PROCEEDINGS OF SCIENCE

Reduction of couplings and its application in particle physics
Finite theories
Higgs and top mass predictions

A guided tour through thirty years of research
based on twenty-three articles

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Preface

In this report we tell the story of the notion *reduction of couplings* as we witnessed it in the course of time. Born as an innocent child of renormalization theory it first served the study of asymptotic behavior of several couplings in a given model. Reduced couplings appeared as functions of a primary one, compatible with the renormalization group equation and thus solutions of a specific set of ordinary differential equations. If these functions have the form of power series the respective theories resemble standard renormalizable ones and thus widen considerably the area covered until then by symmetries as a tool for constraining the number of couplings consistently. Still on the more abstract level reducing couplings enabled one to construct theories with β-functions vanishing to all orders of perturbation theory. Reduction of couplings became physicswise truely interesting and phenomenologically important when applied to the standard model and its possible extensions. In particular in the context of supersymmetric theories it became the most powerful tool known today once it was learned how to apply it also to couplings having dimension of mass and to mass parameters. Technically this all relies on the basic property that reducing couplings is a renormalization scheme independent procedure. Predictions of top and Higgs mass prior to their experimental finding highlight the fundamental physical significance of this notion. Twenty-two original articles and one set of lectures are being commented, put into historical perspective and interrelated with each other.

I would like to thank all authors for their contributions which constitute the core of the present book.
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Klaus Sibold

Leipzig, Germany
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Geleitwort

In spite of their limitations, perturbative local field theories are still of prominent practical value.

It is remarkable that the intrinsic ambiguities connected with locality and causality – most of the time associated with ultraviolet infinities – can be summarized in terms of a formal group which acts in the space of the coupling constants or coupling functions attached to each type of local interaction.

It is therefore natural to look systematically for stable submanifolds. Some such have been known for a long time: e.g., spaces of renormalizable interactions and subspaces characterized by systems of Ward identities mostly related to symmetries.

A systematic search for such stable submanifolds has been initiated by W. Zimmermann in the early eighties.

Disappointing for some time, this program has attracted several other active researchers and recently produced physically interesting results.

It looks at the moment as the only theoretically founded algorithm potentially able to decrease the number of parameters within the physically favoured perturbative models.

Raymond Stora, CERN (Switzerland), December16, 2013

Zueignung

We dedicate this work to *Reinhard Oehme* – friend and colleague.
The authors

PoS(Higgs & top)001

List of papers treated in this work

W. Zimmermann
Reduction in the number of coupling parameters
Commun. Math. Phys. **97** (1985) 211-225

R. Oehme, W. Zimmermann
Relation between effective couplings for asymptotically free models
Commun. Math. Phys. **97** (1985) 569-582

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Phys. Letts. **B153** (1985)142-146

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Nucl. Phys. **B259** (1985) 331-350

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Phys. Letts. **B427** (1998) 291-299

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Journ. of High Energy Physics 0807 (2008) 135-164

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Phys. Letts. **B718** (2013) 1430-1435

M. Mondragón, N.D. Tracas, G. Zoupanos
Reduction of Couplings in the MSSM
Phys. Letts. **B728** (2014) 51-57

Contents

1 Introduction

Particle physics of today is well described by relativistic quantum field theory (QFT) based on flat Minkowski spacetime. Comparison with experiment works astonishingly well within the context of a gauge theory based on the group $SU(3) \times SU(2) \times U(1)$, the so called standard model (SM). Although quarks and gluons are confined to form baryons and mesons a perturbative treatment of the SM yields predictions which are in excellent agreement with experiment and in practical terms one is able to separate quite well the nonperturbative aspects from the perturbative ones. Similarly gravitational effects do not yet seriously require to be considered in particle physics although astrophysical results clearly point to the existence of dark matter and pose the "missing mass" problem to which the SM does not give an answer. If one is interested in the description of particle physics only, one may thus hand over these fundamental problems to string theory, quantum gravity or (non-commutative) extensions of spacetime and study the SM and its extensions on flat spacetime in their own right.

It is precisely the outcome of such studies which we present here in its historical context. The SM requires as input from experiment many parameters: couplings, masses and mixing angles. Too many – according to the taste of quite a few people – to be considered as being fundamental. Hence one calls for ideas to restrict the number of parameters without spoiling the successes of the SM. The two main principles which we invoke here are: reduction of coupling parameters and finiteness. The first one relies on the discovery that parameters which are a priori independent permit ordering according to the degree with which they die out when performing an asymptotic expansion for small coupling in agreement with renomalization group equations. So, to be specific, one may express "secondary" couplings as power series in a "primary" one, hence the secondaries go together with the primary to zero (obviously faster). The general solutions of the renormalization group equations for the secondaries are then found as deviations from their power series in the primary coming with some arbitrary coefficient: the integration constant which carries the information that the secondary can also be an independent coupling in its own right. But again: "reduction" works if this additional contribution (which can depend on logarithms also) goes faster to zero than the power series. The concept "finiteness" is most easily realized in the context of supersymmetric theories and is in its truely physical form understood as the vanishing of β-functions, because those can be constructed as gauge parameter independent quantities. Anomalous dimensions to the contrary are usually gauge parameter dependent, hence only their gauge parameter independent parts may be considered as physical and required to vanish.

Although we stressed here the notion of reduction and finiteness as convenient tools to search for a theoretically appealing and experimentally satisfactory theory of particle physics it is clear that they are interesting areas of research in their own right.

The present report is not to be understood as a traditional review paper, but rather as a guide to existing literature in which these principles have been developped and fused to the aim spelled out above: enriching the SM without loosing its benefits. We therefore have first chosen those original papers where the respective ideas have been worked out; then we put them into a logical order (which is almost the same as time ordering) and – hopefully the most valuable contribution – commented them, in particular by relating them amongst each other.

The outline of the report is as follows. The papers of section 2 introduce the notion of "reduction of couplings". In the examples treated there it becomes in particular clear that

a stability analysis of (power series) solutions of the reduction equations is the appropriate tool for embedding them in an enlightening neighbourhood. Many more examples have been worked out, they can be found in reviews which we quote. Section 3 is devoted to the application of the reduction method to the SM. It turns out that a refined notion, called “partial reduction”, is needed in order to deal with the problem of different asymptotic behavior (UV- versus IR-freedom) of the couplings. It was possible to give either values or bounds to the Higgs and top mass. In section 4 two topics are introduced: finiteness in $N = 1$ supersymmetric gauge theories and an extension of the reduction method for including parameters carrying mass dimension together with the proof that the reduction method is renormalization scheme independent. Whereas the finiteness papers provide simple necessary and sufficient criteria for vanishing β-functions operating at one-loop order the other paper is crucial for correctly and efficiently controlling all types of susy breaking needed later on. Based on values of α_s etc. around 1990 reduction of couplings in the SM eventually predicted for the Higgs mass roughly 65 GeV, for the top mass roughly 100 GeV. Cancellation of quadratical divergencies was already at the borderline of being compatible with these numbers. Soon later precision experiments pointed towards higher mass values. Trusting the reduction method, i.e. the relevance of asymptotic expansions it was tempting to go one step further and to ask for finiteness. Thus, section 5 has been devoted to the development of this line of thought and some of its ramifications. The key notion here became reduction of parameters carrying dimension. It is based on the observation that also such parameters can give rise to closed renormalization orbits which can be found this way.
Still one remark for reading. Every section starts with an introduction putting the subsections which consist of an original paper plus comment into the respective context. Section 6 contains discussion and conclusions for the whole set of papers.

2 Fundamentals: Asymptotic freedom, reduction of couplings

Klaus Sibold
In the context of QCD an important property of the gauge coupling has been found: introducing an effective coupling which depends on the characteristic energy scale of some process under consideration it is seen that this coupling **de**creases in strength when the energy **in**creases. So, for infinite energy the coupling vanishes and the theory becomes free: this behaviour has been coined (UV-) asymptotic freedom. This observation has first been made in the context of perturbation theory but also non-perturbatively it played an important role in the study of QCD.
It is then a natural question to ask in theories of more than one coupling for a criterion that guarantees asymptotic freedom for all couplings. This analysis has been performed by Zimmermann and Oehme and lead Zimmermann by eliminating the running parameter in terms of one – the "primary" – coupling to a set of ordinary differential equations, the "reduction equations". Those are therefore to be studied and solved. The special case of asymptotic freedom suggests to demand that all couplings vanish together with the primary one in the limit of weak coupling. One may hope that the model being considered in perturbation theory has a non-perturbative analogue to which it is a reasonable approximation.

2.1 Reduction in the number of coupling parameters

Title: Reduction in the number of coupling parameters
Author: W. Zimmermann
Journal: Commun. Math. Phys. **97** (1985) 211-225

Comment (*Wolfhart Zimmermann*)
The standard model of elementary particles involves a large number of parameters which are not constrained by any symmetry. Therefore, it is of considerable interest to find general concepts in quantum field theory which can be used for reducing the number of independent parameters even in cases where no suitable symmetry is available.
In the present work renormalizable models of quantum field theory are considered which describe massless particles with an interaction given by several coupling terms in the Lagrangian. A normalization mass is introduced for the purpose of normalizing fields and defining finite coupling parameters. The renormalized Green's functions of the model can be expanded as power series in the coupling parameters at any given value of the normalization mass.
Field operators are normalized by their propagators at the normalization mass. Coupling parameters are conveniently defined by specific values of appropriate vertex functions at the normalization mass. The normalization mass is an auxiliary parameter which may be chosen arbitrarily. A change of the normalization mass merely implies a redefinition of fields and coupling parameters without affecting the model as such. So the field operators are multiplied by positive factors. The coupling parameters are modified by their defining vertex function at the new value of the normalization mass. Thus an equivalent

description of the model is obtained. These equivalence transformations constitute the renormalization group under which the system stays invariant.
The reduction principle proposed in this paper requires that all couplings can be expressed as functions of one of them, the primary coupling, such that the resulting system is again invariant under the renormalization group. Moreover, the following requirements are imposed on the reduced couplings as functions of the primary one:

(i) The dependence should not involve the normalization mass,
(ii) in the weak coupling limit the reduced couplings should vanish together with the primary coupling,
(iii) the reduced couplings can be expanded with respect to powers of the primary coupling.

