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Non-perturbative Renormalization of Four-Fermion Operators Relevant to B_K with Staggered Quarks

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We present preliminary results of matching factors of the four-fermion operators relevant to B_K , which are obtained using the non-perturbative renormalization (NPR) method in the RI-MOM scheme with HYP-smeared improved staggered fermions. We use the MILC asqtad coarse ($a \cong 0.12\,\text{fm}$) ensembles with $20^3 \times 64$ geometry and $am_\ell/am_s = 0.01/0.05$. We compare NPR results with those of one-loop perturbative matching.

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1. Introduction

The indirect CP violation parameter, ε_K in the neutral kaon system is very well known with $\approx 0.5\,\%$ precision from experiments [1]. Our theoretical estimate of ε_K directly from the standard model (SM) has 3.4σ tension with the experiment in the exclusive V_{cb} channel [2]. In order to widen the gap in the unit of σ, we need to increase the precision of the calculation of B_K and V_{cb} in lattice QCD. In our calculation, one of the dominant source of error comes from the matching factor for B_K ($\approx 4.4\%$) using the one-loop perturbation theory. Hence, it becomes essential to reduce the matching factor error. The non-perturbative renormalization method (NPR) with the RI-MOM [3] can reduce this error down to the $\approx 2\%$ level. In the previous work of Refs. [4, 5], the NPR method has been applied to the staggered bilinear operators. Here, we present preliminary results of the renormalization factors of four-fermion operators relevant to B_K operator obtained using NPR in the RI-MOM scheme with improved staggered fermions.

2. Four-fermion operator renormalization in the RI-MOM scheme

There are two kinds of color contraction of four-fermion operators. A general one-color trace four-fermion operator is defined as follows.

$$O_{\alpha,\mathbf{I}}(y) = \sum_{\substack{A,B,\\C,D}} \sum_{\substack{c_1,c_2,\\c_3,c_4}} \left[\overline{\chi}_{c_1}(y_A)\overline{(\gamma_{S_1}\otimes\xi_{F_1})}_{AB}\chi_{c_2}(y_B)\right]\left[\overline{\chi}_{c_3}(y_C)\overline{(\gamma_{S_2}\otimes\xi_{F_2})}_{CD}\chi_{c_4}(y_D)\right] U_{AD;c_1c_4}(y)U_{BC;c_2c_3}(y)\,, \tag{2.1}$$

and a general two-color trace four-fermion operator is defined as follows.

$$O_{\alpha,\mathbf{II}}(y) = \sum_{\substack{A,B,\\C,D}} \sum_{\substack{c_1,c_2,\\c_3,c_4}} \left[\overline{\chi}_{c_1}(y_A)\overline{(\gamma_{S_1}\otimes\xi_{F_1})}_{AB}\chi_{c_2}(y_B)\right]\left[\overline{\chi}_{c_3}(y_C)\overline{(\gamma_{S_2}\otimes\xi_{F_2})}_{CD}\chi_{c_4}(y_D)\right] U_{AB;c_1c_2}(y)U_{CD;c_3c_4}(y)\,, \tag{2.2}$$

where $\alpha = [S_1\otimes F_1][S_2\otimes F_2]$ is an operator index, and c_i are color indices. The y represents a coordinate of the hypercube with its lattice spacing $2a$. The indices A, B, C and D are hypercubic vectors: for example, $A=(1,1,0,0)$. Here, we use the notation of $y_A = 2y+A$. $U_{AB;c_1c_2}(y)$ is a gauge link, an average of the shortest paths which connect y_A and y_B as products of HYP-smeared fat links. γ_S represents the spin and ξ_F the taste. Here, $\chi(y_B)$ represents HYP-smeared staggered quark field. We calculate the amputated Green's function using same method introduced in Ref. [4].

