Feynman integrals via hyperlogarithms

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This talk summarizes recent developments in the evaluation of Feynman integrals using hyperlogarithms. We discuss extensions of the original method, new results that were obtained with this approach and point out current problems and future directions.

[*]Speaker.

1. Schwinger parameters and hyperlogarithms

Many talks at this conference demonstrate the remarkable progress in the exact evaluation of Feynman integrals that was achieved during the past few years. A key element shared by many of these advances is an improved understanding of multiple polylogarithms

$$\mathrm{Li}_{n_1,\dots,n_r}(z_1,\dots,z_r) = \sum_{0 < k_1 < \dots < k_r} \frac{z_1^{k_1} \cdots z_r^{k_r}}{k_1^{n_1} \cdots k_r^{n_r}}, \quad \text{where} \quad n_1,\dots,n_r \in \mathbb{N}, \qquad (1.1)$$

which suffice to express a wide class of Feynman integrals.[1] They admit representations as iterated integrals [19] of the form

$$L_{\sigma_n,\dots,\sigma_1}(z) = \int\cdots\int\limits_{0 < z_1 < \dots z_n < z} \frac{\mathrm{d}z_n}{z_n - \sigma_n} \cdots \frac{\mathrm{d}z_1}{z_1 - \sigma_1} \quad \text{with} \quad \sigma_1,\dots,\sigma_n \in \mathbb{C} \quad \text{and} \quad \sigma_1 \neq 0, \quad (1.2)$$

called hyperlogarithms [23], and can be manipulated symbolically very efficiently. In these proceedings we consider the method put forward by Francis Brown [13], that aims at integrating out Schwinger parameters α_e one by one in the representation

$$\Phi(G) = \frac{\Gamma(\omega)}{\prod_e \Gamma(\nu_e)} \int_{\mathbb{R}_+^N} \frac{\delta(1 - \alpha_{e_N})}{\psi^{D/2}} \left(\frac{\psi}{\varphi}\right)^\omega \prod_{e \in E} \alpha_e^{\nu_e - 1} \mathrm{d}\alpha_e, \quad \omega = \sum_{e \in E} \nu_e - \frac{D}{2} h_1(G) \qquad (1.3)$$

of Feynman integrals associated to a graph G with $h_1(G)$ loops and N edges E. This formula assumes scalar propagators $(k_e^2 + m_e^2)^{-\nu_e}$ of mass m_e at each edge e (raised to some power ν_e called *index*), but generalizations to tensor integrals exist [25]. The *Symanzik polynomials* [9] sum all spanning trees T and 2-forests F in

$$\psi = \sum_T \prod_{e \notin T} \alpha_e \quad \text{and} \quad \varphi = \sum_F q(F)^2 \prod_{e \notin F} \alpha_e + \psi \sum_{e \in E} \alpha_e m_e^2, \qquad (1.4)$$

where $q(F)^2$ is the square of the momentum flowing between the two components of F. We choose an order $e_1,\dots,e_N$ of edges and compute $I_k := \int_0^\infty I_{k-1} \, \mathrm{d}\alpha_{e_k}$ iteratively, starting with the original integrand I_0 of (1.3). Details of the involved algorithms are given in [12, 13] and a public implementation in `Maple` [24] is available [28]. Also the more general approach presented at the preceeding conference [7], based on iterated integrals in several variables (instead of just one), is going to be published in the nearest future.

The crucial limitation of this method is that all $I_k(\alpha_{e_{k+1}},\dots,\alpha_{e_N})$ must be expressible as $\mathbb{C}(\alpha_{e_{k+1}},\dots,\alpha_{e_N})$-linear combinations hyperlogarithms (1.2) with respect to the next integration variable $z = \alpha_{e_{k+1}}$. If this holds for some order on E, we call G *linearly reducible*. This criterion depends only on the Symanzik polynomials and reduction algorithms are available [12, 13] that check conditions sufficient for linear reducibility.[2]

In particular this discussion is unaffected by infinitessimal expansions in analytic regulators, like the popular shift $D = \widehat{D} - 2\varepsilon$ away from an even dimension $\widehat{D} \in 2\mathbb{N}$ of spacetime[3]

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[1]Counterexamples are known in massive [3, 5], massless [15] and even supersymmetric theories [17, 26].