The first condition is obvious, since the normalization mass is only an auxiliary parameter. Requirement (ii) also seems natural, but is already quite restrictive. It cannot be imposed for many models. If the reduced model should resemble a renormalizable theory, all couplings should have power series in the primary coupling (requirement (iii)). Under this condition there is usually only a finite number of solutions, if any.
Invariance under the the renormalization group leads to partial differential equations for the Green's functions with respect to the couplings and the normalization mass. Comparing these equations for the original and the reduced system one finds a set of ordinary differential equations for the coupling parameters as functions of the primary coupling. Its solutions should satisfy the requirements (i) – (iii). These are the reduction equations which form the basis for the studies in this work.
Any symmetry of a system by which all couplings can be expressed in terms of a single one certainly leads to a solution of the reduction equations provided the symmetry can be implemented in all orders of perturbation theory. In cases where a symmetry cannot be established in higher orders the reduction method may still lead to a corresponding solution valid in all orders. But the main purpose of this work is to provide the basis for finding reductions of a system which are not related to any symmetry.
An example is the Yukawa interaction of a spinor and a pseudoscalar field with a quartic interaction of the pseudoscalar field in addition. Here the reduction equation has a unique solution which expresses the coupling of the quartic interaction as a function of the Yukawa coupling. No symmetry seems to be involved in this case.
Finally the massless Wess-Zumino model is treated with two independent couplings, the Yukawa coupling and the coupling of the quartic interaction of the scalar and the pseudoscalar field. One solution of the reduction equation corresponds to the supersymmetric case considered by Wess and Zumino. In addition one finds a family of solutions with an arbitrary parameter – an exceptional case with an infinite number of reduction solutions. A corresponding symmetry is not known.

Commun. Math. Phys. 97, 211–225 (1985)

Communications in
Mathematical
Physics

Reduction in the Number of Coupling Parameters

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Dedicated to the memory of Kurt Symanzik

Abstract. A method is developed for reducing the formulation of massless models with several independent couplings to a description in terms of a single coupling parameter. The original as well as the reduced system are supposed to be renormalizable and invariant under the renormalization group. For most models there are, if any, only a finite number of reductions possible including those which correspond to symmetries of the system. The reduction method leads to a consistent formulation of the reduced model in any order of perturbation theory even in cases where it is difficult to establish a symmetry in higher orders. An example where no symmetry seems to be involved is the interaction of a spinor field with a pseudoscalar field. For this model the reduction method determines the quartic coupling constant uniquely as a function of the Yukawa coupling constant. The Wess-Zumino model is an exceptional case for which the reduction method admits an infinite number of solutions besides the supersymmetric one.

1. Introduction

Symmetry considerations provide a natural method of reducing the number of independent parameters in models of quantum field theory. If a symmetry is imposed, otherwise unconstrained coupling parameters become related among each other so that the number of independent parameters is decreased. Renormalizability of the model is maintained provided anomalies are absent and the symmetry can be implemented in all orders of perturbation theory.

In this paper a more general approach for reducing the number of coupling parameters is taken which is based on the principles of renormalizability and invariance under the renormalization group. It turns out that these requirements severely limit the possibilities of constraining the coupling parameters to a single independent one. The method is developed for the reduction of massless models from $n+1$ coupling parameters $\lambda_0, \lambda_1, \ldots, \lambda_n$ to a description in terms of λ_0 only. Any symmetry requirement leading to a renormalizable formulation is certainly included by this treatment. In fact, hidden symmetries could be detected in this way. On the other hand there are cases where the general reduction is possible and unique, but no symmetry is known to be involved. It is also conceivable that a symmetry can only be implemented in low orders while the general reduction method leads to a unique prescription in all orders of perturbation theory. In such a case a renormalizable formulation of the reduced model is obtained for which, however, the relevant symmetry is only realized in low orders.

In Sect. 2 the general conditions are studied under which a reduction is possible. For the coupling parameters λ_j as functions of λ_0 the ordinary differential equations

$$\beta_0 \frac{d\lambda_j}{d\lambda_0} = \beta_j \tag{1.1}$$

with

$$\lim_{\lambda_0 \to 0} \lambda_j = 0 \tag{1.2}$$

are found. β_j denotes the β-function corresponding to λ_j. Equation (1.1) can be derived either from the Callan-Symanzik equations [1, 2] or the evolution equations of the effective couplings. An interesting possibility is the special case that the β-function of the reduced system vanishes identically[1]. Then, after inserting the functions $\lambda_j(\lambda_0)$ the β-functions of the original system also vanish identically

$$\beta_i \equiv 0, \qquad i = 0, 1, \dots, n, \tag{1.3}$$

and the system (1.1) is trivially satisfied.

Renormalizability for the original as well as the reduced system implies that the functions $\lambda_j(\lambda_0)$ allow for power series expansions in λ_0. In lowest order one finds a system of quadratic equations for the constant lowest order approximations $\varrho_0^{(j)}$ of the ratios

$$\frac{\lambda_j}{\lambda_0} = \varrho_0^{(j)} + o(\lambda_0). \tag{1.4}$$

These are the eigenvalue conditions proposed by Chang for the ratios of coupling constants [4][2]. They form necessary conditions for the possibility of reducing the system. But without further restrictions they are not sufficient. For sometimes higher order effects prevent the extension of (1.4) to power series solutions of (1.1).

In Sect. 3 the case of two coupling parameters g^2 and λ is treated in detail by applying results from [6] and [7][3]. The β-functions are assumed to be of the form

$$\beta_0 = b_0 g^4 + \dots, \qquad \beta_1 = c_1 \lambda^2 + c_2 \lambda g^2 + c_3 g^4 + \dots. \tag{1.5}$$

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1 For some models arguments have been given indicating that the β-function vanishes to all orders of perturbation theory. See for instance [3]

2 Chang et al. applied the eigenvalue conditions to grand unification in order to build asymptotically free models with only one coupling constant. Unfortunately this program turned out to be too ambiguous due to the freedom in introducing heavy particles. See [5] which contains further references

3 The purpose of [6] was to find all asymptotically free solutions of the evolution equations with two coupling parameters. In this context the solutions of (1.1–2) were constructed by asymptotic expansions. Among the solutions found only the power series solutions are relevant for the present paper

A reduction to a renormalizable description in terms of g^2 is only possible if the quadratic equation

$$c_1\varrho_0^2+(c_2-b_0)\varrho_0+c_3=0 \tag{1.6}$$

has real roots, i.e. if the discriminant

$$\Delta=(c_2-b_0)^2-4c_1c_3\geqq 0 \tag{1.7}$$

is non-negative. For asymptotically free gauge theories with a Higgs coupling (1.7) coincides with the condition for asymptotic freedom found by Gross and Wilczek [8]. It is always satisfied for supersymmetric gauge theories where $\lambda=h^2$ with h describing a matter or Higgs interaction. In lowest order the ratio of the coupling parameters is given by one of the roots $\varrho_\pm$ of (1.6)

$$\frac{\lambda}{g^2}=\varrho_\pm+o(g^2)\,. \tag{1.8}$$

Unless

$$\xi=-\frac{c_1}{b_0}(\varrho_+-\varrho_-) \qquad (\varrho_+\geqq\varrho_-) \tag{1.9}$$

is an integer the lowest order term (1.8) can be completed to a power series expansion in g^2. The precise conditions under which an expansion for integral ξ is possible are stated in Sect. 4. It is further shown that by a reparametrization it can be arranged that the lowest order of a power series (1.8) becomes exact

$$\lambda'=\varrho_\pm g^2 \quad \text{if} \quad \varrho_\pm\neq 0 \tag{1.10}$$

and

$$\lambda'=\varrho_n g^{2n+2} \quad \text{if} \quad \lambda=o(g^{2n+2}) \tag{1.11}$$

with a suitably defined new coupling parameter λ'.

In the remainder of the paper the reduction method is applied to two models of special interest: Sect. 4 concerns the interaction of a spinor field with a pseudoscalar field. For a consistent formulation of the renormalization it is necessary to introduce a quartic selfinteraction of a scalar field since the Yukawa interaction alone would not render the four pseudoscalar vertex part convergent. The model thus involves two independent coupling constants, g for the Yukawa coupling and λ for the quartic interaction. No symmetry is known which would relate the two coupling constants. While the bare scalar coupling constant cannot be dropped, one might think of setting the renormalized coupling constant λ equal to zero in order to eliminate the additional parameter. However, formulations with different normalization points would then be inequivalent. On the other hand, the general reduction method leads to a unique power series expansion

$$\lambda=\varrho_\pm g^2+\varrho_1 g^4+\dots \tag{1.12}$$

of λ, thus providing a consistent renormalizable description with one coupling constant g only. The two values ϱ_+ and ϱ_- correspond to different signs of λ.

