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The leading disconnected contribution to the anomalous magnetic moment of the muon

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The hadronic vacuum polarization can be determined from the vector correlator in a mixed time-momentum representation. We explicitly calculate the disconnected contribution to the vector correlator, both in the $N_f = 2$ theory and with an additional quenched strange quark, using non-perturbatively $O(a)$-improved Wilson fermions. All-to-all propagators are computed using stochastic sources and a generalized hopping parameter expansion. Combining the result with the dominant connected contribution, we are able to estimate an upper bound for the systematic error that arises from neglecting the disconnected contribution in the determination of $(g-2)_\mu$.

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Figure 1: The connected and the disconnected contribution to the hadronic vacuum polarization.

1. Introduction

The anomalous magnetic moment of the muon a_μ is one of the most precisely measured quantities in particle physics. A deviation of $\approx 3\sigma$ between the experimental and the theoretical value has persisted for many years. From the theory side, the largest fraction of the error comes from the hadronic vacuum contribution (hvp), which is the leading order QCD contribution to a_μ. Currently, the best estimate of the hvp relies on a semi-phenomenological approach using the cross section of $e^+e^- \to$ hadrons. In the past few years, a lot of effort has been undertaken to calculate the hvp from first principles using lattice techniques [1, 2, 3, 4]. However, the quark-disconnected contribution to the hvp is generally neglected. This may be a significant source of systematic error, since in partially quenched chiral perturbation theory, it was estimated that the disconnected contribution could be as large as -10% of the connected one [5].

We explicitly compute the disconnected contribution to the hvp with $\mathcal{O}(a)$-improved Wilson fermions using the mixed-representation method [6, 7], where the hadronic vacuum polarization is calculated using the vector correlator

$$G^{\gamma\gamma}(x_0) = -\frac{1}{3}\int \mathrm{d}^3x \left\langle j_k^\gamma(x) j_k^\gamma(0)\right\rangle \quad \text{with} \quad j_k^\gamma = \frac{2}{3}\overline{u}\gamma_k u - \frac{1}{3}\overline{d}\gamma_k d + \dots \tag{1.1}$$

as follows:

$$\hat{\Pi}(Q^2) = 4\pi^2 \int_0^\infty \mathrm{d}x_0\, G^{\gamma\gamma}(x_0)\left[x_0^2 - \frac{4}{Q^2}\sin^2\left(\frac{1}{2}Qx_0\right)\right]. \tag{1.2}$$

The vector correlator $G^{\gamma\gamma}(x_0)$ receives a connected and a disconnected contribution as shown in figure 1. We calculate the required disconnected quark loops using stochastic sources and a hopping parameter expansion as described in [8].

2. Results for the vector correlator

In the following we will concentrate on the vector correlator for light and strange quarks combined. The corresponding electromagnetic current

$$j_\mu^{\ell s} = j_\mu^\ell + j_\mu^s = \underbrace{\frac{1}{2}\left(\overline{u}\gamma_\mu u - \overline{d}\gamma_\mu d\right)}_{I=1,\ j_\mu^\rho} + \underbrace{\frac{1}{6}\left(\overline{u}\gamma_\mu u + \overline{d}\gamma_\mu d - 2\overline{s}\gamma_\mu s\right)}_{I=0} \tag{2.1}$$

can be split into an isovector part corresponding to the ρ-current and an isoscalar part. Performing the Wick contractions one finds for the light and strange vector current

$$G^{\ell s}(t) = \frac{5}{9}G^\ell_{\text{con}}(t) + \frac{1}{9}G^s_{\text{con}}(t) + \frac{1}{9}G^{\ell s}_{\text{disc}}(t) \qquad \text{with} \qquad G^\ell_{\text{con}}(t) = 2G^{\rho\rho}(t) \tag{2.2}$$

