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Quantum Finite Elements: 2D Ising CFT on a Spherical Manifold

Richard C. Brower[*†]
Department of Physics
Boston University, 590 Commonwealth Ave, Boston, MA 02215, USA
E-mail: brower@bu.edu

Michael Cheng
Center for Computational Science
Boston University, 3 Cummington Mall, Boston , MA 02215, USA
E-mail: micheng@bu.edu

George T. Fleming
Department of Physics
Yale University, Sloane Laboratory, New Haven, CT 60520, USA
E-mail: George.fleming@yale.edu

Quantum field theories on curved manifolds have many interesting applications, but viable non-perturbative methods are difficult. Here we propose a lattice field theory method, we refer to as the Quantum Finite Element Method (QFEM), which adapts features from both the traditional finite element methods (FEM) and simplicial Regge calculus. A critical quantum component is the need to introduce counter terms in the classical FEM Lagrangian to cancel the ultraviolet distortions in the simplicial lattice and allow the continuum limit to approach the renormalized quantum theory. To test QFEM, we report initial simulations of the 2D $\lambda\phi^4$ on the simplicial lattice of the Riemann sphere compared to the analytical solution at the Wilson-Fisher (or 2D Ising) fixed point.

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[*]Speaker.

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1. Introduction

Lattice field theory provides a powerful *ab intio* approach to quantum field theory. However present practices are generally restricted to flat Euclidean space $\mathbb{R}^D$, employing a uniform cut-off on hypercubic lattices and finite difference discretization of the continuum Lagrangian. Nonetheless there are problems that require non-perturbative methods on nontrivial manifolds. One example, the use of radial quantization to study conformal behavior in field theory, lies at the heart of many challenging theoretical and phenomenological problems. Models for possible strong dynamics for electro-weak symmetry breaking as a replacement of the elementary Higgs of the Standard Model are often built on near-conformal theories. In a recent paper Brower, Fleming and Neuberger(BFN) suggested replacing the traditional Euclidean lattice in favor of one suited to *Radial Quantization* [1] . The potential advantage of a radial lattice is now the dilatation operator generates translations in $\log r$ so a finite radial lattice separates scales *exponentially* in the number of lattice sites.

Radial quantization replaces the Euclidean $\mathbb{R}^D$ manifold by the cylinder on $\mathbb{R}\times\mathbb{S}^{D-1}$. The transformation starts by a conformal diffeomorphism of the Euclidean field theory on $\mathbb{R}^D$ to radial coordinates,

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = r_0^2 e^{2t}(dt^2 + d\Omega_{D-1}^2)\ , \tag{1.1}$$

where $t = \log(r/r_0)$, introducing an arbitrary reference scale r_0. The crucial next step, permitted in conformal field theories, is the Weyl scaling, $g_{\mu\nu}(x) \to b^2(x)g_{\mu\nu}(x)$, canceling the factor e^{2t}. The new manifold $\mathbb{R}\times\mathbb{S}^{D-1}$ has a D-1 dimensional sphere with fixed radius and non-zero curvature for $D > 2$. The challenge for lattice field theory on a sphere, unlike $\mathbb{R}^D$, is the lack of an infinite sequence of lattices accommodating a finite subgroup of the isometries of the manifold with increasing fidelity. For example the largest subgroup of $\mathbb{S}^2$ is the isometries of the icosahedron. There is no position independent uniform cut-off approaching the continuum limit.

To explore radial lattice quantization for $D > 2$, BFN [1] simulated the Wilson-Fisher fixed point for the 3D Ising model by placing it on a simplicial cylindrical lattice with s^2 equilateral triangular mesh refinement of the 20 fundmental faces of an **icosahedron** as an approximation to $\mathbb{S}^2$. While the numerical results were in general encouraging, a small breaking of conformal symmetry was observed for the third descendant even as one approached the continuum. To remove this discrepancy they recommend using finite element methods (FEM) [2] for the ϕ^4 theory and a first implementation was presented by Brower, Cheng and Fleming [3]. A similar approach has been proposed by Neuberger [4].

