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Nonperturbative renormalisation for low moments of light-meson distribution amplitudes

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We discuss nonperturbative renormalisation of the leading-twist flavour non-singlet operators needed for the calculation of the first and second moments of light-meson distribution amplitudes. On the lattice we use a regularisation-independent symmetric (or non-exceptional) momentum scheme, RI/SMOM, which, for the second moment, allows us to include mixing with a total-derivative operator. We calculate the conversion functions needed to connect the RI/SMOM results to MSbar.

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1. Introduction

Parton distribution amplitudes (PDAs) are relevant for exclusive QCD processes at large momentum transfers, near the light cone. They provide process-independent nonperturbative information on the bound-state structure of hadrons, in particular the momentum-fraction distribution of partons in a particular Fock state of a hadron. They have been calculated in three main approaches: extraction from experimental form factor data; QCD sum rules; lattice QCD. We are here concerned with the last of these.

Low moments of PDAs can be computed from non-forward local matrix elements with momentum transferred at the operator insertion. For example, for a pseudoscalar meson P the first and second moments $\langle\xi^1\rangle_P$ and $\langle\xi^2\rangle_P$ are determined by

$$\langle 0|S\bar{q}_a\gamma_\mu\gamma_5\overleftrightarrow{D}_\nu q_b|P(p)\rangle = \langle\xi^1\rangle_P\, f_P\, S p_\mu p_\nu$$
$$\langle 0|S\bar{q}_a\gamma_\mu\gamma_5\overleftrightarrow{D}_\nu\overleftrightarrow{D}_\rho q_b|P(p)\rangle = \langle\xi^2\rangle_P\, f_P\, S p_\mu p_\nu p_\rho$$

where S means symmetrised and traceless in Lorentz indices Bare lattice operators need renormalisation and matching to a continuum scheme like $\overline{\text{MS}}$. For the second moment, because there is non-zero momentum transfer, there will be mixing of the double-covariant-derivative operator with a double-total-derivative operator,

$$S\bar{q}_a\gamma_\mu\gamma_5\overleftrightarrow{D}_\nu\overleftrightarrow{D}_\sigma q_b \qquad \text{and} \qquad S\partial_\mu\partial_\nu(\bar{q}_a\gamma_\sigma\gamma_5 q_b)$$

Hence, on the lattice we will need to compute

$$\langle\xi^1\rangle^{\overline{\text{MS}}} = \frac{Z_{D,D}}{Z_A}\langle\xi^1\rangle^{\text{bare}} \qquad \text{and} \qquad \langle\xi^2\rangle^{\overline{\text{MS}}} = \frac{Z_{DD,DD}}{Z_A}\langle\xi^2\rangle^{\text{bare}} + \frac{Z_{DD,\partial\partial}}{Z_A}$$

where $Z_{D,D}$, $Z_{DD,DD}$ and $Z_{DD,\partial\partial}$ are renormalisation constants to be determined, ideally nonperturbatively (Z_A is the renormalisation constant for the light-quark axial vector current, which we determine elsewhere [1]).

2. Previous RBC/UKQCD calculation

In our previous work ([2, 3] and in preparation), we used two lattice spacings, $a^{-1} = 1.73$ GeV and 2.28 GeV. Figure 1 shows results for the kaon 1st moment and pion 2nd moment. The kaon 1st moment has little lattice spacing dependence, while there is more visible a-dependence for the pion second moment. However, in these calculations the mixing with the double total-derivative operator is perturbative [3], since our previous nonperturbative renormalisation was performed in the RI′/MOM scheme (see below) with zero momentum transfer at the operator. Where we had both, the perturbative and nonperturbative renormalisation constants differed, as shown in table 1.

