - E}$ associated with the entrance channel
  $R_{ff}(E) = \frac{\gamma_{02}^2}{E_0 - E}$ associated with the exit channel
  $R_{if}(E) = \frac{\gamma_{01}\gamma_{02}}{E_0 - E}$ associated with the transfer
- Scattering matrix: 2x2: U_{11}, U_{22} → elastic cross sections
  U_{12},→transfer cross section
- More parameters, but some are common to elastic scattering (E_0, γ_{01}^2)
  → constraints with elastic scattering

8. The R-matrix method

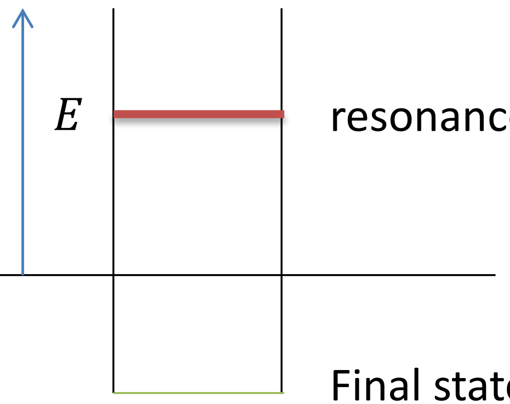
Recent application to $^{18}F(p,p)^{18}F$ and $^{18}F(p,\alpha)^{15}O$
D. Mountford et al, *Phys. Rev. C 85 (2012) 022801*



simultaneous fit of both cross sections
angle: 176°
for each resonance: $J\pi, E_R, \Gamma_p, \Gamma_\alpha$
8 resonances → 24 parameters

8. The R-matrix method

Extension to radiative capture



resonance: energy E_R, reduced γ^2, gamma width Γ_γ ="observed parameters"
"calculated" parameters: E_0, γ_0^2, $\Gamma_{0\gamma}$

threshold

Final state: ANC C_f

Capture reaction= transition between an initial state at energy E to bound states

Cross section $\sigma_C(E) \sim \left|< \Psi_f|H_\gamma|\Psi_i(E) >\right|^2$

Additional pole parameter: gamma width $\Gamma_{\gamma i}$

$$< \Psi_f|H_\gamma|\Psi_i(E) > = < \Psi_f|H_\gamma|\Psi_i(E) >_{int} + < \Psi_f|H_\gamma|\Psi_i(E) >_{ext}$$

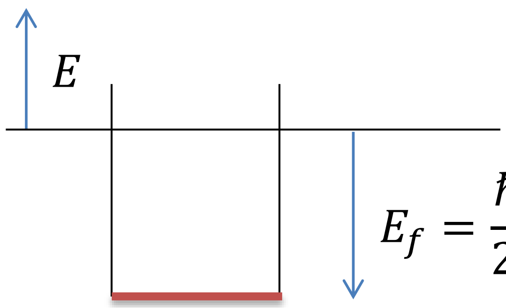
internal part: $< \Psi_f|H_\gamma|\Psi_i(E) >_{int} \sim \sum_{i=1}^{N} \frac{\gamma_i \sqrt{\Gamma_{\gamma i}}}{E_i - E}$

external part: $< \Psi_f|H_\gamma|\Psi_i(E) >_{ext} \sim C_f \int_a^\infty W(2k_f r) r^\lambda \big(I_i(kr) - U O_i(kr)\big) dr$

8. The R-matrix method

External part: $< \Psi_f|H_\gamma|\Psi_i(E) >_{ext} \sim C_f \int_a^\infty W(2k_f r) r^\lambda (I_i(kr) - U O_i(kr)) dr$

Essentially depends on k_f



$$E_f = \frac{\hbar^2}{2\mu} k_f^2$$

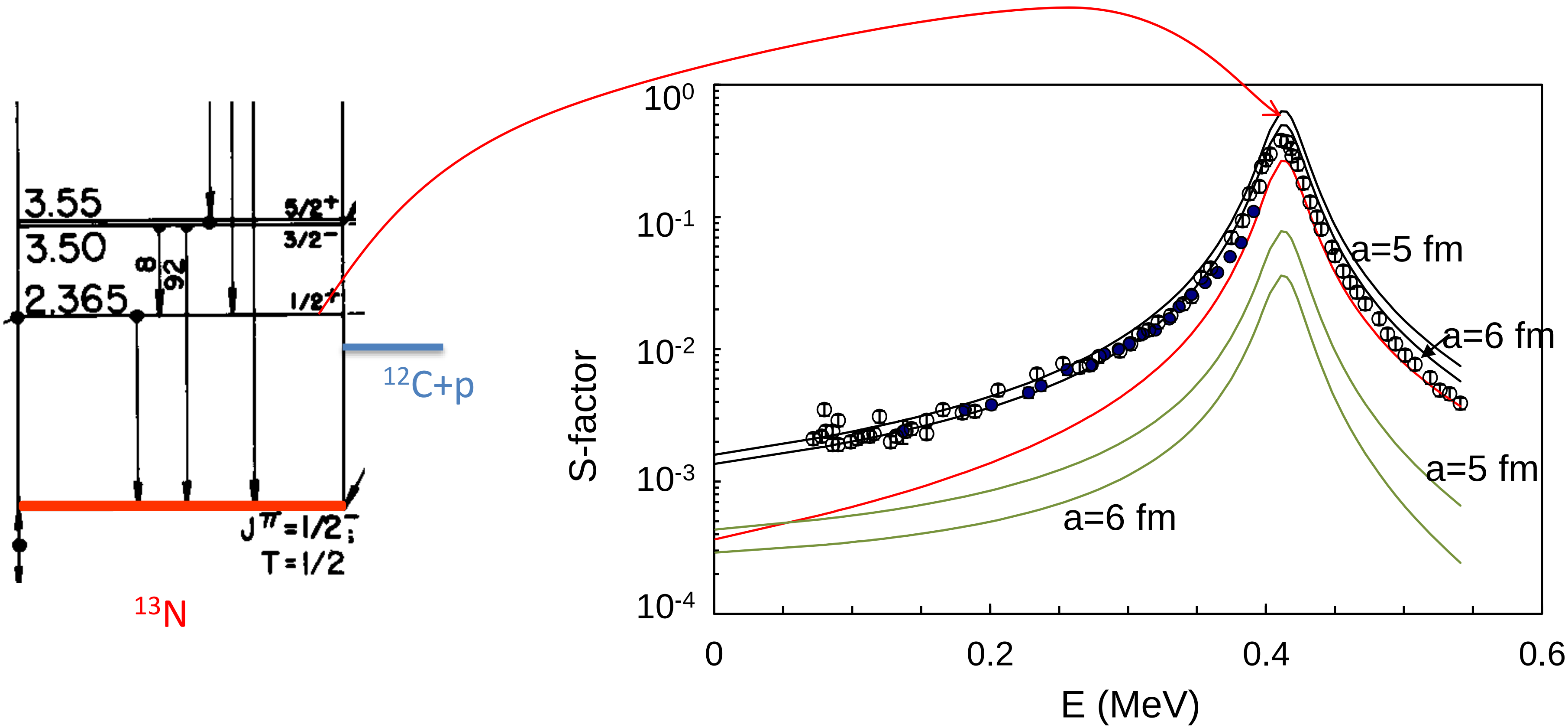
Witthaker function $W(2k_f r) \sim \exp(-k_f r)$