As an example, we choose the four fermion operators used to calculate B_K in order to illustrate how the NPR method produces the matching factors. We introduce the following simple notations for the operators.

$$O_1 \equiv O_{[V\otimes P][V\otimes P],\mathbf{I}}\,, \qquad O_2 \equiv O_{[V\otimes P][V\otimes P],\mathbf{II}}\,,$$
$$O_3 \equiv O_{[A\otimes P][A\otimes P],\mathbf{I}}\,, \qquad O_4 \equiv O_{[A\otimes P][A\otimes P],\mathbf{II}}\,. \tag{2.3}$$

First, we divide the lattice operators into two classes: (C) the diagonal operators defined in Eq.(2.3), $\{O_1,\ O_2,\ O_3,\ O_4\}$, which have the ξ_5 tastes in both bilinears, and (D) the off-diagonal operators which are remaining operators with taste different from ξ_5. The tree level B_K operator is sum of the operators in the (C) class.

$$O_{B_K}^{\text{tree}} = O_1^{\text{tree}} + O_2^{\text{tree}} + O_3^{\text{tree}} + O_4^{\text{tree}} \tag{2.4}$$

The projection operators are also defined in the same way in Eq.(2.3) as follows.

$$\begin{aligned}
\mathbb{P}_1 &\equiv \frac{1}{N}\overline{\overline{(\gamma_\mu^\dagger \otimes \xi_5^\dagger)}}_{BA}\overline{\overline{(\gamma_\mu^\dagger \otimes \xi_5^\dagger)}}_{DC}\delta_{c_4c_1}\delta_{c_3c_2}\,, \qquad \mathbb{P}_2 \equiv \frac{1}{N}\overline{\overline{(\gamma_\mu^\dagger \otimes \xi_5^\dagger)}}_{BA}\overline{\overline{(\gamma_\mu^\dagger \otimes \xi_5^\dagger)}}_{DC}\delta_{c_2c_1}\delta_{c_4c_3}\,,\\
\mathbb{P}_3 &\equiv \frac{1}{N}\overline{\overline{(\gamma_{\mu 5}^\dagger \otimes \xi_5^\dagger)}}_{BA}\overline{\overline{(\gamma_{\mu 5}^\dagger \otimes \xi_5^\dagger)}}_{DC}\delta_{c_4c_1}\delta_{c_3c_2}\,, \qquad \mathbb{P}_4 \equiv \frac{1}{N}\overline{\overline{(\gamma_{\mu 5}^\dagger \otimes \xi_5^\dagger)}}_{BA}\overline{\overline{(\gamma_{\mu 5}^\dagger \otimes \xi_5^\dagger)}}_{DC}\delta_{c_2c_1}\delta_{c_4c_3}\,.
\end{aligned} \tag{2.5}$$

Here, N is normalization factor given as follows.

$$N = 3072 = \underbrace{4^4}_{\text{spin}} \times \underbrace{4^4}_{\text{taste}} \times (\underbrace{3}_{\text{1-color trace}} + \underbrace{9}_{\text{2-color trace}}) \tag{2.6}$$

We fix the normalization factor N such that, when we apply the projection operators to the tree level amputated Green's function, it satisfies the following conditions.

$$\text{tr}[\Lambda_{B_K}^{\text{tree}}\mathbb{P}_1] = \text{tr}[\Lambda_{B_K}^{\text{tree}}\mathbb{P}_2] = \text{tr}[\Lambda_{B_K}^{\text{tree}}\mathbb{P}_3] = \text{tr}[\Lambda_{B_K}^{\text{tree}}\mathbb{P}_4] = 1 \tag{2.7}$$

$$\text{tr}[\Lambda_{B_K}^{\text{tree}}\mathbb{P}_{(D)}] = \text{tr}[\Lambda_{(D)}^{\text{tree}}\mathbb{P}_{(C)}] = 0 \tag{2.8}$$

Here, note that the diagonal terms equal to one and the off-diagonal terms becomes zero.

The renormalized B_K operator is defined as follows.

$$O_{B_K}^R = z_1 O_1^B + z_2 O_2^B + z_3 O_3^B + z_4 O_4^B + \sum_{\alpha\in(D)} z_\alpha O_\alpha^B\,, \tag{2.9}$$

where the superscript R (B) denotes renormalized (bare) quantity, and the coefficients z_i are renormalization factors. The renormalization of quark fields is defined as follows.

$$\chi^R = Z_q^{1/2}\chi^B \tag{2.10}$$

The amputated Green's function is obtained by multiplying the inverse propagators to the unamputated Green's function. Hence the renormalized amputated Green's function is as follows.

