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Generalized Loop Space and TMDs

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Recast of a gauge theory in the Wilson loop space representation, where the degrees of freedom are absorbed in the path/loop dependence, allows one, in principle, to relate observables (field correlators) with fully gauge invariant fundamental variables. Over-completeness of this space requires the introduction of an equivalence relation which is provided by Wilson loop functionals operating on piecewise regular paths. On the other hand, certain classes of the Wilson loops possess the same singularity structure as some Transverse Momentum Dependent PDFs (TMDs), which are not renormalizable by the common methods due to exactly this singularity structure. By introducing geometrical operators, like the area-derivative, we derive an evolution equation for these Wilson loops and propose to develop further this method to construct appropriate renormalization scheme and full set of evolution equations for the TMDs.

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1. Introduction

Wilson lines (also called gauge links or eikonal lines) emerge naturally in gauge theory if one considers a directional derivative:

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(\psi(x+\varepsilon n)-\psi(x)\right), \tag{1.1}$$

of a Dirac field $\psi(x)$. In the form (1.1) this derivative is not well defined due to the fact that gauge fields are local and this is not taken into account. To solve this problem one needs to make use of the parallel transporter (1.2):

$$U(y,x;\Gamma)=\exp\left(\frac{i}{\hbar}g\int_\Gamma A_\mu \mathrm{d}x^\mu\right), \tag{1.2}$$

which is *path dependent*. Applying this operator to parallel transport the Dirac field $\psi(x)$ to $x+\varepsilon n$, and expanding to first order in ε results in:

$$\psi(x+\varepsilon n) = \psi(x)+\varepsilon n^\mu \partial_\mu \psi(x)+\mathscr{O}(\varepsilon^2) \tag{1.3}$$
$$D_\mu \psi(x) = \partial_\mu \psi(x)+igA_\mu(x)\psi(x), \tag{1.4}$$

the usual covariant derivative.

In the calculation of scattering amplitudes where hadrons or mesons are involved one often makes use of the hadronic tensor which operator definition is shown in Eq. (1.5) and is written in function of the correlator $\Phi^{\Gamma}_{ss'}(x,\vec{k}_\perp)$ shown in (1.6). Here we have explicitly written the spin and transverse momentum dependence, which give rise to the polarization dependent and transverse momentum distribution (TMDs) functions for (SI)DIS.

$$W_{\mu\nu}=e_q^2\int\frac{d^4k}{(2\pi)^4}\mathrm{Tr}\left(\gamma^\nu(\not{k}+\not{q})\gamma^\mu\Phi^{\Gamma}_{ss'}(x,\vec{k}_\perp)\right)\delta\left((k+q)^2\right). \tag{1.5}$$

$$\Phi^{\Gamma}_{ss'}(x,\vec{k}_\perp)=\int d^4x\, e^{-ikx}\langle P,s|\bar{\psi}(x)\,\Gamma\,\psi(0)|P,s'\rangle. \tag{1.6}$$

From (1.6) it is easy to see that the correlator is not gauge invariant and depends on fields at two different space-time points. Similar to the case of the directional derivative this problem is solved by making use of the parallel transporter (1.2), in this form also referred to as the Wilson line $\mathscr{W}_{\mathrm{TMD}}=U(x,0;\Gamma)$. In the TMD case these Wilson lines can consist of different segments, which are now no longer restricted to the light like directions as in the collinear case, but now can also contain transverse parts (see for instance [1, 2]). Using these, possibly complicated, Wilson lines we can construct a gauge-invariant correlator $\Phi^{\Gamma}_{ss'}(x,\vec{k}_\perp)$ shown in (1.7). For a discussion on the physical interpretation and structure of this gauge link and a general introduction to TMDs I refer to [3–5].

$$\Phi^{\Gamma}_{ss'}(x,\vec{k}_\perp)=\frac{1}{2}\int\frac{dz^-d^2z_\perp}{(2\pi)(2\pi^2)}e^{ik\cdot z}\left\langle P,s|\overline{\psi}(x)\,\Gamma\,\mathscr{W}_{\mathrm{TMD}}\,\psi(0)|P,s'\right\rangle_{z^+=0} \tag{1.7}$$

2. Generalized Loop Space

The Ambrose-Singer theorem allows the rewriting of a gauge theory in function of the holonomy of the gauge connection 1-forms, which depends on the loop over which this holonomy is calculated, and thus naturally introduces a loop space [6–8]. One of the issues with this approach is that a naive loop space is over-complete, which can be solved by the introduction of an equivalence relation that in our case will be the Wilson Loop Functional (WLF), the trace of the holonomy over the loop Γ, defined in (2.1). To consistently be able to reconstruct the gauge theory extra constraints, algebraic, unitarity and Mandelstam constraints, are necessary. They have their origin in the fact that one needs to be able to combine the loops algebraically and that it needs to be possible to write the product of two traces in the WLF as a single trace over some loop.