- k_f large: fast decrease
  example ^{12}C(α,γ)^{16}O, E_f=7.16 MeV, $\mu = 3$ → external term negligible
  → insensitive to C_f

- k_f small: slow decrease
  example: ^{7}Be(p,γ)^{8}B, E_f=0.137 MeV, $\mu = 7/8$ → external term dominant
  → mainly given by C_f

- Contribution of internal/external terms depends on energy (external larger at low energies)

Example 1: $^{12}C(p,\gamma)^{13}N$: R-matrix calculation with a single pole



Experiment: $E_R = 0.42\ MeV, \Gamma_p = 31\ keV, \Gamma_\gamma = 0.4\ eV$

Red line: internal contribution, pure Breit-Wigner approximation

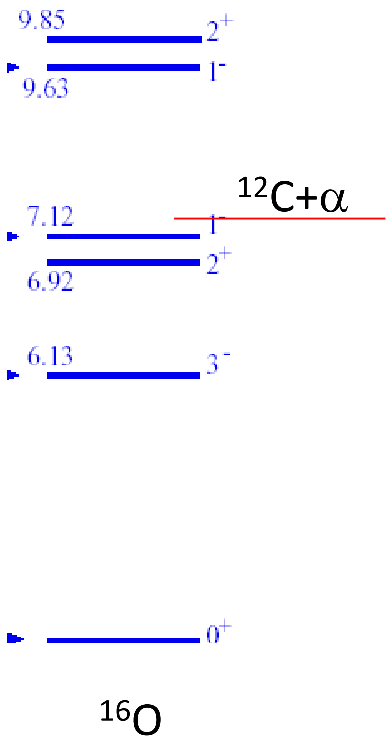
Green lines: external contribution: important at low energies, sensitive to the ANC

Example 2: $^{12}C(\alpha,\gamma)^{16}O$

General presentation of $^{12}C(\alpha,\gamma)^{16}O$

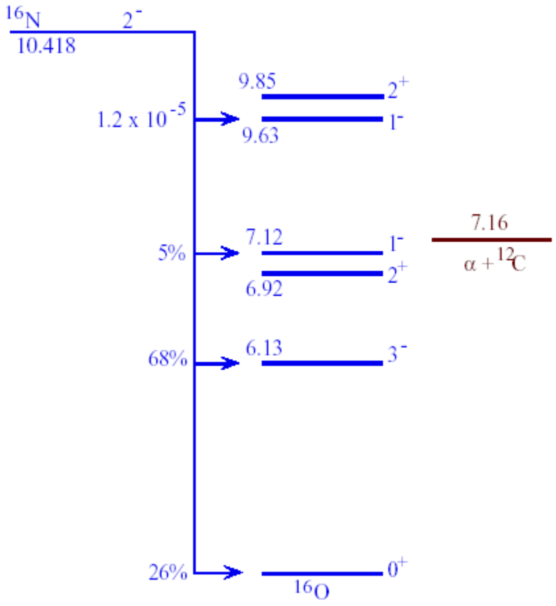
- Determines the $^{12}C/^{16}O$ ratio
- Cross section needed near E_{cm}=300 keV (barrier ~2.5 MeV)
  → cannot be measured in the Gamow peak
- 1^- and 2^+ subthreshold states

  → extrapolation difficult
- E1 and E2 important (E1 forbidden when T=0)
- Interferences between $1^-_1, 1^-_2$ and between 2^+_1, 2^+_2
- Capture to gs dominant but also cascade transitions