For convenience, we consider the disconnected correlator $G^{\ell s}_{\rm disc}(t)$ for light and strange quarks combined, since one can write the disconnected Wick contractions as

$$\begin{aligned} G^{\ell s}_{\rm disc}(x_0) &= -\int \mathrm{d}^3x \left\langle j_k^{\ell s}(x)\, j_k^{\ell s}(0) \right\rangle_{\rm disc} \\ &= -\int \mathrm{d}^3x \left\langle \left(j_k^{\ell}(x) - j_k^{s}(x)\right)\left(j_k^{\ell}(0) - j_k^{s}(0)\right) \right\rangle_{\rm disc} , \end{aligned} \tag{2.3}$$

i.e. we only need differences of light and strange quark loops. Thus, we expect that stochastic noise can be canceled when light and strange quark loops are calculated using the same stochastic sources. Figure 2 shows our results for the disconnected correlator for light quarks only in red and for combined light and strange quarks in green for the E5 ensemble (cf. table 1). As expected, we find that the stochastic error for the combined light and strange disconnected correlator is significantly smaller than the error on the light quark correlator alone. Although we can reduce the

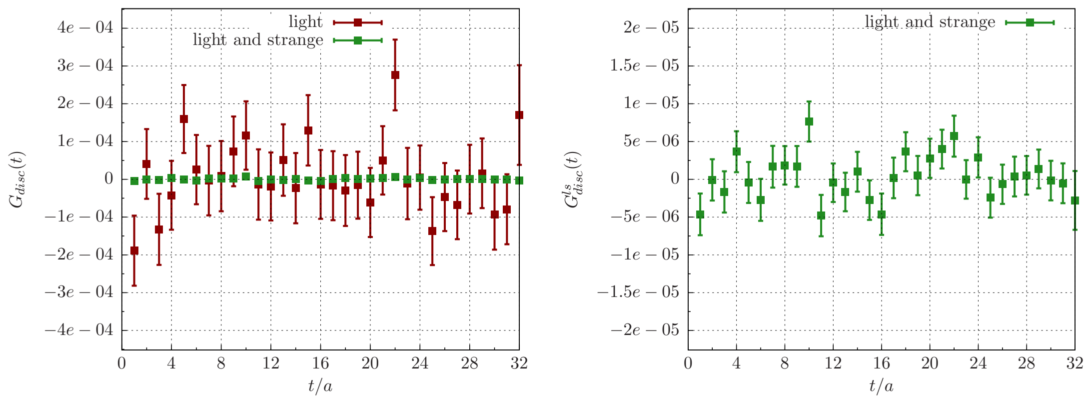

Figure 2: The disconnected vector correlator for light quarks (red) and combined light and strange quarks (green). Note, that the scales on both plots are different.

statistical error significantly when light and strange loops are calculated with the same stochastic sources, we find that the disconnected correlator $G^{\ell s}_{\rm disc}(x_0)$ is still consistent with zero within our current accuracy.

We can add the disconnected correlator to the connected one to obtain the total vector correlator. Figure 3 shows the connected (red) and the total vector correlator (yellow) for the E5 ensemble. Results for light quarks as well as light and strange quarks combined are shown on the left- and the right-hand side, respectively. The horizontal line in both plots shows the level of the statistical error on the disconnected contribution, i.e. it indicates the point from which on our total vector correlator is dominated by the noise of the disconnected contribution. This point sets in for significantly larger euclidean times in the case of the combined light and strange quark correlator.

Although we do not find a non-vanishing signal for the disconnected correlator, we can still use our results to give a limit for the maximum possible contribution to the hadronic vacuum polarization from quark-disconnected diagrams. Here, we will solely consider the case of combined light and strange quarks, for which the statistical error is significantly smaller.

