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The chiral transition as an Anderson transition

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At low temperature the low-lying QCD Dirac spectrum obeys random matrix statistics. Recently we found that above T_c the lowest part of the spectrum consists of localized modes that obey Poisson statistics. An interesting implication of this is that as the system crosses T_c from above, the spectral statistics at $\lambda = 0$ changes from Poisson to random matrix. Here we study this transition and its possible implications for the finite temperature transition of QCD-like theories.

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1. Introduction

It is well known that at low temperature the low-lying eigenmodes of the QCD Dirac operator are delocalized, and their statistics can be described by random matrix theory [1, 2]. This fact is utilized also in practice: low energy constants of chiral perturbation theory can be determined by comparing lattice data and random matrix predictions. Recently it was found that the high-temperature Dirac spectrum is more "complex" than the low temperature one [3 – 6]. This means that the eigenmodes of the Dirac operator are either localized or delocalized depending on where they are located in the spectrum. The low end of the spectrum consists of localized, statistically uncorrelated eigenmodes, while modes in the bulk still remain delocalized and statistically correlated. The QCD Dirac operator is not the only physical system having such a "complex" spectrum. In solid state physics, the Hamiltonian of the Anderson model provides an example of this kind of spectrum. The Anderson model is the simplest model for Anderson localization [7]. The one-electron Hamiltonian of the Anderson model describes non-interacting electrons and has the form

$$H = \sum_i \varepsilon_i |i\rangle\langle i| + \sum_{\langle i,j\rangle} |i\rangle\langle j|, \tag{1.1}$$

where i represents an atomic orbital on a lattice site. The first term in eq. (1.1) represents an on-site interaction with randomly distributed site energies ε_i. The second term is responsible for nearest-neighbor hopping with constant coupling strength. In the case when the disorder is zero, i.e., the variance $\sigma(\varepsilon_i)$ of the random on-site energies is zero, the eigenfunctions of H are the well-known, delocalized Bloch wavefunctions. If the disorder is turned on then localized eigenmodes will appear around the spectrum edges, while states near the band center remain delocalized, and there will be a boundary between the localized and delocalized modes in both the positive and negative part of the spectrum called mobility edge. If the disorder increases the mobility edges move further towards the band center, and eventually at a critical disorder strength all eigenmodes of H will be localized. This transition is known as the Anderson transition.

The Anderson transition in solid-state physics is a real second order phase transition [8]. Some of us found that the finite volume crossover in the high-temperature Dirac spectrum from localized eigenmodes to delocalized eigenmodes also becomes a genuine second-order phase transition in the thermodynamic limit [9]. In the same work we also determined the critical exponent $\nu_{QCD} = 1.43(6)$ which is consistent with the one found in the three dimensional unitary Anderson model $\nu_{And} = 1.43(4)$ [10]. A possible explanation for this consistency, and a more detailed description of the similarities of the spectra of the Anderson model and of the QCD Dirac operator can be found in [11]. In the Anderson model the control parameter which drives the transition is the disorder strength $\sigma(\varepsilon_i)$. On the other hand, the transition can be also examined at fixed disorder strength by scanning the spectrum. Physically, this approach arises more naturally, since one can control the position of the Fermi energy by changing the chemical potential and examine the transition as the Fermi energy crosses the mobility edge. Unfortunately, in QCD (as far as we know) there is no physical control parameter which enables us to change the position in the Dirac spectrum. In fact, the mobility edge is only a point in the spectrum, while all local thermodynamical quantities (for example the quark condensate $\langle\bar{\psi}\psi\rangle$) are obtained by integrating over the full spectrum. Nevertheless, the presence of localized low-lying modes might affect these observables.

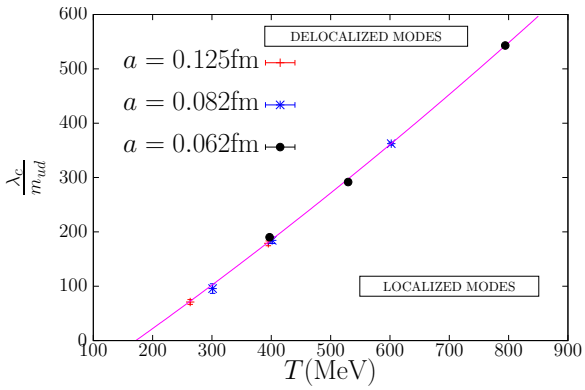

Figure 1: Phase diagram of *QCD* in the eigenvalue-temperature plane.