Simplicial Lattice Method: In the classic finite element method, the central idea is to replace the continuum field in the Lagrangian by an expansion over a basis of compact elements on a simplicial complex: e.g. $\phi(x) = \sum_i \phi_i W_i(x)$. The Regge calculus [5] also introduces a simplicial expansion as discrete representation of the ensemble of geometries. Although the two formalisms have many common features, traditionally they pursue very different goals. In the finite element method, to ensure convergence to the exact solution of partial differential equations, the simplicial complex must be constrained to be and infinite mesh refinement sequence of "shape regular" simplices, whereas in the Regge calculus there are sums over random set of simplicies, weighted by the Einstein action to explore a discrete approximation to quantum gravity. In our quantum field thory application the FEM "shape regular" constraint does indeed improve the infrared behavior [3], but as reported below this still fails to give the correct continuum limit of the quantum path integral, due to ultraviolet (UV) divergences which probe the lattice at the cut-off. In this article we trace these steps employing the ϕ^4 on a sphere as an illustrative example. Two solutions to correct the Lagrangian in the UV are given, a procedure we designate as Quantum Finite Element Method (**QFEM**): The first introduces a Puali-Villars (PV) field and the second determines an explicit QFEM counter term .

2. Geometry of FEM

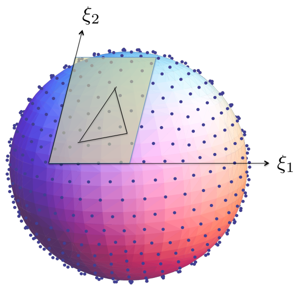
To be concrete, consider the Euclidean action for the ϕ^4theory on a smooth manifold,

$$S = \int d^D x \sqrt{-g}[\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \lambda (\phi^2 - \frac{\mu^2}{2\lambda})^2] \,. \tag{2.1}$$

For radial quantization this manifold will be $\mathbb{R} \times \mathbb{S}^{D-1}$. On the 2D sphere, we construct our simplicial lattice starting with a s^2 equilateral triangular refinement of the 20 faces of the icosahedron radially projected to the unit sphere. As a consequences each simplex becomes a distorted spherical triangle. We introduce barycentric co-ordinats, $\xi_1 + \xi_2 + \xi_3 = 1$, on each triangle and finite elements given by

$$E_k(x) = \xi^k \equiv A_{ij}(x)/A_{123} \tag{2.2}$$

as illustrated in Fig. 1, where the interior point, $\vec{x}(\xi) = \xi^1 \vec{x}_1 + \xi^2 \vec{x}_2 + \xi^3 \vec{x}_3$, breaks each simplex into 3 subtriangles with areas: A_{12},A_{23},A_{31}.



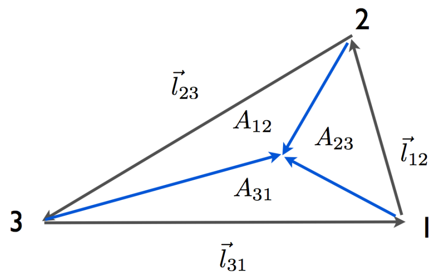


Figure 1: . On the left, the coordinates on the sphere x^μ for each spherical triangle are mapped on to the tangent plane with barycentrc coordinates $\xi^a(x)$. On the right, the geometry of triangular simplex: The vectors on the slides are $\vec{l}_{ij} = \vec{x}_j - \vec{x}_i$ where $\vec{l}_{12} + \vec{l}_{23} + \vec{l}_{32} = 0$.

Now the field $\phi(x)$ is approximated by expanding in the elements, with compact support on each triangle (i, j, k). Namely for the (1,2,3) triangle we have

$$\phi(x) = \phi_1 E_1(x) + \phi_2 E_2(x) + \phi_3 E_3(x) \equiv \xi^1 \phi_1 + \xi^2 \phi_2 + \xi^3 \phi_3 \tag{2.3}$$

where the coefficients of the expansion are $\phi(\vec{x}_i) = \phi_i$. For simplicity the areas on each triangle are approximated by area of a flat triangle [1]. It is also straightforward in principle to conform to

[1]The FEM literature has a vast repertoire of approximations. A google search of "Finite Elements Method" gives 2.7 Million hits! This simples linear element, used in Regge calculus [5] and in the classic paper by Christ, Friedberg and T. D. Lee [6] is adequate for our current problem with scalar fields.

the geometry of a spherical manifold, using the area rule for spherical triangle in Eq. 2.2. For flat subtriangles, a direct (brute force) evaluation of the integral over a single simplex gives,

$$S_{\triangle_{123}} = A_{123} \iint_{\xi^1+\xi^2 \le 1} d\xi^1 d\xi^2\, \vec{\nabla}\phi(\xi^i)\cdot\vec{\nabla}\phi(\xi^i) = \frac{l_{31}^2 + l_{32}^2 - l_{12}^2}{16A_{123}}(\phi_1-\phi_2)^2 + (23) + (31)$$
$$= \frac{1}{2}\sum_{\langle i,j\rangle} A^D_{ij}\Big(\frac{\phi_i-\phi_j}{l_{ij}}\Big)^2 , \tag{2.4}$$

using the identity $l_{12}^2(l_{31}^2 + l_{23}^2 - l_{12}^2) = 16A_{123}A^D_{12}$, where A^D_{ij} is the dual (Voronoi) area associated with the edge $\langle i,j\rangle$.