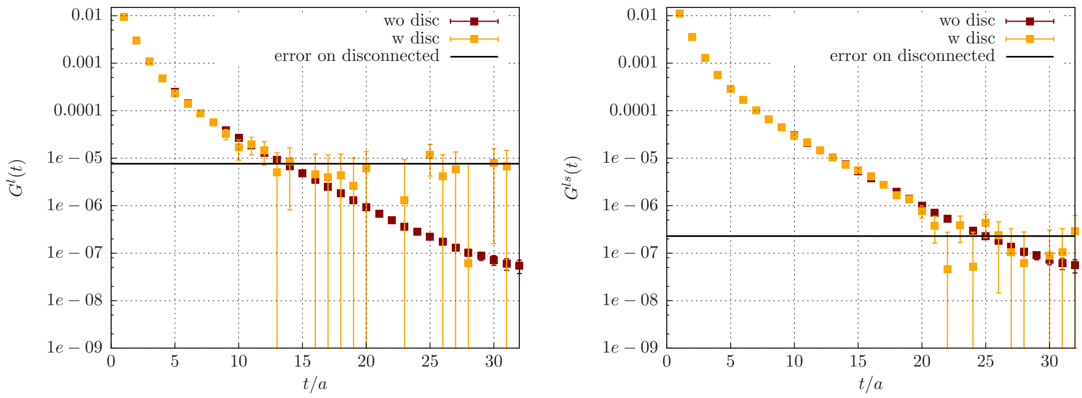

Figure 3: The connected (red) and the total (yellow) vector correlator for light quarks (left) and light and strange quarks (right). The horizontal line in both plots shows level of the statistical error on the disconnected contribution.

3. The vector correlator for large euclidean times

In order to estimate the maximum possible contribution from quark-disconnected diagrams we require information about the behavior of the vector correlator for large euclidean times in addition to our data. For large euclidean times, the vector correlator is dominated by the isovector part [6], due to its lower threshold:

$$G^{\gamma\gamma}(t) = G^{\rho\rho}(t)\left(1 + \mathscr{O}(e^{-m_\pi t})\right) . \tag{3.1}$$

If we rewrite equation (2.2) as

$$\frac{1}{9}\frac{G^{\ell s}_{\text{disc}}(t)}{G^{\rho\rho}(t)} = \underbrace{\frac{G^{\gamma\gamma}(t) - G^{\rho\rho}(t)}{G^{\rho\rho}(t)}}_{\to 0 \quad \text{for } t\to\infty} - \frac{1}{9}\underbrace{\left(1 + 2\frac{G^{s}_{\text{con}}(t)}{G^{\ell}_{\text{con}}(t)}\right)}_{\to 1 \quad \text{for } t\to\infty} \quad \longrightarrow \; -\frac{1}{9} , \tag{3.2}$$

we find an asymptotic value of $-1/9$ for the ratio of the light and strange disconnected correlator $G^{\ell s}_{\text{disc}}(t)$ to the ρ-correlator for large euclidean times. This ratio (3.2) is plotted against t in figure 4. The green line on the left-hand side shows the asymptotic value $-1/9$. As one can see, we can clearly distinguish the ratio from its asymptotic value up to $t \approx 15a$.

To give a conservative upper limit for the disconnected contribution, we assume that the ratio (3.2) falls monotonically from zero to $-1/9$ at some point. Furthermore, our estimate for $G^{\ell s}_{\text{disc}}(t)$ has to be consistent with both our data and with its theoretical asymptotic value. Thus, the disconnected contribution would be maximized if the the ratio were basically zero up to $t \approx 15a$ and then suddenly dropped to $-1/9$, as indicated by the blue line. If we take this as an estimate of the disconnected vector correlator, we can give a conservative upper bound for the magnitude of the disconnected contribution to a_μ.

4. Hadronic vacuum polarization and a_μ

From the vector correlator, one can calculate the hadronic vacuum polarization (cf. equation (1.2)). We calculate $\hat{\Pi}^{\ell s}(Q^2)$ once only for the connected vector correlator (for the details of the analysis, see [9]) and once with the disconnected estimate as described above, i.e.

