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The bottom-quark mass from Υ sum rules at NNNLO

Thomas Rauh[*]
Physik Department T31, James-Franck-Straße,
Technische Universität München, D-85748 Garching, Germany
E-mail: thomas.rauh@mytum.de

We discuss a recent determination of the bottom-quark mass from non-relativistic sum rules at next-to-next-to-next-to leading order. PNRQCD is used to resum Coulomb singularities that appear close to threshold. The effects of a non-zero charm-quark mass are taken into account. The final result is compared to other recent precision determinations.

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Introduction. Due to confinement the direct measurement of quark masses at collider experiments is impossible, or is at least systematically limited by an uncertainty of the order of the confinement scale $\Lambda_{\rm QCD}$. A precise determination of the bottom-quark mass therefore must involve a, ideally strongly mass dependent, theory prediction for a well-known experimental observable. The analysis [1] presented below is based on moments of the inclusive bottom-pair production cross section in e^+e^- collisions [2].

The sum rule. The bottom production cross section is normalized to the muon pair production and can be related to the bottom-quark contribution to the vacuum polarization function of the photon[1] Π_b by the optical theorem $R_b = 12\pi \,\mathrm{Im}\, \Pi_b$. Its moments $\mathscr{M}_n$ are defined by the dispersion relation

$$\mathscr{M}_n \equiv \int_0^\infty ds \frac{R_b(s)}{s^{n+1}} = \frac{12\pi^2}{n!}\left(\frac{d}{dq^2}\right)^n \Pi_b(q^2)\Big|_{q^2=0}, \tag{1}$$

which follows from the requirement of analyticity of the vacuum polarization function. The left-hand side of the sum rule (1) can be evaluated from experimental data. The right-hand side is infrared safe due to $q^2 = 0$ being far off-shell. Therefore, assuming quark hadron duality, it can be computed as an operator product expansion (OPE) in $\Lambda_{\rm QCD}/(m_b/n)$, where m_b/n is the effective smearing range of the moments. The leading nonperturbative contribution is given by the gluon condensate and is of dimension four. It has been shown to be tiny [1, 5] and is neglected in the following. Equating both sides of relation (1) then gives a prediction for the bottom-quark mass.

We focus here on the large-n moments with $n \approx 10$, which are dominated by the threshold region $\sqrt{s} \approx 2m_b$. The experimental moments contain contributions from the $\Upsilon(NS)$ resonances and the open $B\bar{B}$ continuum

$$\mathscr{M}_n^{\rm exp} = 9\pi \sum_{N=1}^{4} \frac{1}{\alpha(M_{\Upsilon(NS)})^2} \frac{\Gamma_{\Upsilon(NS)\to l^+l^-}}{M^{2n+1}_{\Upsilon(NS)}} + \int_{s_{\rm cont}}^{\infty} ds \frac{R_b(s)}{s^{n+1}}. \tag{2}$$

The resonances and the continuum for $\sqrt{s} \leq 11.20$ GeV are measured to great accuracy [7, 8], but there is no data at higher energies. This favors large n, because the respective moments are saturated by the lowest states and we can make a conservative assumption $R_b(\sqrt{s} > 11.20\text{ GeV}) = 0.3 \pm 0.2$ without forfeiting precision. For $n \gg 10$ however, the OPE expansion parameter is no longer much smaller than one and the nonperturbative effects can't be estimated reliably [1].

Theory moments. The characteristic bottom-quark velocity for the large-n moments is given by $v \sim 1/\sqrt{n}$ [5] and for the choice $n \approx 10$ the velocity is of the same size as the strong coupling $v \sim \alpha_s$. This implies that Coulomb potential interactions between the $b\bar{b}$ pair, which generate terms that scale as $(\alpha_s/v)^k$, must be treated in a nonperturbative way. This can be achieved by effective field theory methods as described in [3]. In the effective theory of potential non-relativistic QCD [6] (PNRQCD) the cross section takes the form

$$R_b = 12\pi e_b^2 \,\mathrm{Im}\left[\frac{N_c}{2m_b^2}\left(c_v\left[c_v - \frac{E}{m_b}\left(c_v + \frac{d_v}{3}\right)\right] G(E) + \dots\right)\right], \tag{3}$$

where c_v, d_v are hard matching coefficients for the vector current and $G(E)$ is the non-relativistic Green function for the $b\bar{b}$ pair evaluated for vanishing spatial separation. Due to the nonperturbative

[1] The Z boson mediated part of the cross section [3, 4] can be neglected due to suppression by m_b^2/m_Z^2.

