

Critical behavior and continuum scaling of $3D Z(N)$ lattice gauge theories

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Three-dimensional $Z(N)$ lattice gauge theories are studied numerically at finite temperature for $N = 5, 6, 8, 12, 13, 20$ and for $N_t=2,4,8$. For each model the location of phase transitions and its critical indices are determined. The scaling of critical points with N is proposed. The data obtained enable us to verify the scaling near the continuum limit for the $Z(N)$ models at finite temperatures.

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1. Introduction

In this paper we focus on the phase structure of $Z(N)$ lattice gauge theories (LGTs), which are interesting on their own and can provide for useful insights into the universal properties of $SU(N)$ LGTs, being $Z(N)$ the center subgroup of $SU(N)$. The most general action for the $Z(N)$ LGT is

$$S_{\text{gauge}} = \sum_x \sum_{n < m} \sum_{k=1}^N \beta_k \cos \left(\frac{2\pi k}{N} (s_n(x) + s_m(x + e_n) - s_n(x + e_m) - s_m(x)) \right). \quad (1.1)$$

Gauge fields take on values $s_n(x) = 0, 1, \dots, N-1$ and are defined on the links of the lattice. $Z(N)$ gauge models can generally be divided into two classes: *standard Potts* models (all β_k equal) and *vector* models (otherwise). The conventional vector model corresponds to $\beta_k = 0$ for all $k > 1$. For $N = 2, 3$ the Potts and vector models are equivalent.

An extended description of the phase structure of $Z(N)$ LGTs in three dimension can be found in [1, 2, 3]. In those papers we explored the phase structure of the vector $Z(N)$ LGTs for $N > 4$. We considered first an anisotropic lattice in the limit where the spatial coupling vanishes [1] and were able to present both renormalization group (RG) and numerical evidences for the existence of two BKT-like phase transitions: a (i) *first transition*, from a symmetric, confining phase to an intermediate phase, where the $Z(N)$ symmetry is enhanced to $U(1)$ symmetry; (ii) a *second transition*, from the intermediate phase to a phase with broken $Z(N)$ symmetry. We computed also some critical indices, which appear to agree with the corresponding indices of 2D $Z(N)$ spin models, thus giving further support to the Svetitsky-Yaffe conjecture [4]. In particular, we found that the magnetic critical index η at the first transition, $\eta^{(1)}$, takes the value $1/4$ as in 2D XY , while its value at the second transition, $\eta^{(2)}$, is equal to $4/N^2$. Then, we extended our analysis to the full isotropic 3D $Z(N)$ LGT at finite temperature [2] and confirmed by numerical Monte Carlo simulations [2] that the full gauge models with $N > 4$ possess two phase transitions of the BKT type, with critical indices coinciding with those of 2D vector spin models.

Here we extend the study of Ref. [2] to other values of N and to $N_t = 8$ and aim at checking the scaling near the continuum limit and at establishing the scaling formula for critical points with N . In particular, the theory of dimensional cross-over [5] explains how critical couplings and indices of a finite temperature LGT (finite N_t) approach critical couplings and indices of the corresponding zero-temperature theory ($N_t \rightarrow \infty$). This provides us with a way to crosscheck our zero-temperature results [3] and thus predict the critical temperature in the continuum limit.

The standard approach for studying a BKT transition consists in using Binder cumulants and susceptibilities of the Polyakov loop to determine critical couplings and critical indices. Here, as in Ref. [2], we follow a different strategy: we move to a dual formulation and use Binder cumulants and susceptibilities of *dual $Z(N)$ spins*. This implies that (i) the critical behavior of dual spins is reversed with respect to that of Polyakov loops, namely the spontaneously-broken ordered phase is mapped to the symmetric phase and *vice versa*; (ii) the magnetic critical indices η are interchanged, whereas the index ν is expected to be the same ($=1/2$) at both transitions (see Ref. [2] for details). The obvious advantage of this approach is that cluster algorithms become available, with considerable speed up in the numerical procedure.