$$\mathscr{W}(\Gamma) = \mathrm{Tr}\, \mathscr{P} \exp\left[-ig \int_{\Gamma} dz^{\mu}\, A_{\mu}(z) \right] \in \mathbb{C}. \tag{2.1}$$

Expanding the exponential in (2.1) returns a sum over integrals (2.2) which were introduced by Chen [9–12] and are referred to as *Chen Iterated Integrals*. These integrals are a special example of the product integrals introduced by Volterra in 1880 [13]. They solve differential equations of the form $S'(t) = S(t) \cdot A(t)$, where the prime denotes the derivative with respect to t, that give rise to the parallel transporter (2.2) in differential geometry [8].

$$U_{\Gamma} = 1 + \int_{\Gamma} \omega + \int_{\Gamma} \omega_1 \omega_2 + \cdots \tag{2.2}$$

Chen Iterated Integrals are defined as an iterative extension of the usual line integrals:

$$X(\gamma) = I_{i_1 \cdots i_p}(\gamma) = \int_a^b I_{i_1 \cdots i_{p-1}}(\gamma^t)\, dx_{i_p}(t), \tag{2.3}$$

or after introduction of coordinates:

$$X^{\omega_1 \cdots \omega_r}(\gamma) = \int_{\gamma} \omega_1 \cdots \omega_r = \int_0^1 \left(\int_{\gamma^t} \omega_1 \cdots \omega_{r-1} \right) \omega_r(t) dt, \tag{2.4}$$

where $\omega_k(t) \equiv \omega_k(\gamma(t)) \cdot \dot{\gamma}(t)$ and γ^t represents the path for $t \in [0,t]$. Note that the path-ordering operator $\mathscr{P}$ is absorbed in the integrals by the way they are defined. Motivated by (2.5) one introduces the shuffle product (definition 2.1) on the set of 1-forms $\Omega = \bigwedge^1 \mathscr{M}$ (Real, Complex or Lie-Algebra valued) of the base manifold $\mathscr{M}$.

$$\int_{\gamma} \omega_1 \cdots \omega_k \int_{\gamma} \omega_{k+1} \cdots \omega_{k+l} = \sum_{\sigma} \int_{\gamma} \omega_{\sigma(1)} \cdots \omega_{\sigma(k+l)}, \tag{2.5}$$

where σ is running over all (k,l)-shuffles.

Definition 2.1. *Let $\omega_1 \cdots \omega_k = \omega_1 \otimes \cdots \otimes \omega_k \in \bigotimes^k \left(\bigwedge^1 \mathscr{M} \right), k \geq 1$ and $\omega_1 \cdots \omega_k = 0$, for $k = 0$, the shuffle multiplication is given by:*

$$\omega_1 \cdots \omega_k \bullet \omega_{k+1} \cdots \omega_{k+l} = \sum_{\sigma}{}' \omega_{\sigma(1)} \cdots \omega_{\sigma(k+l)}$$

with $\sum_{\sigma}'$ the sum over all (k,l)-shuffles.

Using this shuffle product Chen [9–12] introduced the notion of an algebraic path, which can be seen as a generalization of the intuitive notion of a path in a manifold in much the same way as distributions in calculus generalize functions. This shuffle algebra is Banach, Hopf, Commutative, Nuclear and Locally Multiplicative-Convex. These porperties then allow for the existence of a Gel'fand space (definition 2.2) so that we can introduce the concept of a generalized loop (definition 2.3).

Definition 2.2. *Let A be a commutative Banach algebra, then we write $\triangle(A)$ (or $\triangle$) for the collection of nonzero complex homomorphisms $h : A \to \mathbb{C}$. Elements of the Gel'fand space are called characters.*

Definition 2.3. *A Generalized Loop based at $p \in \mathscr{M}$ is a character of the algebra $\mathscr{A}_p$ or, equivalently, a continuous complex algebra homomorphism $\tilde{\alpha} : Sh(\mathscr{M}) \to \mathbb{C}$, that vanishes on the ideal $\mathbf{J}_p$. For the details on the ideal $\mathbf{J}_p$ I refer to [9–12, 8].*

It should be clear from (2.1) and the above discussion that the Wilson Loop Functionals form such a complex algebra homomorphism $\mathscr{A}_p$ depending on the (algebraic) path/loop under consideration. The continuity follows from introducing the Gel'fand topology on the set of Wilson Loop Functionals, which is the weak*-induced topology of these homomorphisms. This topology can be shown to be Hausdorff allowing for a consistent definition of convergence. Introducing the product $\tilde{\alpha} \star \tilde{\beta} = \widetilde{\alpha \cdot \beta}$ turns the generalized loop space in a topological group, with which one can associate an infinite dimensional Lie Algebra. The pointed differentiations (definition 2.4) form a tangent space to the generalized loop space and can be shown to be isomorphic to this Lie Algebra.

Definition 2.4. *A pointed differentiation is a pair* (d, p) *where* $d : \mathfrak{U} \to \Omega$ *is a differentiation and* $p \in Alg(\mathfrak{U}, k)$.