In lattice QCD the control parameter is the temperature, which plays a similar role to the disorder strength $\sigma(\varepsilon_i)$ in the Anderson model. In this paper we analyze the localization properties of low-lying modes as the temperature is varied, both in QCD and in a QCD-related toy model displaying a true first order phase transition. We also study the relation between the appearance of localized modes and the presence of a dramatic change in the thermodynamic properties of the system.

2. Phase diagram of QCD

To analyze the temperature dependence of the mobility edge, λ_c, the appropriate quantity to consider is its ratio to the bare quark mass. Since λ_c acts effectively as a bare quark mass, it is expected to renormalize similarly, so that λ_c/m_{ud} is renormalization group invariant. A more formal argument can be found in [6]. In Fig. 1 we show the phase diagram of QCD in the (λ, T) plane. To a good precision, at high temperature, well above the pseudo-critical QCD transition temperature T_c, the mobility edge depends linearly on the temperature. There is a critical line in the phase diagram along which there is a second order phase transition in the spectrum. If we extrapolate this critical line to vanishing mobility edge, we get a temperature consistent with the pseudo-critical QCD crossover temperature. However, the relationship between the Anderson transition and the QCD chiral crossover is not straightforward, because in the latter case there is no unique transition temperature. In order to investigate the relationship between the chiral transition and the Anderson transition in a clear-cut setting we will consider a simple model which shows a real first order phase transition in the thermodynamic limit. In fact our work has also practical motivations and might suggest new order parameters for locating the chiral transition. Usually, the chiral transition is analyzed by comparing the value of the quark condensate at different temperatures. A priori, calculating the bare quark condensate at just one temperature will not tell us to which phase the ensemble of configurations belongs. In this work our aim is also to look for quantities that can be easily computed (without any renormalization or zero temperature simulation) and can be used to identify the phase to which the ensemble of configurations belongs.

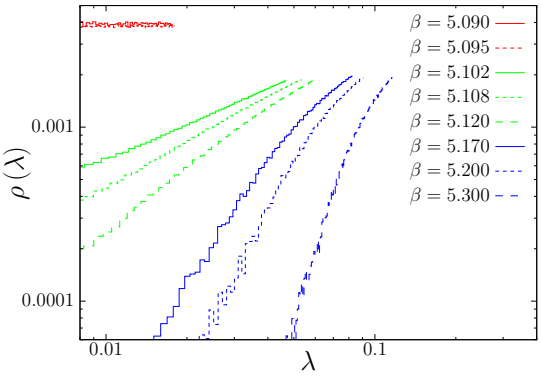

Figure 2: Spectral density (normalized by the lattice volume) of our toy model for different values of β on a log-log scale.

3. A toy model: three flavors of unimproved staggered quarks on $N_t = 4$ lattices

In order to investigate the relationship between the chiral transition and the Anderson transition in the spectrum we simulate $N_f = 3$ unimproved staggered quarks with the Wilson plaquette action. At $N_t = 4$ this toy model has a real first order phase transition in the thermodynamic limit for sufficiently small quark masses [12]. In our case we use $m_{bare} = 0.01$. To check the finite volume effects we use three different lattice volumes: $24^3, 32^3, 48^3$. Since N_t is fixed we use the β gauge coupling to change the temperature. Although, from the point of view of QCD, this first order phase transition is a coarse lattice artifact, our toy model is nevertheless a legitimate statistical physics model, which allows to compare, in a clear-cut setting, the temperature at which a genuine phase transition occurs, and the temperature at which localized modes appear in the Dirac spectrum.

4. Numerical results

Let us first check how the spectral density changes when the first order transition happens, and then to examine how the spectral statistics change along the spectrum at temperatures which definitely belong to the "chirally symmetric" phase. Establishing the transition in the spectrum in the "chirally symmetric" phase we define a mobility edge and map out the phase diagram of our toy model in the (λ, β) plane.