Finally the reduction method is discussed for models which become supersymmetric by imposing relations among the coupling constants. Special problems may occur for models which are not asymptotically free. In Sect. 5 the massless Wess-

Zumino model[4] [9] is treated with independent coupling constants g for the Yukawa and λ for the quartic coupling. A peculiar situation is found due to the fact that ξ is a negative integer

$$\xi = -3. \tag{1.13}$$

This leads to an asymptotic expansion of λ in the form[5]

$$\lambda = \varrho_+ g^2 + \varrho_1 g^4 + \varrho_2 g^6 + \varrho_3 g^8 + d g^8 \ln g^2 + \dots. \tag{1.14}$$

The coefficient ϱ_3 is arbitrary and d is determined uniquely by lower orders including the order g^6. For $d=0$ logarithms are absent and (1.14) represents a power series with arbitrary ϱ_3. Without using supersymmetry, calculations of order g^6 would be required to check whether or not $d=0$. But the existence of a renormalized supersymmetric formulation excludes the occurrence of logarithms so that $d=0$. With suitable supersymmetric normalization conditions one has

$$\lambda = \varrho_+ g^2 \tag{1.15}$$

for the supersymmetric solution and

$$\lambda = \varrho_+ g^2 + \varrho_3 g^8 + \sum_{n=4}^{\infty} \varrho_n g^{2n+2} \tag{1.16}$$

with arbitrary ϱ_3. Thus the general reduction method is not unique in this special case, but also admits infinitely many asymmetric reduced systems[6]. Even the relation (1.15) is not characteristic for the supersymmetric case since by an asymmetric redefinition of λ the relation (1.15) can always be restored.

No such problems seem to occur for supersymmetric models where the primary β-function is negative or vanishes in lowest order. For the $N=2$ and $N=4$ super Yang-Mills theories it was found that the relevant lowest order solutions can indeed be uniquely extended to power series expansions in the primary coupling constant [11]. General statements can be made about two-parametric models with β-functions of the form (1.5) and $b_0 \leq 0$. If $b_0 = 0$ and $\Delta > 0$ two power series can be constructed for λ with uniquely determined coefficients [7]. One of the expansions corresponds to the supersymmetric case. This includes a variety of models, in particular those which may have vanishing β-functions in any order of perturbation theory. If $b_0 < 0$ and $\Delta > 0$ the model is asymptotically free. Usually supersymmetric models with asymptotic freedom are unstable against perturbations of the symmetry [7, 12]. In the unstable case a unique power series for λ can be constructed [6, 7]. Thus in all these cases the general reduction method provides a unique formulation of the reduced model in every order of perturbation theory even though the symmetry may have been established for low orders only.

4 For the renormalization of the massless model see [10]

5 There is also a power series for λ which is not related to supersymmetry

6 Recently it has been shown by O. Piguet and K. Sibold that there is only one realization of supersymmetry in the perturbative treatment of the massless Wess-Zumino model [17]. Therefore, the additional reduced systems do not seem to be supersymmetric

2. General Method of Reduction

We consider a massless model of quantum field theory described by $n+1$ dimensionless coupling parameters $\lambda_0, \lambda_1, \ldots, \lambda_n$ and a normalization mass κ. The model is supposed to be invariant under the renormalization group. Our aim is to express $\lambda_1, \ldots, \lambda_n$ as functions of λ_0 so that a model involving a single coupling parameter λ_0 is obtained which is again invariant under the renormalization group. Accordingly we write each λ_j as a function of

$$\lambda_j = \lambda_j(\lambda_0), \tag{2.1}$$

independent of the normalization mass κ. The functions $\lambda_j(\lambda_0)$ should be differentiable in the domain of λ_0 considered and vanish in the weak coupling limit[7]

$$\lim_{\lambda_0 \to 0} \lambda_j(\lambda_0) = 0 . \tag{2.2}$$

For the Green's functions of the original system the invariance under the renormalization group implies the Callan-Symanzik equations

$$\left(\kappa^2 \frac{\partial}{\partial \kappa^2} + \sum \beta_j \frac{\partial}{\partial \lambda_j} + \gamma\right) \tau = 0 , \tag{2.3}$$

while for the Green's functions of the reduced system the equations

$$\left(\kappa^2 \frac{\partial}{\partial \kappa^2} + \beta' \frac{\partial}{\partial \lambda_0} + \gamma'\right) \tau' = 0 \tag{2.4}$$

follow. The β- and γ-functions depend on the coupling parameters only. β' and γ' are functions of the single variable λ_0. γ and γ' are additive in the contributions from the field operators occurring in the Green's functions. τ is a function of the momenta, the coupling parameters and the normalization mass κ. τ' is obtained from τ by substituting the functions (2.1) for the parameters λ_j. Accordingly,

$$\frac{\partial \tau'}{\partial \lambda_0} = \frac{\partial \tau}{\partial \lambda_0} + \sum_{j=1}^{n} \frac{\partial \tau}{\partial \lambda_j} \frac{d \lambda_j}{d \lambda_0} .$$

Linear independence of the Green's functions and their derivatives leads to the relations

$$\beta' = \beta_0, \ \gamma' = \gamma, \ \beta' \frac{d\lambda_j}{d\lambda_0} = \beta_j .$$

Hence the functions (2.1) must satisfy the following system of ordinary differential equations

$$\beta_0 \frac{d\lambda_j}{d\lambda_0} = \beta_j . \tag{2.5}$$

On the other hand, if the functions (2.1) satisfy (2.5), the reduced form (2.4) of the Callan-Symanzik equations follows. Thus the system (2.5) forms a necessary and sufficient condition for reducing the original system by the functions $\lambda_j(\lambda_0)$.

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7 The condition of renormalizability requiring that the functions λ_j can be expanded with respect to powers of λ_0 will not be used for the time being

It is instructive to use an alternative method for the derivation of (2.5) by eliminating the scale variable from the evolution equations of the effective couplings. At a normalization mass κ'^2 we impose on the coupling parameters that the values λ'_j of λ_j are given functions of the value λ'_0 of λ_0,

$$\lambda'_j = \lambda_j(\lambda'_0) \quad \text{at} \quad \kappa^2 = \kappa'^2 . \tag{2.6}$$

We want to investigate under which restrictions on the functions the same dependence holds at other normalization points:

$$\lambda_j = \lambda_j(\lambda_0) \quad \text{at} \quad \kappa^2 . \tag{2.7}$$

If the normalization mass is changed from κ'^2 to κ^2 the field operators φ_i of the system undergo a transformation of the renormalization group,

$$\varphi_j(x, \lambda_0, \dots, \lambda_n, \kappa^2) = z_j^{1/2} \varphi_j(x, \lambda'_0, \dots, \lambda'_n, \kappa'^2)$$

with positive z_j[8]. The new values of the coupling parameters are given by

$$\lambda_0 = \bar{\lambda}_0(u, \lambda'_0, \lambda'_1, \dots, \lambda'_n) , \tag{2.8}$$

$$\lambda_j = \bar{\lambda}_j(u, \lambda'_0, \lambda'_1, \dots, \lambda'_n) , \tag{2.9}$$

$$u = \kappa^2/\kappa'^2, \qquad j = 1, \dots, n .$$

The functions $\bar{\lambda}_0$, $\bar{\lambda}_j$ denote effective couplings suitably defined as analytic functions of u which are regular at any positive value of u[9].

In order to determine the constraints on the functions λ_j we take a fixed initial value $\lambda'_0 \neq 0$, and first discuss the case where

$$\beta_0(\lambda'_0, \lambda'_1, \dots, \lambda'_n) \neq 0 . \tag{2.10}$$

Expression (2.10) equals the value of $\partial\bar{\lambda}_0/\partial u$ at $u=1$, $\lambda_0 = \lambda'_0$, $\lambda_j = \lambda'_j$. Since $\bar{\lambda}_0$ is regular analytic at $u=1$ the derivative $\partial\bar{\lambda}_0/\partial u$ is continuous near $u=1$ so that

$$\frac{\partial\bar{\lambda}_0}{\partial u}(u, \lambda'_0, \lambda'_1, \dots, \lambda'_n) \neq 0 \tag{2.11}$$

in a neighborhood of $u=1$. Therefore, Eq. (2.8) can be inverted with respect to u. Inserting the inversion

$$u = u(\lambda_0; \lambda'_0, \lambda'_1, \dots, \lambda'_n)$$

into (2.9) we find that the λ_j necessarily become functions of λ_0 which are independent of the normalization mass κ. By definition they represent the functions λ_j in (2.6–7):

$$\lambda_j(\lambda_0) = \bar{\lambda}_j(u(\lambda_0; \lambda'_0, \dots, \lambda'_n), \lambda'_0, \dots, \lambda'_n) . \tag{2.12}$$

With the help of the evolution equations

$$u\frac{\partial\bar{\lambda}_i}{\partial u} = \beta_i(\bar{\lambda}_0, \lambda_1(\bar{\lambda}_0), \dots, \lambda_n(\bar{\lambda}_0)), \qquad i = 0, 1, \dots, n , \tag{2.13}$$

8 For the concept of the renormalization group used here see [13]

9 For the possibility of defining effective couplings as analytic functions see [14]

the scale variable u can be eliminated near $u=1$. We thus obtain (2.5) in the form

$$\frac{d\lambda_j}{d\lambda_0}=\frac{\beta_j(\lambda_0,\lambda_1,\ldots,\lambda_n)}{\beta_0(\lambda_0,\lambda_1,\ldots,\lambda_n)},\quad j=1,\ldots,n,$$

valid in the neighborhood of $\lambda_0=\lambda_0'$.

We next discuss the case where

$$\beta_0(\lambda_0',\lambda_1',\ldots,\lambda_n')=0. \tag{2.14}$$

In this case the function $\beta_0(\lambda_0,\lambda_1(\lambda_0),\ldots,\lambda_n(\lambda_0))$ has a zero at $\lambda_0=\lambda_0'$. Then (2.13) implies

$$\frac{\partial\bar{\lambda}_0}{\partial u}=0\quad\text{at}\quad u=1 \tag{2.15}$$

for the function

$$\bar{\lambda}_0=\bar{\lambda}_0(u,\lambda_0',\lambda_1',\ldots,\lambda_n'). \tag{2.16}$$

Since $\bar{\lambda}_0$ is regular analytic in u at $u=1$ it is

$$\frac{\partial\bar{\lambda}_0}{\partial u}\neq 0\quad\text{for}\quad u\neq 1\quad\text{near}\quad u=1 \tag{2.17}$$

or

$$\frac{\partial\bar{\lambda}_0}{\partial u}\equiv 0. \tag{2.18}$$

Hence the function (2.16) is either variable in u and stationary at $u=1$ or it is constant[10].