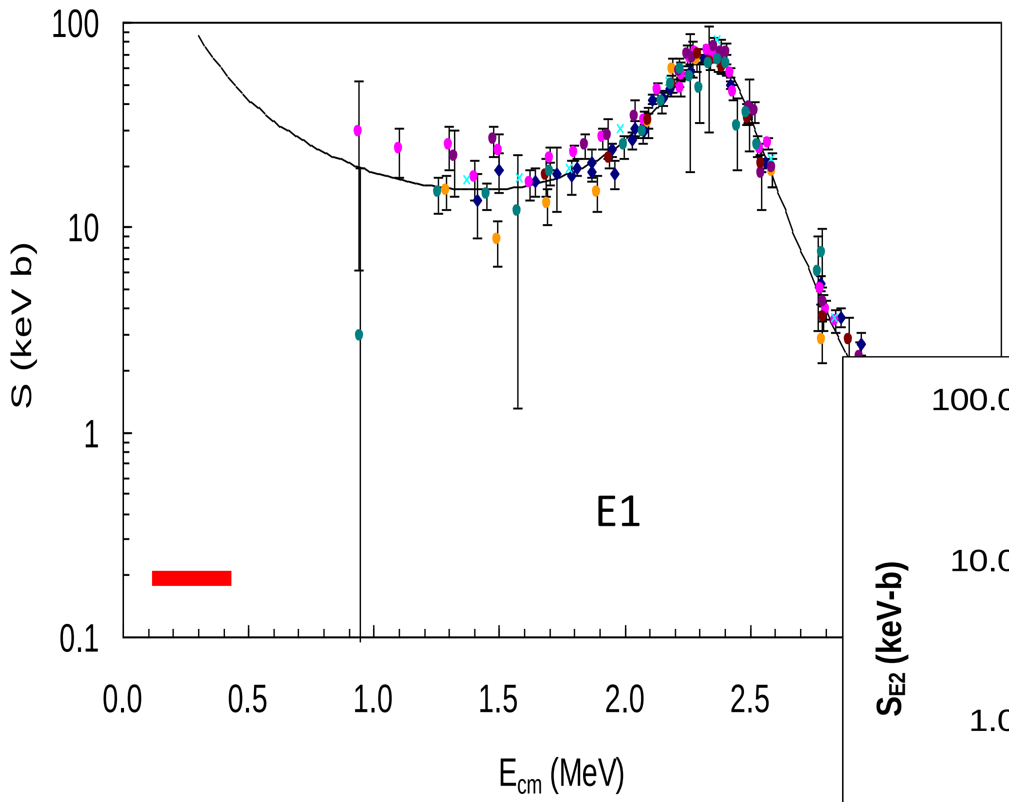


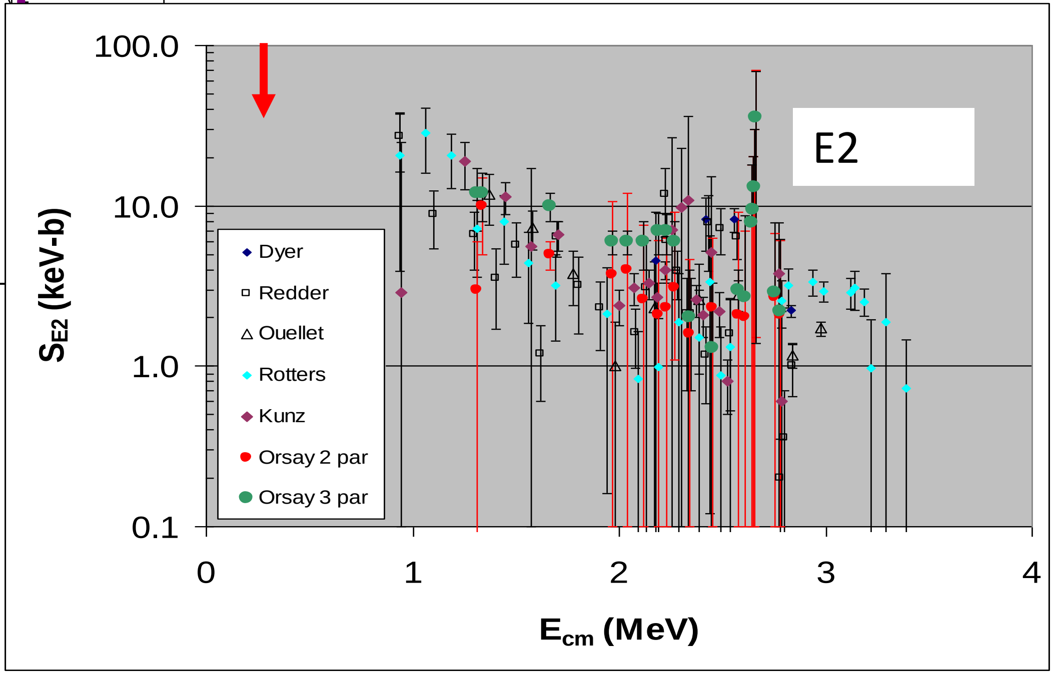
Many experiments

- Direct $^{12}C(\alpha,\gamma)^{16}O$ (angular distributions are necessary: E1 and E2)
- Indirect: spectroscopy of 1^-_1 and 2^+_1 subthreshold states
- Constraints
  - $\alpha+^{12}C$ phase shifts ($1^- \rightarrow$ E1, $2^+ \rightarrow$ E2)
  - E1: ^{16}N beta decay *(Azuma et al, Phys. Rev. C50 (1994) 1194)* probes $J=1^- \rightarrow$ E1
  - E2: ???