The usual way to examine a chiral transition is to study the temperature dependence of the quark condensate. The latter can be expressed in terms of the spectral density of the lattice Dirac operator with the help of the Banks-Casher formula [13] as follows,

$$\langle \bar{\psi}\psi(m) \rangle = \int_0^\infty \mathrm{d}\lambda \, \frac{2m}{\lambda^2 + m^2} \rho(\lambda) \,. \tag{4.1}$$

A sizeable contribution to the integral comes from the low modes of the Dirac operator, so a large change in the spectral density around zero will reflect in a large change of the condensate. In Fig. 2 we show how the spectral density near zero changes as β is varied and the system is driven through the chiral phase transition. Below $\beta = 5.099$ the spectral density near the origin is essentially constant in λ, which leads to a large contribution to the chiral condensate, and changes only slightly

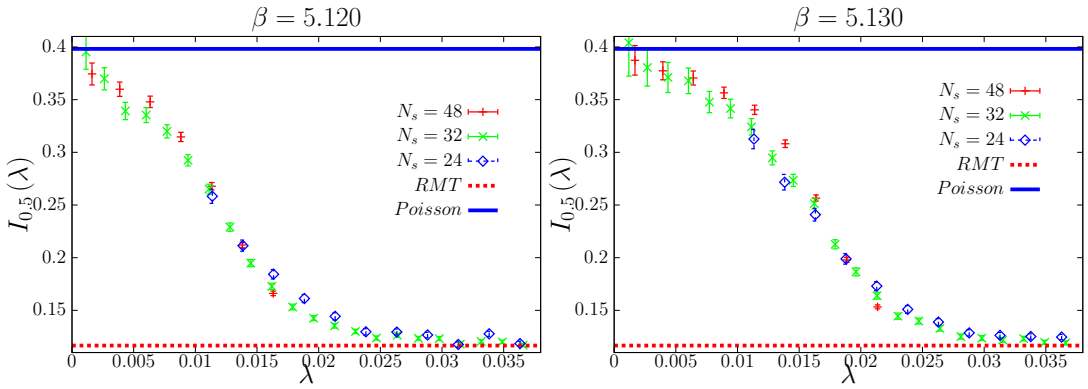

Figure 3: Poisson-RMT transition in high-temperature Dirac spectra for two different temperatures.

with β. Above $\beta = 5.099$ the spectral density vanishes continuously as a power law approaching the origin, so the contribution of the low modes to the condensate drops considerably. As a matter of fact, the chiral condensate has a jump around $\beta = 5.099$ (see Fig. 4), so that the two different behaviors of ρ near the origin correspond to the two phases of the system.

The presence of localized modes in the spectrum can be conveniently detected by analyzing the spectral statistics. Localized modes usually appear in a region of low spectral density, and so they are unlikely to be mixed by the gauge field fluctuations, as they have large eigenvalue differences and small eigenmode overlaps. They are therefore expected to be statistically independent, and the corresponding eigenvalues should follow Poisson statistics. On the other hand, when the spectral density is large the fluctuations can easily mix the modes, leading to delocalised and correlated eigenmodes, with eigenvalues distributed according to random matrix theory.

In order to determine what kind of statistics the eigenvalues obey, it is easier to work with the so-called unfolded spectrum, obtained by properly rescaling the eigenvalues to have unit spectral density throughout the spectrum. In practice, this is done by replacing each eigenvalue with its rank in the full sequence of eigenvalues. A commonly used spectral statistics is the so-called unfolded level spacing distribution (*ULSD*) $p(s)$. In the case of uncorrelated eigenvalues the *ULSD* is an exponential distribution: $p_{Poisson}(s) = \exp(-s)$, and in the case of correlated eigenvalues the *ULSD* can be well approximated by the so-called Wigner surmise of the corresponding random matrix ensemble $p_{RMT}(s) = \frac{32}{\pi^2} s^2 \exp\left(-\frac{4}{\pi} s^2\right)$. First we check whether the low modes of the high-temperature Dirac spectrum are also statistically independent in our toy model. To show this we choose some parameter of the *ULSD*, compute it locally in the spectrum and monitor how it changes between the two limiting cases. We choose the integral of the *ULSD* up to the point where the exponential and the Wigner surmise first cross (approximately):

$$I_{0.5} = \int_0^{0.5} \mathrm{d}s \; p(s) \,. \tag{4.2}$$

In Fig. 3 we show how this statistic changes locally in the spectrum for two different temperatures. For both temperatures we see qualitatively the same behavior: moving from the edge towards the bulk of the spectrum the spectral statistics changes from Poisson to random matrix. For each temperature we show the continuous transition in the spectrum for three different lattice volumes.