2. Theoretical setup

The 3D $Z(N)$ gauge theory on an anisotropic 3D lattice Λ can generally be defined as

$$Z(\Lambda; \beta_t, \beta_s; N) = \prod_{l \in \Lambda} \left(\frac{1}{N} \sum_{s(l)=0}^{N-1} \right) \prod_{p_s} Q(s(p_s)) \prod_{p_t} Q(s(p_t)), \quad (2.1)$$

where the link angles $s(l)$ are combined into the conventional plaquette angle

$$s(p) = s_n(x) + s_m(x + e_n) - s_n(x + e_m) - s_m(x). \quad (2.2)$$

Here, e_n ($n = 0, 1, 2$) denotes a unit vector in the n -th direction and the notation p_t (p_s) stands for the temporal (spatial) plaquettes. Periodic boundary conditions (BC) on gauge fields are imposed in all directions. The most general $Z(N)$ -invariant Boltzmann weight with $N - 1$ different couplings is

$$Q(s) = \exp \left[\sum_{k=1}^{N-1} \beta_p(k) \cos \frac{2\pi k}{N} s \right]. \quad (2.3)$$

The Wilson action corresponds to the choice $\beta_p(1) = \beta_p$, $\beta_p(k) = 0, k = 2, \dots, N - 1$, which is the one adopted in this work. Furthermore, we will consider an isotropic lattice: $\beta_s = \beta_t = \beta$.

Our study is based on the mapping of the gauge model to a generalized 3D $Z(N)$ spin model on a dual lattice Λ_d , whose action is

$$S = \sum_x \sum_{n=1}^3 \sum_{k=1}^{N-1} \beta_k \cos \left(\frac{2\pi k}{N} (s(x) - s(x + e_n)) \right). \quad (2.4)$$

The dual mapping is realized once one specifies the relationship between the original gauge coupling β and the dual effective couplings β_k . This has been done in Ref. [2] (see also Ref. [6]) and the result is

$$\beta_k = \frac{1}{N} \sum_{p=0}^{N-1} \ln \left[\frac{Q_d(p)}{Q_d(0)} \right] \cos \left(\frac{2\pi pk}{N} \right). \quad (2.5)$$

For $N = 5$ it can be seen explicitly [2] that $|\beta_1| \gg |\beta_2|$, thus suggesting that the 3D vector spin model with only β_1 non-vanishing gives already a reasonable approximation of the gauge model. Moreover the weak and the strong coupling regimes are interchanged, *i.e.* when $\beta \rightarrow \infty$ the effective couplings $\beta_k \rightarrow 0$ and, therefore, the ordered symmetry-broken phase is mapped to a symmetric phase with vanishing magnetization of dual spins, whereas the symmetric phase at small β becomes an ordered phase where the dual magnetization is non-zero. The interchange of phases under the dual mapping is not a special feature of $N = 5$, but is rather a general property valid for any N . In Ref. [2] it was also discussed that at the critical point $\beta_c^{(1)}$ of the first transition of the LGT (from the symmetric to the intermediate phase), the *dual* correlation function scales with a critical index η equal to the index $\eta^{(2)} = 4/N^2$ of the Polyakov loop correlator in the LGT, while at the critical point $\beta_c^{(2)}$ of the second transition in the LGT (from the intermediate to the broken phase), it scales with a critical index η equal to the index $\eta^{(1)} = 1/4$ of the Polyakov loop correlator in the LGT. This can be proved in the Villain formulation of the 2D theory and only conjectured (but confirmed numerically) in the 3D case [2].

Table 1: Values of $\beta_c^{(1)}$ and $\beta_c^{(2)}$ obtained for various N_t in 3D Z(N) with $N = 5, 6, 8, 12, 13$ and 20.

N	N_t	$\beta_c^{(1)}$	$\beta_c^{(2)}$
5	2	1.617(2)	1.6972(14)
5	4	1.943(2)	1.9885(15)
5	6	2.05(1)	2.08(1)
5	8	2.085(2)	2.1207(9)
5	12	2.14(1)	2.16(1)
6	2	-	2.3410(15)
6	4	-	2.725(12)
6	8	-	2.899(4)
8	2	-	3.8640(10)
8	4	2.544(8)	4.6864(15)
8	8	3.422(9)	4.9808(5)

N	N_t	$\beta_c^{(1)}$	$\beta_c^{(2)}$
12	2	-	8.3745(5)
12	4	-	10.240(7)
12	8	-	10.898(5)
13	2	1.795(4)	9.735(4)
13	4	2.74(5)	11.959(6)
13	8	3.358(7)	12.730(2)
20	2	-	22.87(4)
20	4	2.57(1)	28.089(3)
20	8	3.42(5)	29.758(6)