A more elegant geometric interpretation follows the Regge calculus approach. Choose one vertex $\vec{x}_k$ (for example k = 3 with $\xi^3 = 1-\xi^1-\xi^2 \to 0$) and define the local vierbeins and metric tensor

$$e^\mu_i(k) = \frac{\partial x^\mu}{\partial \xi^i} \quad , \quad g(k)_{ij} = e^\mu_i(k) e^\mu_j(k) \quad i \neq k\,. \tag{2.5}$$

It follows immediately that $S_{\triangle 123} = \frac{1}{2}\sqrt{g_k} g_k^{ij}(\phi_i-\phi_k)(\phi_j-\phi_k)$, where $\sqrt{g_k} = 2A_{123}$ and $g^{ij}_k = g^{-1}(k)_{ij}$. In spite of the lack of explicit permutation symmetry, this result is in fact identical to Eq. 2.4. The geometrical approach has other strengths, extending to a discrete differential calculus and illuminating intrinsic vs extrinsic geometry.

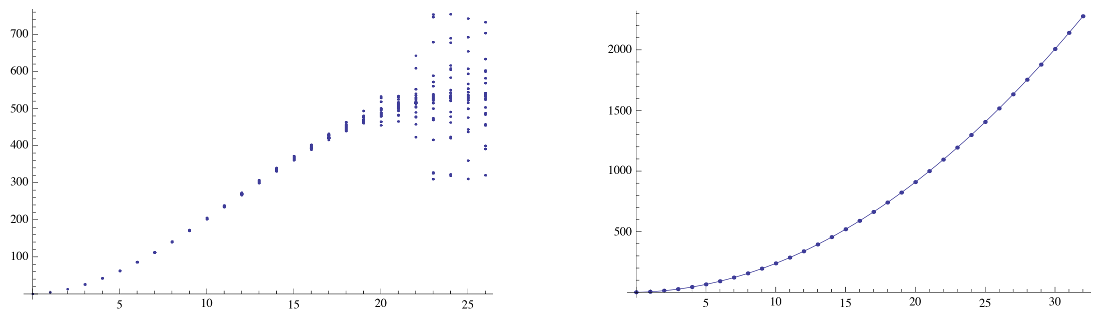


Figure 2: . On the left, all the spectral values $2l+1$ for $m \in [-l,l]$ are plotted against l for $s=8$. On the right, they are averaged over m and fitted to $l + 1.00012\, l^2 - 1.34281\times 10^{-7}\, l^3 - 0.57244\times 10^{-7}\, l^4$ for $s = 128$ and $l \le 32$.

Spectral Convergence in the IR: FEM theory gives a rigorous theoretical foundation to prove convergence to the classical EOM by imposing conditions on simplicial geometry. Namely "shape regular" elements and uniform refinement. These two conditions can be stated as bounds on the radius ρ of the circumscribed circle of all triangle such that $\rho^2_{ijk}/A_{ijk} < \beta$ and $\max[\rho_{ijk}] < O(a)$. Here $a \sim 1/s \to 0$ is a "lattice spacing" taken to zero in the continuum limit. In the present context, convergence can be expressed in interms of the spectral properties of the Laplacian on the sequence of refinements defined in Ref.[3]. Convergence is guaranteed for any fix mode $l < l_{max} = O(1/a)$ analogous to the convergence for momenta in the Brillouin zone $k_\mu \in [-\pi/a, \pi/a]$ for the Laplacian on a hypercubic lattice. This is illustrated n Fig. 2, for refinements with $F = 20s^2$ triangles. More generally, the FEM theoretical framework proves, under suitable conditions, that all solutions to the classical Equations of Motion (i.e. the non-linear PDE's) converge to the exact solution. These conditions are not meet in the random lattice [6] or Regge calculus [5] approach, where instead it is conjectured that ensemble averages over simplicial lattice may define a quantum theory. Here we propose a single sequence of refined lattices. The convergence in the classical limit is an essential ingredient for a FEM Lagrangian but it is insufficient for quantum field theory as discussed next.

