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The $\Delta N\gamma^*$ transition form factors on the lattice

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We generalize the Lüscher approach to the calculation of the *matrix elements* of unstable states and formulate a framework for the extraction of the $\Delta N\gamma^*$ transition form factors from the lattice data. Analytic continuation of the matrix element to the resonance pole position is considered in detail.

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1. Introduction

Recently, the form factors of unstable states have been studied in lattice QCD by several collaborations. For example, the $\Delta N\gamma^*$ form factors were calculated, see Ref. [1]. The electromagnetic, axial and pseudoscalar form factors of the Δ-resonance have been also studied [2]. We would like to also mention the investigation of the electromagnetic form factor of the ρ-meson [3]. Last but not least, a pioneering attempt is made to address the calculation of the electromagnetic form factors of the $\Lambda(1405)$-resonance which, for many reasons, represents a greater challenge than the cases listed above [4]. It should be however noted that the challenges that one encounters in these calculations are not purely technical ones related to the simulations. The presence of the unstable states represents a conceptual challenge as well, since such states do not belong to the set of the eigenstates of the QCD Hamiltonian. Consequently, matrix elements of the currents, which define the resonance form factors, ought to be properly defined in the continuum as well as on the lattice.

In short, the following conceptual issues should be addressed in the calculations:

1. Since the resonances do not belong to the Fock space of the QCD Hamiltonian, the resonance matrix elements can not be defined in a standard manner even in the continuum QFT. One has to consider a consistent definition of this quantity in terms of the Green functions and establish its connection to the experimentally measured quantities.

2. Even given a consistent definition of this quantity in the continuum, it remains to be shown, how it can be calculated from the Euclidean Green functions in a finite volume that are obtained from simulations in lattice QCD. Performing an infinite-volume limit here is a highly non-trivial enterprise and can not be done merely by brute force.

Of course, such problems do not emerge, if simulations are carried out for large quark masses, when the pertinent resonances do not decay. However, since the simulations with the quark masses close to the physical are starting to emerge, this problem needs to be urgently clarified.

Recently, in a series of papers [5, 6], we have addressed this problem in the framework of the non-relativistic EFT in a finite volume (for the alternative approaches, see, e.g., Refs. [7, 8, 9]). The present work is mainly based on the material contained in Ref. [6], where the extraction of the $\Delta N\gamma^*$ transition form factors from the lattice data is considered.

2. Resonance form factors in the infinite volume

As it is well known, a resonance state in QFT is not contained in the basis vectors of the Fock space. Such a state emerges as a pole in the S-matrix elements on the unphysical sheets in the complex energy plane. The real and imaginary parts of the pole position, by definition, are the energy and the half-width of a resonance. With this definition, the parameters of a given resonance are universal (process-independent), i.e., all S-matrix elements for different processes, after analytic continuation, have a pole exactly at the same place. Further, the resonance matrix elements are defined through the residues of the pertinent Green functions, continued to the resonance pole. The quantities defined in this way are unique, i.e., do not depend on a particular process chosen for the extraction.

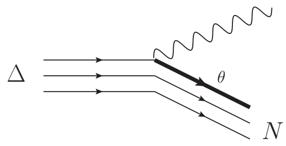

Figure 1: Twisting a single quark in the nucleon.

Historically, however, the resonance position has often been identified with the bump in the amplitude, where the phase shift passes through $\pi/2$. In the context of the present problem, the $\Delta N\gamma^*$ form factors have been algebraically related to the helicity amplitudes $\mathscr{A}_i$ in the pion photo(electro)production at the (real) resonance energy and determined from the experiment (see, e.g., Refs. [10, 11]). Such a procedure has an obvious advantage of operating only with the experimentally observable amplitudes at real energies. However, the form factors, obtained with the use of this method, contain contributions from the background processes, which are different in different reactions. Note also that recently the extraction of the residues of the photoproduction amplitude at the complex pole positions corresponding to the different baryon resonances has been carried out from the experimental data [12]. Theoretically, both methods should give identical results in the limit of an infinitely narrow resonance. The results of Ref. [12] however indicate that, for some resonances, the effect of analytic continuation could be sizable. To summarize, the method based on the analytic continuation is the only theoretically sound method that gives the value of a resonance matrix element in QFT devoid of an if and a maybe. The lattice simulations must be able to predict the value of this quantity, and the aim of our investigation is to formulate this procedure in detail. In addition, in order to have a possibility to compare with the data obtained without the use of the analytic continuation, we formulate the method of determining the photo(electro)production amplitudes from the lattice data.

