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Testing volume independence of large N gauge theories on the lattice

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For a pure SU(N) gauge theory on the lattice we test if the expectation values of small Wilson loops become volume independent in the large N limit.

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1. The Question

Are gauge theories in the large N limit volume independent? Or put in different words: Are finite volume corrections subleading in the large N limit? These questions were advanced many years ago by Eguchi and Kawai [1] (EK), when they realized that, under certain hypothesis, the loop equations did not depend on the size of the box. Furthermore, the arguments were valid for the lattice model itself. No doubt that if the statement is true at all values of the lattice coupling, the result should also hold in the continuum limit.

For years the previous question has been debated and different ways have been put forward to try to realize this idea. A good deal of effort has been put into understanding whether the specific conditions on which Eguchi and Kawai proof is based were met. Most importantly there is the question of center symmetry: invariance under a Z(N) transformation of the Polyakov loops in each direction. The symmetry guarantees that the expectation values of (non-zero winding) Polyakov loops vanish, so that these observables act as order parameters.

Recently [2] we have analysed the problem in a direct way: computing lattice observables for different sizes and values of N and testing if the size dependence goes to zero at large N. The advantage of this idea is that we test volume independence directly and not indirectly via the conditions of EK proof. Hence, it allows to measure the size of the corrections, namely the finite volume dependence at finite N. From a practical viewpoint, this is basic information one needs when making use of the volume independence property. In this talk we will present a brief summary of our results.

2. The models

In Ref. [1] the simplest model with Wilson action in a periodic hypercubic lattice was adopted. Volume independence meant that the single-site model was equivalent to the infinite volume one at large N. This is certainly false as pointed out in Ref. [3]. The failure lies at weak coupling where perturbative methods are good approximations. The classical vacua of the model (generically) break the $Z^4(N)$ symmetry spontaneously. In a trivial fashion one can verify that the plaquette expectation value, for example, has a finite volume correction which does not go to zero at large N.

Several ideas have been put forward over the years to find valid implementations of the volume independence idea. One of the earliest proposals, introduced by the present authors, was based on a very simple modification of the original proposal [4, 5]. The point is to use twisted boundary conditions instead of periodic ones which is perfectly compatible with EK proof. Furthermore, it does not add any computational or fundamental complication to the EK proposal. With suitable choices of the twist tensor one can verify that at weak coupling centre symmetry is broken down to a subgroup large enough to preserve the validity of volume reduction at large N. In particular, if one opts for preserving as much as possible the isotropy among the directions of space-time, one should take $N = \hat{L}^2$ where $\hat{L}$ is an integer. Then the classical minima respect $Z^4(\hat{L})$ symmetry which ensures that all Polyakov loops with windings that are not multiples of $\hat{L}$ should vanish. This choice of twist called *symmetric twist*, depends only on a single integer k coprime with $\hat{L}$, which specifies the flux through all planes. Indeed, a direct calculation shows that to leading order in perturbation theory the volume dependent term of the plaquette vanishes exactly, and for other loops the correction vanishes at large N. In Ref. [5] we gave general arguments why this should also happen at higher orders of perturbation theory. A detailed analysis of the next-to-leading order is currently under way [6], which would allow us to quantify the rate at which volume independence is achieved as a function of N within the perturbative regime.