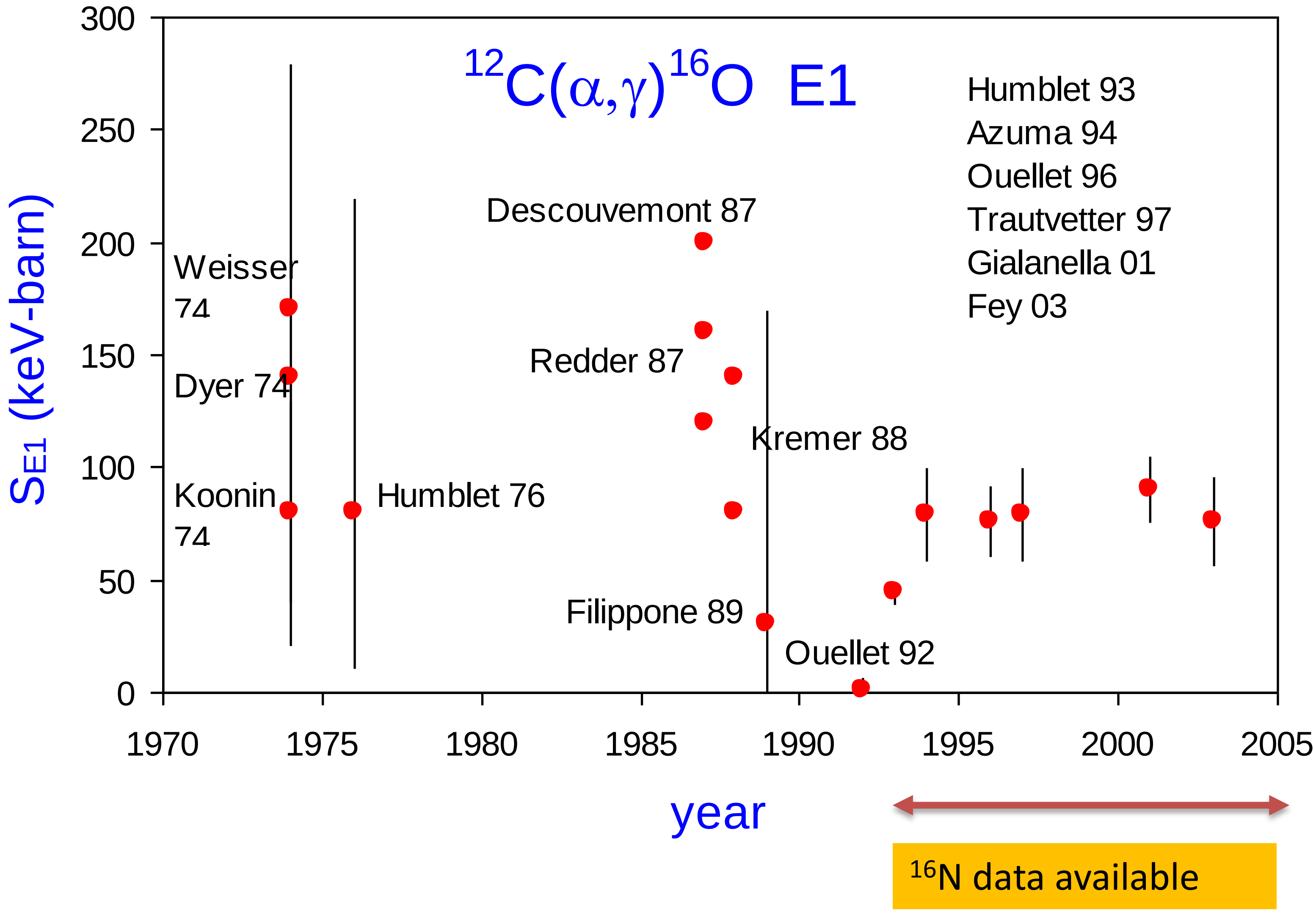


Current situation

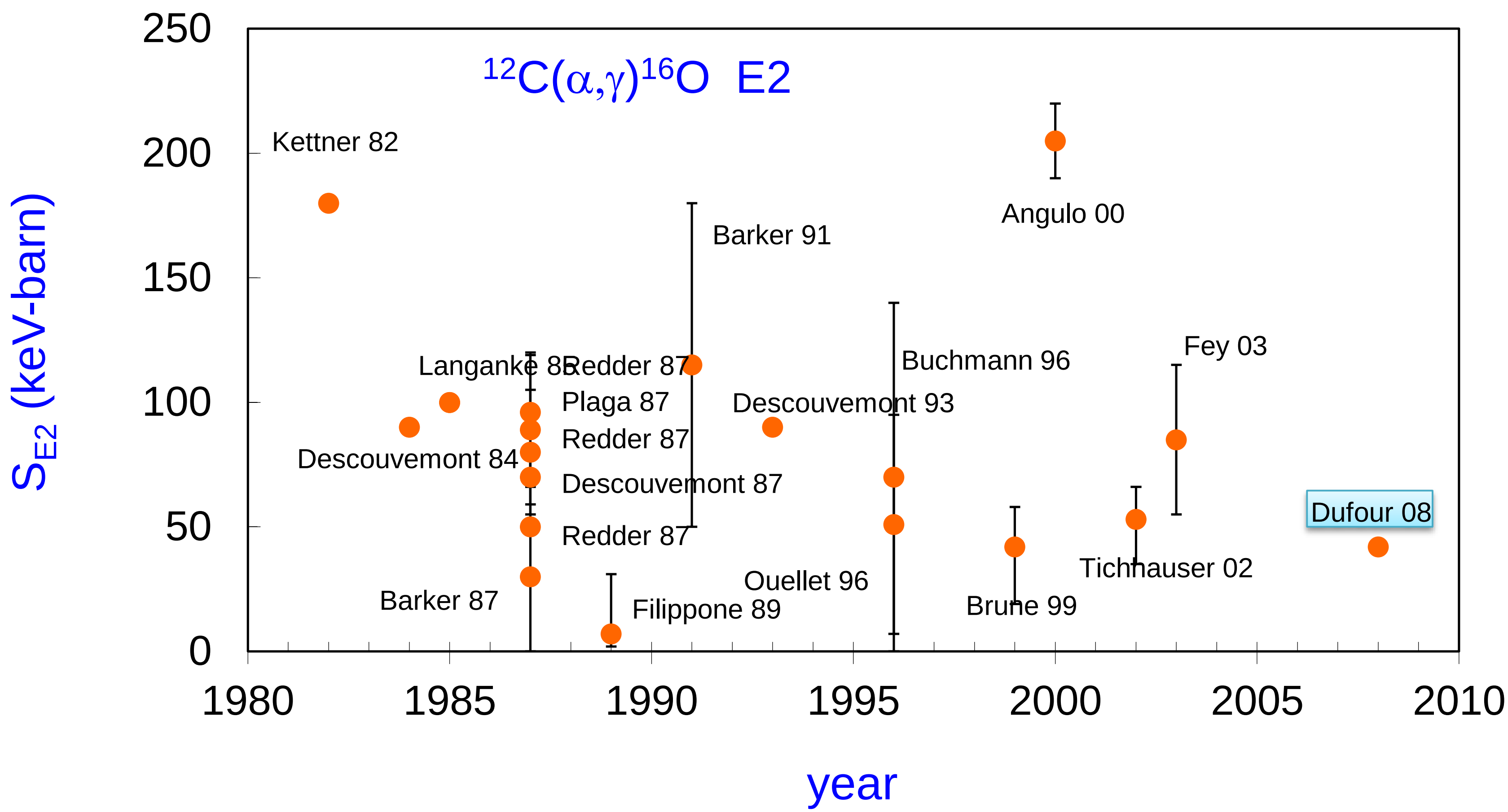




S(300 keV): current situation for E1



S(300 keV): current situation for E2



9. Microscopic models

9. Microscopic models

- Goal: solution of the Schrödinger equation $H\Psi = E\Psi$
- Hamiltonian: $H = \sum_i T_i + \sum_{j>i} V_{ij}$