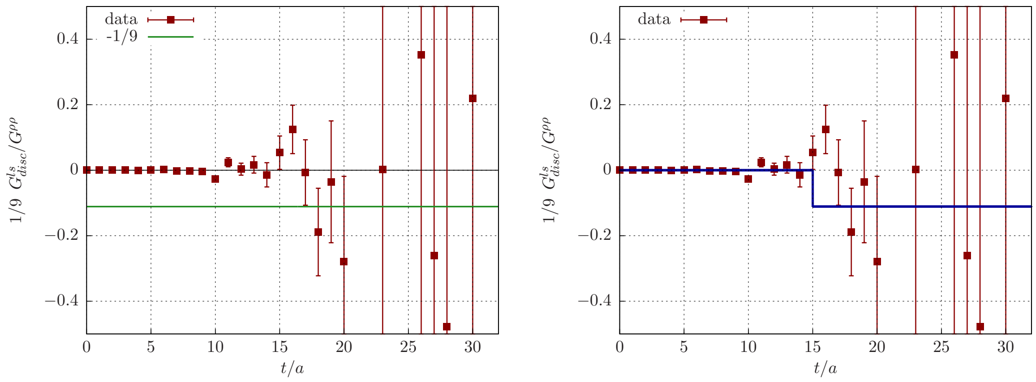

Figure 4: The ratio of the disconnected correlator and the ρ-correlator. The green line on the left-hand side shows the asymptotic value. The blue line on the right-hand side shows our conservative estimate for the disconnected correlator.

- for $t \leq 15a \approx 1$ fm, the vector correlator is well described by the connected part, i.e. we use

$$G^{\ell s}(t) = \frac{5}{9} G^{\ell}_{\text{con}}(t) + \frac{1}{9} G^{s}_{\text{con}}(t) \tag{4.1}$$

- for $t > 15a$ we use the asymptotic value $\frac{1}{9}\, G^{\ell s}_{\text{disc}}(t)/G^{\rho\rho}(t) = -1/9$ as an upper bound for the disconnected part,

$$G^{\ell s}(t) = \frac{5}{9} G^{\ell}_{\text{con}}(t) + \frac{1}{9} G^{s}_{\text{con}}(t) - \frac{1}{9} G^{\rho\rho}(t)\,. \tag{4.2}$$

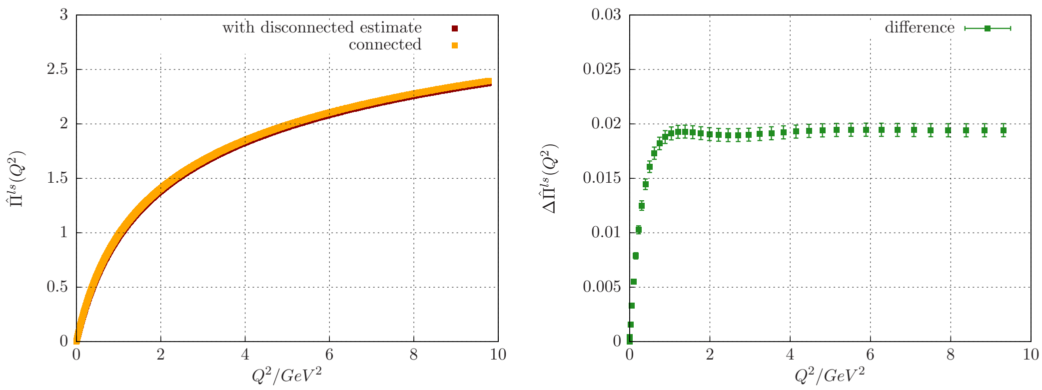

Figure 5: The plot on the left-hand side shows the vacuum polarization from the connected correlator (yellow) and for the correlator with an estimate for the disconnected contribution. The plot on the right-hand side shows their difference.

The left-hand side of figure 5 shows the vacuum contribution for both cases. As expected, the vacuum polarization with the disconnected estimate is smaller than the vacuum polarization from the connected contribution only, since for large euclidean times $G^{\ell s}_{\text{disc}}(t)$ has the opposite sign than the connected correlator. Since the difference between the two curves is small, the right hand side of figure 5 shows their difference, which is larger than the statistical error on $\hat{\Pi}^{\ell s}(Q^2)$.

From the vacuum polarization, one can now calculate the hadronic contribution to the anomalous magnetic moment of the muon [10, 11],

$$a_\mu^{\rm hvp} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty {\rm d}Q^2 \, \frac{1}{Q^2} K(Q^2)\, \hat{\Pi}(Q^2)\,, \tag{4.3}$$

with an electromagnetic kernel function $K(Q^2)$. We calculate $a_\mu^{\rm hvp}$ once for the vacuum polarization for the connected part only, and once for the vacuum polarization which includes the disconnected estimate. For the E5 ensemble, we find that with the disconnected estimate the result for $a_\mu^{\rm hvp}$ is $\approx 3.5\%$ smaller. One has to keep in mind that this is a conservative upper limit, and that the disconnected contribution to $a_\mu^{\rm hvp}$ could also be much smaller. We use the 3.5% as an upper bound for a systematic error that arises when the disconnected contribution is neglected.

