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The Dyson-Schwinger equation of a link variable in lattice Landau gauge theory

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We derive the Dyson-Schwinger equation of a link variable in $SU(n)$ lattice gauge theory in minimal Landau gauge and confront it with Monte-Carlo data for the different terms. Preliminary results for the lattice analog of the Kugo-Ojima confinement criterion is also shown.

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1. Introduction

The Dyson-Schwinger equation (DSE) of a gauge boson was recently revisited [1] in the context of the Kugo-Ojima (KO) confinement criterion [2]. It was found that the term which saturates the rhs. of the DSE for $p \to 0$ depends on the phase of a gauge theory [1]. In the Higgs phase, for example, physical states saturate the transverse DSE, while in the confining phase only unphysical degrees of freedom contribute to the saturation of the DSE. Interestingly, this corollary to the KO criterion is not only true for linear covariant gauges, but can also be applied to other gauges and models like maximal Abelian gauge, non-covariant Coulomb gauge and in the Gribov-Zwanziger theory [1].

In an earlier attempt [3] we have tried to verify the KO criterion directly on the lattice. There however we could not confirm the desired result for the KO function: $u(p) \to -1$ for $p \to 0$. Our data was more in favor of $u(p)$ reaching a limit somewhere between -0.6 and -0.8. In the light of [1] it was thus natural to revisit this earlier calculation and to derive the exact DSE for a link variable in Landau gauge on the lattice. With this one could check if $u(p)$ also on the lattice saturates the DSE in the infrared limit. Here we report on our first findings of a still on-going investigation.

2. Dyson-Schwinger equation of a lattice link variable in Landau gauge

For the following it is advantageous to use a notation in which each link variable is assigned its defining sites x and $y = x \pm \hat{\mu}$, i.e., we use the notation $U_{xy} \in SU(n)$ rather than the usual $U_{x\mu}$. The Wilson gauge action for an $SU(n)$ lattice gauge theory then reads

$$S_W[U] = \frac{1}{4g_0^2} \sum_{i,j,k,l}^{N} P_{ijkl} \quad \text{with} \quad P_{ijkl} = \operatorname{Tr}\left(U_{ij}U_{jk}U_{kl}U_{li}\right) \tag{2.1}$$

where P is a plaquette variable and $g_0^2 = 2n/\beta$. Since we are interested in Landau gauge we will consider links which minimize the (real-valued) Morse potential

$$V[U] = -\frac{1}{2}\sum_{i,j}^{N} \operatorname{Tr} U_{ij}\,. \tag{2.2}$$

These minima fulfill the Landau gauge condition (we use anti-hermitian generators t^a of $SU(n)$)

$$0 \stackrel{!}{=} f_i^a = \sum_j \mathfrak{Re}\operatorname{Tr}\left(t^a U_{ij}\right) \qquad \forall i, a\,. \tag{2.3}$$

Starting with a thermalized (non-gauged) configuration U, gauge-fixing on the lattice is commonly performed by an iterative procedure which consecutively gauge-transforms $U \to U^g_{ij} = g_i U_{ij} g_j^\dagger$ until Eq. (2.3) is satisfied to numerical precision.

For the derivation of the DSE we now define an infinitesimal left-variation $\underline{\delta}^b_{lm}$ of such a *gauge-fixed* configuration $U = \{U_{rs}\}$:

$$\underline{\delta}^b_{lm} U_{rs} \;\equiv\; t^b U_{lm}\delta_{rl}\delta_{sm} - U_{ml}t^b\delta_{rm}\delta_{sl} + \theta_r U_{rs} - U_{rs}\theta_s\,. \tag{2.4}$$