  T_i = kinetic energy of nucleon *i*

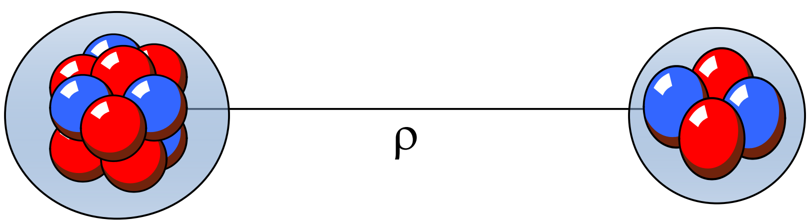
  V_{ij} = nucleon-nucleon interaction

- **Cluster** approximation $\Psi = \mathcal{A}\phi_1\phi_2 g(\rho)$

  with ϕ_1, ϕ_2 = internal wave functions (**input, shell-model**)

  $g(\rho)$ =relative wave function (**output**)

  $\mathcal{A}$ = antisymmetrization operator



- Generator Coordinate Method (GCM): the radial function is expanded in Gaussians
  → Slater determinants (well adapted to numerical calculations)
- Microscopic R-matrix: extension of the standard R-matrix → reactions

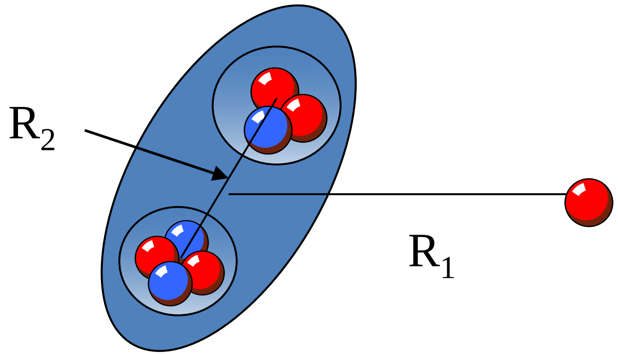
9. Microscopic models

Many applications: not only nuclear astrophysics

spectroscopy, exotic nuclei, elastic and inelastic scattering, etc.

Extensions:

- Multicluster calculations: → deformed nuclei (example: ^{7}Be+p)

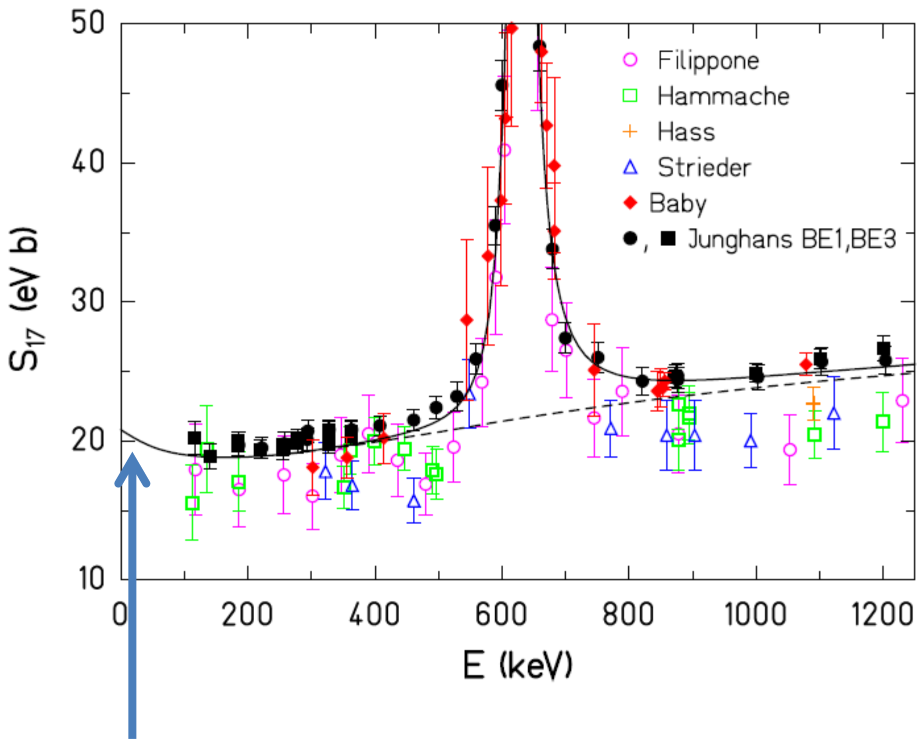


- Multichannel calculations: $\Psi = \mathcal{A}\phi_1\phi_2 g(\rho) + \mathcal{A}\phi_1^*\phi_2^* g^*(\rho) + \cdots$
  → better wave functions
  → inelastic scattering, transfer

- Ab initio calculations: no cluster approximation
  →very large computer times
  → limited to light nuclei
  → difficult for scattering (essentially limited to nucleon-nucleus)

Example: $^{7}Be(p,\gamma)^{8}B$

- Important for the solar-neutrino problem
- Since 1995, many experiments:
    - Direct (proton beam on a ^{7}Be target)
    - Indirect (Coulomb break-up)
- Extrapolation to zero energy needs a theoretical model (energy dependence)



astrophysical energies

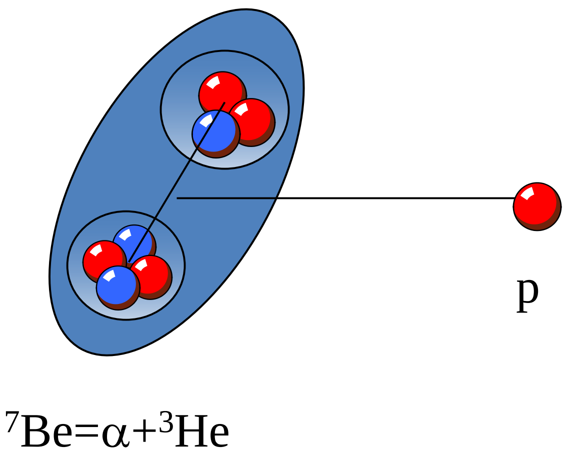
From E. Adelberger et al., Rev. Mod. Phys.83 (2011) 196

