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Status of the lambda lattice scale for the SU(3) Wilson gauge action

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With the emergence of the Yang-Mills gradient flow technique there is renewed interest in the issue of scale setting in lattice gauge theory. Here I compare for the SU(3) Wilson gauge action non-perturbative scale functions of Edwards, Heller and Klassen (EHK), Necco and Sommer (NS), both relying on Sommer's method using the quark potential, and the scale function derived by Bazavov, Berg and Velytsky (BBV) from a deconfining phase transition investigation by the Bielefeld group. It turns out that the scale functions are based on mutually inconsistent data, though the BBV scale function is consistent with the EHK data when their low β ($\beta = 5.6$) data point is removed. Besides, only the BBV scale function is consistent with three data points calculated from the gradient flow by Lüscher. In the range for which data exist the discrepancies between the scale functions are only up to $\pm 2\%$ of their values, but clearly visible within the statistical accuracy.

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1. Introduction

With the emergence of the Yang-Mills gradient flow technique [1] there is renewed interest into the issue of scale setting in lattice gauge theory. For a review see [2]. Therefore, it appears to be worthwhile to analyze the status of the lambda scale for the SU(3) Wilson gauge action from previous literature. Based on the Sommer scale [3] there are two estimates (parametrizations) of the SU(3) scaling function, a paper by Edwards, Heller and Klassen [4] (EHK) and another by Necco and Sommer [5] (NS). Independently an estimate of the SU(3) scaling function was later extracted by Bazavov, Berg and Velytsky [6] (BBV) using deconfining transition coupling β_t estimates and other information from a paper by the Bielefeld group [7] ($\beta = 6/g^2$, where g is the bare coupling of the SU(3) Wilson gauge action). In addition to the data points on which these scale function estimates are based, we include three data points calculated by Lüscher [1] with the gradient method and two recent large lattice estimates of β_t [8]. Summary and conclusions follow in the final section 3.

2. Definition and comparison of the scales

Sommer [3] proposed to set a hadronic scale r_i/a through the force $F(r)$ between static quarks at intermediate distances r by $r_i^2 F(r_i) = c_i$ (Sommer scale). For their SU(3) investigations NS [5] use the values

$$r_0^2 F(r_0) \;=\; 1.65 \;\text{ and }\; r_c^2 F(r_c) \;=\; 0.65\,. \tag{2.1}$$

The r_0 value was suggested in the original paper by Sommer. It is used by NS for their smaller lattices and also by EHK, who employ also larger values for c_i, which we do not discuss here. The r_c definition is used by NS for their set of large lattices. While a number of choices have to be made when calculating r_i/a (for details see the EHK and NS papers), estimations of the deconfining transition temperatures $T_t = 1/[a(\beta_t)N_t]$ are in essence free of ambiguities when one uses maxima of the Polyakov loop susceptibility on $N^3 N_t$ lattices to determine $\beta_t(N_t)$ for the limit $N \to \infty$. In particular, when refining the lattice a switch of a reference value, like from r_0 to r_c (2.1), is unwarranted when T_t is used.

In the following we compile the analytical expressions of the three scaling functions. The EHK scaling function, the second of Eqs. (4.4) in their paper [4] with $\hat{a}$ defined by their Eq. (4.1), is given by

$$[a\Lambda_L]^{EHK} = f_\lambda^{EHK}(\beta) = \lambda^{EHK}(g^2)\, f_\lambda^{as}(g^2)\,, \tag{2.2}$$

and derived from data in the range $5.6 \le \beta \le 6.5$. Here $f^{as}(g^2)$ is the universal two-loop scaling function of SU(3) gauge theory,

$$f^{as}(g^2) \;=\; \left(b_0\, g^2\right)^{-b_1/(2b_0^2)}\, e^{-1/(2b_0\, g^2)} \quad \text{with} \quad b_0 \;=\; \frac{11}{3}\,\frac{3}{16\pi^2}\,,\; b_1 \;=\; \frac{34}{3}\left(\frac{3}{16\pi^2}\right)^2\,. \tag{2.3}$$

Higher perturbative and non-perturbative corrections are parametrized by

$$\lambda^{EHK}(g^2) \;=\; (1 + a_1\,\hat{a}^2 + a_2\,\hat{a}^4)/a_0 \;\text{ with }\; \hat{a} = \hat{a}(g^2) = f^{as}(g^2)/f^{as}(1) \tag{2.4}$$