It generates a left-variation of the link variable U_{lm} followed by an infinitesimal gauge transformation that returns the configuration to lattice Landau gauge (LLG). The anti-hermitian traceless $n \times n$ matrices $\{\theta_i\}$ in (2.4) ensure that the configuration remains in LLG. That is, the left-variation of the gauge condition is zero:

$$0 = \underline{\delta}^b_{lm} f^a_i = \mathfrak{Re}\,\mathrm{Tr}\left[t^a t^b U_{lm}(\delta_{il} - \delta_{im}) + \sum_j t^a \left(\theta_i U_{ij} - U_{ij}\theta_j\right)\right] .$$

and the components of $\theta_i = \sum_c t^c \theta^{c;b}_{i;\,lm}$ solve the linear system,

$$\sum_{c,j} M^{ac}_{ij}\, \theta^{c;b}_{j;lm} = \mathscr{U}^{ab}_{lm}(\delta_{il} - \delta_{im}) , \tag{2.5}$$

where M is the Faddeev-Popov (FP) matrix

$$M^{ab}_{ij} := \mathscr{U}^{ba}_{ij} - \delta_{ij}\sum_k \mathscr{U}^{ab}_{ik} , \tag{2.6}$$

given in terms of the real symmetric matrix,

$$\mathscr{U}^{ab}_{ij} = \mathscr{U}^{ba}_{ji} := \mathfrak{Re}\ \mathrm{Tr}\left(t^a t^b U_{ij}\right) . \tag{2.7}$$

The Faddeev-Popov matrix $M^{ab}_{ij} = M^{ba}_{ji}$ is symmetric under the simultaneous exchange of color and site indices when Eq. (2.3) holds. Global gauge invariance of V implies that $\sum_i M^{ab}_{ij} = 0$, i.e., M^{ab}_{ij} has $n^2 - 1$ generic zero-modes. The remaining $(N-1)(n^2-1)$ eigenvalues are positive at each local minimum of V, and Eq. (2.6) implies that one may choose the solution $\theta^{a;b}_{i;lm}$ of (2.5) orthogonal to the zero modes.

If no gauge was fixed, the integration measure would be the Wilson measure $d\mu_W = D[U]e^{S_W[U]}$. $D[U]$ is the product of Haar measures dU_{ij} for each oriented link variable which is invariant under left- as well as right- group multiplication, $d(gU_{ij}) = d(U_{ij}g) = dU_{ij}$ for any $g \in SU(n)$, whereas S_W is invariant only under lattice gauge transformations $U^g_{ij} = g_i U_{ij} g^\dagger_j$.

To account for (minimal) Landau gauge we introduce (in a Faddeev-Popov-like manner) a density

$$\rho_\alpha(U) := \mathscr{D}_\alpha[U] e^{-\alpha(S_{LG}[U]-S_{LG}[\bar{U}])} \quad \text{with} \quad \mathscr{D}^{-1}_\alpha[U] \equiv \int \prod_i dg_i\, e^{-\alpha(S_{LG}[U^g]-S_{LG}[\bar{U}])} , \tag{2.8}$$

which in the limit $\alpha \to \infty$ has support only at the absolute minima $\bar{U}$ of S_{LG} and whose integral over the gauge orbit $\int \prod_i dg_i\, \rho_\alpha(U^g) = 1$. In a first attempt we set $S_{LG}[U] = V[U]$, which for $\alpha \to \infty$ gives support only for $\bar{U}$, the global minimum of V on the gauge orbit of U. We will later see that for any of the standard lattice implementations of minimal Landau gauge (these typically find only local minima of V) the identification $S_{LG}[U] = V[U]$ accounts only for the leading contribution in the DSE.[1] Anyhow in the limit $\alpha \to \infty$ it gives for $\mathscr{D}_\alpha[U]$

$$\lim_{\alpha\to\infty} \mathscr{D}_\alpha[U] = \sqrt{\det M[\bar{U}]} \qquad (\text{if } S_{LG} \equiv V) \tag{2.9}$$

[1]The effective form of $S_{LG}[U]$ was unclear to us at the time of the conference, but it will be further discussed and specified in a forthcoming publication.

and the expectation value of a *gauge-variant* quantity $\mathscr{O}$ in minimal Landau gauge (MLG) becomes

$$\langle \mathscr{O} \rangle_{MLG} = \lim_{\alpha\to\infty} \frac{1}{Z_W} \int D[U]\, \rho_\alpha(U)\, \mathscr{O}[U]\, e^{S_W[U]} \tag{2.10}$$

with $Z_W = \int D[U]\, \rho_\alpha(U)\, e^{S_W[U]}$. Note that in lattice perturbation theory one commonly sets $S_{LG}[U] = f_i^a f_i^a$ which gives $|\det M[U]|$ for $\mathscr{D}_\alpha[U]$ (see, e.g.,[4]).