3. Numerical setup and results

The 3D Z(N) spin model, dual of the 3D Z(N) Wilson LGT, has been simulated by means of a cluster algorithm on $N_t \times L \times L$ lattices with periodic BC. The system has been studied for $N = 5, 6, 8, 12, 13$ and 20 on lattices with the temporal extension $N_t=2, 4, 8$. With respect to our previous work [2], we considered new values of N (6, 8, 12, 20) and included also $N_t = 8$. We focused on the following observables:

- complex magnetization $M_L = |M_L|e^{i\psi}$, with $M_L = \sum_{x \in \Lambda} \exp\left(\frac{2\pi i}{N}s(x)\right)$, where we stress that $s(x)$ is a dual spin variable;
- real part of the rotated magnetization, $M_R = |M_L|\cos(N\psi)$, and normalized rotated magnetization, $m_\psi = \cos(N\psi)$;
- susceptibilities of M_L and M_R , $\chi_L^{(M)}$, $\chi_L^{(M_R)}$: $\chi_L^{(\cdot)} = L^2 N_t \left(\langle \cdot^2 \rangle - \langle \cdot \rangle^2 \right)$;
- Binder cumulants $U_L^{(M)}$ and $B_4^{(M_R)}$: $U_L^{(M)} = 1 - \frac{\langle |M_L|^4 \rangle}{3 \langle |M_L|^2 \rangle^2}$, $B_4^{(M_R)} = \frac{\langle |M_R - \langle M_R \rangle|^4 \rangle}{\langle |M_R - \langle M_R \rangle|^2 \rangle^2}$.

To determine the critical couplings of the *second transition point*, $\beta_c^{(2)}$, we have looked for the value of β at which the curves giving the Binder cumulant $U_L^{(M)}(\beta)$ on lattices with different size L “intersect” (see Ref. [7] for details). The same method can in principle be used for the couplings of the *first transition*, $\beta_c^{(1)}$, using either the Binder cumulant $B_4^{(M_R)}$ or m_ψ ; it turned out, however, that the precision required by this method on these observables could not be met with a sensible simulation time. For this reason, as the position of the first critical point we used our previous determinations given in Ref. [2], where $\beta_c^{(1)}$ was taken as the value of β at which $B_4^{(M_R)}$ and m_ψ plotted *versus* $(\beta - \beta_c^{(1)})\ln L^{1/\nu}$ show the best overlap for different values of L . The results of the determinations of $\beta_c^{(1)}$ and $\beta_c^{(2)}$ are summarized in Table 1.

For the critical couplings at the second transition, $\beta_c^{(2)}$, where determinations for many values of N are available, we tried to find a simple scaling dependence with N at fixed N_t . From the

Table 2: Parameters of the scaling with N of the second transition point, $\beta_c^{(2)} = A/(1 - \cos 2\pi/N) + B(1 - \cos 2\pi/N)$ at fixed N_t .

N_t	A	B	χ_r^2
2	1.1194(11)	0.141(24)	209
4	1.37440(60)	-0.0046(88)	18.2
8	1.45745(57)	0.0155(53)	16.1

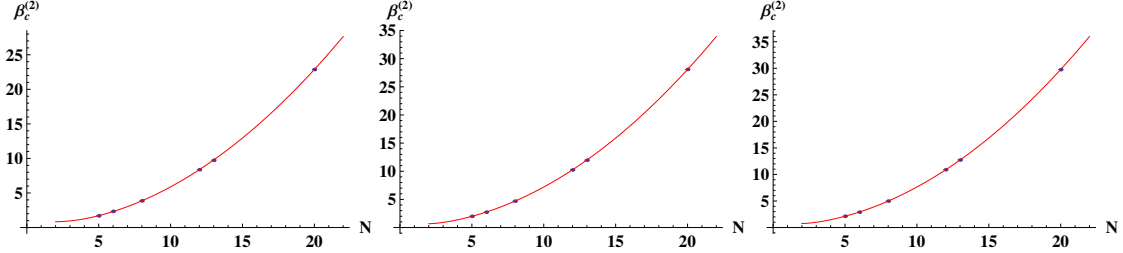


Figure 1: Scaling function $A/(1 - \cos 2\pi/N) + B(1 - \cos 2\pi/N)$ versus data for $\beta_c^{(2)}$ at $N_t = 2, 4, 8$ (from left to right).