It could be argued, that our method is a generalization of the Lüscher-Lellouch method [7] for the calculation of the resonance matrix elements. While in that paper a resonance decays only weakly, here we consider the resonances with a non-zero strong width (e.g., the Δ), so that the analytic continuation to the resonance pole is inevitable.

3. Lattice calculations: the kinematics

The matrix element, describing the $\Delta N\gamma^*$ vertex, depends on two momenta. In the CM frame $\mathbf{p}_\Delta = 0$, it is convenient to choose two independent variables: the total energy of the $N\gamma^*$ system E and the magnitude of the three-momentum of the nucleon $|\mathbf{Q}|$. All other kinematical variables can be expressed in terms of these two quantities.

Since the Δ is a resonance, one has to perform a lattice measurement at different values of E in the vicinity of the resonance energy, and then extrapolate the result of the measurement of the matrix element to the complex pole position. In order to get a meaningful procedure, another kinematical variable, namely $|\mathbf{Q}|$, should be fixed. This can be achieved in different ways:

- Choose $\mathbf{Q}$ in the CM frame along the third axis, and consider asymmetric boxes, where the side length along the third axis is different from other two; vary the side length only in the first two directions.

- Carry out the calculation of the matrix element at different volumes, but using twisted boundary condition for the quark propagator in the nucleon which is attached to the external photon (see Fig. 1). One may adjust the twisting angle θ so that the momentum $\mathbf{Q}$ stays fixed as the box size varies. It is important that the pertinent value of θ can be determined from kinematics alone, prior to any simulations.

In practice, a combination of these two strategies can be also applied. Note also that fixing $|\mathbf{Q}|$ is equivalent to fixing the Lorentz-invariant variable $t = Q^2$ since, for a fixed E in the CM frame, the energy of the photon $Q_0 = E - (m_N^2 + \mathbf{Q}^2)^{1/2}$ is also fixed (here, m_N denotes the nucleon mass).

Further, the $\Delta N\gamma^*$ matrix elements are described by the three Lorentz-invariant form factors $G_M(t)$, $G_E(t)$, $G_C(t)$. In order to project out these, we construct the local operators that correspond to a definite spin projection of the nucleon and the Δ on the third axis. Let $\mathscr{O}^\mu(x)$ and $\psi(x)$ denote the Δ and the nucleon interpolating fields, respectively, and $J^\rho(x)$ is the electromagnetic current. Let us introduce the following operators:

$$\begin{aligned}
\mathscr{O}_{3/2}(t) &= \sum_{\mathbf{x}} \frac{1}{2}(1+\Sigma_3)\frac{1}{2}(1+\gamma_4)\frac{1}{\sqrt{2}}(\mathscr{O}^1(\mathbf{x},t) - i\Sigma_3\mathscr{O}^2(\mathbf{x},t)),\\
\mathscr{O}_{1/2}(t) &= \sum_{\mathbf{x}} \frac{1}{2}(1-\Sigma_3)\frac{1}{2}(1+\gamma_4)\frac{1}{\sqrt{2}}(\mathscr{O}^1(\mathbf{x},t) + i\Sigma_3\mathscr{O}^2(\mathbf{x},t)),\\
\tilde{\mathscr{O}}_{1/2}(t) &= \sum_{\mathbf{x}} \frac{1}{2}(1+\Sigma_3)\frac{1}{2}(1+\gamma_4)\mathscr{O}^3(\mathbf{x},t)\\
\bar{\psi}^{\mathbf{Q}}_{\pm 1/2}(t) &= \sum_{\mathbf{x}} e^{i\mathbf{Q}\mathbf{x}}\bar{\psi}(\mathbf{x},t)\frac{1}{2}(1\pm\Sigma_3)\frac{1}{2}(1+\gamma_4),\\
J^{\pm}(0) &= \frac{1}{\sqrt{2}}(J^1(0) \pm iJ^2(0)),
\end{aligned} \tag{3.1}$$

where $\Sigma_3 = \mathrm{diag}(\sigma_3, \sigma_3)$. Consider now the following three-point functions:

$$\begin{aligned}
\tilde{R}_{1/2}(t',t) &= \langle 0|\tilde{\mathscr{O}}_{1/2}(t')J^3(0)\bar{\psi}^{\mathbf{Q}}_{1/2}(t)|0\rangle,\\
R_{1/2}(t',t) &= \langle 0|\mathscr{O}_{1/2}(t')J^+(0)\bar{\psi}^{\mathbf{Q}}_{-1/2}(t)|0\rangle,\\
R_{3/2}(t',t) &= \langle 0|\mathscr{O}_{3/2}(t')J^+(0)\bar{\psi}^{\mathbf{Q}}_{1/2}(t)|0\rangle,
\end{aligned} \tag{3.2}$$