The Dyson-Schwinger equation (DSE) of a link variable in minimal Landau gauge we now obtain from a left-variation (see Eq. (2.4)) of the expectation value $\langle U_{rs}\rangle_{MLG}$. This expectation value must not change under the left-variation and so

$$\underline{\delta}^b_{lm} \left(\lim_{\alpha\to\infty} \frac{1}{Z_W} \int D[U]\, \rho_\alpha(U)\, e^{S_W[U]}\, U_{rs} \right) = 0\,. \tag{2.11}$$

Applying $\underline{\delta}^b_{lm}$ to each term gives the DSE in an implicit form

$$\lim_{\alpha\to\infty} \left\langle \underline{\delta}^b_{lm} U_{rs} + U_{rs} \underline{\delta}^b_{lm} \left(S_W[U] + \ln \rho_\alpha[U] \right) \right\rangle_{MLG} = 0. \tag{2.12}$$

which after some algebra becomes

$$c_f \, \langle \operatorname{Tr} U_{lm} \rangle_{MLG} \, (\delta_{rl}\delta_{sm} - \delta_{rm}\delta_{sl}) = \mathfrak{Re} \sum_a \langle K^a_{lm} \operatorname{Tr} t^a U_{rs} \rangle_{MLG} + \sum_{ab} \left\langle \theta^{a;b}_{r;lm} \mathscr{U}^{ba}_{rs} - \theta^{a;b}_{s;lm} \mathscr{U}^{ba}_{sr} \right\rangle_{MLG} . \tag{2.13}$$

Here $c_f = (n^2-1)/(2n)$ is the quadratic Casimir invariant of the fundamental representation of $SU(n)$, and the components of $\theta_j = \sum_a t^a \theta^{a;b}_{j;lm}$ solve Eq. (2.5). The (conserved) current K^a_{lm} is of the form

$$K^a_{lm} := \Sigma^a_{lm} + \Phi^a_{lm}\,. \tag{2.14}$$

where Σ^a_{lm} arises from varying the Wilson action,

$$\Sigma^a_{lm} := \underline{\delta}^a_{lm} S[U] = \frac{2}{g_B^2} \mathfrak{Re} \sum_{j,k} \operatorname{Tr} t^a U_{lm} U_{mj} U_{jk} U_{kl} \,, \tag{2.15}$$

and Φ from the variation of the induced measure ρ_α

$$\Phi^a_{lm}[U] := \lim_{\alpha\to\infty} \underline{\delta}^a_{lm} \ln(\rho_\alpha[U]) = \frac{1}{2} \mathfrak{Re} \sum_{ij,bc} \tilde{M}^{-1\,cb}_{ji} (\delta_{im} - \delta_{il}) \operatorname{Tr}(t^b t^c \delta_{jl} - t^c t^b \delta_{jm}) t^a U_{lm}\,. \tag{2.16}$$

Here we have used that $\rho_\infty[\bar{U}^g]$ is stationary with respect to gauge transformations. This also implies that Φ^a_{lm} is transverse, $\sum_m \Phi^a_{lm}[\bar{U}] = 0$, in fact this is true at any minimum of V. Also Σ^a_{lm} is transverse in the sense that $\sum_m \Sigma^a_{lm} = 0$.

The longitudinal part of the lattice DSE is *algebraically* satisfied by the last term of (2.13): Summing on the index "s" and using (2.3) and (2.6) we have at any minimum of $V[U]$

$$\sum_{ab,s} \left(\theta^{a;b}_{r;lm} \mathscr{U}^{ba}_{rs} - \theta^{a;b}_{s;lm} \mathscr{U}^{ba}_{sr} \right) = -\sum_{ab,s} M^{ba}_{rs} \theta^{a;b}_{s;lm} = c_f (\delta_{rl} - \delta_{rm}) \mathfrak{Re} \operatorname{Tr} U_{lm}\,. \tag{2.17}$$