3. Nonperturbative renormalisation

We use a Rome–Southampton regularisation independent (RI) momentum subtraction (MOM) scheme. For operator O, the renormalisation constant Z_O is determined by

$$\Lambda_R^O = \frac{1}{Z_q} Z_O\, \Lambda_B^O$$

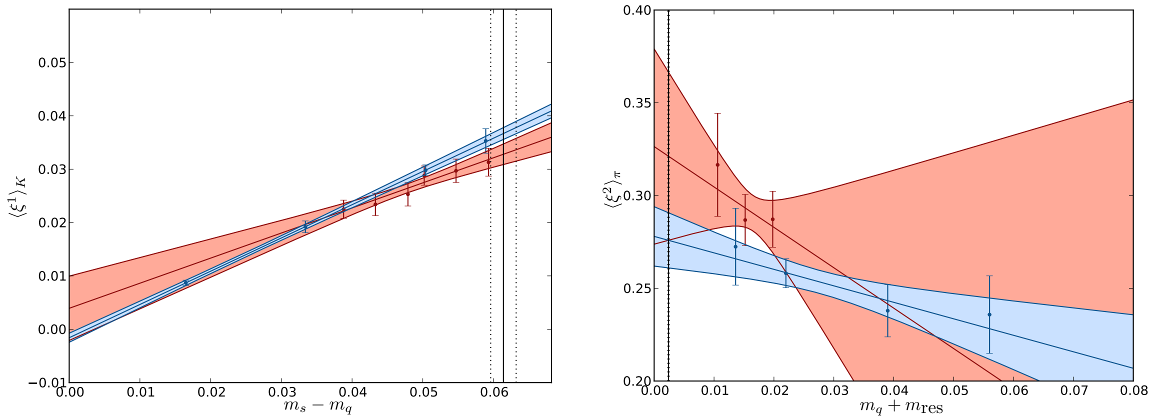

Figure 1: Chiral extrapolations of results for the kaon 1st moment, left, and pion 2nd moment, right, at two different lattice spacings: $a^{-1} = 1.73$ GeV blue, 2.28 GeV red. The 1st moment result, with nonperturbative renormalisation, shows little lattice-spacing dependence. For the 2nd moment, where the double covariant derivative operator is nonperturbatively renormalised but the total-derivative operator is perturbatively renormalised, there is more a-dependence.

	$Z_{D,D}/Z_A$	$Z_{DD,DD}/Z_A$	$Z_{DD,\partial\partial}/Z_A$
nonperturbative	1.50(2)	1.97(5)	—
mean-field imp PT	1.28(4)	1.51(6)	0.015(4)

Table 1: Perturbative and nonperturbative renormalisation constants on the $a^{-1} = 2.28$ GeV lattice with $\overline{\rm MS}$ scale $\mu = 2$ GeV.

where Λ^O is an amputated quark two-point function with an insertion of O (Lorentz indices implicit). The subscripts R and B denote renormalised and bare respectively. For operators which mix, as is the case for the 2nd moment PDA calculation, Z_O will be a matrix. Z_q is the quark wavefunction renormalisation constant.

We impose the renormalisation condition (or conditions for operators which mix) at a particular momentum configuration with associated scale μ, satisfying $\Lambda_{\rm QCD} \ll \mu \ll 1/a$. In the SMOM or *symmetric momentum* scheme, in Euclidean space, the incoming quark momentum q, incoming antiquark momentum p and momentum transfer $q + p$ satisfy

$$q^2 = p^2 = (q+p)^2 = \mu^2, \qquad q \cdot p = -\mu^2/2$$

as indicated in figure 2.

Previously we used the RI′/MOM scheme with an *exceptional* momentum choice $q^2 = p^2 = \mu^2$ and no momentum transfer, $q + p = 0$. The SMOM scheme allows mixing with total derivative operators (needed in our case), suppresses contamination from IR effects and is expected to be better-behaved (or at least not worse behaved) in the perturbative series needed for conversion to the $\overline{\rm MS}$ scheme. Here we are concerned with the *conversion functions* from SMOM to $\overline{\rm MS}$ for the PDA 1st and 2nd moments. Evaluating those conversions is done in the continuum and the relevant

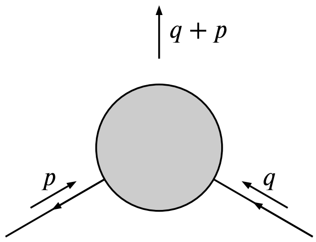

Figure 2: Momentum configuration for the SMOM scheme applied to a quark two-point function with operator insertion.

calculations have been performed by J Gracey to 3 loops in $\overline{\text{MS}}$ and to 2 loops in SMOM [4, 5, 6, 7]. The amputated two-point quark Green functions are matrices in colour and spin indices and the choice of how those are traced into the scalars $\Lambda^{O}_{R,B}$ will be reflected in the precise values of the conversion functions.