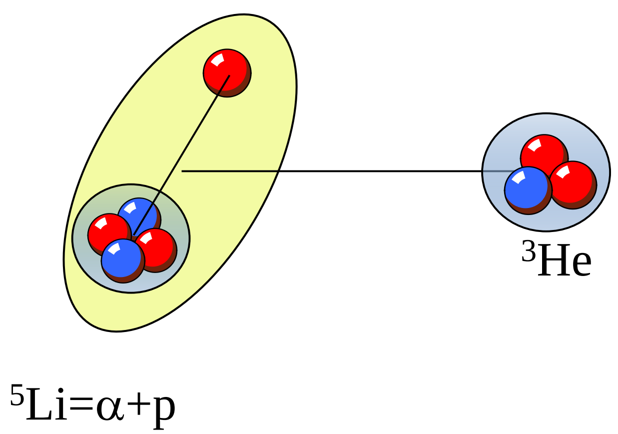
9. Microscopic models

Example: ^{7}Be(p,γ)^{8}B

- Microscopic cluster calculations: 3-cluster calculations
  - P. D. and D. Baye, Nucl. Phys. A567 (1994) 341
  - P.D., Phys. Rev. C 70, 065802 (2004)

- Includes the deformation of ^{7}Be: cluster structure a+^{3}He
- Includes rearrangement channels ^{5}Li+^{3}He
- Can be applied to ^{8}B/^{8}Li spectroscopy
- Can be applied to ^{7}Be(p,γ)^{8}B and ^{7}Li(n,γ)^{8}Li





Spectroscopy of ^{8}B

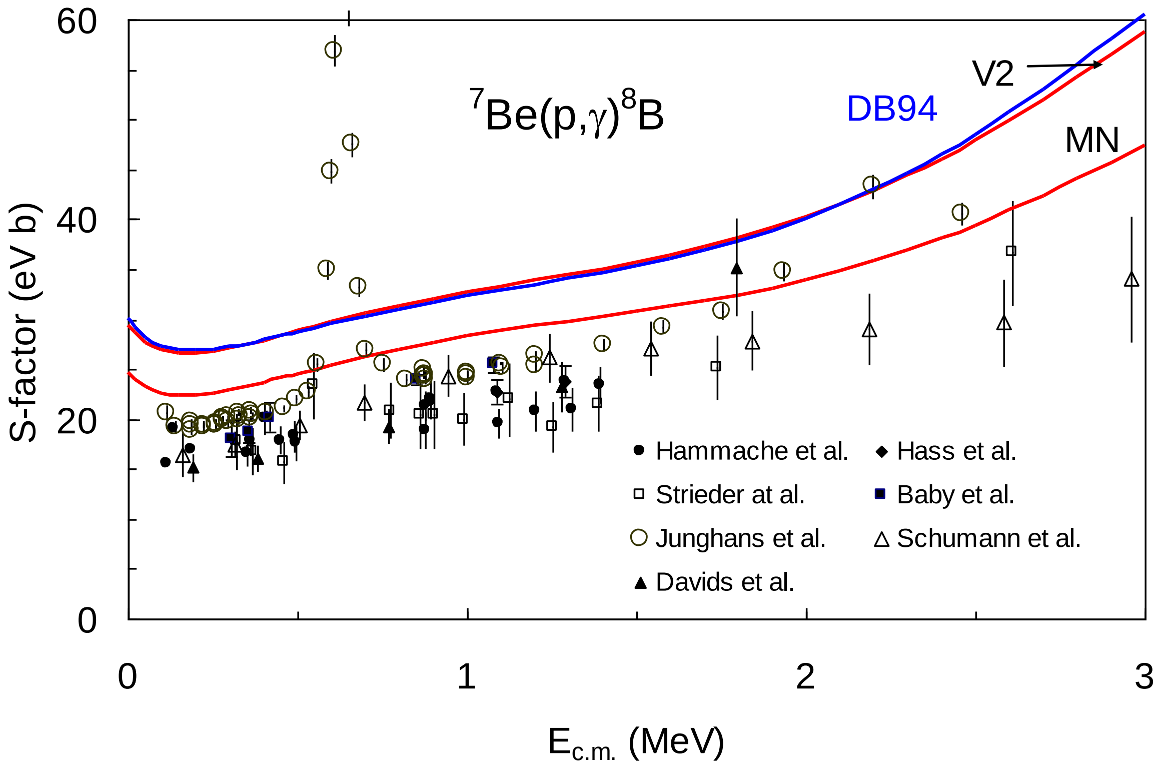
	experiment	Volkov	Minnesota
μ (2^+) (μ_N)	1.03	1.48	1.52
$Q(2^+)$ ($e.fm^2$)	6.83 ± 0.21	6.6	6.0
$B(M1, 1^+ \rightarrow 2^+)$ (W.u.)	5.1 ± 2.5	3.4	3.8

Channel components in the ^{8}B ground state

$^{7}Be(3/2^-)+p$	47%
$^{7}Be(1/2^-)+p$	9%
$^{5}Li(3/2^-)+^{3}He$	34%
$^{5}Li(1/2^-)+^{3}He$	3%

$\Rightarrow$ Important role of the 5+3 configuration

$^7Be(p,\gamma)^8B$ S factor



- Low energies (E<100 keV): energy dependence given by the Coulomb functions
- 2 NN interactions (MN, V2): → the sensitivity can be evaluated
- Overestimation: due to the ^8B ground state (cluster approximation)

9. Microscopic models

Cluster models

- In general a good approximation, but do not allow the use of realistic NN interactions
- Example: α particle described by 4 0s orbitals

  → intrinsic spin =0

  → no spin-orbit, no tensor force, no 3-body force

  → these terms are simulated by (central) NN interactions

Ab initio models

- No cluster approximation
- Use of realistic NN interactions (fitted on deuteron, NN phase shifts, etc.)
- Application: d+d systems $^{2}H(d,\gamma)^{4}He$, $^{2}H(d,p)^{3}H$, $^{2}H(d,n)^{3}He$