In case (2.17) of variable $\bar{\lambda}_0$ we may invert (2.8) for $u<1$ as well as $u>1$, obtaining

$$u=u_-(\lambda,\lambda_0',\lambda_1',\ldots,\lambda_n')\quad\text{for}\quad u<1, \tag{2.19}$$

$$u=u_+(\lambda,\lambda_0',\lambda_1',\ldots,\lambda_n')\quad\text{for}\quad u>1. \tag{2.20}$$

If $\bar{\lambda}_0$ has an extremal value at $u=1$ the inversions u_- and u_+ denote different branches of u both defined for $\lambda_0<\lambda_0'$ in case of a maximum or $\lambda_0>\lambda_0'$ in case of a minimum. Inserting (2.19) and (2.20) into (2.9) we find two sets of functions λ_j of λ_0 which must be identical to (2.6) and thus to each other. For $\lambda_0\neq\lambda_0'$ again (2.5) follows. Equation (2.5) can be extended to $\lambda_0=\lambda_0'$ by taking the limit $\lambda_0\to\lambda_0'$.

We now turn to the case (2.18) of constant $\bar{\lambda}_0$. Equations (2.7) and (2.8) imply that λ_j does not depend on u either, so that by (2.13) also the other β-functions vanish. The system (2.5) is then trivially satisfied.

We summarize the results as follows: In all cases the functions satisfy the system (2.5) of ordinary differential equations in agreement with the derivation given in the first part of this section. If λ_0 is a zero of one of the β-functions – with the other coupling parameters expressed as functions (2.6) of λ_0' – the system (2.5) implies that λ_0' is a zero of all β-functions. It follows that the effective couplings are either variable in u and stationary at $u=1$ or they are all independent of u.

We provide some further information on the zeroes of the β-functions considered. Zeroes of the first type with variable effective couplings are always

10 For a discussion of zeroes of the β-function which correspond to stationary values of the effective coupling see [14]

isolated. For it is

$$\beta(\bar{\lambda}_0, \lambda_1(\bar{\lambda}_0), \ldots, \lambda_n(\bar{\lambda}_0)) = \frac{\partial \bar{\lambda}_0}{\partial u} \neq 0 \quad \text{for} \quad \bar{\lambda}_0 \neq \lambda'_0$$

in a neighborhood of λ'_0. We further observe that a zero must necessarily be of the second type with constant effective couplings if all derivatives

$$\frac{d^m}{d\lambda_0^m} \beta(\lambda_0, \lambda_1(\lambda_0), \ldots, \lambda_n(\lambda_0)) \tag{2.21}$$

exist at $\lambda_0 = \lambda'_0$. For then all derivatives of $\bar{\lambda}_0$ with respect to u vanish at $u=1$ as follows by differentiating the evolution equation (2.13) of $\bar{\lambda}_0$. Sufficient for the existence of the derivatives (2.21) is the existence of all partial derivatives of β[11]. The functions $\lambda_j(\lambda_0)$ were assumed to be differentiable and the existence of their higher derivatives follows by differentiating the system (2.5).

Zeroes of the β-functions with the effective couplings independent of the scale variable need not be isolated. In fact, arguments have been given for some supersymmetric models that reduced forms exist with β-functions vanishing in any order of perturbation theory. If this should prevail independent of perturbation theory the relations

$$\beta_i(\lambda_0, \lambda_1(\lambda_0), \ldots, \lambda_n(\lambda_0)) = 0, \qquad i = 0, 1, \ldots, n, \tag{2.22}$$

would vanish identically in λ_0 for some functions $\lambda_j(\lambda_0)$.

The reducibility condition (2.5) allows for a large class of solutions unless further restrictions are imposed. In a region of non-vanishing β_0 the Lipschitz condition can be verified for the ratios β_j/β_0 provided certain differentiability assumptions on the β-functions are made. With this the Picard-Lindelöf theorem applies according to which exactly one solution $\lambda_j(\lambda_0)$ of (2.5) passes through any point $\lambda'_0, \lambda'_1, \ldots, \lambda'_n$. Due to the singular nature of the system (2.5) at $\lambda_0 = \lambda_j = 0$, the standard existence theorems cannot be applied there. On the other hand it is difficult to gain control over the asymptotic behavior in the weak coupling limit for solutions with prescribed non-vanishing initial values $\lambda'_0, \ldots, \lambda'_n$. In general, uniqueness properties do not hold for solutions passing through the origin $\lambda_0 = \lambda_j = 0$: For some systems there are no solutions of (2.5) which satisfy (2.2). For others there are infinitely many such solutions.

Further constraints are imposed if we require renormalizability for the original as well as the reduced system. Then the Green's functions of the original system have power series expansions in $\lambda_0, \lambda_1, \ldots, \lambda_n$ and the Green's functions of the reduced system can be expanded with respect to powers of λ_0[12]. This leads to the requirement that the solutions $\lambda_j(\lambda_0)$ of (2.5) possess power series expansions in λ_0.

It is easy to work out the conditions necessary for the renormalizable reduction of a system in lowest order of the primary coupling constant. As example we

11 A stationary value of the effective coupling indeed leads to a singular behavior for the derivatives of the β-function (see [14])

12 We do not consider here expansions with respect to fractional powers or logarithms of coupling constants which may arise due to infrared singularities of conventional perturbation theories (see for instance [15, 16])

consider β-functions with the expansions

$$\beta_0 = b_0\lambda_0^2 + \sum_{n=3}^{\infty}\sum_{m=0}^{n-1}\sum_{j_1\dots j_m} b_{n-m,j_1\dots j_m}\lambda_0^{n-m}\lambda_{j_1}\dots\lambda_{j_m}, \tag{2.23}$$

$$\begin{aligned}\beta_j = {} & \sum_{ik} c_{ik}^{(j)}\lambda_i\lambda_k + \sum_i c_i^{(j)}\lambda_i\lambda_0 + c^{(j)}\lambda_0^2 \\ & + \sum_{n=3}^{\infty}\sum_{m=0}^{n}\sum_{j_1\dots j_m} c_{n-m,j_1\dots j_m}\lambda_0^{n-m}\lambda_{j_1}\dots\lambda_{j_m}\end{aligned} \tag{2.24}$$

where λ_0 is the square $\lambda_0 = g^2$ of the primary coupling parameter g. Since all β-functions are even functions of g it is natural to require that the coupling parameters λ_j of the reduced system are also even in g. Renormalizability combined with the condition (2.2) implies that the coupling parameters of the reduced system have power series expansions

$$\lambda_j = \varrho_0^{(j)} g^2 + \sum_{n=2}^{\infty} \varrho_n^{(j)} g^{2n+2}. \tag{2.25}$$

Comparing the coefficients of g^4 in (2.5) we find the quadratic equations [4]

$$\sum_{ik} c_{ik}^{(j)}\varrho_0^{(i)}\varrho_0^{(k)} + \sum_i (c_i^{(j)} - \delta_{ij}b_0)\varrho_0^{(i)} + c^{(j)} = 0. \tag{2.26}$$

Its solutions $\varrho_0^{(j)}$ represent the lowest order values of the ratios λ_j/g^2. As such they should be real and – if required by the model – satisfy constraints like the positivity of coupling parameters. The equations (2.26) are necessary for the renormalizability of the reduced system, but not always sufficient. For in some cases the lowest order approximation based on a solution of (2.26) cannot be extended to power series expansions. Examples for that will be found in the following section.

3. Two Coupling Parameters

We are going to discuss in some detail the reduction of systems involving two coupling constants. The notation used is

$$\left(\kappa^2\frac{\partial}{\partial\kappa^2} + \beta_0(g^2,\lambda)\frac{\partial}{\partial g^2} + \beta_1(g^2,\lambda)\frac{\partial}{\partial\lambda} + \gamma\right)\tau = 0 \tag{3.1}$$

for the Callan-Symanzik equations. The β-functions are assumed to have expansions of the form

$$\beta_0 = b_0 g^4 + \sum_{n=3}^{\infty}\sum_{m=0}^{n-1} b_{n-m,m} g^{2(n-m)}\lambda^m, \tag{3.2}$$

$$\begin{aligned}\beta_1 = {} & c_1\lambda^2 + c_2\lambda g^2 + c_3 g^4 \\ & + \sum_{n=3}^{\infty}\sum_{m=0}^{n} c_{n-m,m} g^{2(n-m)}\lambda^m,\end{aligned} \tag{3.3}$$

which cover a large variety of models. We want to investigate under which conditions the model can be reduced by

$$\lambda = \lambda(g^2) \tag{3.4}$$

to a renormalizable system involving a single coupling constant g. The reducibility condition (2.5) takes the form

$$\beta_0 \frac{d\lambda}{dg^2} = \beta_1 . \tag{3.5}$$

Renormalizability and condition (2.2) impose on the solutions that they can be expanded in the form

$$\lambda = \varrho_0 g^2 + \sum_{j=1}^{\infty} \varrho_j g^{2j+2} . \tag{3.6}$$

The first coefficient ϱ_0 is determined to be a root of the quadratic equation

$$c_1 \varrho_0^2 + (c_2 - b_0)\varrho_0 + c_3 = 0 . \tag{3.7}$$

ϱ_0 is only real if the discriminant

$$\Delta = (c_2 - b_0)^2 - 4c_1 c_2 \tag{3.8}$$

is non-negative

$$\Delta \geqq 0 . \tag{3.9}$$

This requirement already precludes the reduction for a large number of models. In the work that follows (3.9) will be assumed. There may be further restrictions on the values of the first coefficients ϱ_0. For instance, in some models λ is the square of a coupling parameter and cannot be negative for that reason. In this case only non-negative values of $\varrho_\pm$ are admissible.