3. Quantum Finite Element Method

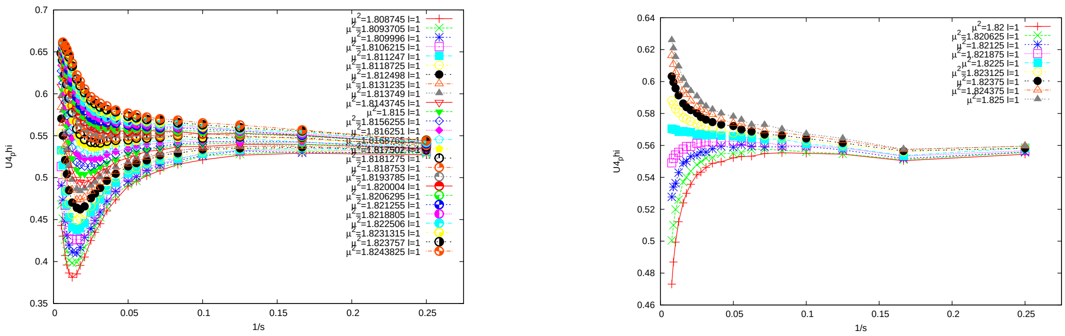


Figure 3: . On the left, the Binder cumulant for the FEM Lagrangian with no quantum counter term. On the right, the Binder cumulant for the QFEM Lagrangian with the analytic counter term shift in the bare mass.

Renormalizable quantum field theory introduces a fundamentally new problem. The fluctuations (even in perturbative theory) probe all configurations up to the lattice cut-off, resulting in ultraviolet divergences as the cut-off is removed. In perturbation theory, these must be cancelled by explicit subtraction of counter terms. On a regular hypercubic lattice, Wilsonian renormalization amounts to tuning to a critical surface replacing bare parameters by holding fixed renormalized (e.g. physical) parameters. The problem we face on an irregular simplicial lattice is that there is no site-independent (translationally or on a sphere rotationally invariant) cut-off. One is confronted with the unhappy possibility of an infinite number of counter terms. Here we show in one simple example that this is not the case and that there is a simple, elegant solution. For our test problem we drop the radial axis, $\mathbb{R}\times\mathbb{S}^2 \to \mathbb{S}^2$, dimensionally reducing radial quantization to the 2D ϕ^2 theory on the projective Riemann sphere. Moreover now the Wilson-Fisher IR fixed point is an exactly solved 2D conformal theory (c = 1/2 minimal model) so comparison to this exact result poses a stringent test of our spherical lattice manifold. In 2D, the one loop graph is the only divergence, introducing a logarithmic mass shift of $O(\lambda)$. This diagram "sees" the UV and amplifies distortions of the FEM cut-off as illustrated in Fig. 4 for our simulation.

We must correct the classical FEM simplicial Lagrangian by adding a finite counter term to form what we refer to as a improved QFEM simplicial Lagrangian. Unlike conventional improvement schemes on a hypercubic lattice, this is essential to the existence of the continuum limit. We have found two solutions for 2D Riemann sphere, which we believe suggests a direction for more general solution for simplicial lattice field theory on smooth manifolds. The first one introduces

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a Pauli-Villars field and the second an explicit counter term to the bare FEM Lagrangian. To understand the necessity of modifying the FEM Lagrangian, we studied the approach to the critical surface using the 4th order Binder cumulant,

$$U_4(\mu,s) = 1 - \frac{\langle\, (\sum_x \phi_x)^4 \,\rangle}{3\langle\, (\sum_x \phi_x)^2 \,\rangle^2} \,, \tag{3.1}$$

whose exact value at the continuum critical point is $U_4^* = 0.567336$. For the FEM lattice Lagrangian, at first for small lattices (or large lattice spacing), the Binder cumulant on the left in Fig. 3 appear reasonable, but in fact we believe there is now no critical surface in the continuum. Different portions of the sphere go "critical" at different bare masses μ, causing the critical surface to blur on larger lattices. While on the right in Fig. 3 with the QFEM Lagrangian, the critical surface is apparently restored with the Binder cumulant approaching the exact value in the continuum.