[2]This means that reductions can reveal linear reducibility, but not disprove it.

[3]Even if one starts out with $\widehat{D} = 4$ dimensions, a reduction of tensor integrals will produce scalar integrals also in even dimensions above $\widehat{D}$.
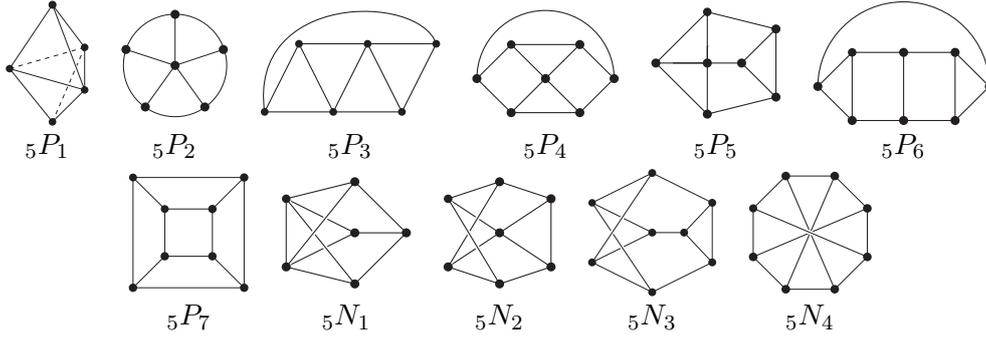
Figure 1: All three-connected five-loop vacuum graphs [27], divided into planar (P) and non-planar (N) ones. The zig-zag $_5P_3 = \mathrm{ZZ}_5$ and $_5N_1$ were considered in [13]. Cutting any edge produces a propagator with four loops, deleting a three-valent vertex creates a three-loop three-point graph.
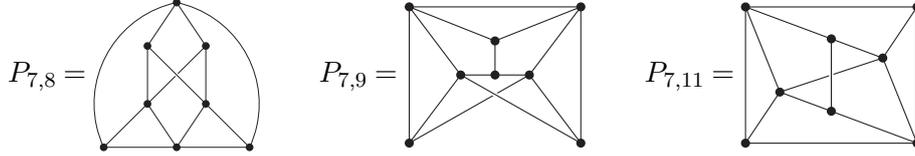

Figure 2: The most complicated primitive periods in ϕ^4-theory at seven loops are given by the three graphs $P_{7,8}$, $P_{7,9}$ and $P_{7,11}$ in the notation of the census [31].

such that $\Phi(G) = \sum_k \Phi_k(G) \cdot \varepsilon^k$ becomes a Laurent series. All of its coefficients can in principle be computed as soon as G is linearly reducible, and remarkably this also applies for the multivariate expansions in the indices $\nu_e = \widehat{\nu}_e + \varepsilon_e$ close to integers $\widehat{\nu}_e \in \mathbb{Z}$.

2. Massless propagators up to 6 loops

Originally, this method was applied to convergent massless propagators G only [13]. These transform into vacuum graphs $\widetilde{G} = G \dot\cup \{e\}$ upon joining the external legs of G by an edge e. Then only the first Symanzik $\psi_{\widetilde{G}} = \psi_G \alpha_e + \varphi_G$ is of interest and a powerful toolbox of algebraic identities becomes available [12, 15, 16].

We verified that all massless propagators up to four loops (e.g. all which arise by cutting an edge of the graphs shown in figure 1) are linearly reducible [27]. The recent computation [4, 33] of the corresponding master integrals (using different methods) can therefore be extended to higher orders in ε and to include self-energy insertions. Such results will be needed for calculations at higher loop orders and are already available for many examples [27].

The traditional benchmark in this field is the evaluation of primitive divergent periods of ϕ^4-theory in $D = 4$ dimensions [11]. This problem is now completely solved up to 7 loops: All primitive graphs of the census [31] to this order are linearly reducible—except for $P_{6,4}$ (that can be computed with *graphical functions* [32]) and $P_{7,11}$ of figure 2 which we address below. These results suggest that all massless propagators with at most 6 loops could be computable using multiple polylogarithms. This is the most optimistic scenario, as some 7 loop massless propagators are known to exceed the world of polylogarithms [14].