For the case

$$b_0 \neq 0, \quad c_1 \neq 0 , \tag{3.10}$$

we may take over the results obtained in [6] concerning power series solutions of (3.5). The following notations will be used. $\varrho_\pm$ denotes the roots of (3.7) with ϱ_+ being the larger value,

$$\varrho_+ \geqq \varrho_- . \tag{3.11}$$

A number ξ is defined by

$$\xi = -\frac{c_1}{b_0}(\varrho_+ - \varrho_-), \quad (b_0 \neq 0) . \tag{3.12}$$

Since usually $c_1 > 0$ positive ξ implies asymptotic freedom.

If $\xi < 0$ a power series solution

$$\lambda_- = \varrho_- g^2 + \sum_{j=1}^{\infty} \varrho_{-j} g^{2j+2} \tag{3.13}$$

of (3.5) exists with uniquely determined coefficients. Further the solution

$$\lambda_+ = \varrho_+ g^2 + \sum_{j=1}^{\infty} \varrho_{+j} g^{2j+2} \tag{3.14}$$

exists if $\xi<0$ is not integral. If ξ is a negative integer one finds the general solution of (3.5) as an expansion involving logarithms

$$\lambda_+ = \varrho_+ g^2 + \sum_{j=1}^{|\xi|-1} \varrho_{+j} g^{2j+2} + \varrho_{+|\xi|} g^{2\xi+2} + d g^{2|\xi|+2} \ln g^2 + \dots . \tag{3.15}$$

The coefficient $\varrho_{+|\xi|}$ is arbitrary, the coefficient d of the first logarithmic term is uniquely determined by lower orders. If $d=0$ no power series solution λ_+ of (3.5) exists. In that case a solution with asymptotic behavior $\varrho_+ g^2$ for $g\to 0$ can only be formed by including logarithmic terms which do not correspond to a renormalizable Lagrangian. If $d=0$ the power series solution (3.15) exists with arbitrary coefficient $\varrho_{+|\xi|}$ and represents the general solution. Thus for negative integral ξ either no power series solution λ_+ exists or λ_+ represents the general solution of (3.5) with an arbitrary parameter.

If $\xi>0$ a power series (3.14) always exists for λ_+. The power series (3.13) for λ_- exists provided ξ is not integral. If ξ is a positive integer either no power series solution (3.13) exists or it represents the general solution with arbitrary coefficient $\varrho_{-(\xi+1)}$.

For $\xi=0$ both expansions (3.13–14) coincide and exist with uniquely determined coefficients.

The case $b_0=0$, $c_1 \neq 0$ was treated in [7]. If $\Delta>0$, there exist two distinct power series solutions λ_+ and λ_- of the form (3.13–14). Although their coefficients are unique, they may include the general solution. For the difference of two solutions with the same weak coupling behavior $\varrho_+ g^2$ (or $\varrho_- g^2$ respectively) is exponentially decreasing for $g\to 0$. If $b_0=0$ and $\Delta=0$ no power series solution of (3.5) exists unless all coefficients of terms g^{2n} in β_1 vanish. In the latter case $\lambda\equiv 0$ is the only power series solution.

We now discuss the simplifications which occur for supersymmetric gauge theories with $\lambda=h^2$, where h describes a matter or Higgs interaction. In that case all coefficients of the terms g^{2n} in β_1 vanish. As the square of a coupling parameter λ is non-negative. We further assume $c_1>0$ which is usually the case.

The absence of a g^4-term in β_1 implies that (3.9) is always satisfied. This eliminates a major obstacle in constructing renormalizable reduced models. The roots of (3.7) become

$$\varrho_0=0 \quad \text{and} \quad \varrho_0=\frac{b_0-c_2}{c_1}. \tag{3.16}$$

Since β_1 vanishes at $\lambda=0$, Eq. (3.5) has the solution $\lambda\equiv 0$. Apart from this trivial solution we list the following power series solutions of (3.5) under the positivity constraint $\lambda\geqq 0$:

(1) $b_0<0$, $c_2<b_0$, or equivalently $\xi>0$, $\varrho_+>0$, $\varrho_-=0$.

There is the expansion (3.14) of λ_+ with unique coefficients. If ξ is an integer there is further an expansion of λ_-,

$$\lambda_- = d g^{2\xi+2} + \sum_{j=\xi+1}^{\infty} \varrho_{-j} g^{2j+2}, \qquad \xi=1,2,\dots, d>0, \tag{3.17}$$

with arbitrary positive coefficient d.

(2a) $b_0>0$, $c_2<b_0$, or equivalently $\xi<0$, $\varrho_+>0$, $\varrho_-=0$.

If ξ is not an integer there is the expansion (3.14) of λ_+ with unique coefficients. If ξ is a negative integer either λ_+ does not exist or the coefficient $d_{+|\xi|}$ is arbitrary.

(2b) $b_0>0$, $c_2>b_0$, or equivalently $\xi<0$, $\varrho_+=0$, $\varrho_-<0$.

If ξ is an integer there is the expansion

$$\lambda_+=dg^{2|\xi|+2}+\sum_{j=|\xi|+1}^{\infty}\varrho_{+j}g^{2j+2}, \qquad \xi=-1,-2,\ldots,d>0. \tag{3.18}$$

(3) $b_0=0$, $c_2<0$ implying $\varrho_+>0$, $\varrho_-=0$.

There is the expansion (3.14) of λ_+ with unique coefficients.

In all other cases, namely $b_0=c_2$ or $b_0\leqq 0$ with $c_2\geqq b_0$, there are no power series solutions except $\lambda\equiv 0$, which have $\lambda\geqq 0$ for sufficiently small g^2.

Finally we remark that the lowest order form $\lambda\approx\varrho_0 g^2$ of a power series expansion can be made exact by reparametrizing λ provided $\varrho_0\neq 0$. For the coefficients of

$$\lambda'=\lambda+a_1\lambda^2+a_2\lambda^3+\ldots \tag{3.19}$$

can be chosen such that

$$\lambda'\equiv\varrho_0 g^2. \tag{3.20}$$

If $\varrho_0=0$ the transformation (3.19) in general does not even lead to a polynomial form of λ'. But

$$\lambda'=\lambda+b_1\lambda g^2+b_2\lambda g^4+\ldots$$

can be used to transform a power series solution

$$\lambda=\varrho_n g^{2n+2}+o(g^{2n+4})$$

into

$$\lambda'\equiv\varrho_n g^{2n+2}.$$

4. Model of a Spinor and Pseudoscalar Field

We consider the massless renormalizable model of a single spinor field ψ interacting with a pseudoscalar field A. The interaction terms are

$$ig\bar{\psi}\gamma_5 A\psi-\frac{\lambda}{4!}A^4.$$

The model contains two independent parameters g and λ. We try to reduce the system to a renormalizable description in terms of g only. In lowest order the β-functions are

$$\begin{aligned}\beta_0&=\frac{1}{16\pi^2}5g^4+\ldots,\\ \beta_1&=\frac{1}{16\pi^2}\left(\frac{3}{2}\lambda^2+4\lambda g^2-24g^4\right)+\ldots.\end{aligned} \tag{4.1}$$

From this the values

$$\begin{aligned}\varrho_+&=\tfrac{1}{3}+\tfrac{1}{3}\sqrt{145}>0,\\ \varrho_-&=\tfrac{1}{3}-\tfrac{1}{3}\sqrt{145}<0,\\ \xi&=-\tfrac{1}{5}\sqrt{145}<0\end{aligned} \tag{4.2}$$

follow. If λ is positive for sufficiently small g there is only one power series solution,

$$\lambda = \tfrac{1}{3}(1+\sqrt{145})g^2 + \varrho_{+1}g^4 + \varrho_{+2}g^6 + \dots, \tag{4.3}$$

uniquely determining λ as a function of g.* By a redefinition of λ, the lowest order can be made exact

$$\lambda = \tfrac{1}{3}(1+\sqrt{145})g^2 . \tag{4.4}$$

For completeness we quote the generalization of (3.15) from [6]

$$\begin{aligned}\lambda = {} & \tfrac{1}{3}(1+\sqrt{145})g^2 + \varrho_{+1}g^4 + \varrho_{+2}g^6 \\ & + d_{11}g^{\frac{2}{5}\sqrt{145+2}} + \varrho_{+3}g^8 + d_{12}g^{\frac{3}{5}\sqrt{145+4}} \\ & + \varrho_{+4}g^{10} + d_{21}g^{\frac{4}{5}\sqrt{145+2}} + \dots .\end{aligned} \tag{4.5}$$

The terms are ordered according to decreasing magnitude for $g \to 0$. ϱ_{+1} and ϱ_{+2} are unique. d_{11} is arbitrary, all other coefficients are determined for given d_{11}. The power series solution (4.3) is stable since the general solution (4.5) has the same asymptotic behavior for $g \to 0$. If λ is negative for sufficiently small g there is the power series

$$\lambda = \tfrac{1}{3}(1-\sqrt{145})g^2 + \varrho_{-1}g^4 + \varrho_{-2}g^6 + \dots \tag{4.6}$$

which is an unstable solution.

