Understanding localisation in QCD through an Ising-Anderson model

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Above the QCD chiral crossover temperature, the low-lying eigenmodes of the Dirac operator are localised, while moving up in the spectrum states become extended. This localisation/delocalisation transition has been shown to be a genuine second-order phase transition, in the same universality class as that of the 3D Anderson model. The existence of localised modes and the effective dimensional reduction can be tentatively explained as a consequence of local fluctuations of the Polyakov loop, that provide 3D on-site disorder, in analogy to the on-site disorder of the Anderson model. To test the viability of this explanation we study a 3D effective, Anderson-like model, with on-site disorder provided by the spins of a spin model, which mimics the Polyakov loop dynamics. Our preliminary results show that localised modes are present in the ordered phase, thus supporting the proposed mechanism for localisation in QCD.

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1. Introduction

Low-lying Dirac eigenmodes are of considerable physical interest, especially in connection with the issue of chiral symmetry breaking. In recent years, it has become clear that the localisation properties of the low-lying modes change across the chiral crossover: while at low temperatures all the Dirac eigenmodes are delocalised, above the chiral crossover temperature T_c the low modes become localised on the scale of the inverse temperature; higher up in the spectrum, above a critical eigenvalue $\lambda_c(T)$, modes remain delocalised also above T_c [1, 2, 3, 4, 5, 6, 7].

The coexistence of localised and delocalised modes in the same spectrum is a well known phenomenon in condensed matter physics, and it is the characteristic feature of the 3D Anderson model for "dirty" conductors [8, 9, 10]. The Hamiltonian of the Anderson model consists of the usual tight-binding Hamiltonian plus a random on-site potential,

$$H_{\vec{x}\vec{y}} = \varepsilon_{\vec{x}}\delta_{\vec{x}\vec{y}} + \sum_{\mu}(\delta_{\vec{x}+\hat{\mu}\,\vec{y}} + \delta_{\vec{x}-\hat{\mu}\,\vec{y}}), \tag{1.1}$$

where $\varepsilon_{\vec{x}}$ provides a diagonal disorder term mimicking the presence of impurities in the crystal. The random potential is usually chosen to be uniformly distributed in an interval $\varepsilon_{\vec{x}} \in \left[-\frac{W}{2}, \frac{W}{2}\right]$, with the width W controlling the amount of disorder in the system. If a magnetic field is applied to the system, the model is modified to

$$H_{\vec{x}\vec{y}} = \varepsilon_{\vec{x}}\delta_{\vec{x}\vec{y}} + \sum_{\mu}(\delta_{\vec{x}+\hat{\mu}\,\vec{y}} + \delta_{\vec{x}-\hat{\mu}\,\vec{y}})e^{i\phi_{\vec{x}\vec{y}}}, \qquad \phi_{\vec{y}\vec{x}} = -\phi_{\vec{x}\vec{y}}, \tag{1.2}$$

where $\phi_{\vec{x}\vec{y}}$ are random phases. This is the unitary Anderson model. The name "unitary" comes from the symmetry class to which the Hamiltonian Eq. (1.2) belongs in the random matrix theory classification. In the same classification, the model Eq. (1.1) belongs to the orthogonal class. In both these models, for any nonzero W, there is a critical energy $E_c(W)$, called "mobility edge", which separates localised and delocalised modes: for energies beyond E_c, the energy eigenmodes at the band edge are localised, while eigenmodes in the band center are delocalised. The transition from localised to delocalised modes as one moves along the spectrum is a true second-order phase transition, known as Anderson transition.

It has been recently shown that the analogous transition in the Dirac spectrum above T_c is also a genuine second-order phase transition [7]. The critical exponent of the correlation length has been determined, and found to be $\nu_{\text{QCD}} = 1.43(6)$, which is compatible with the one found in the 3D unitary Anderson model, $\nu_{\text{UAM}} = 1.43(4)$ [11]. While the appearence of the unitary class is expected, since it is the symmetry class of QCD in the language of random matrix models, the fact that a phase transition in a 4D theory seems to belong to the same universality class as a phase transition in a 3D model needs to be explained. The dimensionality of the system is not the only difference between high-temperature QCD and the Anderson model. While in the Anderson model disorder is diagonal[1] and uncorrelated, in QCD it is off-diagonal, i.e., appearing in the hopping terms, and correlated. Correlations are short-range, so they should not be relevant; on the other hand, how off-diagonal disorder in QCD can produce the same effects as diagonal disorder in the Anderson model requires an explanation.