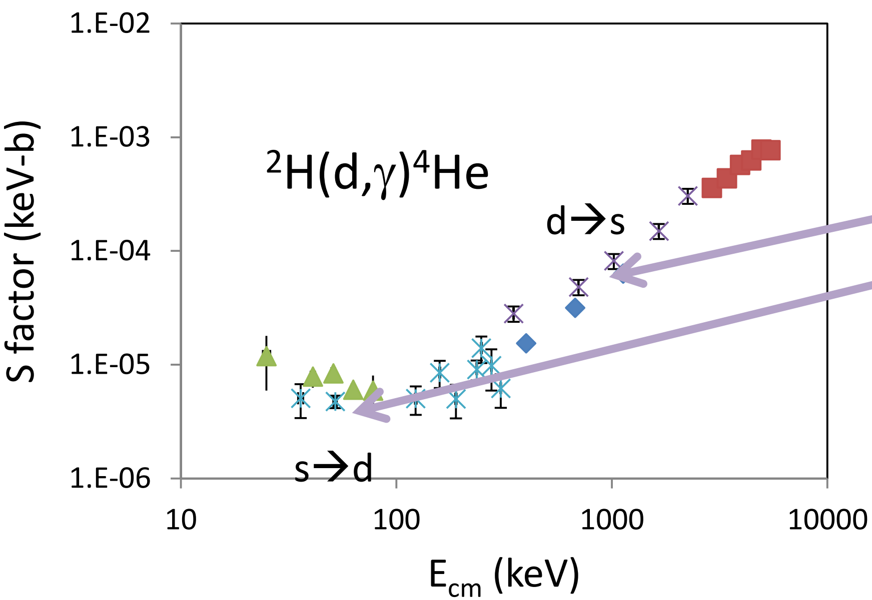
  two physics issues

  - Analysis of the d+d S factors (Big-Bang nucleosynthesis)
  - Role of the tensor force in $^{2}H(d,\gamma)^{4}He$

9. Microscopic models

$^2H(d,\gamma)^4He$ S factor

- Ground state of ^4He=0+
- E1 forbidden→ main multipole is E2 → 2^+ to 0^+ transition→ d wave as initial state
- Experiment shows a plateau below 0.1 MeV: typical of an s wave
- Interpretation : the ^4He ground state contains an admixture of d wave
  final 0^+ state: $\Psi^{0+} = \Psi^{0+}(L=0, S=0) + \Psi^{0+}(L=2, S=2) = | 0^+, 0 > + | 0^+, 2 >$
  initial 2^+ state: $\Psi^{2+} = \Psi^{2+}(L=2, S=0) + \Psi^{2+}(L=0, S=2) = | 2^+, 0 > + | 2^+, 2 >$



E2 matrix element $< \boldsymbol{\Psi^{0+}} \mid \boldsymbol{E2} \mid \boldsymbol{\Psi^{2+}} >$

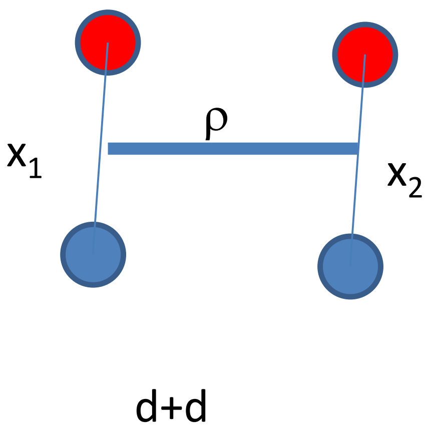
$\approx < 0^+, 0 \mid \mathrm{E2} \mid 2^+, 0 >$: d→s, dominant E>100 keV

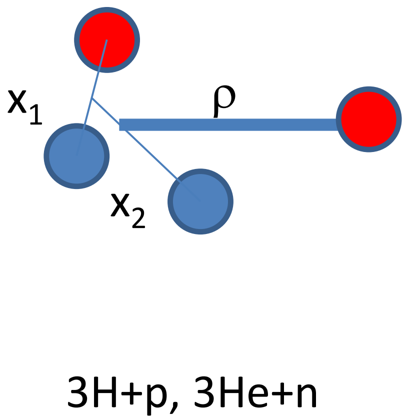
$+ < 0^+, 2 \mid \mathrm{E2} \mid 2^+, 0 >$: s→d, tensor (E<100 keV)

→ direct effect of the tensor force

Application: d+d systems

- Collaboration Niigata (K. Arai, S. Aoyama, Y. Suzuki)-Brussels (D. Baye, P.D.)
  Phys. Rev. Lett. 107 (2011) 132502
- Mixing of d+d, ^{3}H+p, ^{3}He+n configurations