β	EHK r_0/a	β	NS r_0/a	β	NS r_c/a	β_t	Bielefeld $(aT_t)^{-1}$
5.60	2.344 (08)	5.70	2.922 (09)	6.57	6.25 (4)	5.6925 (05)*	4.0000 (18)
5.70	2.990 (24)	5.80	3.673 (05)	6.69	7.29 (5)	5.8941 (05)	6.0000 (55)
5.85	4.103 (12)	5.95	4.898 (12)	6.81	8.49 (5)	6.0609 (09)	8.000 (12)
6.00	5.3681 (86)	6.07	6.033 (17)	6.92	9.82 (6)	6.3331 (13)	12.000 (22)
6.20	7.368 (30)	6.20	7.380 (26)	Lüscher $\sqrt{8t_0}/a$		–	–
6.20	7.368 (30)	6.20	7.380 (26)	5.96	4.7205 (53)	Francis et al. $(aT_t)^{-1}$	
6.40	9.82 (12)	6.40	9.74 (05)	6.17	6.6266 (85)	6.4488 (59)	14.00 (12)
6.50	11.23 (21)	–	–	6.42	9.4830 (97)	6.5509 (39)	16.000 (82)

Table 1: Data used. *The statistical error bar of this data point has been increased, so that it does not dominate the whole T_c set, when the overall constant is adjusted to fit to the NS or EHK scale function.

and the coefficients are given by $a_0 = 0.01596$, $a_1 = 0.2106$, $a_2 = 0.05492$. Up to the over-all constant $1/a_0$, the asymptotic scale $f^{as}(g^2)$ is approached for $\beta \to \infty$. In contrast to that NS present their scale in form of a polynomial fit, Eq. (2.6) in their paper [5], which is supposed to be valid in the region $5.7 \le \beta \le 6.92$: $[a\Lambda_L]^{NS} = f_\lambda^{NS}(\beta)$ with

$$f_\lambda^{NS}(\beta) = \exp\left[-1.6804 - 1.7331\,(\beta-6) + 0.7849\,(\beta-6)^2 - 0.4428\,(\beta-6)^3\,\right]. \tag{2.5}$$

The BBV scaling function, Eq. (19) in their paper [6], is given by[1]

$$[a\Lambda_L]^{BBV} = f_\lambda^{BBV}(\beta) = 10 \times \lambda^{BBV}(g^2)\, f^{as}(g^2)\,, \tag{2.6}$$

where f^{as} is again the asymptotic scaling function (2.3) and higher perturbative and non-perturbative corrections are parametrized by

$$\lambda^{BBV}(g^2) \;=\; 1 + e^{\ln a_1}\, e^{-a_2/g^2} + a_3\, g^2 + a_4\, g^4 \tag{2.7}$$

with the coefficients $\ln a_1 = 18.08596$, $a_2 = 19.48099$, $a_3 = -0.03772473$, $a_4 = 0.5089052$. As the EHK scale, the BBV scale approaches up to a constant factor $f^{as}(g^2)$ for $\beta \to \infty$.

In table 1 data are compiled on which the scales rely. As usual error bars are given in parenthesis and apply to the last digits. The EHK data are from table 4 of their paper [4], which includes also results from other groups. Thus several data point exists at some β, which are here combined into one estimate per β value. Their $\beta = 5.54$ data point is omitted, because it is not used for the determination of their r_0/a scaling function (2.2). The NS data are from table 1 of their paper [5]. The Bielefeld data are from table 2 of their paper [7]. We also list the three gradient flow data points from Lüscher [1] and two recent large-lattice β_t estimates from Francis et al. [8]. As these data are not used for the determination of the scaling functions they provide independent tests. The statistical errors for estimates of deconfining transition transition temperatures are in β_t with N_t fixed. To allow for direct comparison with the statistical accuracy of the Sommer method, we attach to $(aT_t)^{-1}$ error bars by means of the equation

$$\triangle(aT_t)^{-1} \;=\; \frac{N_t}{f_\lambda^{BBV}(\beta_t)}\left[f_\lambda^{BBV}(\beta_t) - f_\lambda^{BBV}(\beta_t - \triangle\beta_t)\right]. \tag{2.8}$$

[1]To get convenient constants in the upcoming table 2, our definition (2.6) differs by a factor 10 from the one in [6].