and define

$$\begin{aligned}
\tilde{D}_{1/2}(t) &= \mathrm{Tr}\,\langle 0|\tilde{\mathscr{O}}_{1/2}(t)\bar{\tilde{\mathscr{O}}}_{1/2}(0)|0\rangle,\\
D_{1/2}(t) &= \mathrm{Tr}\,\langle 0|\mathscr{O}_{1/2}(t)\bar{\mathscr{O}}_{1/2}(0)|0\rangle,\\
D_{3/2}(t) &= \mathrm{Tr}\,\langle 0|\mathscr{O}_{3/2}(t)\bar{\mathscr{O}}_{3/2}(0)|0\rangle,\\
D^{\pm}_{\mathbf{Q}}(t) &= \mathrm{Tr}\,\langle 0|\psi^{\mathbf{Q}}_{\pm 1/2}(t)\bar{\psi}^{\mathbf{Q}}_{\pm 1/2}(0)|0\rangle.
\end{aligned} \tag{3.3}$$

It can be seen that, in the limit $t' \to +\infty$, $t \to -\infty$,

$$
\begin{aligned}
&\mathscr{N}\frac{\mathrm{Tr}\,(\tilde{R}_{1/2}(t',t))}{\tilde{D}_{1/2}(t'-t)}\left(\frac{D^+_{\mathbf{Q}}(t')\tilde{D}_{1/2}(-t)\tilde{D}_{1/2}(t'-t)}{\tilde{D}_{1/2}(t')D^+_{\mathbf{Q}}(-t)D^+_{\mathbf{Q}}(t'-t)}\right)^{1/2} \to \langle 1/2|J^3(0)|1/2\rangle\,,\\
-&\mathscr{N}\frac{\mathrm{Tr}\,(R_{1/2}(t',t))}{D_{1/2}(t'-t)}\left(\frac{D^-_{\mathbf{Q}}(t')D_{1/2}(-t)D_{1/2}(t'-t)}{D_{1/2}(t')D^-_{\mathbf{Q}}(-t)D^-_{\mathbf{Q}}(t'-t)}\right)^{1/2} \to \langle 1/2|J^+(0)|-1/2\rangle\,,\\
-&\mathscr{N}\frac{\mathrm{Tr}\,(R_{3/2}(t',t))}{D_{3/2}(t'-t)}\left(\frac{D^+_{\mathbf{Q}}(t')D_{3/2}(-t)D_{3/2}(t'-t)}{D_{3/2}(t')D^+_{\mathbf{Q}}(-t)D^+_{\mathbf{Q}}(t'-t)}\right)^{1/2} \to \langle 3/2|J^+(0)|1/2\rangle\,,
\end{aligned}
\tag{3.4}
$$

where $\mathscr{N}=(4E\sqrt{m_N^2+\mathbf{Q}^2})^{1/2}$. Note that the above relations can be straightforwardly generalized to include the excited states with the quantum numbers of the Δ. We are interested in those states which, at a given volume, have the energies close to the Δ-mass.

The right-hand side of the above equations denotes the matrix elements that are directly measured on the lattice. The spin projection of the nucleon and Δ on the third axis is explicitly shown. We further use a shorthand notation $F_i,\ i=1,2,3$ for these matrix elements. As already mentioned, the F_i are measured in a finite volume. In case of an unstable Δ, the infinite volume limit in these matrix elements can not be performed straightforwardly. Our main aim is to show, how the measured F_i can be related to the form factors in the infinite volume.

4. Extraction of the form factors

The extraction of the form factors is described in Ref. [6], where further details can be found. In order to set up a framework for the extraction, the non-relativistic EFT in a finite volume is used. The essence of the method is to carry out calculations twice: calculate the quantities F_i in a finite volume and the form factors in the infinite volume. Since in these calculations the same effective Lagrangian is used, these two quantities are related. Reading off this relation, we ensure that it does not depend on the explicit form of the Lagrangian which thus plays only an auxiliary role in the derivation and disappears from the final results.

Our findings can be summarized as follows:

1. In order to calculate the $\Delta N\gamma^*$ transition form factors, first evaluate the finite-volume matrix elements $F_i = F_i(E,|\mathbf{Q}|)$ on the lattice for a fixed $\mathbf{Q}$ and different values of E.