5. Wess-Zumino Model

We study the massless Wess-Zumino model with the coupling constants g and λ of the interaction terms

$$g\bar{\psi}(A + i\gamma_5 B)\psi - \frac{\lambda}{2}(A^2 + B^2)^2$$

treated as independent parameters. In lowest order the β-functions are

$$\begin{aligned}\beta_0 &= \frac{1}{16\pi^2} 12g^4 + \dots \\ \beta_1 &= \frac{1}{16\pi^2}(20\lambda^2 + 8\lambda g^2 - 16g^4) + \dots .\end{aligned} \tag{5.1}$$

From this the values

$$\varrho_+ = 1, \qquad \varrho_- = -\tfrac{4}{5}, \qquad \xi = -3 \tag{5.2}$$

follow. The solutions corresponding to the supersymmetric ratio $\lambda/g^2 \approx \varrho_+$ have the asymptotic expansion (3.15)

$$\lambda = g^2 + \varrho_{+1}g^4 + \varrho_{+2}g^6 + \varrho_{+3}g^8 + dg^8 \ln g^2 + \dots . \tag{5.3}$$

ϱ_{+1}, ϱ_{+2}, and d are uniquely determined. ϱ_{+3} is arbitrary. The higher order coefficients are determined for given ϱ_{+3}. The existence of a renormalized version of the supersymmetric model implies that a power series solution of λ exists.

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* See Note added in proof on p. 225

Therefore $d=0$, so that (5.3) takes the form

$$\lambda=g^2+\sum_{j=1}^{\infty}\varrho_{+j}g^{2j+2} \tag{5.4}$$

with arbitrary ϱ_{+3}. Only one of those corresponds to the supersymmetric case. With suitable supersymmetric normalization conditions it is

$$\lambda=g^2 \tag{5.5}$$

for the supersymmetric system and

$$\lambda=g^2+\varrho_{+3}g^8+\sum_{j=4}^{\infty}\varrho_{+j}g^{2j+2} \tag{5.6}$$

for the asymmetric reduced systems with arbitrary $\varrho_{+3}\neq 0$. The solution (5.3) is stable since its asymptotic behavior is the same as for the general solution (5.6). In addition there is the power series solution starting with $\varrho_{-}g^2$,

$$\lambda=-\frac{4}{5}g^2+\sum_{j=1}^{\infty}\varrho_{-j}g^{2j+2},$$

which is unstable and not related to supersymmetry.

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Note added in proof. This relation between the quartic and the Yukawa coupling constant was found independently by E. de Rafael and R. Stora (private communication).

2.2 Relation between effective couplings for asymptotically free models

Title: Relation between effective couplings for asymptotically free models
Authors: R. Oehme, W. Zimmermann
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Comment (*Wolfhart Zimmermann*)
Massless models of quantum field theory involving two couplings g and λ are considered which are renormalizable and asymptotically free. Momentum dependent effective couplings $\bar{g}$ and $\bar{\lambda}$ (also called running coupling parameters) are introduced by appropriate vertex functions at suitably chosen momentum configurations. By the principle of asymptotic freedom the effective couplings vanish in the high momentum limit. The purpose of this paper is to derive relations between the effective couplings which aymptotically hold for large momenta or small coupling values.
The momentum dependence of the effective couplings is controlled by the evolution equations which are ordinary differential equations whith respect to the momentum variable. By eliminating the momentum variable one obtains an ordinary differential equation for $\bar{\lambda}$ as a function of $\bar{g}$ which has the form of a reduction equation with the corresponding β-functions as coefficients. For studying the high momentum behavior the β-functions are expanded with respect to powers of $\bar{g}$ and $\bar{\lambda}$. It is assumed that powers of $\bar{\lambda}$ only are absent in the expansion of the β-function associated with the coupling $\bar{g}$. This should cover most applications. With the β-functions approximated to lowest order the differential equation $\bar{\lambda}(\bar{g})$ can be solved exactly. Including all higher powers one finds asymptotic expansions for $\bar{\lambda}(\bar{g})$ involving powers (including fractional or irrational exponents) and possibly logarithmic terms. The solutions obtained are complete in the sense as they generalize the exact solutions found in lowest order.

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Relation Between Effective Couplings for Asymptotically Free Models

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Abstract. For asymptotically free models with two independent couplings asymptotic expansions are constructed which express one effective coupling in terms of the other. The expansions involve powers (including fractional or irrational exponents) and logarithms. All orders of the β-functions are taken into account. The expansions found are complete in the sense that they represent solutions (exact to any order) which generalize all the solutions obtained with the β-functions approximated to second order. It is shown that higher orders are relevant since it is not possible in general to reparametrize the system such that the β-functions become polynomials of the coupling parameters. The simplifications in case of supersymmetric models are discussed.

1. Introduction

In this paper asymptotic properties of effective couplings will be studied for massless field theoretical models which are asymptotically free and involve two coupling constants. As example may serve a non-Abelian gauge field of coupling constant g to which a Higgs field with interaction constant λ is coupled. The effective coupling parameters $\bar{g}$ and $\bar{\lambda}$ are defined as functions of the coupling constants, a Euclidean momentum variable $k^2<0$ and a normalization mass $\kappa^2<0$. In terms of dimensionless variables,

$$\bar{g}=\bar{g}(u,g,\lambda)\,,\qquad \bar{\lambda}=\bar{\lambda}(u,g,\lambda)\,,\qquad u=\frac{k^2}{\kappa^2}\,. \tag{1.1}$$

A model is called asymptotically free if both effective couplings vanish in the limit of large Euclidean momenta [1–3]

$$\lim_{u\to\infty}\bar{g}=0\,,\qquad \lim_{u\to\infty}\bar{\lambda}=0\,. \tag{1.2}$$

Only solutions with bounded ratio $\bar{\lambda}/\bar{g}^2$ will be considered.

The effective couplings satisfy the differential equations [4]

$$u\frac{\partial \bar{g}^2}{\partial u}=\beta_1(\bar{g}^2,\bar{\lambda}), \tag{1.3}$$

$$u\frac{\partial \bar{\lambda}}{\partial u}=\beta_2(\bar{g}^2,\bar{\lambda}). \tag{1.4}$$

For the asymptotic expansions of the β-functions we consider the following forms of the β-functions which should cover most applications

$$\beta_1=b_1g^4+\sum_{n=3}^{\infty}\sum_{m=0}^{n-1} b_{n-m,m}\,g^{2(n-m)}\lambda^m, \tag{1.5}$$

$$\beta_2=c_1\lambda^2+c_2\lambda g^2+c_3g^4+\sum_{n=3}^{\infty}\sum_{m=0}^{n} c_{n-m,m}\lambda^{n-m}g^{2m}. \tag{1.6}$$

Though there are important models with vanishing lowest order of β_1 we assume $b_1, c_1 \neq 0$ throughout the present paper. Since terms of the form λ^m are not included in β_1 we have

$$\beta_1=0 \quad \text{at} \quad g=0. \tag{1.7}$$

Therefore, Eq. (1.3) admits the trivial solution

$$\bar{g}\equiv 0, \tag{1.8}$$

leaving the differential equation (1.4) for $\bar{\lambda}$ alone. This case in which the primary coupling g is turned off will not be considered any further.

Of particular interest are supersymmetric gauge theories with $\lambda=h^2$, where h describes a matter or Higgs interaction. For such models all coefficients c_3 and c_{0n} of terms g^{2n} vanish in β_2 so that

$$\beta_2=0 \quad \text{at} \quad \lambda=0. \tag{1.9}$$

Then (1.4) allows for the trivial solution $\bar{\lambda}\equiv 0$, in which case the secondary coupling h is turned off.

The ordinary differential equation

$$\beta_1\frac{d\bar{\lambda}}{d\bar{g}^2}=\beta_2 \tag{1.11}$$

follows from (1.3) and (1.4) by eliminating u. Apart from the trivial solution (1.8) u can always be eliminated since $d\bar{g}^2/du \neq 0$ as a consequence of (1.2), (1.3), and (1.5) for large enough u. Thus except for (1.8) all asymptotically free solutions satisfy (1.11) in a sufficiently small neighborhood of $\bar{\lambda}=\bar{g}=0$. The purpose of this paper is to derive asymptotic expansions which express $\bar{\lambda}$ as a function of small values $\bar{g}$.

In case of the lowest order approximation

$$\begin{aligned} &\beta_1=b_1\bar{g}^4, && b_1\neq 0,\\ &\beta_2=c_1\bar{\lambda}^2+c_2\bar{\lambda}\bar{g}^2+c_3\bar{g}^4, && c_1\neq 0, \end{aligned} \tag{1.12}$$

the exact solutions of (1.11) are well-known.

For

$$\varDelta=(c_2-b_1)^2-4c_1c_3>0, \tag{1.13}$$

the solutions are

$$\bar{\lambda}=\bar{g}^2\frac{\varrho_-+A\varrho_+\bar{g}^{2\xi}}{1+A\bar{g}^{2\xi}}=\bar{g}^2\frac{\varrho_++B\varrho_-\bar{g}^{-2\xi}}{1+B\bar{g}^{-2\xi}} \tag{1.14}$$

with an arbitrary constant of integration A or B. The exponent ξ is defined by

$$\xi=-\frac{c_1}{b_1}(\varrho_+-\varrho_-). \tag{1.15}$$

$\varrho_\pm$ denotes the roots of

$$c_1x^2+(c_2-b_1)x+c_3=0, \tag{1.16}$$

with ϱ_+ being the larger value

$$\varrho_+\geqq\varrho_-. \tag{1.17}$$

ξ is non-vanishing and in sign opposite to c_1/b_1 if $\varDelta>0$. For vanishing A or B there are the special solutions

$$\bar{\lambda}_\pm=\varrho_\pm\bar{g}^2. \tag{1.18}$$

In the limit $\bar{g}\to 0$ the general solution (1.14) approaches

$$\bar{\lambda}\to\bar{\lambda}_- \quad \text{if} \quad \xi>0, \quad B\neq 0$$

and (1.19)

$$\bar{\lambda}\to\bar{\lambda}_+ \quad \text{if} \quad \xi<0, \quad A\neq 0.$$

Hence for $\xi>0$ the special solution $\bar{\lambda}_-$ which corresponds to the smaller root of (1.16) is stable while the solution $\bar{\lambda}_+$ is unstable provided $\varrho_+\neq\varrho_-$.