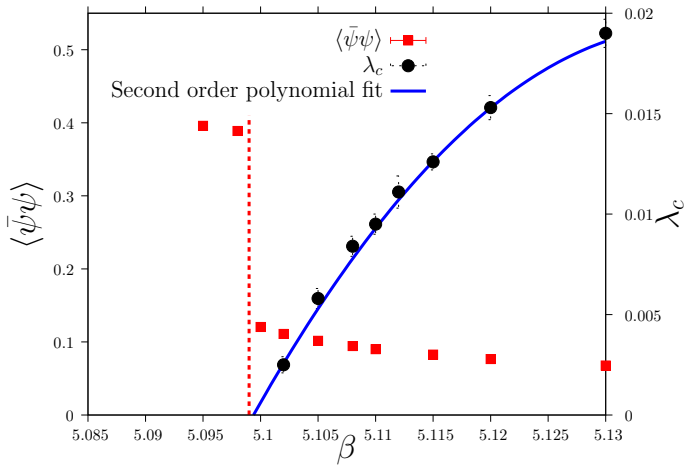

Figure 4: The phase diagram of our toy model in the (λ,β) plane. We show the mobility edge with filled circles and the value of $\langle\bar{\psi}\psi\rangle$ with filled squares. The smooth line represents the critical line in the (λ,β) plane. The vertical dotted line corresponds to $\beta_c = 5.099$ where the (extrapolated) mobility edge vanishes.

It is clear that the transition becomes sharper on larger volumes, suggesting a singular behavior in the thermodynamic limit. For the QCD Dirac spectrum we showed that this transition is a second order phase transition. Most likely this is the case for our toy model too, since around the critical point in the spectrum (the mobility edge), where $I_{0.5}$ is volume independent, this quantity takes the same value as in QCD, $I_{0.5}(\lambda_c) = 0.187$ [14], both for $\beta = 5.120$ and 5.130. We also see from Fig. 3 that the mobility edge decreases as the temperature decreases. In the following we examine how this mobility edge goes to zero as a function of the temperature and how this correlates with the location of the first order chiral phase transition.

5. Phase diagram of the toy model

For any value of β, the mobility edge can be identified as the point in the spectrum where local spectral statistics are volume-independent in the large-volume limit. With decreasing β the mobility edge goes towards the spectrum edge and we need larger and larger volumes to capture the volume dependence of the transition. At large lattice volume, however, we can determine the mobility edge using simulations at a single volume only. We identify the mobility edge with the location in the spectrum where the spectral statistics takes its critical value i.e. $I_{0.5}(\lambda) = 0.187$. In Fig. 4 we show how the mobility edge changes as a function of the temperature. In the same figure we also show a second-order polynomial fit to the data points. The fit can be used to read off the critical β_c where the mobility edge becomes zero. In the same figure we also show how $\langle\bar{\psi}\psi\rangle$ changes as a function of β. We calculated $\langle\bar{\psi}\psi\rangle$ with the stochastic method. This quantity, being the first derivative of the partition function, exhibits a discontinuity at a real first order phase transition. Our most important result is that the mobility edge becomes zero around the point $\beta_c = 5.099$, where the value of the $\langle\bar{\psi}\psi\rangle$ jumps. This means that the localized modes appear at the temperature where the chiral transition occurs.

6. Conclusions and outlook

The question whether the effect of the chiral crossover in QCD can be seen also in terms of localization of the eigenmodes of the Dirac operator is a rather old one [3]. At that time high statistics simulations with physical quark masses were unfeasible. In QCD with physical quark masses we found that the Anderson transition in the Dirac spectrum starts to appear around 163 MeV [15], which is consistent with the pseudo-critical QCD transition temperature determined from other quantities [16]. In this paper we deal with a model in which there is a real first order phase transition in the thermodynamic limit. Our main result is that in this model the chiral transition and the Anderson transition occur at the same temperature. The next step would be to understand if and how the Anderson transition can possibly explain the chiral transition. To this extent, it would be interesting to extrapolate the spectral statistics to $\lambda = 0$ at a given value of β at a fixed lattice volume V and then do a finite-size scaling analysis. However, to get a reliable extrapolation to $\lambda = 0$ around β_c we need even larger volumes. Another interesting possibility is to study the statistical behavior of the first and the second eigenvalue of the lattice Dirac operator. For that aspect the overlap Dirac operator would be much better, since it does not suffer from taste-symmetry breaking of staggered fermions.

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