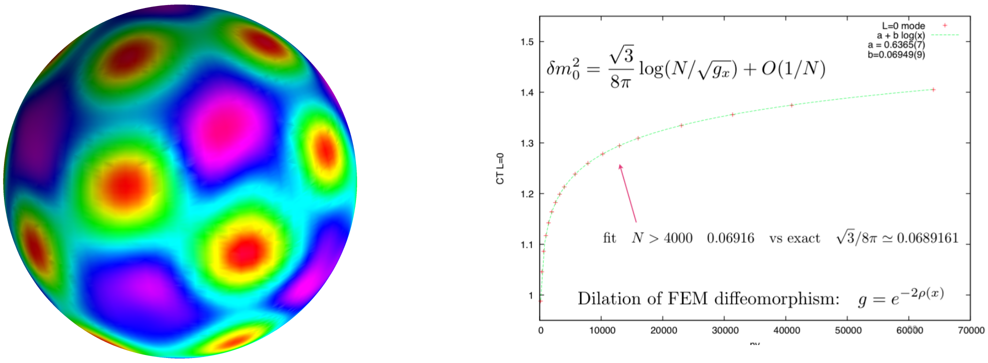

Figure 4: On the left, the amplitude of $\langle\phi_x^2\rangle$ for our lattice simulation for the unrenormalized FEM Lagrangian on the 2 D Riemann Sphere. On the right, the comparison of numerical one loop Counter Term (CT) from lattice perturbation theory vs the analytic expression.

QFEM Counter Terms: Let us describe briefly the quantum counter terms. The Pauli-Villars (or Feynman-Stuekelberg) approach is straight forward. It amounts to the addition of a ghost field giving a lattice equivalent of the continuum propagator,

$$\frac{1}{p^2} \to \frac{1}{p^2} - \frac{1}{p^2 + M_{PV}^2} = \frac{1}{p^2 + p^4/M_{PV}^2} \,, \tag{3.2}$$

in the massless limit. In principle to reach the continuum requires a double limit: $a \to 0$ at fixed aM_{PV} followed by $aM_{PV} \to \infty$. Our preliminary simulations with PV regulator demonstrates the restoration of full rotational symmetry. The Pauli-Villars mass protect the theory from the irregular lattice cut-off. This is easily generalized to other theories but in out current ϕ^4 simulations, it has the technical disadvantage in the PV term prevents the use of the very efficient cluster algorithm simulation of Ref. [7]. Moreover explicit computation of the one loop divergence is suggestive of a more fundamental solution to the problem. The one loop lattice calculations of mass shift in Fig.4 on the right is numerically well fit by

$$\delta\mu^2 = \frac{\sqrt{3\lambda}}{8\pi}\log(N/\sqrt{g_x}) + O(1/N) \tag{3.3}$$

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A critical consequence, seen in our numerical simulations and proven by FEM methods, is that the divergent term for the one loop Greens function is universal in spite of the variation in the effective lattice spacing. Consequently there is only a finite local scheme dependence renormalization constant at each lattice site, requiring only a finite local shift FEM ϕ_x^2 mass term.

The pattern of distortions look very similar to the patterns seen in Fig 4 on the left and the result is remarkably simple as illustrated in Fig. 4 on the right. In fact there is an analytical derivation based on the following observation. In the Regge calculus representation of a 2D manifold the arc lengths, l_{ij}, parameterize both intrinsic geometry and a discrete subgroup of the diffeomorphism invariance of the manifold. Our choice of the simplicial lattice began with flat equilateral triangles on each of the 20 faces of the original icosahedron. The radial projection onto the sphere is a Weyl transformation to constant curvature but in addition it **induces a new conformal diffeomorphism**. To achieve this our choice of the mesh refinement sequence had to approaches a smooth conformal map. The conformal factor (or Jacobian of the map) is

$$\sqrt{g} = e^{\rho} = \frac{(x^2 + y^2 + 1 - R^2)^{3/2}}{\sqrt{1 - R^2}} \tag{3.4}$$

where R is the circumradius for one of the 20 icosahedral faces and x, y the flat co-ordinates on that face. In Fig 4, we see a fit of the analytical factor vs the numerical computation of the one loop lattice counter term.

As we have shown by a comparing the two graphs in Fig 3, our QFEM corrected Lagrangian has apparently completely solved the numerical problem of reaching the continnum limit. The lattice QFEM Lagrangian does have a critical surfaace as illustrated numerically in Fig. 3 on the right. The fit gives $U_4^* = 0.5661(2)$ relative to the exact value 0.567336 and approachs it with the scaling exponent $\nu = 0.978(25)$ relative to $\nu = 1$ exact. We are in the process of computing correlations function in comparison with the exact solution of this 2D conformal field theory on a sphere to further verify numerically that this does indeed have the correct continuum limit. These results will be reported soon in a longer publication. Future work will also begin to generalize this to full radial quantization with the inclusion of Dirac and gauge fields. None of these steps are trivial but it is hoped that this 2D toy problem provides the basic strategy on how to proceed.

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