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Properties of Homothetic Solutions in Bigravity

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We consider the properties of solutions in the bigravity theory for general models, which are parametrized by two parameters α_3 and α_4. We assume that two metric tensors $g_{\mu\nu}$ and $f_{\mu\nu}$ satisfy the condition $f_{\mu\nu} = C^2 g_{\mu\nu}$ where C is a constant. We call this class of solutions as homothetic solutions. Then we can find the solutions $f_{\mu\nu} = g_{\mu\nu}$ for arbitrary parameters. We investigate the conditions for the parameters so that the solutions with $C \neq 1$ could exist.

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1. Introduction

Bigravity is a recently proposed theory, which is nonlinear massive gravity that can be free of ghost with the dynamical reference metric [1, 2, 3, 4, 5]. This gravity model is called bigravity or bi-metric gravity because the model contains two symmetric tensor fields $g_{\mu\nu}$ and $f_{\mu\nu}$ and a massive spin-2 field appears in addition to the massless spin-2 field corresponding to the graviton.

In this work, we consider the general model of bigravity without specifying the parameters, and study the properties of the solutions [6, 7]. In order to obtain the solution, we only assume that one metric is proportional to another, $f_{\mu\nu} = C^2 g_{\mu\nu}$ with a constant C, as background solutions. And we investigate the parameter region which gives non-trivial solutions with $C \neq 1$.

2. Bigravity and the Equations of Motion

The action of bigravity is given by

$$\begin{aligned} S_{\text{bigravity}} =& M_g^2 \int d^4x \sqrt{-\det(g)} R(g) + M_f^2 \int d^4x \sqrt{-\det(f)} R(f) \\ & - 2m_0^2 M_{\text{eff}}^2 \int d^4x \sqrt{-\det(g)} \sum_{n=0}^{4} \beta_n e_n \left(\sqrt{g^{-1} f} \right) \end{aligned} \tag{2.1}$$

Here, g and f are dynamical variables and rank-two tensor fields which have properties as metrics, $R(g)$ and $R(f)$ are the Ricci scalars for $g_{\mu\nu}$ and $f_{\mu\nu}$, respectively, M_g and M_f are the two Planck mass scales for $g_{\mu\nu}$ and $f_{\mu\nu}$ as well, and the scale M_{eff} is the effective Planck mass scale defined by $1/M_{\text{eff}}^2 = 1/M_g^2 + 1/M_f^2$.

The quantities β_ns and m_0 are free parameters, and the formers define the form of interactions and the latter expresses the mass of the massive spin-2 field. The matrix $\sqrt{g^{-1}f}$ is defined by the square root of $g^{\mu\rho} f_{\rho\nu}$. For general matrix $\mathbf{X}$, $e_n(\mathbf{X})$s are polynomials of the eigenvalues of X:

$$\begin{aligned} & e_0(\mathbf{X}) =1, \quad e_1(\mathbf{X}) = [\mathbf{X}], \quad e_2(\mathbf{X}) = \frac{1}{2}\left([\mathbf{X}]^2 - [\mathbf{X}^2]\right), \quad e_3(\mathbf{X}) = \frac{1}{6}\left([\mathbf{X}]^3 - 3[\mathbf{X}][\mathbf{X}^2] + 2[\mathbf{X}^3]\right), \\ & e_4(\mathbf{X}) = \frac{1}{24}\left([\mathbf{X}]^4 - 6[\mathbf{X}]^2[\mathbf{X}^2] + 3[\mathbf{X}^2]^2 8[\mathbf{X}][\mathbf{X}^3] - 6[\mathbf{X}^4]\right) = \det(\mathbf{X}), \quad e_k(\mathbf{X}) = 0 \quad \text{for } k > 4, \end{aligned} \tag{2.2}$$

where the square brackets denote traces of the matrices, that is, $[X] = X^\mu_\mu$.

Now we consider the variation of the action (2.1) with respect to $g_{\mu\nu}$ and $f_{\mu\nu}$, and we find the equations of motion for are given by

$$0 = R_{\mu\nu}(g) - \frac{1}{2} R(g) g_{\mu\nu} + \frac{1}{2} \left(\frac{m_0 M_{\text{eff}}}{M_g} \right)^2 \sum_{n=0}^{3} (-1)^n \beta_n \left[g_{\mu\lambda} Y^\lambda_{(n)\nu}(\sqrt{g^{-1}f}) + g_{\nu\lambda} Y^\lambda_{(n)\mu}(\sqrt{g^{-1}f}) \right] \tag{2.3}$$

$$0 = R_{\mu\nu}(f) - \frac{1}{2} R(f) f_{\mu\nu} + \frac{1}{2} \left(\frac{m_0 M_{\text{eff}}}{M_f} \right)^2 \sum_{n=0}^{3} (-1)^n \beta_{4-n} \left[f_{\mu\lambda} Y^\lambda_{(n)\nu}(\sqrt{f^{-1}g}) + f_{\nu\lambda} Y^\lambda_{(n)\mu}(\sqrt{f^{-1}g}) \right]. \tag{2.4}$$

Here, for a matrix $\mathbf{X}$, $Y_n(\mathbf{X})$s are defined by

$$Y^\lambda_{(n)\nu}(\mathbf{X}) = \sum_{r=0}^{n} (-1)^r \left(X^{n-r} \right)^\lambda{}_\nu \, e_r(\mathbf{X}), \tag{2.5}$$

3. Homothetic Solutions and Their Properties

3.1 Equations of Motion

Now, we consider the case where $f_{\mu\nu} = C^2 g_{\mu\nu}$ and C is a constant. This class of solutions is called homothetic solutions. By putting this condition on (2.3) and (2.4), we obtain the two Einstein equations with cosmological constant as follows:

$$0 = R_{\mu\nu}(g) - \frac{1}{2}R(g)g_{\mu\nu} + \Lambda_g(C)g_{\mu\nu}, \tag{3.1}$$

$$0 = R_{\mu\nu}(f) - \frac{1}{2}R(f)f_{\mu\nu} + \Lambda_f(C)f_{\mu\nu}. \tag{3.2}$$

Here the dynamics of two metric tensors $g_{\mu\nu}$ and $f_{\mu\nu}$ are separated from each other, and the Bianchi identity is automatically satisfied. This structure of dynamics means that if $f = C^2 g$, the solutions of bigravity is vacuum solutions of the general relativity, and we can use the solutions in the general relativity.