	EHK r_0	EHK $r_0 - 1$	NS r_0	NS r_c	Bielefeld+	Lüscher
E	0.9994 (14)	0.9996 (15)	0.99204 (97)	0.5172 (17)	1.35102 (81)	0.94272 (62)
N	1.0055 (14)	1.0031 (15)	0.99995 (98)	0.5140 (17)	1.36108 (81)	0.94420 (62)
B	0.21566 (28)	0.21646 (31)	0.21415 (21)	0.11024 (35)	0.29146 (17)	0.20388 (13)

Table 2: Scale constants c from fitting Eq. (2.10) to the data (E for EHK, N for NS and B for BBV).

	EHK r_0	EHK $r_0 - 1$	NS r_0	NS r_c	Bielefeld+	Lüscher
EHK	0.83	0.66	10^{-7}	0.035	10^{-15}	10^{-3}
NS	10^{-6}	10^{-3}	0.12	0.52	0	10^{-8}
BBV	10^{-9}	0.45	0	0.54	0.31	0.80

Table 3: Probabilities Q that the discrepancy between scale and data set is due to chance. Zero indicates a positive number smaller than 10^{-12}.

For each of the the three scaling functions we perform one-parameter fits of the form

$$c/f_\lambda(\beta) \tag{2.9}$$

to altogether six data sets: EKH r_0 data, EHK r_0 data with the data point for $\beta = 5.6$ removed (the lowest β entering the determination of their scaling function) and denoted EHK $r_0 - 1$, NS r_0 data, NS r_c data, combined Bielefeld and Francis et al. data denoted Bielefeld+ and Lüscher's data points. The NS data are split, because their r_0 and r_1 data require independent determinations of the over all constant in (2.9), while the Bielefeld and Francis et al. data are combined by the opposite reason. The results for the twelve constants are compiled in table 2.

Even more interesting than the constants are the thus obtained goodness of fit values Q, which are given in table 3. We see that the EHK r_0 data are only consistent with the EHK scale, similarly the NS r_0 data are only consistent with the NS scale and the Bielefeld+ data only with the BBV scale. The NS r_c data from large lattices are rather inaccurate. They are consistent with the NS and BBV scales and almost consistent with the EHK scale. Leaving the $\beta = 5.6$ EHK data point out, because we may not expect universal scaling at such a small β value, the EHK $r_0 - 1$ data are then consistent with the BBV scale, but still in disagreement with the NS scale. In the last column it is seen that only the BBV scale is consistent with Lüscher's data points.

Using the best fits to the BBV scale, regardless of good or bad Q values, Fig. 1 is obtained for the differences between the data and the BBV scale function divided by this function (relative deviation). Correspondingly, the relative deviations to the EHK and NS scale functions are calculated and shown in the figure. Rotating the scale functions around, the relative deviations from the NS and EHK scales are found in the same way and shown in Figs. 2 and 3.

The ratio between the NS data sets r_0 and r_c changes when different scale functions are used. From the constants of table 2 one finds