treatment of the leading order Coulomb potential the Green function contains bound-state poles below threshold

$$G(E) \stackrel{E\to E_N}{\longrightarrow} \frac{|\psi_N(0)|^2}{E_N - E - i\varepsilon}, \tag{4}$$

where E_N is the binding energy and ψ_N the wave function of the $\Upsilon(NS)$ resonance. The theory moments also contain a resonance and a continuum contribution

$$\mathscr{M}_n^{\rm th} = \frac{12\pi^2 N_c e_b^2}{m_b^2} \sum_{N=1}^{\infty} \frac{c_v\left[c_v - \frac{E_N}{m_b}\left(c_v + \frac{d_v}{3}\right)\right]|\psi_N(0)|^2}{(2m_b + E_N)^{2n+1}} + \int\limits_{4m_b^2}^{\infty} ds \frac{R_b(s)}{s^{n+1}}. \tag{5}$$

Thus their evaluation at NNNLO requires the matching coefficients c_v at order α_s^3 [9] and d_v at order α_s [10], the non-relativistic potentials up to NNNLO [3, 11–15] and corrections to the energy levels and wave functions at the origin [16–20] as well as to the continuum Green function $G(E)$ [18, 20] from the potentials and from ultrasoft gluons [21–23]. These quantities are available for top-pair production and therefore require two modifications for the bottom case. Firstly, the sizeable top width provides an effective IR cutoff on the top-pair cross section. The bottom-quark width however is negligible in the evaluation of the cross section and many expressions have to be analytically continued in the limit $\Gamma \to 0$, i.e. approaching the branch cut. Secondly, the light quarks have been treated as massless for top-pair production, whereas the charm-quark mass is of the order of the soft scale $m_c \sim m_b v$. An expansion in the charm-quark mass is not justified in this case and it has to be integrated out in the soft matching to PNRQCD. Using results for the charm-mass corrections to the Coulomb potential [24–26] we have computed the contributions to the energy levels and wave functions at the origin up to NNLO and discussed the effects on large-n moments in [1]. The insertion of a charm loop provides an effective infrared cutoff on the respective gluon lines and we expect charm-mass effects to be large for quantities with large infrared sensitivity. In accordance with the expectation we find charm-mass effects to be large for the moments expressed in terms of the bottom-quark pole mass, but negligible once the spurious infrared sensitivity introduced by the use of the pole mass [27–29] is removed. The latter step is achieved by the use of the potential-subtracted (PS) mass [30]. The $\overline{\rm MS}$ mass is also renormalon free, but not adequate for threshold problems, where the bottom quarks are very close to the mass shell. At the Lagrangian level this manifests by that fact that the residual $\overline{\rm MS}$ mass term $\psi^\dagger \delta m_b^{\overline{\rm MS}} \psi$ is of order v^4 in the non-relativistic power counting, whereas the LO PNRQCD Lagrangian, that contains the kinetic terms, is of order v^5, which is clearly inconsistent. Thus, we first extract a PS mass, which is then converted to the $\overline{\rm MS}$ scheme. For the analysis we use moments defined in the PS-shift scheme of [20], where the cross section is not expanded in $\delta m_b^{\rm PS} = m_b^{\rm pole} - m_b^{\rm PS}$, which was observed in [20] to avoid unphysical behaviour close to the threshold. The requirement of renormalon cancellations in the PS-shift scheme requires that a pole resummation is performed in the non-relativistic Green function. As a consequence the factor $(2m_b + E_N)^{-2n-1}$ in the moments (5) is not expanded around the PS mass. Furthermore we have considered moments $\widetilde{\mathscr{M}}_n^{\rm th}$, where the factor $1/s$ in the vacuum polarization function is not expanded in the kinetic energies. These are used for the final numerics, since they produce results that are more stable under variation of n. The differences between $\mathscr{M}_n^{\rm th}$ and $\widetilde{\mathscr{M}}_n^{\rm th}$ are large, possibly due to subleading renormalon contributions, but they are within our estimate for the perturbative uncertainty.

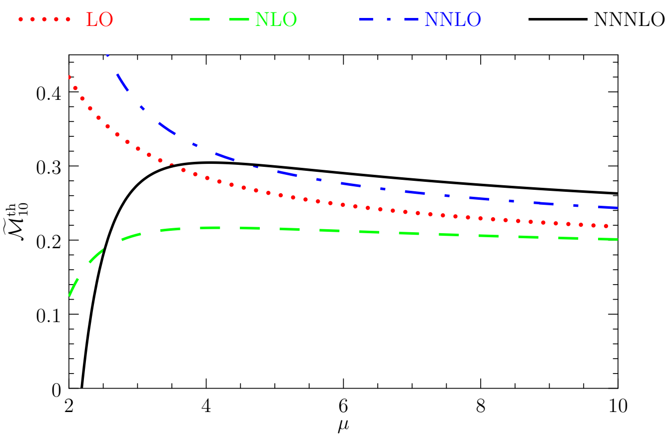

Figure 1: Scale dependence of the tenth moment $\widetilde{\mathcal{M}}_{10}^{\rm th}$ in units of $(10\ \mathrm{GeV})^{-20}$ for $m_b^{\rm PS} = 4.5$ GeV.