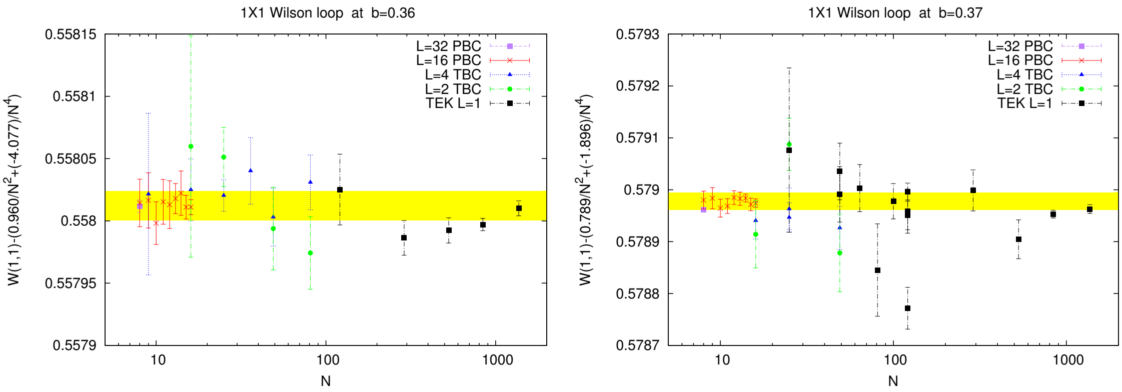
The previous paragraphs define our setting. We will be studing SU(N) lattice gauge theory with Wilson action on a finite box of size L^4 with symmetric twisted boundary conditions and flux k. We will also include periodic boundary conditions, which can be incorporated within the same formalism but with $k = 0$. The only continuous parameter in our study is the lattice coupling $b = \beta/(2N^2)$, the inverse of 't Hooft coupling λ on the lattice. Our study includes the single site ($L = 1$) twisted model, called Twisted Eguchi-Kawai model (TEK) [5], but we also allow for the corresponding models with $L = 2,4$. For the periodic boundary condition situation ($k = 0$) we have studied various sizes $L = 4,8,16,32$. This allows us to explore the regime studied by Narayanan and Neuberger [7]. These authors proposed that volume independence holds even for periodic boundary conditions beyond a certain threshold $L > L_c(b)$.

The choice of k could be crucial, even for values coprime with $\hat{L}$. Results obtained by several authors [8, 9, 10] showed that at certain values of the coupling b centre symmetry can break and invalidate the EK proof. This problem was analysed by the present authors in Ref. [11], where we concluded that to avoid the problem one should take the large N limit keeping $k/\hat{L}$ larger than a threshold value of ~ 0.1. Other problems associated with tachyonic instabilities require that the limit should be taken keeping $\bar{k}/\hat{L}$ large enough too. The integer $\bar{k}$ is defined in terms of k and $\hat{L}$ as the one satisfying $k\bar{k} = 1 \bmod \hat{L}$. These requirements match perfectly with corresponding conditions found in a detailed analysis of the $2+1$-dimensional case [12, 13]. This analysis explores the connection between finite volume and finite N effects. It seems that the main part of these effects combine into finite volume effects with an effective volume of $L_{\text{eff}} = L\hat{L}$. This result is compatible with volume independence in the large N limit, but goes beyond since it predicts that the main correction depends on L_{eff} (with coefficients that depend on $\bar{k}/\hat{L}$). A particular consequence is the possibility of replacing finite-size scaling by finite-N scaling for the model with $L = 1$, which seems to work remarkably well as seen in a presentation at this conference [14].

3. Results

We focused our study upon the behaviour of expectation values of small $R \times R$ Wilson loops, from $R = 1$ (the plaquette) up to $R = 4$. These observables are well measured lattice quantities. We explored the dependence of these values upon the parameters of the models: the coupling b, the matrix rank N, the lattice linear size L and the flux k. Here we will restrict to the results obtained in the region of relatively strong coupling $0.35 \leq b \leq 0.385$ which corresponds with that from which continuum limit results are usually extracted. Our results were obtained with standard Monte Carlo techniques. Computer resources limits the total number of degrees of freedom that can be studied. Thus, for $L = 32$ we could only go up to $N = 8$, for $L = 16$ up to $N = 16$ and for $L = 1$ up to $N = 1369$.

A systematic analysis was performed at two values of the coupling $b = 0.36$ and $b = 0.37$. We started by studying the system with periodic boundary conditions ($k = 0$). At fixed N, volume dependence is clearly observed in the expectation values for $L < 16$. The correction turns out to



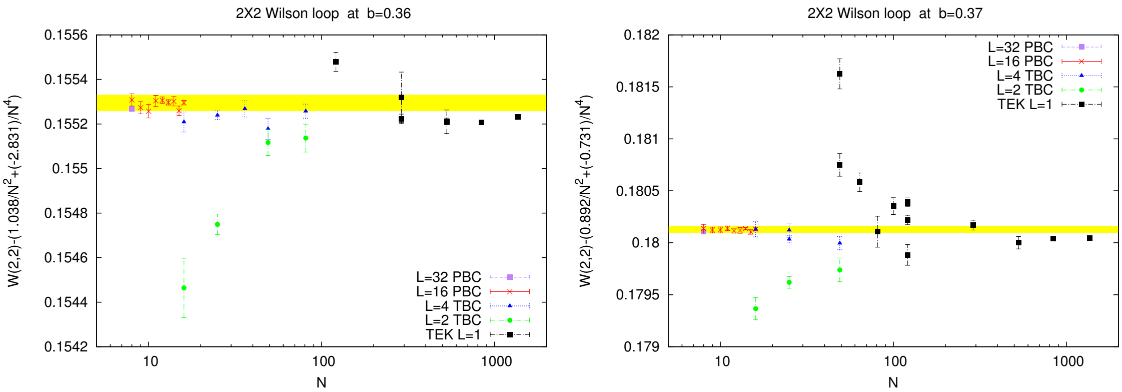
(a) (b)