$$(r_c/r_0)^{BBV} = 0.11024\,(35)/0.21415\,(21) = 0.5148\,(18)\,, \tag{2.10}$$

$$(r_c/r_0)^{NS} = 0.5140\,(17)/0.99995\,(98) = 0.5140\,(18)\,, \tag{2.11}$$

$$(r_c/r_0)^{EHK} = 0.5172\,(17)/0.99204\,(97) = 0.5214\,(18)\,. \tag{2.12}$$

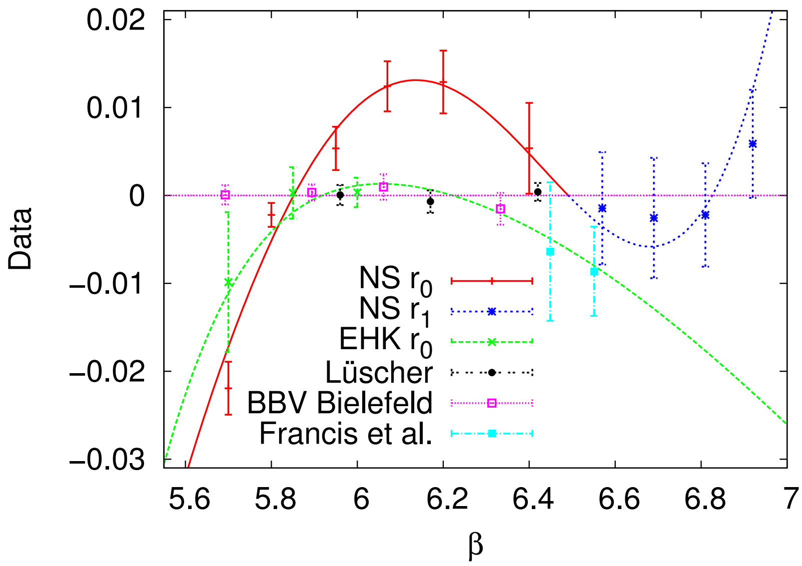

Figure 1: Relative deviations after the best fit of each data set to the BBV scale function.

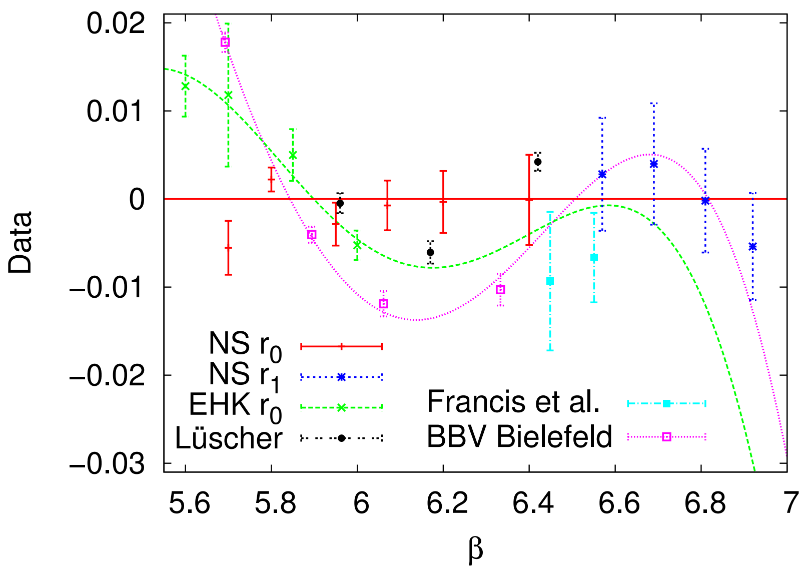

Figure 2: Relative deviations after the best fit of each data set to the NS scale.

The first values for $(r_c/r_0)^{BBV}$ and $(r_c/r_0)^{NS}$ are in statistical agreement with one another as well as with the ratio $r_c/r_0 = 0.5133\,(24)$, which is given in Eq. (2.5) of the NS paper and used to determine the NS scale function. For $(r_c/r_0)^{NS}$ this is obvious in Fig. 2, where the NS scale function (i.e., the zero-line) fits both NS data sets well. All other data are in disagreement with this scale. The BBV reference scale of Fig. 1 fits the EHK $r_0 - 1$ data[2], the T_c, the NS r_c and Lüscher's data well and

[2]The fit drawn is for the EHK r_0 data. It becomes good for the EHK $r_0 - 1$ data (omission of the $\beta = 5.6$ data point implies also small changes for the EHK data coefficients as listed in table 2).

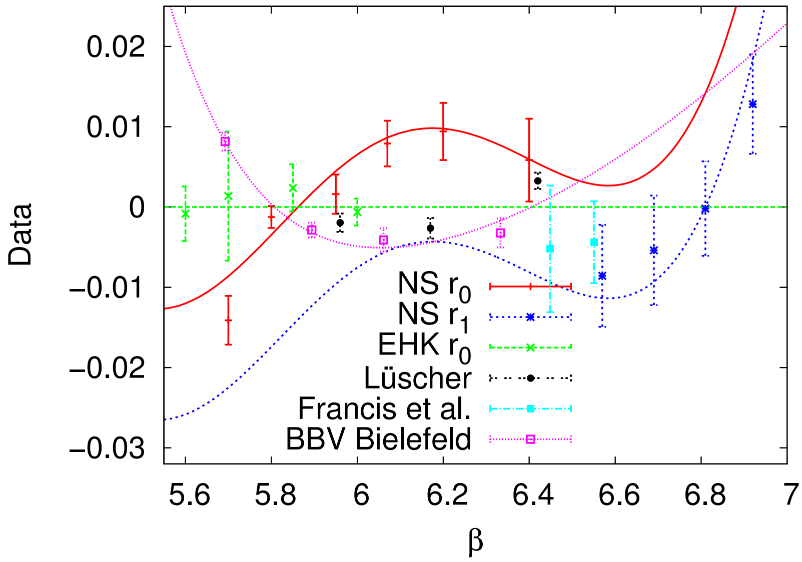

Figure 3: Relative deviations after the best fit of each data set to the EHK scale function.