$$\Lambda_{B_K}^R = \frac{z_1}{z_q^2}\Lambda_1^B + \frac{z_2}{z_q^2}\Lambda_2^B + \frac{z_3}{z_q^2}\Lambda_3^B + \frac{z_4}{z_q^2}\Lambda_4^B + \sum_{\alpha\in(D)}\frac{z_\alpha}{z_q^2}\Lambda_\alpha^B \tag{2.11}$$

The RI-MOM scheme prescription is that the renormalized quantity is equal to its tree level value.

$$\text{tr}[\Lambda_\alpha^R(\widetilde{p},\widetilde{p},\widetilde{p},\widetilde{p})\mathbb{P}_\beta] = \text{tr}[\Lambda_\alpha^{\text{tree}}(\widetilde{p},\widetilde{p},\widetilde{p},\widetilde{p})\mathbb{P}_\beta]\,, \tag{2.12}$$

where $\widetilde{p}$ is a momentum defined in the reduced Brillouin zone.[a] We define the projected amputated Green's function as follows.

$$\Gamma_{\alpha\beta}^B \equiv \frac{1}{z_q^2}\text{tr}[\Lambda_\alpha^B\mathbb{P}_\beta] \tag{2.13}$$

[a]Please refer to Ref. [4] for more details.

Hence, from Eq. (2.7), Eq. (2.8), Eq. (2.11) and Eq. (2.12), we obtain the following relations.

$$1 = z_1\Gamma^B_{1\alpha} + z_2\Gamma^B_{2\alpha} + z_3\Gamma^B_{3\alpha} + z_4\Gamma^B_{4\alpha} + \sum_{\gamma\in(D)} z_\gamma \Gamma^B_{\gamma\alpha}\,, \quad \alpha \in (C) \tag{2.14}$$

$$0 = z_1\Gamma^B_{1\beta} + z_2\Gamma^B_{2\beta} + z_3\Gamma^B_{3\beta} + z_4\Gamma^B_{4\beta} + \sum_{\gamma\in(D)} z_\gamma \Gamma^B_{\gamma\beta}\,, \quad \beta \in (D) \tag{2.15}$$

We can express these equations as a matrix equation.

$$\vec{z}_{\rm tree} = \vec{z}\cdot\hat{\Gamma}^B\,, \tag{2.16}$$

where $\vec{z}_{\rm tree}$ and $\vec{z}$ are vectors as follows.

$$\vec{z}_{\rm tree} = (1,1,1,1,0,\cdots,0)\,, \quad \vec{z} = (z_1,z_2,z_3,z_4,z_5,z_6,\cdots) \tag{2.17}$$

where z_i with $i \geq 5$ are renormalization factors of off-diagonal operators. The $\hat{\Gamma}^B$ is a matrix as follows. The upper-left (red) block elements are diagonal terms and others are off-diagonal terms.

$$\hat{\Gamma}^B = \left(\begin{array}{cccc|ccc}
\Gamma^B_{11} & \Gamma^B_{12} & \Gamma^B_{13} & \Gamma^B_{14} & \Gamma^B_{15} & \Gamma^B_{16} & \cdots \\
\Gamma^B_{21} & \Gamma^B_{22} & \Gamma^B_{23} & \Gamma^B_{24} & \Gamma^B_{25} & \Gamma^B_{26} & \cdots \\
\Gamma^B_{31} & \Gamma^B_{32} & \Gamma^B_{33} & \Gamma^B_{34} & \Gamma^B_{35} & \Gamma^B_{36} & \cdots \\
\Gamma^B_{41} & \Gamma^B_{42} & \Gamma^B_{43} & \Gamma^B_{44} & \Gamma^B_{45} & \Gamma^B_{46} & \cdots \\
\hline
\Gamma^B_{51} & \Gamma^B_{52} & \Gamma^B_{53} & \Gamma^B_{54} & \Gamma^B_{55} & \Gamma^B_{56} & \cdots \\
\Gamma^B_{61} & \Gamma^B_{62} & \Gamma^B_{63} & \Gamma^B_{64} & \Gamma^B_{65} & \Gamma^B_{66} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right) \tag{2.18}$$

Hence, we can compute z-factors from the inverse of $\hat{\Gamma}^B$ matrix as follows.