2.1 Changing variables for linear reductions

After ten integrations for $P_{7,11}$, the partial integral $I_{10} = L/d_{10}$ consists of a hyperlogarithm L of weight 8 and the irreducible, totally quadratic denominator

$$
\begin{aligned}
d_{10} = {}& \alpha_2 \alpha_4^2 \alpha_1 + \alpha_2 \alpha_4^2 \alpha_3 - \alpha_1 \alpha_2 \alpha_3 \alpha_4 + \alpha_2^2 \alpha_4 \alpha_1 + \alpha_2^2 \alpha_4 \alpha_3 - 2\alpha_2 \alpha_3^2 \alpha_4 - \alpha_2^2 \alpha_3^2 \\
& - 2\alpha_2^2 \alpha_3 \alpha_1 - 2\alpha_2 \alpha_3^2 \alpha_1 - \alpha_3^2 \alpha_4^2 - 2\alpha_3^2 \alpha_4 \alpha_1 - \alpha_2^2 \alpha_1^2 - 2\alpha_2 \alpha_3 \alpha_1^2 - \alpha_3^2 \alpha_1^2.
\end{aligned}
\tag{2.1}
$$

Further integration would introduce square roots of the discriminant of this polynomial and therefore escape the space of hyperlogarithms with rational arguments; $P_{7,11}$ is not linearly reducible as such. But (2.1) can be linearized: If we change variables according to $\alpha_3' \alpha_1 = \alpha_3(\alpha_1 + \alpha_2 + \alpha_4)$, $\alpha_4 = \alpha_4'(\alpha_2 + \alpha_3')$ and $\alpha_1 = \alpha_1' \alpha_4'$, the new denominator

$$
d_{10}' = (\alpha_2 + \alpha_3')(\alpha_2 + \alpha_2 \alpha_4' - \alpha_1')(\alpha_1' \alpha_4' + \alpha_2 + \alpha_2 \alpha_4' + \alpha_3' \alpha_4')
\tag{2.2}
$$

factorizes linearly such that α_1', α_3' and α_4' can indeed be integrated without further complications ($\alpha_2 = 1$). Interestingly, the final period is not a multiple zeta value but a linear combination of multiple polylogarithms (1.1) evaluated at a sixth roots of unity.

2.2 Missing alternating sums

The other two of the most complicated vacuum graphs with 7 loops (shown in figure 2) are linearly reducible without further ado and we succeeded to compute

$$
P_{7,9} = \tfrac{92943}{160} \zeta_{11} + \tfrac{3381}{20}\left(\zeta_{3,5,3} - \zeta_3 \zeta_{3,5}\right) - \tfrac{1155}{4}\zeta_3^2 \zeta_5 + 896 \zeta_3 \left(\tfrac{27}{80}\zeta_{3,5} + \tfrac{45}{64}\zeta_3 \zeta_5 - \tfrac{261}{320}\zeta_8\right)
\tag{2.3}
$$

using hyperlogarithms. This result was recently proposed by David Broadhurst [10] and is now confirmed. Interestingly, for both $P_{7,8}$ and $P_{7,9}$ the last integrand I_{12} has denominator $(\alpha_1 + \alpha_2)(\alpha_1 - \alpha_2)$ and produces a result involving alternating sums. Only after reducing these to the data mine basis [6] we arrived at the multiple zeta value (2.3). So far, the reason for this absence of alternating sums in massless ϕ^4-theory still eludes us.

3. Non-trivial kinematics

With non-trivial kinematics, the second Symanzik φ makes linear reducibility a more restrictive criterion that depends sensitively on the distribution of external momenta and internal masses. In particular it requires at least one propagator to be massless.[4]

Some examples of reducible graphs with masses are shown in figure 3 and some explicit results were computed [29]. However, no combinatorial characterization of this class of graphs is known so far. We now specialize to two particular kinematic configurations in order to state more general results.