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[1] Although in the unitary model there is also off-diagonal disorder, this is known to be much less effective in inducing localisation [12].

In this contribution we propose a mechanism for localisation in QCD, which is able to explain the apparent differences between the two models. The argument consists in a refinement of the proposal of Ref. [13]. To test this mechanism, we introduce and justify an effective model ("Ising-Anderson model"), which should produce localisation precisely through this same mechanism. We also show some preliminary numerical results that support the viability of our explanation.

2. Polyakov lines and localisation

In order to better understand the relation between QCD and the Anderson model, the key observation is that QCD above T_c is effectively a 3D model, at least as far as the qualitative features of quark eigenfunctions are concerned. Indeed, as the size of the temporal dimension is finite and smaller than the correlation length, the time slices are strongly correlated, and so quark eigenfunctions will look qualitatively the same on all time slices, in particular for what concerns the localisation properties.

Let us discuss this point in more detail. It is convenient to work in the temporal gauge, $U_4(t,\vec{x}) = 1$ for $t = 0,\ldots,N_T - 2$ and $\forall \vec{x}$, in which $U_4(N_T - 1,\vec{x})$ equals the local Polyakov line $P(\vec{x}) = \prod_{t=0}^{N_T-1} U_4(t,\vec{x})$. A further time-independent gauge transformation allows to diagonalise each local Polyakov line: we will refer to this as the diagonal temporal gauge. In any temporal gauge, covariant time differences are replaced by ordinary differences; the price to pay is that the antiperiodic boundary conditions become effective, $\vec{x}$-dependent boundary conditions, which involve the local Polyakov line,

$$\psi(N_T,\vec{x}) = -P(\vec{x})\psi(0,\vec{x}).\tag{2.1}$$

Since the time slices are strongly correlated, these effective, $\vec{x}$-dependent boundary conditions will affect the behaviour at the spatial point $\vec{x}$ for all times t. Furthermore, $P(\vec{x})$ fluctuates in space (and obviously from one configuration to another). From the point of view of a disordered system, QCD above T_c therefore contains effectively a diagonal (on-site), 3D source of disorder.

To see how the effective boundary conditions affect the quark wave functions, let us first discuss a simplified setting in which the Dirac equation can be explicitly solved, generalising the argument of Ref. [13] to $SU(3)$. Consider configurations with constant temporal links U_4 and trivial spatial links $U_j = 1$. In the diagonal temporal gauge, one has $P(x) = P = \mathrm{diag}(e^{i\varphi_1}, e^{i\varphi_2}, e^{i\varphi_3})$ with $\varphi_3 = -\varphi_1 - \varphi_2$, so the Dirac operator is diagonal in colour, and the colour components ψ_k of the quark eigenfunctions decouple. The eigenfunctions of the Dirac operator are plane waves,

$$\slashed{D}\psi_k = i\lambda\psi_k, \quad \psi_k(t,\vec{x}) \propto e^{i\omega_k t + i\vec{p}\cdot\vec{x}}, \quad \lambda = \pm\sqrt{\sin^2\omega_k + \sum_{j=1}^3 \sin^2 p_j},\tag{2.2}$$

with $\frac{L}{2\pi}p_j = 0, 1, \ldots, L-1$ to fulfill the spatial periodic boundary conditions, and

$$\omega_k = \tfrac{1}{N_T}(\pi + \varphi_k \mod 2\pi) = aT(\pi + \varphi_k \mod 2\pi),\tag{2.3}$$

to fulfill the effective temporal boundary conditions. We will refer to ω_k as the effective Matsubara frequencies. For a given value of the phase φ_k, $N_T/2$ (twofold degenerate) different branches for ω_k appear, corresponding to increasing values of λ. For the lowest branch, ω_k decreases as φ_k moves away from zero, and it is minimal when $e^{i\varphi_k} = -1$.

Let us now consider the general case. Above T_c, in a typical gauge configuration the Polyakov line $P(\vec{x})$ gets ordered along $\mathbf{1} = \mathrm{diag}(1,1,1)$ with "islands" of "wrong" $P(\vec{x}) \neq \mathbf{1}$. If it were completely ordered, there would be a sharp gap in the spectrum at $\lambda_g = \omega_k(0) = \pi aT$ (and a symmetric one at $-\lambda_g$), but fluctuations (both of $P(x)$ and of the spatial links) smoothen it out. In particular, living on the "islands" of "wrong" $P(\vec{x})$ is "energetically" favourable for the quark eigenfunctions, yielding $|\lambda| < \lambda_g$ as long as the momentum required to localise the state does not overbalance the gain. The gap becomes therefore an effective gap λ_c, identified as the "mobility edge" separating localised and delocalised modes. This effective gap is furthermore displaced by the presence of spatial fluctuations, which most likely have a delocalising effect on the eigenmodes, thus pushing the effective gap down.