4. Operator basis and scalar coefficients

We use the following basis of C-eigenstate operators[1]:

$$\begin{aligned} X_2 &= S\bar\psi\gamma_\mu \overleftrightarrow{D}_\nu\psi \\ \partial X_2 &= S\partial_\mu(\bar\psi\gamma_\nu\psi) \end{aligned} \qquad \begin{aligned} X_3 &= S\bar\psi\gamma_\mu \overleftrightarrow{D}_\nu \overleftrightarrow{D}_\sigma\psi \\ \partial\partial X_3 &= S\partial_\mu\partial_\nu(\bar\psi\gamma_\sigma\psi) \\ \partial X_3 &= S\partial_\mu(\bar\psi\gamma_\nu \overleftrightarrow{D}_\sigma\psi) \end{aligned}$$

X_2, ∂X_2, ∂X_3 are all multiplicatively renormalised while X_3 and $\partial\partial X_3$ mix. For the PDA 1st and 2nd moments we need to renormalise X_2, X_3 and $\partial\partial X_3$, but we include the single total-derivative operators ($\partial X_{2,3}$) in order to make some checks: ∂X_2 and $\partial\partial X_3$ are total derivatives of vector current and should have the same anomalous dimension; similarly, ∂X_3 is a total derivative of X_2.

Following Gracey [4, 5, 6, 7], we expand amputated quark two-point Green functions of these operators in bases of Lorentz tensor structures with scalar coefficients. For example, for the first moment operators (suppressing spin and colour indices):

$$\Lambda^{\mu\nu}(p,q)_{\text{sym}} = \sum_{i=1}^{10} P^{\mu\nu}_{(i)}(p,q)\Sigma_i(\mu^2)$$
$$\Sigma_i(\mu^2) = \frac{1}{\mu^2}\,\text{Tr}\left[M_{ij}P^{\mu\nu}_{(j)}(p,q)\Lambda_{\mu\nu}(p,q)_{\text{sym}}\right]$$

where 'sym' means evaluated at an SMOM symmetric momentum configuration and $P^{\mu\nu}_{(i)}(p,q)$ are 10 Lorentz tensors with

$$N_{ij} = \frac{1}{\mu^2}\,\text{Tr}\left[P^{\mu\nu}_{(i)}P_{(j)\mu\nu}\right]_{\text{sym}} \qquad M = N^{-1} \tag{4.1}$$

[1]Operators with and without γ_5 renormalise in the same way if chiral symmetry is respected. Our lattice simulations use a domain wall fermion action with good chiral symmetry properties.

There are similar decompositions for bilinear and second moment operators with 6 and 14 Lorentz structures respectively.

Gracey used a different basis of operators. Changing to the C-conserving basis above leads to relations between Gracey's $\overline{\text{MS}}$ anomalous dimensions which are all satisfied[2]. For the amputated Green functions, charge-conjugation implies a set of relations between the scalar coefficients in our basis. These are satisfied by the Gracey continuum calculations (after the change of basis) and by lattice data for a unit gauge field. They are also well-satisfied by our lattice data at the two lattice spacings.

5. SMOM renormalisation conditions

A specific SMOM renormalisation scheme is fixed by demanding that after tracing with some 'projector' P, the renormalised amputated Green function should give the tree-level result

$$\frac{1}{Z_q}\,\mathrm{Tr}(Z_O \Lambda^O_{B,\mathrm{sym}} P) = \mathrm{Tr}(\Lambda^O_{\mathrm{tree,sym}} P)$$

We aim to choose P's to respect the charge-conjugation properties of the operators and, for operators which are total derivatives of vector current, to maintain the Ward identity.