$$\vec{z} = \vec{z}_{\rm tree}\cdot(\hat{\Gamma}^B)^{-1} \tag{2.19}$$

The $\hat{\Gamma}^B$ can be rewritten by sub-matrices as follows.

$$\hat{\Gamma}^B = \begin{pmatrix} X_{4\times4} & Y_{4\times20} \\ Z_{20\times4} & W_{20\times20} \end{pmatrix} \tag{2.20}$$

Here, X is diagonal terms, and Y is off-diagonal terms.

$$X = \begin{pmatrix} \Gamma^B_{11} & \Gamma^B_{12} & \Gamma^B_{13} & \Gamma^B_{14} \\ \Gamma^B_{21} & \Gamma^B_{22} & \Gamma^B_{23} & \Gamma^B_{24} \\ \Gamma^B_{31} & \Gamma^B_{32} & \Gamma^B_{33} & \Gamma^B_{34} \\ \Gamma^B_{41} & \Gamma^B_{42} & \Gamma^B_{43} & \Gamma^B_{44} \end{pmatrix}\,, \qquad Y = \begin{pmatrix} \Gamma^B_{15} & \Gamma^B_{16} & \cdots \\ \Gamma^B_{25} & \Gamma^B_{26} & \cdots \\ \Gamma^B_{35} & \Gamma^B_{36} & \cdots \\ \Gamma^B_{45} & \Gamma^B_{46} & \cdots \end{pmatrix} \tag{2.21}$$

The number of the off-diagonal operators is 20 and they are $\{O_5, O_6, \cdots\} = \{(S\otimes V)(S\otimes V)_{\mathbf{I}}, (S\otimes V)(S\otimes V)_{\mathbf{II}}, (S\otimes A)(S\otimes A)_{\mathbf{I}}, \cdots\}$. We assume that $Z \simeq Y^T \simeq \mathscr{O}(\alpha_s)$[b] and $W \simeq 1 + \mathscr{O}(\alpha_s)$. The inverse of block matrix is

$$(\hat{\Gamma}^B)^{-1} = \begin{pmatrix} (X - YW^{-1}Z)^{-1} & -X^{-1}Y(W - ZX^{-1}Y)^{-1} \\ -W^{-1}Z(X - YW^{-1}Z)^{-1} & (W - ZX^{-1}Y)^{-1} \end{pmatrix}. \tag{2.22}$$

[b]Note that this approximation is good within factor of 3.

Using power series expansion in Y and Z, it becomes as follows.

$$(\hat{\Gamma}^B)^{-1} \simeq \begin{pmatrix} X^{-1}+X^{-1}YW^{-1}ZX^{-1} & -X^{-1}Y(W^{-1}+W^{-1}ZX^{-1}YW^{-1}) \\ -W^{-1}Z(X^{-1}+X^{-1}YW^{-1}ZX^{-1}) & W^{-1}+W^{-1}ZX^{-1}YW^{-1} \end{pmatrix} \tag{2.23}$$

With our assumption, $Z \simeq Y^T$ and $W \simeq 1$,

$$(\hat{\Gamma}^B)^{-1} \simeq \begin{pmatrix} X^{-1}+X^{-1}YY^TX^{-1} & -X^{-1}Y(1+Y^TX^{-1}Y) \\ -Y^T(X^{-1}+X^{-1}YY^TX^{-1}) & 1+Y^TX^{-1}Y \end{pmatrix}. \tag{2.24}$$

3. Results

First, we present the data analysis of diagonal terms. Let us consider the element Γ^B_{11} of $\hat{\Gamma}^B$ matrix. We measure the data for 5 valence quark masses and 9 external momenta. The scale of raw data is determined by external momentum ($\mu = |\widetilde{p}|$). Hence, we convert the scale of raw data to the common scale $\mu_0 = 3\,\text{GeV}$ using two-loop RG evolution [6]. We fit the data with respect to quark mass for a fixed momentum to the following function f suggested from Ref. [7] based on the Weinberg theorem [8].

$$f(m,a,\widetilde{p}) = c_1 + c_2\cdot(am) + c_3\cdot\frac{1}{(am)} + c_4\cdot\frac{1}{(am)^2}, \tag{3.1}$$

where m is a valence quark mass. After m-fit, we take the $c_1(a\widetilde{p})$ as chiral limit values. Because of the sea quark determinant contributions ($c_3, c_4 \propto (m_\ell^2 m_s)^\nu$) with ν the number of zero modes, these pole terms contributions vanish in the chiral limit. The fitting results are presented in Table 1 and Fig. 1(a).