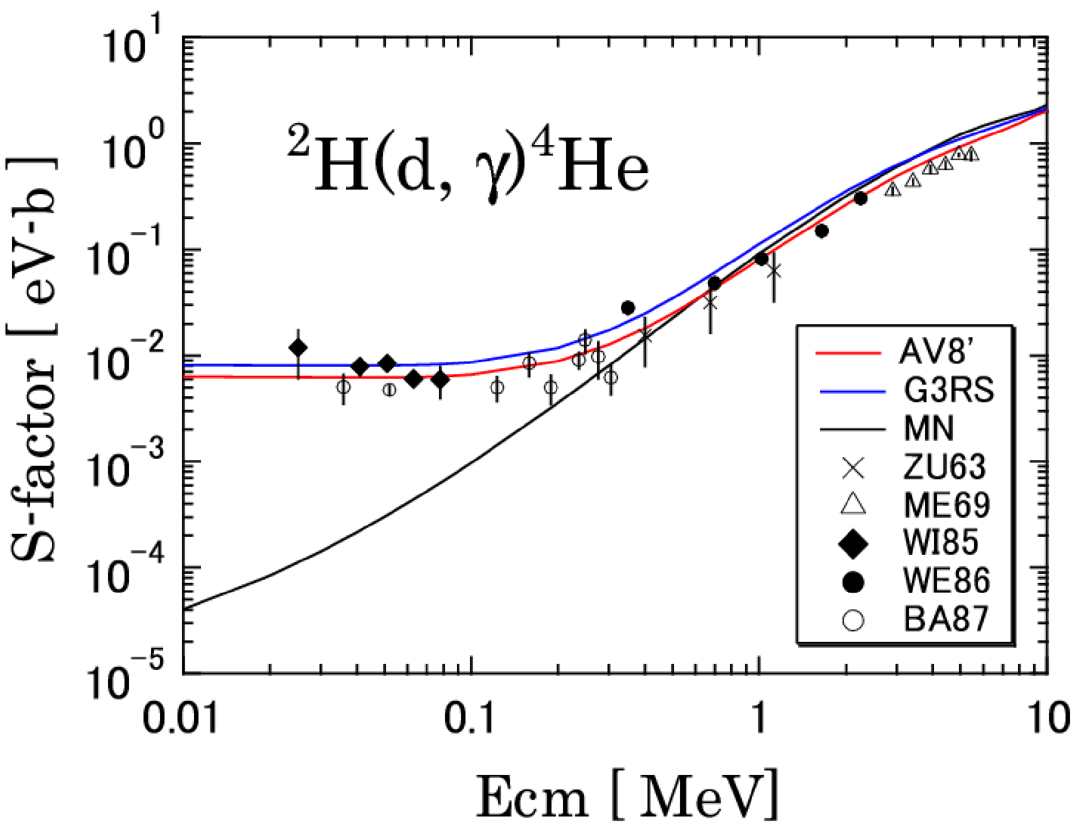


- The total wave function is written as an expansion over a gaussian basis
- Superposition of several angular momenta
- 4-body problem (in the cluster approximation we would have: $x_1=x_2=0$)

9. Microscopic models

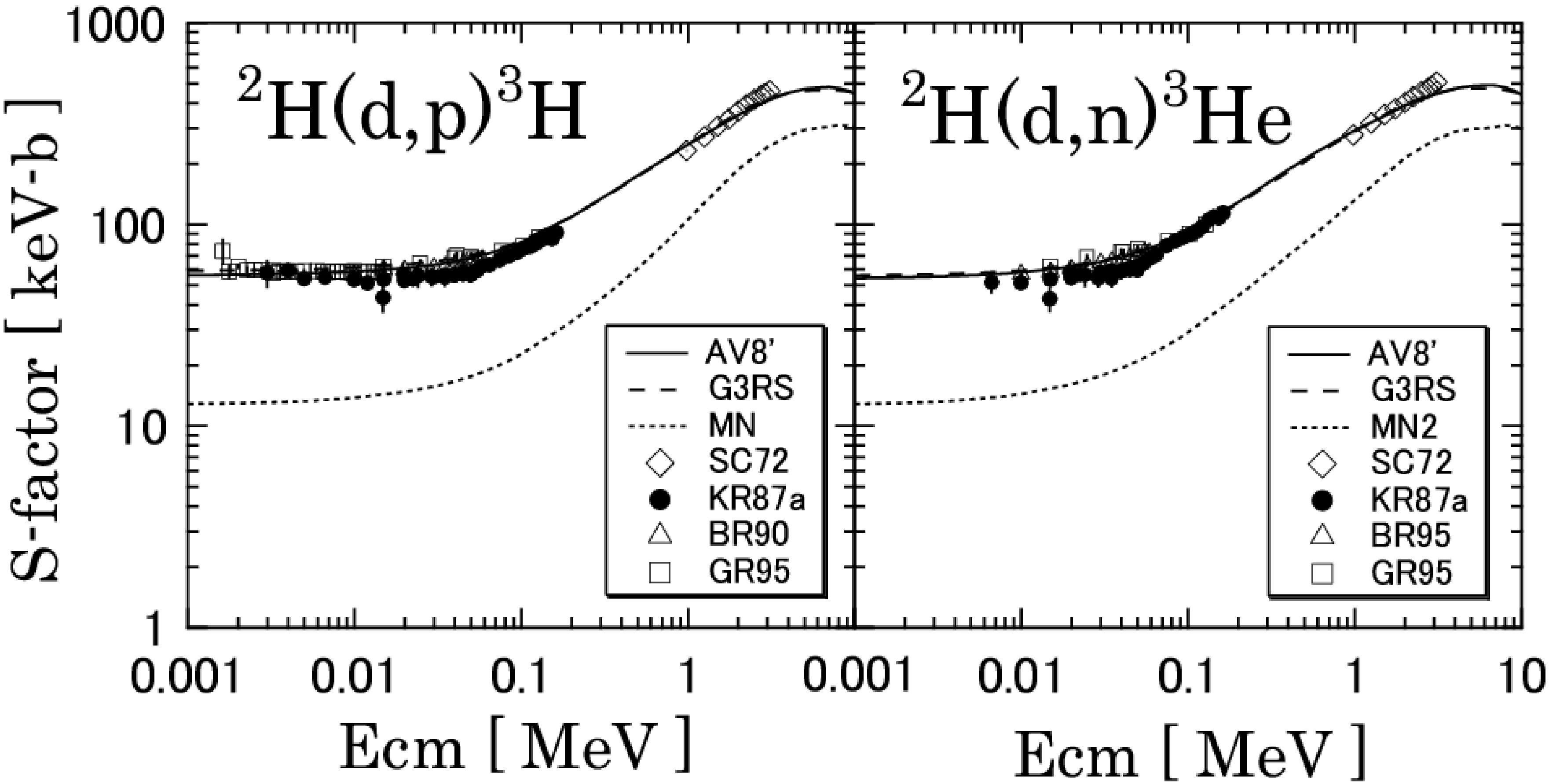
We use 3 NN interactions:

- Realistic: Argonne AV8', G3RS
- Effective: Minnesota MN



- No parameter
- MN does not reproduce the plateau (no tensor force)
- D wave component in ^{4}He:
  13.8% (AV8')
  11.2% (G3RS)

Transfer reactions $^2H(d,p)^3H$, $^2H(d,n)^3He$



10. Conclusions

Needs for nuclear astrophysics:

- low energy cross sections
- resonance parameters

Experiment: direct and indirect approaches

Theory: various techniques

- fitting procedures (R matrix)→ extrapolation
- non-microscopic models: potential, DWBA, etc.
- microscopic models:
  - cluster: developed since 1960's, applied to NA since 1980's
  - ab initio: problems with scattering states, resonances→ limited at the moment
- Current challenges: new data on $^{3}He(\alpha,\gamma)^{7}Be$, triple α process, $^{12}C(\alpha,\gamma)^{16}O$, etc.
  $D(d,\gamma)^{4}He$: 4 nucleons → 4 clusters