For example, the SMOM renormalisation condition for the vector current [8]

$$\frac{1}{12\mu^2}\frac{Z_V}{Z_q}\,\mathrm{Tr}(k_\mu \Lambda^\mu_{V,B} \not{k}) = 1 \qquad \text{where} \qquad k = q + p$$

maintains the Ward identity $k_\mu \Lambda^\mu_{V,R} = S_R^{-1}(-p) - S_R^{-1}(q)$, where S_R is the renormalised quark propagator. We choose renormalisation conditions for total derivatives of the vector current

$$\frac{Z_{\partial X_2}}{Z_q}\,\mathrm{Tr}\left[\left(\mathcal{S} k_\mu k_\nu\right)\not{k}\Lambda^{\mu\nu}_{\partial X_2,B}\right] = 9i(\mu^2)^2$$

$$\frac{Z_{\partial\partial X_3}}{Z_q}\,\mathrm{Tr}\left[\left(\mathcal{S} k_\mu k_\nu k_\rho\right)\not{k}\Lambda^{\mu\nu\rho}_{\partial\partial X_3,B}\right] = -6(\mu^2)^3$$

and confirm that the conversion functions from SMOM to $\overline{\text{MS}}$ for all three operators are then 1.

As another example, for the second moment the operator $X_3 = \mathcal{S}\bar\psi\gamma_\mu \overleftrightarrow{D}_\nu \overleftrightarrow{D}_\sigma \psi$ mixes with $\partial\partial X_3 = \mathcal{S}\partial_\mu\partial_\nu(\bar\psi\gamma_\sigma\psi)$. The tree-level matrix elements are

$$\Lambda^{\overleftrightarrow{D}\overleftrightarrow{D}}_{\mu\nu\sigma}(p,q)_{\mathrm{tree}} = -\mathcal{S}(q_\mu - p_\mu)(q_\nu - p_\nu)\gamma_\sigma = \frac{\mu^2}{3}(P_{3,\mu\nu\sigma} + P_{1,\mu\nu\sigma} - P_{2,\mu\nu\sigma})$$

$$\Lambda^{\partial\partial}_{\mu\nu\sigma}(p,q)_{\mathrm{tree}} = -\mathcal{S}(q_\mu + p_\mu)(q_\nu + p_\nu)\gamma_\sigma = \frac{\mu^2}{3}(P_{3,\mu\nu\sigma} + P_{1,\mu\nu\sigma} + P_{2,\mu\nu\sigma})$$

where $P_{1,2,3}$ are three of 14 possible Lorentz structures. To fix the renormalisation constants $Z_{DD,DD}$ and $Z_{DD,\partial\partial}$ to get from the bare lattice results to SMOM, we impose renormalisation conditions

$$\frac{1}{Z_q}\,\mathrm{Tr}\left[\left((MP)_3 + (MP)_1 - (MP)_2\right)(Z_{DD,DD}\Lambda_B^{\overleftrightarrow{D}\overleftrightarrow{D}} + Z_{DD,\partial\partial}\Lambda_B^{\partial\partial})\right]$$
$$= \mathrm{Tr}\left[\left((MP)_3 + (MP)_1 - (MP)_2\right)\Lambda^{\overleftrightarrow{D}\overleftrightarrow{D}}_{\mathrm{tree}}\right] = \mu^2$$

[2] In fact they determine one of the second moment anomalous dimensions to one higher power in g^2. In the notation of [7], the relation $\gamma_{11}^{W_3} + \gamma_{12}^{W_3} - \gamma_{22}^{W_3} = 0$ fixes the $(g^2)^3$ term in $\gamma_{12}^{W_3}$.

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$$\frac{1}{Z_q}\,\mathrm{Tr}\left[\big((MP)_3+(MP)_1+(MP)_2\big)(Z_{DD,DD}\Lambda_B^{\overleftrightarrow{D}\overleftrightarrow{D}}+Z_{DD,\partial\partial}\Lambda_B^{\partial\partial})\right]$$

$$=\mathrm{Tr}\left[\big((MP)_3+(MP)_1+(MP)_2\big)\Lambda_{\mathrm{tree}}^{\overleftrightarrow{D}\overleftrightarrow{D}}\right]=\frac{\mu^2}{3}$$

In these expressions the trace is on spin and colour indices. There is also a summation on the (suppressed) Lorentz indices. The notation $(MP)_i$ denotes the summation $M_{ij}P_{(j)}$ where M is the matrix defined in equation 4.1.