3. Numerical verification

To numerically verify our DSE we have Fourier-transformed Eq. (2.13) to momentum space. In this way we take advantage of its translation invariance to maximally reduce the statistical noise of the various Monte-Carlo expectation values. In momentum space the DSE reads[2]

$$\mathscr{V}\delta_{\mu\nu} = \Sigma_{\mu\nu}(p) + \Phi_{\mu\nu}(p) + L_{\mu\nu}(p) \tag{3.1}$$

where $\mathscr{V} = \langle V\rangle/2n$ is the (momentum-independent) expectation value of the Morse potential in minimal Landau gauge, while the terms on the rhs. are momentum-dependent. To verify that these dependences exactly cancel in the sum we have calculated (on several gauge-fixed ensembles) the term ($n_g = n^2-1$)

$$L_{\mu\nu}(p) = \frac{1}{Nn_g}\sum_{ab,xy} e^{-ip(x-y)}\left\langle \theta^{a;b}_{y;x\mu}\,\mathscr{U}^{ba}_{y\nu} - \theta^{a;b}_{y+\nu;x\mu}\,\mathscr{U}^{ab}_{y\nu}\right\rangle \tag{3.2}$$

with θ solving Eq. (2.5) and the transverse terms

$$\Sigma_{\mu\nu}(p) = \frac{2}{g_0^2}\frac{1}{Nn_g}\sum_{a,xy} e^{-ip(x-y)}\mathfrak{Re}\sum_a\left\langle\left(\mathfrak{Re}\,\mathrm{Tr}\, t^a U_{x\mu}W_{x\mu}\right)\cdot \mathrm{Tr}\, t^a U_{y\nu}\right\rangle \tag{3.3}$$

and

$$\Phi_{\mu\nu}(p) = \frac{1}{Nn_g}\sum_{a,xy} e^{-ip(x-y)}\left\langle \Phi^a_{x\mu}\cdot \mathrm{Tr}\, t^a U_{y\nu}\right\rangle . \tag{3.4}$$

$W_{x\mu}$ is the sum of staples attached to the link $U_{x\mu}$ and $\Phi^a_{x\mu}$ can be read off from Eq. (2.16) identifying $l = x$ and $m = x+\hat{\mu}$. For its evaluation we need different elements of $M^{-1}[U]$ which we estimate with the stochastic noise technique. It is remarkable, that a number of 8 to 32 Gaussian noise vectors for each U is sufficient to provide us with a good signal for $\Phi^a_{x\mu}$.

In the continuum limit $L_{\mu\nu}$ becomes longitudinal. On the lattice this is not strictly the case, numerically however, the transverse contribution is negligible at the considered values of β.

In Fig.1 we show the longitudinal (left panel) and transverse (right panel) terms of the DSE in momentum space. To this end, we have projected all terms with the respective longitudinal and transverse projectors, $P_L = \hat{p}_\mu\hat{p}_\nu/\hat{p}^2$ and $P_T = 1-P_L$ where $a\hat{p}_\mu = 2\sin(\pi k_\mu/L_\mu)$. The figure hence shows the "form factors" of $L_{\mu\nu}$, $\Sigma_{\mu\nu}$ and $\Phi_{\mu\nu}$ versus momentum $a^2\hat{p}^2$. This is similar to what one typically does for the gluon propagator.

In the left panel of Fig.1 one clearly sees that the longitudinal part of the DSE is fulfilled for all momenta, as expected. Only $L_{\mu\nu}(p)$ gives a contribution, and the longitudinal part of $L_{\mu\nu}(p) - \mathscr{V}\delta_{\mu\nu} = 0$ (see the small panel on top).

The transverse channel of the DSE has contributions from $\Sigma_{\mu\nu}(p)$, $\Phi_{\mu\nu}(p)$ and $L_{\mu\nu}$. Their sum however does not equal $\mathscr{V}\delta_{\mu\nu}$. In fact, we see a clear deviation from zero for the difference $\mathscr{V}\delta_{\mu\nu} - (\Sigma_{\mu\nu}+\Phi_{\mu\nu}+L_{\mu\nu})$ which can be compensated if one rescales Φ by 1.3 (see top right panel of Fig.1). Without rescaling the deviation is proportional to the gluon propagator $D_{\mu\nu}(p)$, calculated for the same lattice parameters. This we verified for SU(2) and SU(3) for different β and lattice sizes (see Fig.2). That is, the DSE would be fulfilled if it included another term $cD_{\mu\nu}$.