2. The multipoles for the pion photoproduction at a given energy E are obtained by multiplying the quantities F_i by the well-known Lüscher-Lellouch factor:

$$
\mathscr{A}_i(p,|\mathbf{Q}|) = e^{i\delta(p)}V^{1/2}\left(\frac{p^2}{2\pi|d\delta(p)/dp+d\phi(q)/dp|}\right)^{-1/2}|F_i(p,|\mathbf{Q}|)|\,. \tag{4.1}
$$

Here, p is the relative momentum in the πN system, corresponding to the total energy E in the CM frame, V is the volume, $\delta(p)$ denotes the phase shift in the P_{33} partial wave (to be measured in the same simulation), $q = pL/2\pi$ (for asymmetric boxes, L is the box size in directions 1,2) and $\phi(q)$ is related to the Lüscher zeta-function in a standard manner so that the Lüscher equation reads $\delta(p)+\phi(q)=n\pi\,,\ n=0,1,\ldots$ (here, the mixing to other partial waves is neglected). The twisting angle, as well as the asymmetry parameter of the box are not explicitly shown. The definition of the multipoles from Ref. [10] is used:

$$\begin{aligned}\tilde{\mathscr{A}}_{1/2} &= -16\pi iE\sqrt{2}\frac{k}{|\mathbf{k}|}S_{1+}\\ \mathscr{A}_{1/2} &= -\frac{1}{2}(3E_{1+}+M_{1+})(-16\pi iE),\\ \mathscr{A}_{3/2} &= \frac{\sqrt{3}}{2}(E_{1+}-M_{1+})(-16\pi iE),\end{aligned} \tag{4.2}$$

where $\mathbf{k}$ is the photon 3-momentum, k_0 is the photon energy, $k^2=\mathbf{k}^2-k_0^2$, and M,E,S denote the pertinent magnetic, electric and scalar multipoles.

3. The resonance matrix elements, defined at real energies, are proportional to the imaginary part of the multipoles at the energies, where the phase shift passes through $\pi/2$. Explicit expressions can be found in Refs. [6, 10].

4. In order to extract the matrix element at the resonance pole, one has to first multiply each F_i by the pertinent Lüscher-Lellouch factor

$$\bar{F}_i(p,|\mathbf{Q}|)=V^{1/2}\left(\frac{\cos^2\delta(p)}{|d\delta(p)/dp+d\phi(q)/dp|}\frac{p^2}{2\pi}\right)^{-1/2}F_i(p,|\mathbf{Q}|) \tag{4.3}$$

and then one should fit the coefficients of the effective-range expansion for the matrix element on the real p^2-axis, $|\mathbf{Q}|$ fixed:

$$p^3\cot\delta(p)\,\bar{F}_i(p,|\mathbf{Q}|)=A_i(|\mathbf{Q}|)+p^2B_i(|\mathbf{Q}|)+\cdots\,. \tag{4.4}$$

5. Finally, the resonance matrix elements at the pole are obtained by the substitution

$$F_i^R(p_R,|\mathbf{Q}|)=i\,p_R^{-3}\,Z_R^{1/2}(A_i(|\mathbf{Q}|)+p_R^2B_i(|\mathbf{Q}|)+\cdots)\,. \tag{4.5}$$

where p_R is the complex momentum corresponding to the pole, and

$$Z_R=\left(\frac{p}{8\pi E}\right)^2\left(\frac{16\pi p^3E^3}{w_Nw_\pi(2p\,d(p^3\cot\delta(p))/dp+3ip^2)}\right)\Bigg|_{p=p_R}. \tag{4.6}$$

Here, w_N,w_π denote the on-shell energies of a nucleon and a pion with the three-momentum p. The quantities p_R and Z_R should be evaluated separately from the measured phase shifts.

6. The form factors are related to the above resonance matrix elements through

$$\begin{aligned}\tilde{F}^R_{1/2} &= \frac{E_R-k_R^0}{E_R}A\,G_C(t_R)\\ F^R_{1/2} &= \sqrt{\frac{1}{2}}A(G_M(t_R)-3G_E(t_R))\,,\\ F^R_{3/2} &= \sqrt{\frac{3}{2}}A(G_M(t_R)+G_E(t_R))\,,\end{aligned} \tag{4.7}$$

where E_R, k_R^0, t_R denote the full energy, the photon energy and the 4-momentum transfer squared at the pole, and

$$A = \frac{E_R + m_N}{2m_N}\sqrt{2E_R(k_R^0 - m_N)}\,. \tag{4.8}$$

5. Conclusions, outlook

Using non-relativistic EFT in a finite volume, we have set up a framework for the extraction of the $\Delta N\gamma^*$ transition form factors from the lattice data. A counterpart of the Lüscher-Lellouch formula for the pion photoproduction multipoles has been derived. The form factors at the resonance pole have been defined.

The above result can be improved in a variety of ways. In particular, partial-wave mixing (ignored so far) should be considered, as well as a resonance in the moving frame. Further, multi-channel resonances deserve special attention (for the related work, see, e.g., Ref. [9]).

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