In summary, the presence of "islands" of "wrong" Polyakov lines provides a localising "trap" for eigenmodes. At fixed lattice spacing, we expect λ_c to increase as the temperature is increased, being "dragged" by the effective Matsubara frequency.

3. Effective 3D model

The considerations above suggest that it should be possible to understand the qualitative features of the Dirac spectrum and eigenfunctions in QCD, in particular concerning spectral statistics and localisation properties, by using a genuinely 3D model. To construct such a model, one has to strip off all the features that are irrelevant to localisation. The first step is to get rid of the time direction, reducing the lattice to three dimensions, and replacing the time covariant derivative in the Dirac operator with a diagonal noise term, intended to mimic the effective boundary conditions. Moreover, it is known that off-diagonal disorder is less effective than diagonal disorder in producing localisation [12], so we can replace spatial covariant derivatives with ordinary derivatives. As a consequence, colour components decouple: this changes the symmetry class (in the sense of random matrix models), but should not affect the presence of localisation. In conclusion, the main features of localisation should still be captured if we replace the 4D lattice with a 3D lattice, and the Dirac operator with the effective "Hamiltonian"

$$i\slashed{D}_{xy} = i\gamma^4(D_4)_{xy} + i\vec{\gamma}\cdot\vec{D}_{xy} \longrightarrow H_{\vec{x}\vec{y}} = \gamma^4 \mathcal{N}_{\vec{x}} \delta_{\vec{x}\vec{y}} + i\vec{\gamma}\cdot\vec{\partial}_{\vec{x}\vec{y}}, \tag{3.1}$$

where $\mathcal{N}_{\vec{x}}$ is the diagonal noise, to be specified later. This Hamiltonian is diagonal in colour, and so has effectively only spacetime and Dirac indices; for lattices of even spatial size, a spin diagonalisation allows to get rid of the latter.

At this point we should specify the diagonal noise $\mathcal{N}_x$ intended to mimic the effective boundary conditions, which has therefore to satisfy a few requirements:

1. it should not be uncorrelated, but rather be governed by Polyakov-loop-like dynamics. This suggests to base it on some spin model in the ordered phase;

2. as it is the phase of $P(\vec{x})$ which enters the effective boundary conditions, which is a continuous variable, we have to use continuous spins;

3. finally, $\mathcal{N}_{\vec{x}}$ should produce an effective gap in the spectrum.
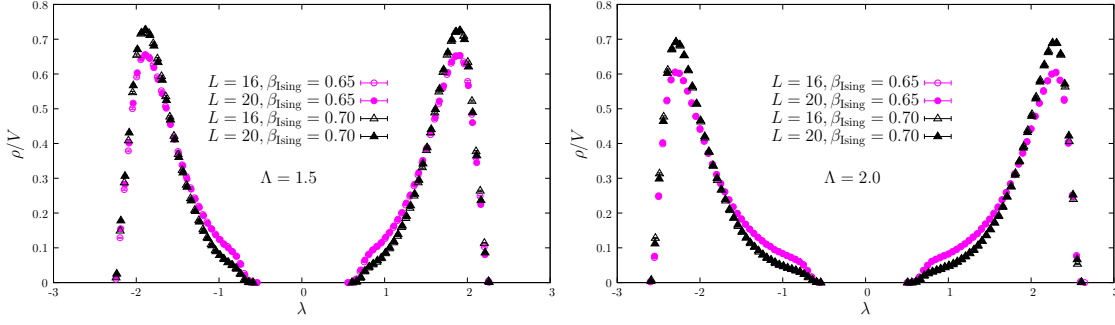
Figure 1: Spectral density of the effective model Eq. (3.3).