[2] Note, in what follows we change our notation for the links back to the usual, i.e., $U_{xy} \to U_{x\mu}$.

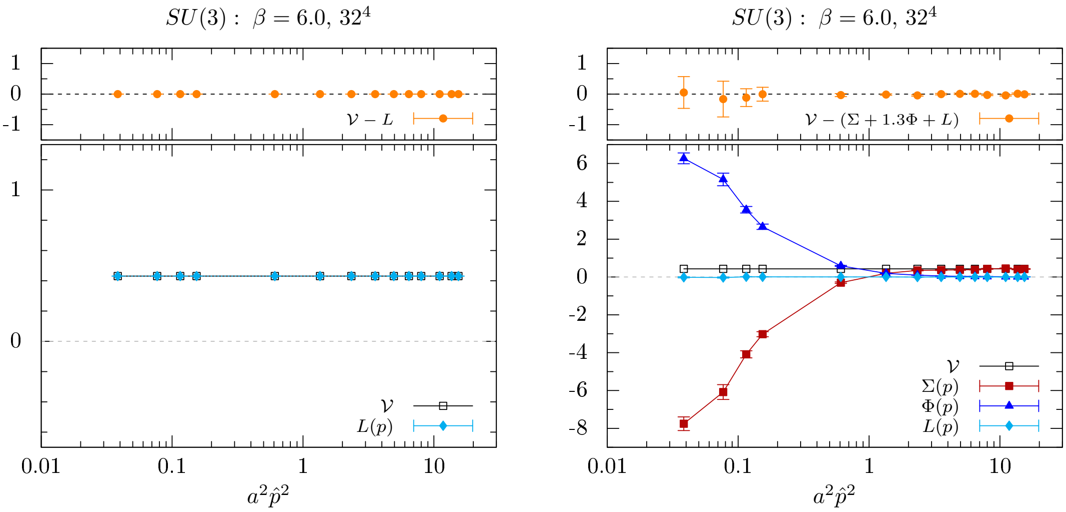

Figure 1: Longitudinal (left) and transverse (right) terms of the DSE as a function of the lattice momentum. The small panels on top show the validity of the (modified) DSE for all momenta.

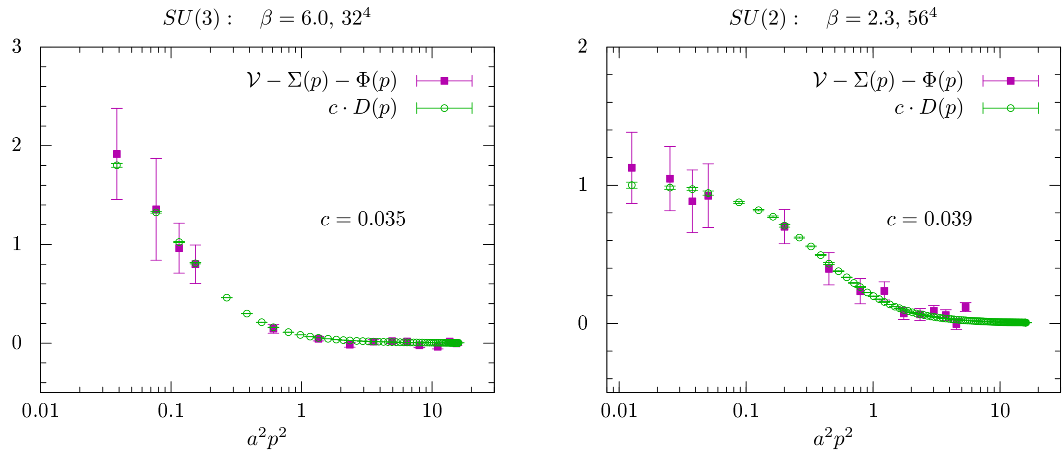

Figure 2: Deviation of the transverse part of our initial ansatz for the DSE (full squares) in comparison to the corresponding gluon propagator times a constant (open diamonds). All in units of the lattice spacing and versus the lattice momentum squared. Left: SU(3) at $\beta = 6.0, 32^4$; Right: SU(2) $\beta = 2.3, 56^4$.