For $\varDelta=0$ the general solution of (1.11) and (1.12) is

$$\bar{\lambda}=\varrho_\pm\bar{g}^2-\frac{b_1}{c_1}\frac{\bar{g}^2}{\ln\bar{g}^2+A}, \tag{1.20}$$

where

$$\varrho_+=\varrho_-=\frac{b_1-c_2}{2c_1} \tag{1.21}$$

is the root of (1.16). In addition there is the special solution

$$\bar{\lambda}_\pm=\varrho_\pm\bar{g}^2, \tag{1.22}$$

which corresponds to infinite A.

The case $\varDelta<0$ will not be considered here. It has first been observed by Gross and Wilczek that a model with $\varDelta<0$ cannot be asymptotically free even if the necessary condition $b_1<0$ is satisfied [5].

The solutions based on the lowest order approximation (1.12) of the β-functions may be misleading. In particular, if the second order approximation of $\bar{\lambda}$ vanishes, the leading asymptotic behavior in $\bar{g}^2$ could be quite different. On the other hand it will be shown that for two independent couplings it will in general not be possible to reparametrize the system by a regular transformation such that the β-functions become polynomial. Therefore the full asymptotic expansions (1.5) and (1.6) of the β-functions will be used in this paper in order to construct expansions of $\bar{\lambda}$ in terms of powers (including fractional or irrational exponents) and logarithms of $\bar{g}^2$ which are valid asymptotically for small $\bar{g}$. The expansions found will be complete in the sense that they represent all possible solutions if applied to the approximated system (1.12).

We briefly state some of the results. A general solution will be constructed involving an arbitrary constant of integration and, in addition, special solutions $\bar{\lambda}_\pm$ which correspond to the solutions (1.18) of the approximate system. The leading term of any expansion is always $\varrho_\pm \bar{g}^2$ provided the roots $\varrho_\pm$ of (1.16) do not vanish.

The expansions found are only meaningful if the coefficients of the β-functions satisfy

$$\Delta \geqq 0 . \tag{1.23}$$

Otherwise $\bar{\lambda}$ is not real. Under the further condition

$$b_1 < 0 , \tag{1.24}$$

the expansions represent effective couplings which are asymptotically free. However, not all models satisfying $\Delta \geqq 0$ and $b_1 < 0$ are covered by the asymptotic expansions obtained. An important restriction is the positivity condition first stated by Browne, O'Raifeartaigh, and Sherry for supersymmetric models [6] [1]. A similar restriction in the general case excludes positive values of λ for asymptotically free models if the roots $\varrho_\pm$ are negative. If $\varrho_- < 0$ but $\varrho_+ \geqq 0$, only an unstable mode of the system can be asymptotically free. There may be other requirements of a related nature. For instance the ratio λ/g^2 may for dynamical reasons be bounded, say by

$$0 < \frac{\lambda}{g^2} \leqq \eta .$$

If the upper bound η is below the two roots $\varrho_\pm$ the model cannot be asymptotically free.

Since asymptotic freedom requires $b_1 < 0$ and c_1 is usually positive, the value of ξ as defined by (1.15) is non-negative. We therefore set

$$\xi \geqq 0 \tag{1.25}$$

in the remainder of the introduction.

Special solutions of (1.11) can be constructed in the form of power series

$$\bar{\lambda}_+ = \varrho_+ \bar{g}^2 + \sum_{n=2}^{\infty} a_{+n} \bar{g}^{2n} , \tag{1.26}$$

$$\bar{\lambda}_- = \varrho_- \bar{g}^2 + \sum_{n=2}^{\infty} a_{-n} \bar{g}^{2n} , \tag{1.27}$$

which correspond to the solutions (1.18) of the approximate system.

1 This work was generalized to models of more than two couplings in [7]

The solution $\bar{\lambda}_+$ always exists with uniquely determined coefficients. A unique solution $\bar{\lambda}_-$ also exists provided ξ is not a positive integer. For positive integral ξ either $\bar{\lambda}_-$ cannot be constructed or represents the general solution with arbitrary coefficient $a_{\xi+1}$.

If $\xi>0$ and not an integer the general solution $\bar{\lambda}$ involves fractional or irrational powers of $\bar{g}^2$. The lowest order contribution of this kind is

$$d\bar{g}^{2(\xi+1)} \tag{1.28}$$

with arbitrary d corresponding to the constant of integration. The other coefficients are uniquely determined. The special solution $\bar{\lambda}_-$ is obtained by setting $d=0$.

If ξ is an integer, logarithms usually appear in the expansion of the general solution. For positive, integral ξ the first logarithm may appear in the order $\bar{g}^{2(\xi+1)}$. In this order the general solution contains the terms

$$a_{\xi+1}\bar{g}^{2(\xi+1)}+d_{\xi+1}\bar{g}^{2(\xi+1)}\ln\bar{g}^2 . \tag{1.29}$$

$a_{\xi+1}$ is arbitrary, $d_{\xi+1}$ and the other coefficients are unique. $d_{\xi+1}$ may vanish in which case the expansion becomes a power series.

For $\xi=0$ the general solution may be expanded with respect to powers of $\bar{g}^2$ and inverse powers of $\ln\bar{g}^2$. The leading terms are

$$\bar{\lambda}=\frac{b_1-c_2}{c_1}\bar{g}^2-\frac{b_1}{c_1}\frac{\bar{g}^2}{\ln\bar{g}^2+A}+o(\bar{g}^4). \tag{1.30}$$

A is an arbitrary integration constant. The coefficients of the higher order terms are unique.

The asymptotic behavior of the solutions obtained is

$$\begin{aligned}&\bar{\lambda}\simeq\bar{\lambda}_-\simeq\varrho_-\bar{g}^2,\\ &\bar{\lambda}_+\simeq\varrho_+\bar{g}^2,\end{aligned}\qquad \varrho_\pm\neq 0,\qquad \xi\geqq 0, \tag{1.31}$$

for non-vanishing roots $\varrho_\pm$. Accordingly $\bar{\lambda}_-$ is a stable solution while $\bar{\lambda}_+$ is unstable if the roots $\varrho_\pm$ are different ($\xi>0$).

For supersymmetric models these results simplify considerably. Because of $c_3=0$ the condition $\Delta\geqq 0$ is always satisfied. The coupling parameter $\lambda=h^2$ cannot be negative. The differential Eq. (1.11) always admits the trivial solution $\bar{\lambda}\equiv 0$ which corresponds to $\lambda=0$. For the interacting case the positivity condition $\lambda>0$ and the condition $b_1<0$ leads to the requirement

$$c_2<b_1<0 \tag{1.32}$$

of Browne, O'Raifearthaigh, and Sherry for asymptotic freedom. It is

$$\varrho_-=0,\qquad \varrho_+=\frac{b_1-c_2}{c_1}>0,\qquad \xi=\frac{c_2}{b_1}-1>0. \tag{1.33}$$

The leading behavior of the general solution is always determined by the value of ξ,

$$\bar{\lambda}\simeq d\bar{g}^{2(\xi+1)} \tag{1.34}$$

with arbitrary coefficient d. If ξ is a positive integer, no logarithms occur and the general solution can be expanded as power series starting with the term (1.34). Apart from the trivial solution

$$\bar{\lambda}_- \equiv 0, \tag{1.35}$$

there is also an unstable solution

$$\bar{\lambda}_+ = \varrho_+ \bar{g}^2 + \sum_{n=2}^{\infty} \varrho_n \bar{g}^{2n} \tag{1.36}$$

with uniquely determined coefficients.

In Sect. 2 of this paper reparametrizations in two variables will be discussed with the result that in general the β-functions cannot be made polynomial by a regular transformation. Asymptotic expansions of $\bar{\lambda}$ in terms of $\bar{g}^2$ are derived in Sect. 3. The special case of supersymmetric models is discussed in Sect. 4.

2. Reparametrization in Two Variables

We consider transformations g^2, $\lambda \to g'^2$, λ' defining new coupling parameters g'^2, λ' by power series expansions

$$g'^2 = g^2 + \sum_{n=2}^{\infty} \sum_{m=0}^{n-1} a_{n-m,m}\, g^{2(n-m)} \lambda^m, \tag{2.1}$$

$$\lambda' = \lambda + \sum_{n=2}^{\infty} \sum_{m=0}^{n-1} d_{n-m,m}\, \lambda^{n-m} g^{2m}. \tag{2.2}$$

In case of a single coupling parameter it has been shown by 't Hooft that the β-function can always be made polynomial by a regular reparametrization [8]. Here the corresponding problem with two coupling parameters will be discussed, as well as the question of the invariance for the coefficients of the β-functions.