For our purposes it is enough to have a continuous spin model which displays an ordered phase, and the simplest choice is the Ising model with continuous spin. A possible choice for the noise term is

$$\mathcal{N}_{\vec{x}} = \Lambda \frac{1+s_{\vec{x}}}{2}, \qquad s_{\vec{x}} \in [-1,1], \tag{3.2}$$

where $s_{\vec{x}}$ is the spin variable at point $\vec{x}$, and Λ is a constant determining the strength of the coupling of the fermions to the spins. Since $\frac{1+s_{\vec{x}}}{2}$ is 1 for "aligned" spins (i.e., for $s_{\vec{x}} = 1$), this choice provides indeed an effective spectral gap when the spins are ordered.[2]

Putting everything together, our effective model reads

$$H_{\vec{x}\vec{y}} = \gamma^4 \Lambda \frac{1+s_{\vec{x}}}{2} \delta_{\vec{x}\vec{y}} + i\vec{\gamma} \cdot \vec{\partial}_{\vec{x}\vec{y}}, \tag{3.3}$$

with $s_{\vec{x}}$ distributed according to the dynamics of the Ising model,

$$\frac{H_{\text{Ising}}}{kT} = -\beta_{\text{Ising}} \sum_{<\vec{x}\vec{y}>} s_{\vec{x}} s_{\vec{y}}, \qquad s_{\vec{x}} \in [-1,1]. \tag{3.4}$$

In the ordered phase of the Ising model there is a "sea" of $s_{\vec{x}} = 1$ spins with "islands" of $s_{\vec{x}} \neq 1$ spins, so the underlying configurations have the same features as the Polyakov loop configurations in QCD. While in QCD we have a single parameter governing the ordering of the configuration and the size of the effective gap, in the effective model the ordering of the spin configuration is governed by β_{Ising}, while the size of the gap is mainly determined by the spin-fermion coupling Λ.

A curious feature of this model is that it belongs to different symmetry classes for lattices of even or odd size *L*, namely to the orthogonal class for *L* even, and to the symplectic class for *L* odd. As we have already said above, it is more convenient to work with even-sized lattices, since one can get rid of the Dirac indices through a spin diagonalisation. Studying the localisation properties of this model would provide a test for the viability of the sea/islands explanation.

4. Numerical results

We have performed numerical simulations of the effective model Eq. (3.3) on medium-size lattices (*L* = 16 and *L* = 20), performing full diagonalisation of the Hamiltonian, for two different temperatures $1/\beta_{\text{Ising}}$ of the Ising system in the ordered phase, and for two choices of the spin-fermion coupling Λ.

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[2]It is understood that we work with magnetic field $h = 0^+$.
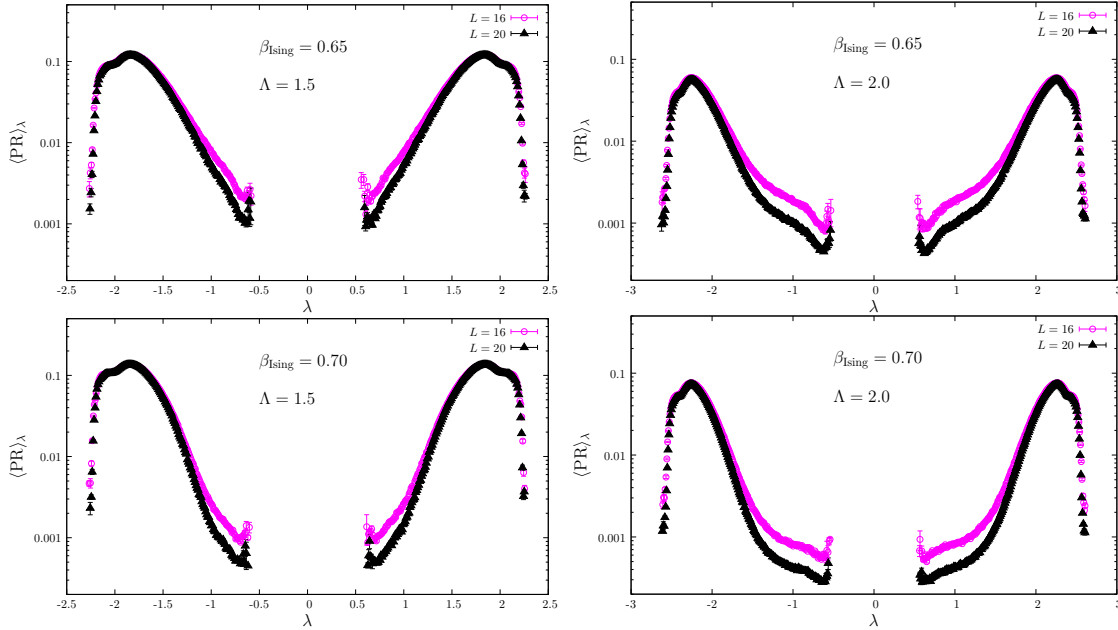
Figure 2: Participation ratio of eigenmodes of the effective model Eq. (3.3).