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[4]This restriction does not apply when there is no momentum dependence at all. Important examples of such graphs that are linearly reducible are known and hyperlogarithms are used to compute generating functions for Mellin moments of massive operator matrix elements [1, 2].
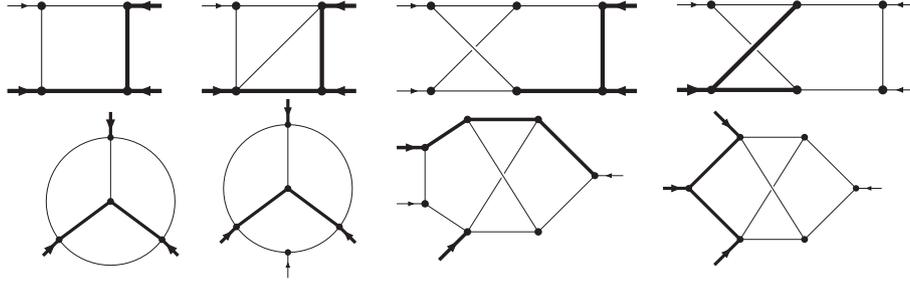
Figure 3: Examples of linearly reducible graphs with some massive internal and off-shell external momenta (thick edges).
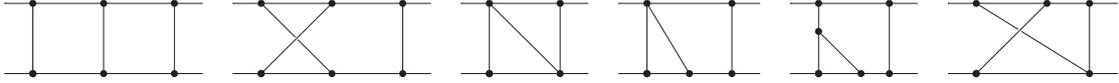

Figure 4: Two-loop four-point graphs without self-energy (propagator) subgraphs.

3.1 Massless three-point functions

Massless three-point functions (depending on arbitrary p_1^2, p_2^2 and p_3^2) seem to be very well suited for parametric integration: Linear reducibility holds for all such graphs at 2 loops [18] and even at 3 loops [29]. This includes all graphs obtained by removing one three-valent vertex from any of the vacuum graphs in figure 1 (external legs are attached to the neighbours of that deleted vertex).

When these integrals are written as hyperlogarithms, one encounters the square root of the Källén function $\lambda = \left(p_1^2 + p_2^2 - p_3^2\right)^2 - 4p_1^2 p_2^2$. The reparametrization given by

$$p_2^2 = p_1^2 \cdot z\bar{z} \quad \text{and} \quad p_3^2 = p_1^2 \cdot (1-z)(1-\bar{z}) \quad \text{rationalizes} \quad \sqrt{\lambda} = \pm p_1^2 \cdot (z - \bar{z}) \qquad (3.1)$$

in terms of two new variables z and $\bar{z}$. Then $\Phi(G)$ is given by $p_1^{-2\omega}$ and a rational linear combination of hyperlogarithms $L_v(z) L_w(\bar{z})$. The polynomial reduction provides an upper bound on the set of letters that may appear in the words v and w; put differently it restricts the entries of the *symbol* [20] of the hyperlogarithms.

Details and explicit results for some integrals can be found in [29]. There are reducible examples at higher loop orders, but non-reducible graphs appear already at 4 loops.

3.2 Massless four-point functions

All massless four-point on-shell graphs ($p_1^2 = p_2^2 = p_3^2 = p_4^2 = 0$) with at most two loops are linearly reducible [8]. In particular these include the graphs of figure 4.

At 3 loops the first graphs that are not linearly reducible occur, including the complete graph K_4 (first graph in bottom row of figure 3 with the fourth external momentum attached to the center) which was recently evaluated [21] to harmonic polylogarithms [30]. For its parametric integration, a change of variables is necessary [29].