is in disagreement with the NS r_0 data and the $\beta = 5.6$ EHK value. Due to the slight difference between the ratios (2.10) and (2.11) the NS scale on its r_0 data should in Fig. 1 be slightly higher than the NS scale on its r_c data. As this stays within statistical errors, we have just averaged the two curves, but use distinct colors, red for the r_0 and blue for the r_c range. Such averaging is not possible when plotting the NS data versus the EHK scale, because the ratio (2.12) is incompatible with the other two ratios. It amounts to the difference between the red and blue curves in Fig. 3.

3. Summary and conclusions

Table 3 shows that the three scale functions (EHK, NS and BBV) are derived from data sets, given in table 1, which are mutually inconsistent in the range up to $\beta = 6.4$, while the NS r_c data for the range $6.57 \le \beta \le 6.92$ are not very restrictive. Only the BBV scaling function is consistent with Lüscher's accurate data (see also Figs. 1 to 3).

In the range $5.65 \le \beta \le 6.92$ the relative discrepancy between the scales is never larger than $\pm 2\%$ as is shown in the upper part of Fig. 4 for ratios of the form $const\, f_\lambda^{EHK}/f_\lambda^{BBV}$ and $const'\, f_\lambda^{NS}/f_\lambda^{BBV}$ (the upper abscissa and the right ordinate apply and the constants (2.9) used from table 2 are the same as those for Fig. 1). Note that the previous figures, which exhibit relative deviations from the scales, cover a corresponding [-0.03:0.02] range.

The lower part of Fig. 4 shows that the EHK and BBV scales approach the universal asymptotic scale (2.3) in rather distinct ways, whereas such a parametrization is not attempted by NS (this part of the figure uses a normalization in which all scales agree at $\beta = 6$). The discrepancy between EHK and BBV with respect to the approach of the asymptotic scale relies on making distinct assumptions which can only be resolved on the basis of more accurate results at larger β values, which could come from calculations of the SU(3) deconfining temperature for $N_t > 12$. This may

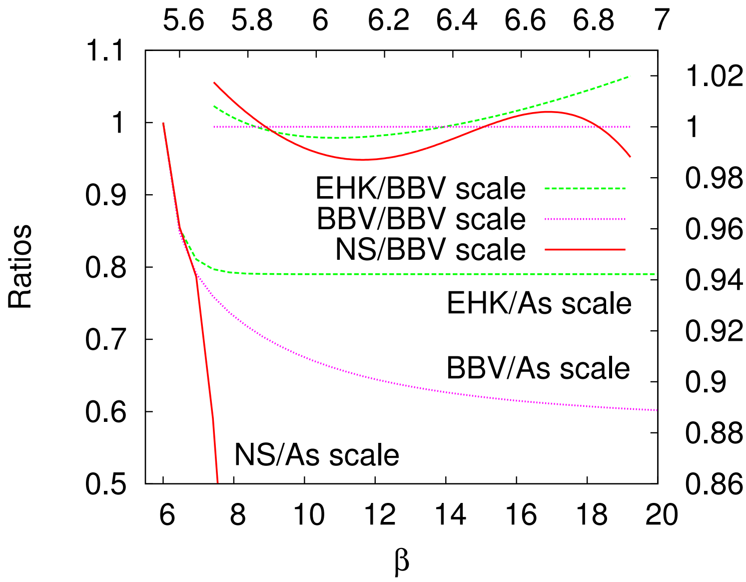

Figure 4: Ratios with respect to the BBV scale (upper part, top abscissa and right ordinate) and asymptotic behavior of the scales (lower part, bottom abscissa and left ordinate).

need some innovative techniques as the $N_t = 14$ and 16 data from Francis et al. are seen to exhibit similar inaccuracies as the large lattice NS data. Most promising may be calculations with the gradient method at larger β values. That this will work is also not obvious. For instance, the sensitivity of the gradient method to topological excitations [1, 9] on periodic lattices turns into a disadvantage when it comes to accurate scale calculations.

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