At $\beta = 6.0$ the proportionality constant c is about 0.035. Our current ansatz for the DSE thus does not fully account for the Monte Carlo data at small momenta.

We have not yet found a full explanation for this deviation. A subsequent analysis suggests that the reason for the deviation is our ansatz for lattice Landau gauge. Above we set $S_{LG} = V$ but our data at finite β effectively appears to favor local minima of $S_{LG} = V + \varepsilon f^2$. Such an effective Morse potential would have the same minima as V but a different Hessian and for small ε this Hessian would result in an additional term in the DSE proportional to the gluon propagator. At $\beta = 0$, however, the deviation is not longer proportional to the gluon propagator. At $\beta = 0$, the rescaling $\Phi \to 2\Phi$ would approximately restore the DSE for the considered range of momenta (12^4). For more details, and hopefully an explanation, we have to refer to a forthcoming publication.

4. Kugo-Ojima

Besides verifying the DSE we also want to check if the Kugo-Ojima correlator satisfies our DSE at $p=0$. In the continuum the Kugo-Ojima correlator is given by the expectation value $\sum_b \left\langle (D^y_\nu c)^b (D^x_\mu \bar{c})^b \right\rangle$. On the lattice this corresponds to $\sum_{abc}\left\langle [\mathscr{U}^{ba}_{y\nu} c^a_y - c^a_{y+\nu}\,\mathscr{U}^{ab}_{y\nu}][\mathscr{U}^{bc}_{x\mu}\bar{c}^c_x - \bar{c}^c_{x+\mu}\,\mathscr{U}^{cb}_{x\mu}]\right\rangle$ where $\langle c^a_x \bar{c}^b_y\rangle_{c\bar{c}} = (M^{-1})^{ab}_{xy}$ is the Faddeev-Popov matrix. We look at this correlator again in momentum space where it reads

$$u(p^2) = \frac{1}{Nn_g}\sum_{xy} e^{ip(x-y)} \sum_{abc}\left\langle [\mathscr{U}^{ba}_{y\nu} c^a_y - c^a_{y+\nu}\,\mathscr{U}^{ab}_{y\nu}][\mathscr{U}^{bc}_{x\mu}\bar{c}^c_x - \bar{c}^c_{x+\mu}\,\mathscr{U}^{cb}_{x\mu}]\right\rangle \,.$$

Since we are interested in the limit $u(p^2\to 0)$, we have calculated $u(p^2)$ for the case of $SU(2)$ on a 56^4 lattice at $\beta=2.3$. This has allowed us to reach relatively low momenta. This data is shown in Fig. 3, and one clearly sees the momentum dependence behaves as expected: $u \propto p^2$ for $p^2 \to 0$. Nonetheless the limit $u(p^2\to 0)$ does not equal $\mathscr{V}$, shown as dashed line in Fig. 3. If this is a feature signaling the non-applicability of the KO criterion for minimal lattice Landau gauge, or related to our ansatz for the DSE needs to be clarified yet.

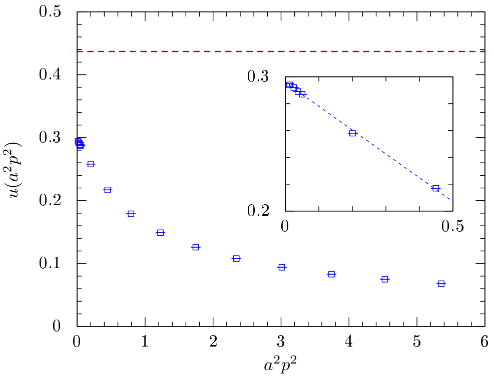

Figure 3: KO function vs. (lattice) momentum squared on a 56^4 lattice; $\beta=2.3$ ($n=2$). The dashed line marks the value of $\mathscr{V}$.

5. Summary

We have developed, for the first time, the Dyson-Schwinger equation for a lattice link variable in minimal Landau gauge. The longitudinal channel of our DSE is algebraically satisfied, but for the transverse channel we see clear deviations whose origin is not fully understood, but will be further analyzed in a forthcoming publication. Once this is settled, our data for the KO function will also be revisited again. At present it signals the non-applicability of the KO criterion for lattice gauge theories in Landau gauge, but this may be related to our present DSE.

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