Two cosmological constants are defined as follows:

$$\Lambda_g(C) = \left(\frac{m_0 M_{\rm eff}}{M_g}\right)^2 (C-1)\left[(\alpha_3 - \alpha_4)C^2 + (-5\alpha_3 + 2\alpha_4 + 3)C + (4\alpha_3 - \alpha_4 - 6)\right], \tag{3.3}$$

$$\Lambda_f(C) = \left(\frac{m_0 M_{\rm eff}}{M_f}\right)^2 \frac{C-1}{C^3}\left[\alpha_4 C^2 + (3\alpha_3 - 2\alpha_4)C + (-3\alpha_3 + \alpha_4 + 3)\right]. \tag{3.4}$$

Here, we take $C > 0$ and express five β_ns in terms of two free parameters α_3 and α_4 [3], as follows:

$$\beta_0 = 6 - 4\alpha_3 + \alpha_4, \quad \beta_1 = -3 + 3\alpha_3 - \alpha_4$$
$$\beta_2 = 1 - 2\alpha_3 + \alpha_4, \quad \beta_3 = \alpha_3 - \alpha_4, \quad \beta_4 = \alpha_4. \tag{3.5}$$

For the consistency, both of Eqs.(3.1) and (3.2) should be identical with each other. By putting $f_{\mu\nu} = C^2 g_{\mu\nu}$, we find $R_{\mu\nu}(f) = R_{\mu\nu}(g)$, $R(f)f_{\mu\nu} = R(g)g_{\mu\nu}$, thus we need $\Lambda_g = C^2\Lambda_f$. From the Eqs.(3.3) and (3.4), we obtain the quartic equation as follows:

$$\begin{aligned} 0 =& (C-1)\left[M_{\rm ratio}^2(\alpha_3 - \alpha_4)C^3 + \{-5M_{\rm ratio}^2\alpha_3 + (2M_{\rm ratio}^2 - 1)\alpha_4 + 3M_{\rm ratio}^2\}C^2\right. \\ &\left. + \{(4M_{\rm ratio}^2 - 3)\alpha_3 - (M_{\rm ratio}^2 - 2)\alpha_4 - 6M_{\rm ratio}^2\}C + (3\alpha_3 - \alpha_4 - 3)\right], \end{aligned} \tag{3.6}$$

where we define $M_{\rm ratio} \equiv M_f / M_g$.

Apparently, we can find that general model with arbitrary α_3 and α_4 has solution where $f_{\mu\nu} = g_{\mu\nu}$, that is $C = 1$, and therefore two cosmological constants vanish, which tells that general model in bigravity has the solution $g_{\mu\nu} = f_{\mu\nu}$ which is asymptotically flat solution in the general relativity. Now, we concentrate on the cubic part in Eq.(3.6) and classify two parameters α_3 and α_4 when $C \neq 1$. If there is no solution which satisfies $C > 0$ and $C \neq 1$, we do not have non-trivial solution in bigravity.

3.2 Classification of Solutions

We classify the parameter region by the existence of the solutions. In the following, we consider the case $M_f = M_g$, that is, $M_{\rm ratio} = 1$ for simplicity, and define a function $F_3(x)$ as follows:

$$F_3(x) \equiv (\alpha_3 - \alpha_4)x^3 - (5\alpha_3 - \alpha_4 - 3)x^2 + (\alpha_3 + \alpha_4 - 6)x + (3\alpha_3 - \alpha_4 - 3). \tag{3.7}$$

We now solve the equation $F_3(x) = 0$ for $x > 0$. Because we find $F_3(1) = -6$, $C = 1$ is not a solution for arbitrary α_3 and α_4. Hence, by solving $F_3(x) = 0$, we may obtain non-trivial solutions with $C \neq 1$ thanks to $M_{\text{ratio}} = 1$.

We can solve the equation $F_3(x) = 0$ analytically, and we find the parameter region where we can obtain non-trivial solutions. The parameter region is described in Figure 1 [7].

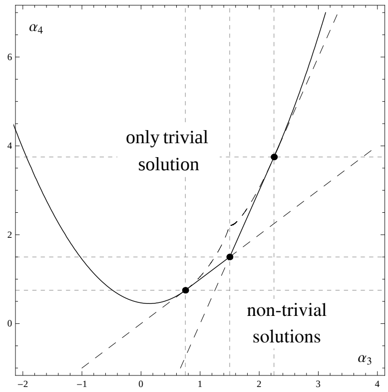

Figure 1: In the region where α_4 is large and $|\alpha_3|$ is small, we do not obtain non-trivial solution. The minimal model $(\alpha_3, \alpha_4) = (1, 1)$ is included in the region where there is only trivial solution.

4. Summary and Discussion

We have studied the properties of general model in bigravity with condition $f_{\mu\nu} = C^2 g_{\mu\nu}$. In this condition, we have shown that the structure of dynamics is not changed from that of the general relativity, the solutions of bigravity are also that of the general relativity as well. We have also investigated the parameter region with condition $M_f = M_g$, where we can obtain the non-trivial solution $C \neq 1$.

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