solution of the renormalization group equations for 2D Z(N) spin model, we know that in that model $\beta_c^{(2)}(N)$ grows as N^2 for large N [8]. In [1] we have found that this is the case also for the 3D Z(N) LGT at finite temperature, at least in the strong coupling limit. Taking inspiration from Ref. [9], we started from a scaling law written in the form $\beta_c^{(2)}(N) = A/(1 - \cos 2\pi/N)$. Then, considering that the next non-negligible correction comes at the order $1/N^2$, we added a second term and ended up with the same scaling function we used in the zero-temperature case [3],

$$\beta_c^{(2)}(N) = \frac{A}{(1 - \cos 2\pi/N)} + B(1 - \cos 2\pi/N).$$

In Table 2 we report the values of the parameters A and B for $N_t = 2, 4, 8$, while Figs. 1 shows the fitting functions against numerical data.

Finding the continuum limit of the finite temperature theory in the first or in the second transition amounts to extrapolate the corresponding critical couplings, $\beta_c^{(1)}$ or $\beta_c^{(2)}$, to the limit $N_t \rightarrow \infty$ at fixed N . The theory of dimensional cross-over [5] suggests the fitting function to be used:

$$\beta_c^{(1,2)}(N_t) = \beta_{c, T=0}^{(1,2)} - (N_t a T_c)^{-1/\nu}, \quad (3.1)$$

where $\beta_{c, T=0}^{(1,2)}$ and ν are the critical couplings and the critical index of the zero-temperature theory. Since we know that, for any N , the 3D Z(N) LGT at zero-temperature exhibits only one phase transition, with the critical index ν depending on the side from which the transition is approached [3], we expect that, for a given N , the fit parameters $\beta_{c, T=0}^{(1)}$ and $\beta_{c, T=0}^{(2)}$ take the same value and agree with the zero-temperature critical coupling at the same N . As for the fit parameter ν , we expect it to agree with the value of the critical index ν at one of the two sides of the zero-temperature transition. We fitted with the function given in (3.1) our data for the critical couplings $\beta_c^{(1)}(N_t)$ at $N=5$ and for the critical couplings $\beta_c^{(2)}(N_t)$ at $N=5, 6, 8, 12, 13, 20$ (see Table 3). In some

Table 3: Results of the fit of $\beta_c^{(1)}(N_t)$ for $N =$ and of $\beta_c^{(2)}(N_t)$ for $N=5, 6, 8, 12, 13, 20$ with the function (3.1). Parameters are given without errors when their values were fixed at the known results of the $T = 0$ corresponding theory [3] (for the ν index we considered both the values at the left and at the right of the $T = 0$ critical point). Parameters are given with a (-) mark when their errors are unavailable and with a * mark when obtained from fits on data with $N_t = 4, 8$ only (in general, $N_t = 2, 4, 8$ were considered).

N	aT_c	$\beta_{c, T=0}^{(1)}$	ν	χ_r^2
5	0.790(5)	2.198(9)	0.84(3)	1.21
	0.764(14)	2.144(9)	0.670	23.1
	0.758(16)	2.135(11)	0.640	33.6
	0.786(7)	2.17961	0.788(10)	2.66
	0.722(16)	2.17961	0.670	105
	0.709(19)	2.17961	0.640	171