6. Conversion functions

Having specified SMOM renormalisation conditions, we need to evaluate the conversion functions to give final results in $\overline{\mathrm{MS}}$. Suppose operators renormalised in SMOM and $\overline{\mathrm{MS}}$ are related by a (matrix) C,

$$O_{\overline{\mathrm{MS}}}=CO_R$$

The corresponding relation for amputated two-point quark Green functions with an insertion of O is

$$\Lambda_{\overline{\mathrm{MS}}}=\frac{1}{C_q}C\Lambda_R\quad\text{where}\quad C_q\equiv\frac{Z_{q,\overline{\mathrm{MS}}}}{Z_q}$$

Expand the Green function Λ_a for operator O_a in terms of tensors P_i with scalar coefficients Σ_{ai}

$$\Lambda_a=\sum_i\Sigma_{ai}P_i\qquad\Sigma_{ai}=\mathrm{Tr}\left[(MP)_i\Lambda_a\right]$$

Our renormalisation prescription is that tracing Λ_{Ra} with some projector P_A gives the tree-level (or other chosen) result, T_{aA}

$$\mathrm{Tr}\left(\Lambda_{Ra}P_A\right)=T_{aA}$$

We may need to choose several P_A's if the operators mix. Let $N^P_{iA}\equiv\mathrm{Tr}(P_iP_A)$ and use $\Lambda_R=C_qC^{-1}\Lambda_{\overline{\mathrm{MS}}}$ to write

$$C_qC^{-1}_{ab}\Sigma^{\overline{\mathrm{MS}}}_{bi}N^P_{iA}=T_{aA}$$

Gracey's $\overline{\mathrm{MS}}$ results give the scalar coefficients $\Sigma^{\overline{\mathrm{MS}}}$ while N^P and T (and C_q) are also known. We can then impose enough conditions to solve for the elements of C. Subsequently, combining C with the Z's determined by our SMOM renormalisation conditions allows us to convert from lattice to $\overline{\mathrm{MS}}$ at scale μ. We can then use $\overline{\mathrm{MS}}$ anomalous dimensions to scale to a common value, say 2 GeV.

Once an SMOM renormalisation prescription has been fixed, the SMOM anomalous dimensions can be found from

$$\gamma_{\mathrm{SMOM}}=C^{-1}\,\gamma_{\overline{\mathrm{MS}}}\,C-\mu\,\frac{dC^{-1}}{d\mu}\,C$$

We close by presenting the conversion functions calculated for our choice of SMOM renormalisation prescription. Here $a=g^2/16\pi^2$, α is the gauge parameter (which will be set to 0 since our lattice Green functions are evaluated in Landau gauge) and N_f is the number of flavours. For the first moment the non-zero elements of C are

$$C_{11}=1-(1.63903\alpha+5.12484)a-(3.8244\alpha^2+6.37866\alpha-12.1458N_f+106.359)a^2$$
$$C_{22}=1$$

The subscript indices 1 and 2 refer to X_2 and ∂X_2 respectively.

For the second moment the non-zero elements are

$$
\begin{aligned}
C_{11} &= 1 - (2.18537\alpha + 8.24516)a - (5.18357\alpha^2 + 2.38666\alpha - 19.8008N_f + 156.444)a^2 \\
C_{12} &= (0.138749\alpha + 1.15755)a + (0.419338\alpha^2 + 1.95065\alpha - 2.31945N_f + 20.0837)a^2 \\
C_{22} &= 1 \\
C_{33} &= 1 - (1.63903\alpha + 5.12484)a - (3.8244\alpha^2 + 6.37866\alpha - 12.1458N_f + 106.359)a^2
\end{aligned}
$$

where now the indices 1, 2 and 3 refer to X_3, $\partial\partial X_3$ and ∂X_3 respectively. The C_{ij} are 1 for operators which are total derivatives of the vector current. We also observe that ∂X_3 is a total derivative of X_2 and has the same conversion coefficient (compare C_{11} for the first moment with C_{33} for the second moment).

7. Summary

We are interested in calculating 1st and 2nd moments of PDAs with fully nonperturbative renormalisation. This involves non-forward matrix elements, allowing mixing with total derivative operators and demanding the use of an SMOM renormalisation scheme. Continuum calculations exist [4, 5, 6, 7] to allow the needed conversion functions from SMOM to $\overline{\text{MS}}$ to be computed once renormalisation conditions have been imposed. This enables a fully nonperturbative lattice computation with continuum perturbation theory needed only for the final conversion to $\overline{\text{MS}}$.

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