Figure 1: The plaquette expectation value for $b = 0.36$ (left) and $b = 0.37$ (right) after subtracting a term $B/N^2 + C/N^4$ plotted as a function of N. The coefficients B and C (appearing in the y-label) are determined by fitting the N dependence of the periodic boundary conditions (PBC) $L = 16$ points (in red). The one-sigma region in the constant coefficient of the fit is shown as a yellow band. In addition, we plot the $N = 8$ $L = 32$ plaquette, and the $L = 1,2,4$ symmetric twist results; all with the same subtraction.

be positive for all our observables. On the other hand, the results at $L = 16$ are compatible within errors with those at $L = 32$ wherever the two were available. Thus, we can take the $L = 16$ results to be a good approximation to those at infinite volume. At $L = 16$ we performed simulations at all values of N in the interval $[8,16]$. The N dependence of the results is sizable at the scale of the errors. Thus, to obtain good estimates of the value of these observables at $N = \infty$, we should extrapolate our results. Good fits are obtained in all cases with a quadratic polynomial in $1/N^2$: $A + B/N^2 + C/N^4$. The correction is dominated by the $1/N^2$ correction with a coefficient B which is positive and of order 1 (dependent on b and R). For display purposes we subtracted $B/N^2 + C/N^4$ from the measured values and displayed them as the red points in Figs. 1-4. The yellow band gives the value of A within one sigma obtained from the fit. This value amounts to the determination of the corresponding observable at $N = \infty$. In the same plot we also display the measurement obtained for $L = 32$ and $N = 8$.

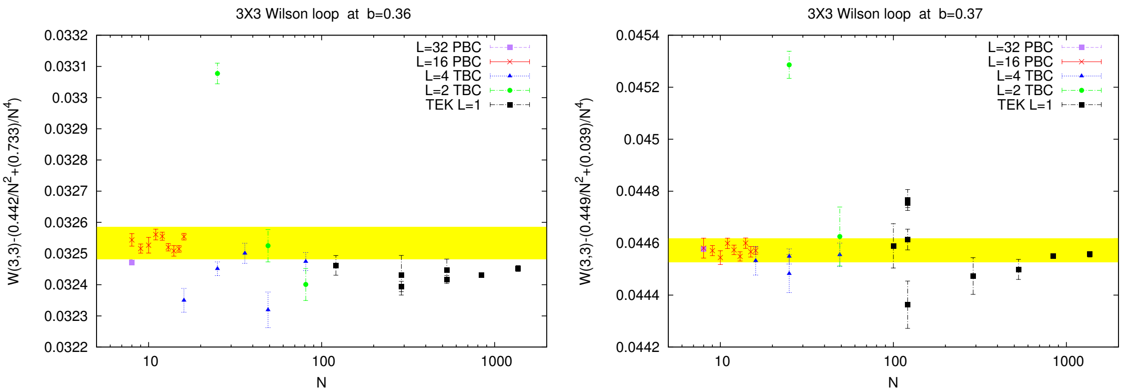
As mentioned earlier the expectation values obtained for $L = 8$ are significantly different than those of $L = 16$ however a similar fit for these data allows to extrapolate to $N = \infty$. Curiously the fit obtained for $b = 0.36$ gives a consistent extrapolation for all loops except $R = 4$. On the contrary the extrapolation for $b = 0.37$ is inconsistent with that of $L = 16$. Repeating the process for $L = 4$ (for which we could explore larger values of N) we found clearly inconsistent extrapolations in all cases. These results are in agreement with the conclusions of Ref. [7] since $L_c(0.37) > L_c(0.36) \sim 8$.