μ_0	c_1	c_2	c_3	c_4	χ^2/dof
3GeV	0.17991(20)	-0.0975(15)	0.0007118(85)	-0.000000790(36)	0.00194(40)

Table 1: m-fit

We fit $c_1(a\widetilde{p})$ to the following fitting function.

$$g(a\widetilde{p}) = b_1 + b_2\cdot(a\widetilde{p})^2 + b_3\cdot((a\widetilde{p})^2)^2 + b_4\cdot(a\widetilde{p})^4 + b_5\cdot((a\widetilde{p})^2)^3 \tag{3.2}$$

To avoid non-perturbative effects at small $(a\widetilde{p})^2 \le 1$, we choose the momentum window as $(a\widetilde{p})^2 > 1$. Because we assume that those terms of $\mathcal{O}((a\widetilde{p})^2)$ and higher order are pure lattice artifacts, we take the b_1 as Γ^B_{11} value at $\mu_0 = 3\,\text{GeV}$ in the RI-MOM scheme. The fitting result and plot are presented in Table 2 and Fig. 1(b).

μ_0	b_1	b_2	b_3	b_4	b_5	χ^2/dof
3GeV	1.088(16)	-0.515(18)	0.0953(74)	0.0020(65)	-0.00663(91)	0.08(17)

Table 2: p-fit

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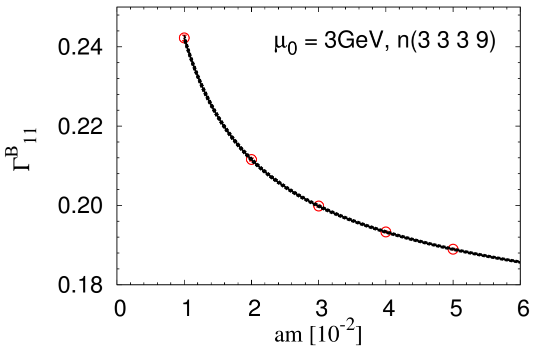
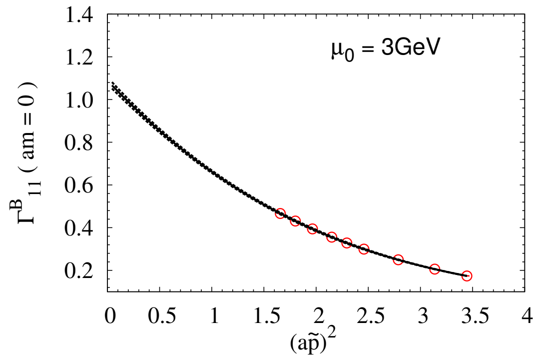


(a) m-fit



(b) p-fit

Figure 1: m-fit and p-fit plot at $\mu_0 = 3\,\text{GeV}$

Similarly, we analyse the whole elements of $\hat{\Gamma}^B$ matrix. Results of diagonal terms in the inverse of $\hat{\Gamma}^B$ are

$$X^{-1} = \begin{pmatrix} 1.333(32) & -0.793(45) & 0.336(21) & 0.007(31) \\ -0.726(44) & 1.940(41) & -0.008(30) & -0.047(33) \\ 0.341(30) & 0.032(45) & 1.222(32) & -0.604(37) \\ 0.018(45) & -0.084(52) & -0.637(39) & 1.543(36) \end{pmatrix} \tag{3.3}$$

$$X^{-1}YY^TX^{-1} = \begin{pmatrix} 0.0127(15) & -0.0079(11) & 0.0020(17) & -0.0001(10) \\ -0.00713(92) & 0.0064(11) & -0.0010(11) & -0.00045(73) \\ 0.0020(18) & 0.0000(15) & 0.0200(90) & -0.0103(48) \\ 0.0001(12) & -0.0012(11) & -0.0109(52) & 0.0059(28) \end{pmatrix} \tag{3.4}$$

We obtain $\vec{z}$ in RI-MOM scheme at $\mu_0 = 3\,\text{GeV}$. We convert the scheme from RI-MOM to $\overline{\text{MS}}$ using two-loop RG evolution. Results are summarized in Table 3.