Numerical analysis and Discussion. The scale dependence of the tenth moment is shown in Figure 1. We observe that an overlap of the values of the moments at subsequent orders of perturbation theory is first achieved at NNNLO. This shows the importance of including these computationally elaborate corrections. We choose a central scale of $\mu = m_b^{\rm PS}$ and vary μ in the range $[3\ \mathrm{GeV}, 10\ \mathrm{GeV}]$ to determine the perturbative uncertainty. This is in contrast to most earlier works on large-n sum rules, which chose μ near the soft scale. In this region $\mu \lesssim 3$ GeV there is however no sign of convergence, which is why larger scales should be the preferred choice. We note that variation of the scale around the soft scale gives a larger perturbative uncertainty. We however conclude that our prescription is sufficiently conservative based on the facts that the NNNLO band spanned by scale variation is completely contained in the NNLO one and that the difference between the moments $\widetilde{\mathcal{M}}_n^{\rm th}$ and $\mathcal{M}_n^{\rm th}$ is also covered.

The final results for the PS and $\overline{\rm MS}$ masses determined from $\widetilde{\mathcal{M}}_{10}^{\rm th}$ are

$$\begin{aligned} m_b^{\rm PS}(2\,\mathrm{GeV}) &= \big[4.532^{+0.002}_{-0.035}(\mu) \pm 0.010(\alpha_s)^{+0.003}_{-0}(\mathrm{res}) \pm 0.001(\mathrm{conv}) \\ &\quad \pm 0.002(\mathrm{charm})^{+0.007}_{-0.013}(n) \pm 0.003(\mathrm{exp})\big]\,\mathrm{GeV} \\ &= 4.532^{+0.013}_{-0.039}\,\mathrm{GeV}\,, \end{aligned} \tag{6}$$

$$\begin{aligned} m_b^{\overline{\rm MS}}(m_b^{\overline{\rm MS}}) &= \big[4.193^{+0.002}_{-0.031}(\mu) \pm 0.001(\alpha_s)^{+0.003}_{-0}(\mathrm{res})^{+0.021}_{-0.010}(\mathrm{conv}) \\ &\quad \pm 0.002(\mathrm{charm})^{+0.006}_{-0.012}(n) \pm 0.003(\mathrm{exp})\big]\,\mathrm{GeV} \\ &= 4.193^{+0.022}_{-0.035}\,\mathrm{GeV}\,. \end{aligned} \tag{7}$$

Besides scale variation the uncertainties from the input value $\alpha_s(m_Z) = 0.1184 \pm 0.0010$, the number of theoretically considered resonances, the mass scheme conversions, unknown NNNLO charm effects, the choice of $n \in [8,12]$ and the experimental data have been considered. Uncertainties due to QED corrections and nonperturbative effects from the dimension four gluon condensate were found to be negligible (less than 1 MeV).

We have provided a detailed comparison with other determinations based on sum rules in [1]. Note that while analyses using small-n sum rules quote smaller uncertainties, they do require strong assumptions on the behaviour of the experimental continuum above threshold. If instead

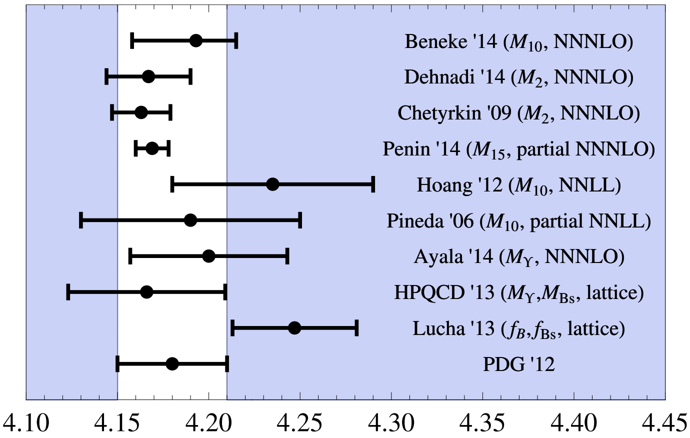

Figure 2: Our result (Beneke '14) for the $\overline{\rm MS}$ bottom-quark mass in GeV is shown in comparison with other recent results [31–38] and the PDG average [7] (white region).

we used our rough estimate, the experimental uncertainty for the mass obtained from $\mathscr{M}_2$ would be ~ 300 MeV, which should not be taken as a suggestion for the error budget, but rather as a demonstration of the complementarity of both methods. Finite-energy sum rules have been used to suppress the dependence on the experimental continuum [39], but have not been included, because no systematic treatment of nonperturbative corrections is available to our knowledge. The result (7) is in good agreement with previous works employing sum rules and other recent precision determinations using various methods as well as the PDG average [7] as shown in Figure 2.

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PoS(FWNP)017

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PoS(FWNP)017