β	a[fm]	lattice	m_π[MeV]	$m_\pi L$	Label	$N_{\rm cnfg}$	$t_{\rm cut}$
5.3	0.063	64×32^3	451	4.7	E5	1000	15
5.3	0.063	96×48^3	324	5.0	F6	300	13
5.3	0.063	96×48^3	277	4.3	F7	250	13

Table 1: The CLS ensembles used for the calculation of the disconnected contribution to the hadronic vacuum polarization.

So far, we have done this calculation for three different gauge ensembles, which are listed in table 1. For the ensembles F6 and F7 we have less statistics than for E5, and we can not resolve the ratio of disconnected correlator and ρ-correlator as well as for E5. Thus, we choose a slightly smaller value $t_{\rm cut}$ up to which we neglect the disconnected correlator and from which on we use the asymptotic value. For both ensembles we find a upper limit for the disconnected contribution of $\approx 5\%$.

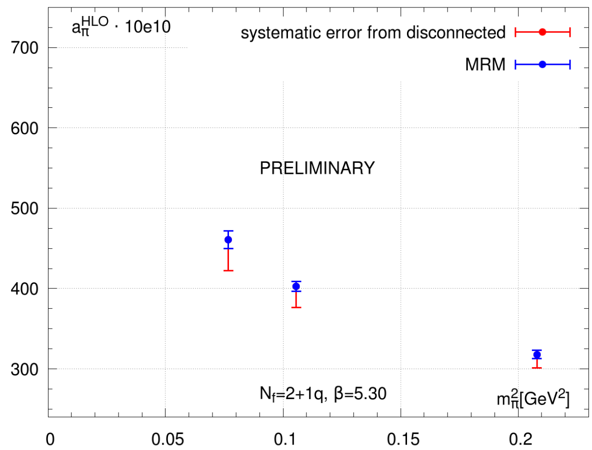

Figure 6: $a_\mu^{\rm hvp}$ plotted against the pion mass. Blue points show the results from the connected correlator. The red error bars show the maximum systematic error from neglecting the disconnected contribution.

Figure 6 shows the results for $a_\mu^{\rm hvp}$ plotted against m_π^2. The blue points show the results for the connected correlator and are the same as in [9]. The red error bars denote the maximum systematic error from neglecting the disconnected contribution. One can see that this systematic error is larger than the statistical error on $a_\mu^{\rm hvp}$. Thus, improving the accuracy of the QCD prediction of the hadronic contribution to a_μ requires improvements to the computation of the disconnected contribution.

5. Conclusions

We have explicitly calculated the disconnected vector correlator for light and strange quarks. Since the disconnected correlator depends only on the difference of light and strange propagators, the statistical error can be significantly reduced when using the same stochastic sources for light and strange loops. However, we still find that the disconnected vector correlator is consistent with zero within our current accuracy. Using the asymptotic behavior of the vector correlator for large euclidean times, we are able to give an upper limit for the disconnected contribution to $a_\mu^{\rm hvp}$. The lattice data shows that up to some time $t_{\rm cut}$ the vector correlator is well described by the connected contribution only, and thus the disconnected one can be neglected. From this time on, we use the asymptotic value for $G_{\rm disc}^{\ell s}(t)$. This allows us to give an upper estimate for the systematic error from neglecting the disconnected contribution, which we find to be of the order of $4-5\%$. We note however, that this estimate is as conservative as possible, and that the disconnected contribution might be much smaller. Nevertheless, any further reduction of the error on the calculation of the hadronic contribution to the anomalous magnetic moment of the muon from lattice QCD requires an improvement of the computation of the disconnected contribution.

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PoS(LATTICE2014)128