N	aT_c	$\beta_{c, T=0}^{(2)}$	ν	χ_r^2	N	aT_c	$\beta_{c, T=0}^{(2)}$	ν	χ_r^2
5	0.868(-)	2.23055(-)	0.877(-)	-	12	0.24728(-)	11.2566(-)	0.674(-)	-
	0.813(27)	2.177(12)	0.670	158		0.24559(13)	11.2640(23)	0.670	0.22
	0.803(30)	2.170(14)	0.640	223		0.25615(72)	11.218(12)	0.640	6.18
	0.825(38)	2.17961	0.692(45)	131		0.2602(32)	11.1962	0.630(11)	14.2
	0.810(13)	2.17961	0.670	81.8		0.24954(28)	11.1962	0.670	89.8
	0.776(31)*	2.17961	0.670	74.2*		0.2619(87)*	11.1962	0.670	55.5*
	0.789(17)	2.17961	0.640	161		0.25742(10)	11.1962	0.640	12.7
	0.731(18)*	2.17961	0.640	31.4*		0.2597(51)*	11.1962	0.640	21.3*
6	0.6814(-)	3.04317(-)	0.876(-)	-	13	0.22433(-)	13.1391(-)	0.654(-)	-
	0.6769(76)	2.977(10)	0.674	5.02		0.21872(53)	13.1656(56)	0.671	5.88
	0.6740(85)	2.969(12)	0.642	6.90		0.22851(40)	13.1199(42)	0.642	3.40
	0.6832(46)	3.00683	0.768(15)	1.14		0.2310(12)	13.1077	0.635(4)	8.86
	0.6573(47)	3.00683	0.674	22.6		0.2225(30)	13.1077	0.671	314
	0.572(13)*	3.00683	0.674	1.44*		0.2342(62)*	13.1077	0.671	113*
	0.6487(60)	3.00683	0.642	40.6		0.22928(67)	13.1077	0.642	16.0
	0.542(21)*	3.00683	0.642	4.48*		0.2311(24)*	13.1077	0.642	19.2*
8	0.42330(-)	5.14422(-)	0.674(-)	-	20	0.144857(-)	30.5427(-)	0.608(-)	-
	0.42378(12)	5.14299(25)	0.672	0.19		0.1297(37)	30.73(10)	0.673	147
	0.4316(22)	5.1225(46)	0.637	66.5		0.1356(24)	30.658(64)	0.647	58.8
	0.4294(12)	5.12829	0.648(6)	33.0		0.1357(26)	30.6729	0.642(19)	58.2
	0.4287(39)	5.12829	0.672	321		0.13171(98)	30.6729	0.673	97.3
	0.4427(39)*	5.12829	0.672	177*		0.13199(13)*	30.6729	0.673	1.57*
	0.4298(19)	5.12829	0.637	86.1		0.13506(54)	30.6729	0.647	31.0
	0.4216(10)*	5.12829	0.637	2.21*		0.13519(49)*	30.6729	0.647	23.9*

cases in the fit we fixed either $\beta_{c, T=0}^{(1,2)}$ or ν , or both, at the values known from the zero-temperature theory [3]. The scenario which emerges from the inspection of Tables 3 is that, despite the large reduced chi-squared obtained in a few cases, the agreement between the fit parameters $\beta_{c, T=0}^{(1,2)}$ and the known zero-temperature critical couplings [3] is satisfactory. As for the value of the fit parameter ν , results are not precise enough to discriminate between the known values of the critical index ν of the zero-temperature theory at one or the other side of the transition [3]. This analysis allows us for the determination of the critical temperature aT_c in the continuum limit for all the values of N considered in this work.

Some critical indices at the two transitions in the 3D $Z(N)$ LGT at finite temperature can be extracted by the standard FSS analysis. In particular, the behavior on the lattice size L of the standard magnetization M_L and of its susceptibility at the second transition allows to extract the indices β/ν and γ/ν through a fit with the functions

$$M_L = A_{M_L} L^{-\beta/\nu}, \quad \chi_{M_L} = A_{\chi_{M_L}} L^{\gamma/\nu}. \quad (3.2)$$

Similarly, the behavior on L of the rotated magnetization M_R and of its susceptibility at the first transition point allow the extraction of the same critical indices at that transition. Thereafter, the hyperscaling relation $2\beta/\nu + \gamma/\nu = 2$ can be checked and the magnetic index $\eta = 2 - \gamma/\nu$ can be extracted at both transitions. Our results are reported in Ref. [7] and show that the hyperscaling relation is generally satisfied and the critical index η generally takes values compatible with $1/4$ at the second transition and with $4/N^2$ at the first transition, in agreement with the expectations.

4. Summary

This paper completes our study of the critical behavior of 3D $Z(N > 4)$ lattice gauge theories both at finite temperatures. We have found that in all $Z(N)$ vector models two BKT-like phase transitions occur at finite temperatures if $N > 4$. In all cases studied, the results for the critical indices suggest that finite-temperature $Z(N)$ lattice models belong to the universality class of two-dimensional $Z(N)$ vector spin models, in agreement with the Svetitsky-Yaffe conjecture. Furthermore, the available results for many values of N allowed us to propose and check some scaling formulas for the critical point of the second phase transition. Combining the results of the present paper with those for the index ν obtained by us at zero temperature in Ref. [3] enabled us to check the continuum scaling and to predict the approximate value for aT_c in the continuum limit.

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