In Fig. 1 we show the spectral density per unit volume of the system. Low modes have small spectral density, which rapidly increases as one goes up in the spectrum, as expected. Decreasing the temperature, thus making the system more ordered, decreases the density of low modes, as one expects if these modes are localised. Increasing the spin-fermion coupling enlargens the region where the spectral density is small, again as expected, since it should push the effective gap up in the spectrum. Notice the symmetry under $\lambda \to -\lambda$ of the spectrum, which can be proved to be a property of the average spectrum, but which does not hold configuration by configuration as it does in QCD with staggered fermions. There is apparently also a sharp gap in the spectrum, which is however essentially insensitive to Λ. It is not clear if this is due to the use of not large enough volumes or if it is indeed a feature of the effective model, possibly due to the absence of disorder in the hopping terms; in any case, it is irrelevant for our purposes.

In Fig. 2 we show the average participation ratio $PR = (\sum_x |\psi(x)|^4)^{-1}/V$, which gives a measure of the fraction of space occupied by a given mode, as a function of its location in the spectrum. The *PR* changes by two orders of magnitude as one moves up in the spectrum starting from the low-density region; more importantly, when increasing the size of the system it remains almost constant in the bulk of the spectrum, while it visibly decreases near the origin. This signals that modes in the bulk are delocalised, while modes near the origin are localised.

Finally, in Fig. 3 we show a suitably defined local spectral statistics I_λ across the spectrum. Spectral statistics can be used to detect a localisation/delocalisation transition, since the eigenvalues corresponding to localised or delocalised eigenmodes are expected to obey different statistics, namely Poisson or Wigner-Dyson statistics, respectively. More precisely, I_λ is defined as

$$I_\lambda = \int_0^{\bar{s}} ds \, p_\lambda(s), \qquad s_i = \frac{\lambda_{i+1} - \lambda_i}{\langle \lambda_{i+1} - \lambda_i \rangle_\lambda}, \qquad \bar{s} \simeq 0.5, \qquad (4.1)$$

where $p_\lambda(s)$ is the probability distribution, computed locally in the spectrum, of the so-called
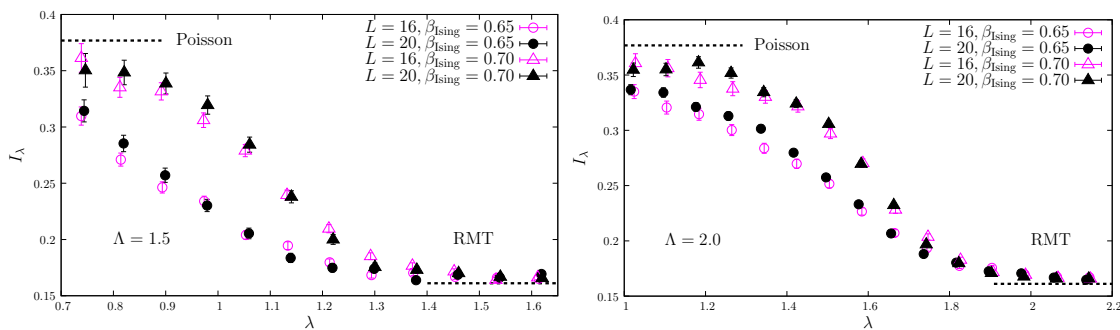
Figure 3: The spectral statistics I_λ, Eq. (4.1), in the effective model Eq. (3.3).

unfolded level spacing s_i, i.e., the level spacing divided by the local average level spacing $\langle \lambda_{i+1} - \lambda_i \rangle_\lambda$. The results confirm that the eigenmodes change from localised to delocalised when one moves up in the spectrum, and that the "mobility edge" separating localised and delocalised modes goes up in the spectrum when the Ising system is made more ordered by decreasing the temperature, or when the spin-fermion coupling is increased. Furthermore, they give also a first indication that the steepness of the curve increases as the volume is increased, thus hinting at the existence of a true phase transition in the spectrum.

5. Conclusions

We have proposed a possible mechanism to explain localisation of quark eigenmodes in QCD above T_c. The mechanism is based on the "trapping" effect of spatial fluctuations of the local Polyakov line on the eigenmodes in the background of a partially ordered gauge configuration. To test the explanation, we have constructed an effective 3D model which should capture the main features of localisation. Preliminary numerical results support the viability of our explanation.

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