While the general type of massless four-point functions is not clear, it seems that at least all of the n-loop box ladders B_n (figure 5) are linearly reducible: For on-shell kinematics we always obtained a result in terms of harmonic polylogarithms. Due to the recent interest [22] into these integrals in $D = 6$ dimensions (where they are finite), we list
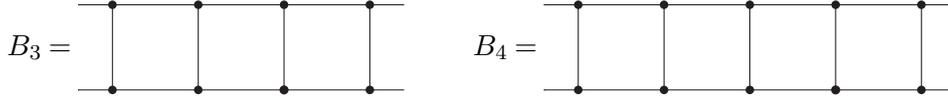
Figure 5: Examples of the *box ladder* graphs B_n of $n = 3$ and $n = 4$ loops.

the values $c_n := \Phi(B_n)|_{s=1,t=0}$ for $n \le 6$ where $s = (p_1 + p_2)^2$ and $t = (p_1 + p_4)^2$ measure the momentum running through B_n in the horizontal and vertical directions:[5]

$$c_2 = 2\zeta_2, \tag{3.2}$$

$$c_3 = 4\zeta_3^2 + \tfrac{124}{35}\zeta_2^3 - 8\zeta_3 - 6\zeta_2, \tag{3.3}$$

$$c_4 = -56\zeta_7 - 32\zeta_2\zeta_5 + 32\zeta_3^2 + \tfrac{8}{5}\zeta_3\left(4\zeta_2^2 - 15\right) + \tfrac{992}{35}\zeta_2^3 - 8\zeta_2^2 - 18\zeta_2, \tag{3.4}$$

$$c_5 = 56\zeta_7\left(\zeta_3 - 5\right) + 26\zeta_5^2 + 4\zeta_5\left(8\zeta_2\zeta_3 + 35\zeta_3 - 40\zeta_2 - 49\right) + \tfrac{4}{5}\zeta_3^2\left(140 - 25\zeta_2 - 4\zeta_2^2\right)$$
$$\quad + 8\zeta_3\left(7\zeta_2 + 4\zeta_2^2 - 14\right) - \tfrac{1168}{385}\zeta_2^5 - \tfrac{24}{7}\zeta_2^4 + \tfrac{496}{5}\zeta_2^3 + 4\zeta_2\left(2\zeta_{3,5} - 21\right) + 20\zeta_{3,5} + 4\zeta_{3,7}, \tag{3.5}$$

$$c_6 = \tfrac{18864}{35}\zeta_2^3 + 336\zeta_{3,5} - 12\zeta_9\left(20\zeta_2 + 161\right) + \tfrac{8}{5}\zeta_7\left(104\zeta_2^2 + 35\zeta_2 + 840\zeta_3 - 1120\right)$$
$$\quad + 624\zeta_5^2 + \tfrac{16}{35}\zeta_5\left(1680\zeta_2\zeta_3 - 3675 - 12\zeta_2^3 - 2240\zeta_2 + 490\zeta_2^2 + 5145\zeta_3\right)$$
$$\quad + 96\left(\zeta_3^2 + \zeta_{3,7}\right) - \tfrac{48}{5}\zeta_3^2\left(35\zeta_2 + 8\zeta_2^2 - 60\right) - \tfrac{32}{5}\zeta_3\left(105 - 32\zeta_2^2 + 3\zeta_2^3 - 75\zeta_2\right)$$
$$\quad + 24\zeta_2\left(8\zeta_{3,5} - 21\right) - \tfrac{28032}{385}\zeta_2^5 - \tfrac{288}{5}\zeta_2^4 - 1320\zeta_{11}. \tag{3.6}$$

4. Problems for parametric integration with hyperlogarithms

1. To compute divergent integrals in dimensional regularization, one first needs to construct a representation involving only convergent integrands. An algorithm that solves this problem was presented in [29], but already for low numbers of divergences it can produce expressions that become untractibly large. It seems very promising to combine this algorithm with programs for integration by parts, in order to obtain a reduction to *finite (convergent) master integrals.*

2. We know many cases (like K_4 and $P_{7,11}$ mentioned above) of integrals that are not linearly reducible in the original Schwinger parameters, but become so after a suitable change of variables. It is unclear under which general circumstances this is possible. So far, there is only one combinatorial analysis available which considers a particular kind of transformation for vacuum integrals [34].

3. Why do alternating sums so far not occur in massless propagators?

4. The implementation [28] works in the Euclidean region. Analytic continuation to various kinematic regimes of the physical region can in general be very cumbersome and should be automated.

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[5]Printing the expressions for $\Phi(B_n)$ including the dependence on s and t would take too much space.

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