We also studied the results obtained with the symmetric twist and $L = 1,2,4$. The raw values are displayed in the same figs. 1-4 after subtracting $B/N^2 + C/N^4$ with B and C obtained from the $L = 16$ periodic data (no new fit). The results for $N > 300$ obtained with the single site TEK model are essentially unaffected by the subtraction. These results include the case of $N = 529$ ($\hat{L} = 23$, $k = 7$, $\bar{k} = 10$), $N = 841$ ($\hat{L} = 29$, $k = 9$, $\bar{k} = 13$), and $N = 1369$ ($\hat{L} = 37$, $k = 11$, $\bar{k} = 10$). The



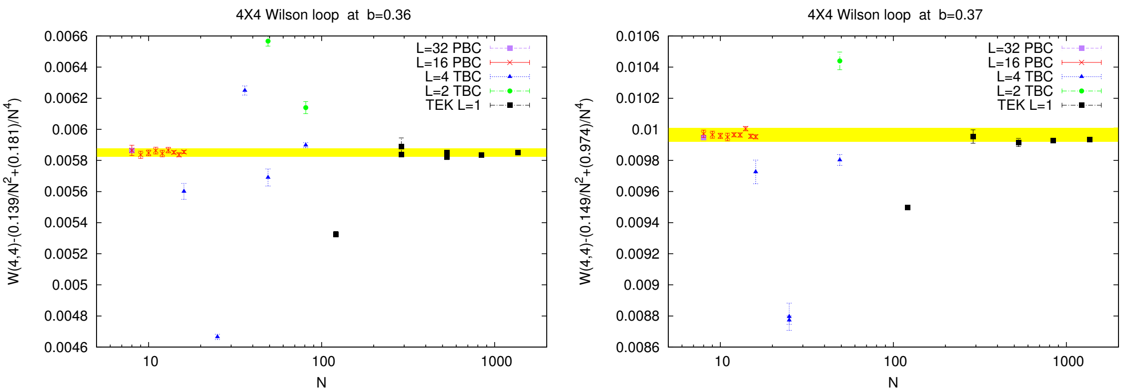
(a) (b)

Figure 2: The same as Fig. 1 but for the 2×2 Wilson loop.



(a) (b)

Figure 3: The same as Fig. 1 but for the 3×3 Wilson loop.



(a) (b)

Figure 4: The same as Fig. 1 but for the 4×4 Wilson loop.

results are compatible with each other and with the extrapolation of the $L = 16$ results. This is quite remarkable given the wide range of N values covered, the small errors of the data and the large relative finite N corrections to the $L = 16$ results. This provides a strong confirmation of volume independence in the extreme case, since we compare results in the infinite volume limit with those of $L = 1$. Obviously twisted boundary conditions are essential for this result. Notice that the values of the flux k lie within the safe region. In some cases (for example the 3×3 loop at $b = 0.36$) the TEK data lies below the yellow band, although these deviations are smaller than 2 sigma. In any case, we believe that the TEK value is a more reliable estimate of the large N result than the extrapolated result, given the characteristic systematic errors of extrapolations.

In Figs. 1-4 we also plot the results obtained for smaller values of N at $L = 1, 2, 4$ and symmetric twisted boundary conditions applying the same subtraction as before. We plotted all the available results which matched the criteria $\frac{\bar{k}}{\hat{L}} > 0.25$ and $L_{\mathrm{eff}} = \hat{L}L > 3R$. Extending the region, the deviations grow and the scale of the plot has to be reduced accordingly. Two comments are in order. The first is that there is no reason why the $1/N^2$ corrections of the twisted data should coincide with those of the $L = 16$ periodic data. The second is that these corrections might depend on the value of k. However, the fact that all the results fall within the scale of the plot implies that the corrections are not terribly different from the periodic ones as long as we stay within the safe k and $\bar{k}$ range. This is particularly remarkable for the plaquette for which almost all the data fit nicely within the extrapolated band. We do not have enough information for a more systematic study of the k dependence of the corrections, but that is not necessary for the specific goals of this work.

We have performed less extensive tests at other values of b and found results consistent with those of $b = 0.36$ and 0.37. Furthermore, we have been able to find interpolating formulas that allow us to use all the available data both for large sizes and periodic boundary conditions and for the TEK model. All the data are consistent with the latter matching the infinite N limit of the former. The full results can be consulted in Ref. [2]

4. Conclusions

We have analyzed Wilson loop expectation values for various values of the lattice size L and N for both periodic and symmetric twisted boundary conditions. The results obtained by large N extrapolation and large volumes match within the tiny errors of the data with those obtained with the $L = 1$ TEK model. This provides a strong direct evidence supporting volume independence at large N.

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