3. Wilson Loops on the Light-Cone - a new derivative

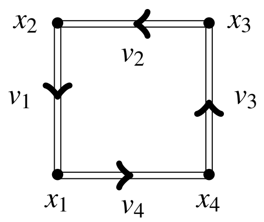

Figure 1: Parametrisation

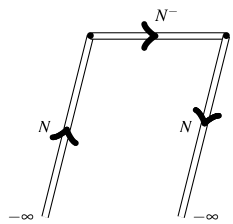

Figure 2: Π shape

As a first example we studied the vacuum expectation value of a Wilson loop quadrilateral on the light-cone (figure 1), which resulted at first order in [14]:

$$W_{\text{L.O.}}(\Gamma_\square) = 1 - \frac{\alpha_s C_F}{\pi}\left(2\pi\mu^2\right)^\varepsilon \Gamma(1-\varepsilon)\left[\frac{1}{\varepsilon^2}\left(-\frac{s}{2}\right)^\varepsilon + \frac{1}{\varepsilon^2}\left(-\frac{t}{2}\right)^\varepsilon - \frac{1}{2}\left(\ln^2\frac{s}{-t} + \pi^2\right)\right] + \mathscr{O}(\alpha_s^2), \quad (3.1)$$

where $s = (v_1 + v_2)^2$ and $t = (v_2 + v_3)^2$. When we tried applying the area derivative, which was also used by Makeenko and Migdal [15] in the derivation of the Makeenko-Migdal equations, to this simple Wilson loop this derivative failed to be well-defined [16]. We thus were forced to introduce a new differential operator (3.2) that for the moment seems to be a special case of the Fréchet derivative [17].

$$\delta\sigma^{+-} = N^+\delta N^- \to v_1\delta v_2 = \frac{1}{2}\delta s\,, \qquad \delta\sigma^{-+} = -N^-\delta N^+ \to -v_2\delta v_1 = \frac{1}{2}\delta t\,,$$
$$\frac{\delta}{\delta \ln\sigma} \equiv \sigma_{+-}\frac{\delta}{\delta\sigma_{+-}} + \sigma_{-+}\frac{\delta}{\delta\sigma_{-+}}. \quad (3.2)$$

4. Conjecture

Applying this new derivative followed by the usual renormalization derivative $\frac{d}{d\ln\mu}$ to (3.1), and taking the large N_c limit results in:

$$\mu\frac{d}{d\mu}\frac{\delta\,\ln W(\Gamma_\square)}{\delta\ln\sigma} = -4\,\Gamma_{\text{cusp}}\,,\ \Gamma_{\text{cusp}} = \frac{\alpha_s N_c}{2\pi} + O(\alpha_s^2), \quad (4.1)$$

where Γ_{cusp} is the quark cusp anomalous dimension. We then propose the generalization (4.2) of (4.1) as an evolution equation for Wilson loops on the light-cone [18–21].

$$\mu\frac{d}{d\mu}\frac{\delta\,\ln W(\Gamma_\square)}{\delta\ln\sigma} = -\sum_{\text{cusps}}\Gamma_{\text{cusp}}. \quad (4.2)$$

5. Other example

To test our conjecture we consider as an example the Pi-shape (figure 2). Calculating the diagram to first order results in [22]:

$$W(\Gamma_\Pi) = 1 + \frac{\alpha_s N_c}{2\pi} + \left[-L^2(NN^-) + L(NN^-) - \frac{5\pi^2}{24} \right] ,$$
$$L(NN^-) = \frac{1}{2}\left(\ln(\mu NN^- + i0) + \ln(\mu NN^- + i0)\right)^2 . \tag{5.1}$$

Applying our derivative, with now $\frac{d}{d\sigma} = \frac{d}{d(2NN^-)}$, and the usual energy scaling derivative $\frac{d}{d\ln\mu}$ to (5.1) shows that this is consistent with our conjecture [18]

$$\mu \frac{d}{d\mu} \left[\frac{d}{d\ln\sigma} \ln W(\Gamma_\Pi) \right] = -2\Gamma_{\text{cusp}} . \tag{5.2}$$

6. Self-intersecting and overlapping paths

In [23] we considered two symmetric extensions of the loop in figure 1. In a first case we added a copy of the original loop, but shifted it to the left resulting in a loop with overlapping paths. Due to the two possible relative orientations between the two constituting loops this gives rise to two possibilities. In a second case we also added a copy of the original loop but now we shifted it left and down, again taking into account the two possible relative orientations, this generates loops with a self-intersection (for figures see [23]). We calculated the leading order contributions of these four diagrams and applied the derivative (3.2) to them in such a way that the structure of the loop is not change (i.e. the number of cusps and self-intersections is unchanged). The results are consistent with our conjecture (4.2) if one counts the number of cusps in a correct way, which is derived from path-reduction. The details on how to count cusps can be found in [23].

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