	RI-MOM(3GeV)	$\overline{\text{MS}}$(3GeV)
z_1	0.9666(78)	0.9812(79)
z_2	1.095(30)	1.111(31)
z_3	0.9139(73)	0.9277(74)
z_4	0.898(28)	0.912(29)

Table 3: The z-factors in RI-MOM and $\overline{\text{MS}}$ scheme. Here, the errors are purely statistical.

Now let us switch the gear to the systematic errors. The first systematic error comes from the diagonal correction terms, $X^{-1}YW^{-1}ZX^{-1} \approx X^{-1}YY^TX^{-1}$. We quote $E_{diag} \equiv \vec{z}_{\text{tree}} \cdot X^{-1}YY^TX^{-1}$ as this error. The second systematic error comes from the off-diagonal correction terms. Their size $(-X^{-1}Y)$ are typically less than 7%. However, thanks to the wrong taste suppression ($\ll 1\%$) [9], their effect becomes $\ll 0.07\%$. Hence we neglect them without loss of generality. Another systematic error comes from truncated higher order of the two-loop RG evolution factor (RI-MOM→ $\overline{\text{MS}}$). We quote $E_t \equiv z_i \cdot \alpha_s^3$ as this error. We add these systematic errors (E_{diag} and E_t) in quadrature as

	$\overline{\text{MS}}$(3GeV)	E_{diag}	E_t	E_{tot}
z_1	0.9812(79)	0.0077	0.0144	0.0163
z_2	1.111(31)	0.0027	0.0163	0.0165
z_3	0.9277(74)	0.0101	0.0136	0.0171
z_4	0.912(29)	0.0050	0.0134	0.0143

Table 4: The systematic errors of z-factors in $\overline{\text{MS}}$ scheme at 3 GeV. E_{tot} represents the total systematic error.

summarized in Table 4.

In addition, we compare the NPR result($\overline{\text{MS}}$ [NDR]) with those of one-loop perturbative matching. We quote truncated two-loop uncertainty: $E_t^{\text{one-loop}} \equiv z_i \cdot \alpha_s^2$ as our estimate of the systematic error of one-loop matching. Here, note that the results of NPR are consistent with those of one-loop matching within 2σ. This indicates that our NPR results are quite reasonable.

	NPR(3GeV)	one-loop(3GeV)	Δ
z_1	0.981(8)(16)	1.035(62)	0.83 σ
z_2	1.111(31)(17)	1.120(67)	0.12 σ
z_3	0.928(7)(17)	1.043(63)	1.75 σ
z_4	0.912(29)(14)	0.953(57)	0.63 σ

Table 5: Comparison NPR results and one-loop perturbative matching factors at 3 GeV.

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References

[1] **Particle Data Group** Collaboration, K. Olive *et al., Chin.Phys.* **C38** (2014) 090001.

[2] J. Bailey, Y.-C. Jang, and W. Lee, *PoS* **LATTICE2014** (2014) 371.

[3] Y. Aoki, P. Boyle, N. Christ, C. Dawson, M. Donnellan, *et al., Phys.Rev.* **D78** (2008) 054510, [0712.1061].

[4] J. Kim, J. Kim, W. Lee, and B. Yoon, *PoS* **LATTICE2013** (2013) 308, [1310.4269].

[5] A. T. Lytle and S. R. Sharpe, *Phys.Rev.* **D88** (2013), no. 5 054506, [1306.3881].

[6] J. Kim, W. Lee, J. Leem, S. R. Sharpe, and B. Yoon, *Phys.Rev.* **D90** (2014) 014504, [1404.2368].

[7] T. Blum, N. H. Christ, C. Cristian, C. Dawson, G. T. Fleming, *et al., Phys.Rev.* **D66** (2002) 014504, [hep-lat/0102005].

[8] S. Weinberg, *Phys.Rev.* **118** (1960) 838–849.

[9] W. Lee, *UMI-96-06917-mc (microfiche) 1995.* Ph.D. Thesis (Columbia University).