Equivalence under a renormalization group transformation requires (2.1) and (2.2) to satisfy the differential equations

$$\beta_1' = \frac{\partial g'^2}{\partial g^2} \beta_1 + \frac{\partial g'^2}{\partial \lambda} \beta_2, \tag{2.3}$$

$$\beta_2' = \frac{\partial \lambda'}{\partial \lambda} \beta_2 + \frac{\partial \lambda'}{\partial g^2} \beta_1. \tag{2.4}$$

β_1', β_2' denote the new β-functions in terms of the new variables,

$$\beta_1' = b_1' g'^4 + \sum_{n=3}^{\infty} \sum_{m=0}^{n-1} b'_{n-m,m} g'^{2(n-m)} \lambda'^m, \tag{2.5}$$

$$\beta_2' = c_1' \lambda'^2 + c_2' \lambda' g'^2 + c_3' g'^4 + \sum_{n=3}^{\infty} \sum_{m=0}^{n} c'_{n-m,m} \lambda'^{n-m} g'^{2n}. \tag{2.6}$$

Comparing coefficients in second order g^4, $g^2\lambda$, λ^2 yields the invariance of all second order coefficiients

$$b_1' = b_1, \quad c_1' = c_1, \quad c_2' = c_2, \quad c_3' = c_3.$$

Comparing third order coefficients of g^6, $g^4\lambda$, $g^2\lambda^2$ in (2.3) we find

$$b'_{30}=b_{30}+a_{11}c_3\,,\qquad b'_{21}=b_{21}+a_{11}(c_2-b_1)\,,$$
$$b'_{12}=b_{12}+a_{11}c_1\,. \tag{2.7}$$

From the third order of (2.4) we get

$$\begin{aligned}
c'_{30}&=c_{30}\,,\\
c'_{21}&=c_{21}+c_2(d_{20}-a_{11})-c_3d_{11}\,,\\
c'_{12}&=c_{12}+2c_3(d_{20}-a_{11})+b_1d_{11}-a_{20}c_2\,,\\
c'_{03}&=c_{03}+c_3(d_{11}-2a_{20})\,.
\end{aligned} \tag{2.8}$$

Hence the invariants in third order are

$$c_{30}\lambda^3\,, \tag{2.9}$$

and the combination

$$b_{30}-\frac{c_3}{c_1}b_{12}\,. \tag{2.10}$$

At this stage we do not dispose of the second order coefficients $a_{20}, a_{11}, d_{20}, d_{11}$ of the transformations (2.1) and (2.2). Instead we try to use them to make all coefficients of β'_1, β'_2 in fourth and higher order vanish.

Six more parameters a_{30}, a_{21}, a_{12}, d_{30}, d_{21}, and d_{12} enter the fourth order check of the conditions (2.3) and (2.4). Together with the four parameters left over from the third order there are ten parameters available for the purpose of making the fourth order coefficients in β'_1, β'_2 vanish. By comparing the coefficients of g^8, $g^6\lambda$, $g^4\lambda^2$, $g^2\lambda^3$ in (2.3) and of λ^4, λ^3g^2, λ^2g^4, λg^6, g^8 in (2.4) one obtains nine constraints if all fourth order terms in β'_1, β'_2 are set zero. Since there is one variable more than there are constraints it is possible to eliminate the fourth order terms of the β-functions – apart from exceptional situations.

In fifth order there are eight more parameters in (2.1) and (2.2) together with one parameter left over from fourth order. On the other hand there are eleven constraints if the fifth order coefficients of β'_1, β'_2 are required to vanish. Hence in general the available parameters are overdetermined.

In n^{th} order we get $2(n-1)$ new parameters of the transformations as compared to $2n+1$ new constraints. Hence it appears impossible to eliminate all fifth and higher order terms in the β-functions for the general case of two variables.

This does not preclude the possibility of rendering the β-functions polynomial in special situation. We have therefore checked the supersymmetric case separately in which all powers g^{2n} in β_2 are absent. This simplification makes it indeed possible to eliminate all fourth and fifth order terms. In sixth order, however, the available parameters are overdetermined. In the n^{th} order there are $2(n-1)$ new parameters for $2n$ new constraints.

In conclusion we remark that there are three third order invariants in the supersymmetric case, namely

$$b_{30}g^6\,,\quad c_{30}\lambda^3\,, \tag{2.11}$$

and the combination

$$b_{21}+\frac{b_1-c_2}{c_1}b_{12}. \tag{2.12}$$

3. Construction of Asymptotic Expansions

We first study the possibility of power series expansions.

$$\bar{\lambda}=a_0+a_1\bar{g}^2+a_2\bar{g}^4+\dots. \tag{3.1}$$

Asymptotic freedom (1.2) requires $a_0=0$. Comparing coefficients of $\bar{g}^4$ in the differential Eq. (1.11) we find for a_1 the condition

$$c_1a_1^2+(c_2-b_1)a_1+c_3=0. \tag{3.2}$$

a_1 can only be real if $\Delta \geqq 0$ which will be assumed in the work that follows.

We begin with the case $\Delta>0$, or equivalently $\xi \neq 0$. Then there are two distinct solutions for

$$a_1=\varrho_+ \quad \text{or} \quad \varrho_- \quad \text{with} \quad \varrho_+>\varrho_-. \tag{3.3}$$

Comparing the coefficients of g^{2n+2} in (1.11) with $n=2, 3, \dots$ we find the condition

$$(b, n-2c_1a_1-c_2)a_n=E_n \tag{3.4}$$

for a_n. E_n only depends on lower order coefficients ($m<n$). If for all $n=2, 3, \dots$ the expression

$$b_1n-2c_1a_1-c_2 \neq 0 \tag{3.5}$$

does not vanish, all coefficients a_n are uniquely determined. A value $n=k$ satisfying

$$b_1k-2c_1a_1-c_2=0 \tag{3.6}$$

may directly be related to ξ by

$$\begin{aligned} \xi&=k-1 \quad \text{if} \quad a_1=\varrho_-, \\ \xi&=1-k \quad \text{if} \quad a_1=\varrho_+, \end{aligned} \tag{3.7}$$

with ξ defined by (1.15). Hence we arrive at the following statement: If ξ is not a positive or negative integer the differential Eq. (1.11) can be solved by two power series,

$$\bar{\lambda}_+=\varrho_+\bar{g}^2+\sum_{n=2}^{\infty} a_{+n}\bar{g}^{2n}, \tag{3.8}$$

$$\bar{\lambda}_-=\varrho_-\bar{g}^2+\sum_{n=2}^{\infty} a_{-n}\bar{g}^{2n}. \tag{3.9}$$

Moreover (3.8) exists for positive integral ξ, (3.9) exists for negative integral ξ. If ξ is a positive integer and $E_{\xi+1}=0$ also (3.9) exists. In that case $a_{\xi+1}$ is not restricted by (3.4), hence may be an arbitrary constant. Similarly (3.8) exists with arbitrary $a_{1-\xi}$ if ξ is a negative integer and $E_{1-\xi}=0$.

In the case $\Delta=0$ (equivalently $\xi=0$) the two expansions (3.8) and (3.9) coincide. The construction is unique since (3.6) can only be solved by $k=\xi+1=1$.

This completes the construction of the power series solutions of (1.11). They represent the special solutions which may be constructed. In addition they also provide the general solution if ξ is an integer and $E_{\xi+1}$ or $E_{1-\xi}$ respectively vanishes.

Logarithms occur if $\xi=\pm 1, \pm 2, \dots$ and $E_n \neq 0$ with $n=\xi+1$ or $1-\xi$, respectively. Then (3.6) holds and (3.4) has no solution a_n. The difficulty can be resolved by adding a logarithmic term

$$\bar{\lambda}=a_1\bar{g}^2+\ldots+a_k\bar{g}^{2k}+d_k\bar{g}^{2k}\ln\bar{g}^2+\ldots.$$

The inclusion of the logarithmic term will automatically lead to the general solution with an arbitrary parameter. Comparing the coefficients of $\bar{g}^{2k}$ and $\bar{g}^{2k}\ln\bar{g}^2$ in (1.11), one finds

$$\begin{aligned}(b_1k-2c_1a_1-c_2)a_k&=-b_1d_k+E_k,\\(b_1k-2c_1a_1-c_2)d_k&=0.\end{aligned} \tag{3.10}$$

Since $b_1k-2c_1a_1-c_2=0$ the two conditions are satisfied by arbitrary a_k and $d_k=E_k/b_1$. In higher orders also powers of logarithms occur. Inductively one finds

$$\begin{aligned}\bar{\lambda}=&\sum_{n=1}^{k-1}a_n\bar{g}^{2n}+a_k\bar{g}^{2k}+d_k\bar{g}^{2k}\ln\bar{g}^2\\&+\sum_{\varrho=k+1}^{\infty}\sum_{\sigma}h_{\varrho\sigma}\bar{g}^{2\varrho}\ln^{\sigma}\bar{g}^2,\\ \left.\begin{matrix}k=\xi+1\\a_1=\varrho_-\end{matrix}\right\}&\text{ if }\xi>0,\ \left.\begin{matrix}k=1-\xi\\a_1=\varrho_+\end{matrix}\right\}\text{ if }\xi<0,\qquad\xi=\pm1,\pm2,\ldots.\end{aligned} \tag{3.11}$$

The exponents σ of the logarithms are restricted by

$$1\leqq\sigma\leqq\frac{\varrho-1}{k-1}.$$

Equation (3.11) represents the general solution in case of integral $\xi\neq 0$.

For $\xi=0$ the solution (1.20) of the approximate systems suggests the ansatz

$$\bar{\lambda}=\sum_{n=1}^{\infty}a_n\bar{g}^{2n}+\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}d_{nj}\bar{g}^{2n}\ln^{-j}\bar{g}^2. \tag{3.12}$$

The logarithmic terms do not affect the recursion formulae of the a_n. Therefore the first series in (3.12) is the power series expansion of the stable solution. In particular

$$a_1=\varrho_\pm=\frac{b_1-c_2}{2c_1}. \tag{3.13}$$

For the logarithmic terms of order g^2 one finds

$$d_{11}=0\quad\text{or}\quad d_{11}=-\frac{b_1}{c_1}.$$

The solution $d_{11}=0$ leads back to the power series expansion since it implies that all $d_{nj}=0$. In the other case d_{12} is arbitrary and all $d_{1j}(j \geqq 3)$ are unique. The logarithmic terms of order g^2 can be summed up in closed form by

$$\sum_{j=1}^{\infty} d_{1j}\bar{g}^2 \ln^{-j}\bar{g}^2 = -\frac{b_1}{c_1}\frac{\bar{g}^2}{\ln g^2 + A},$$

with the integration constant A. This follows by making the ansatz

$$\bar{\lambda} = \varrho(\ln\bar{g}^2)\bar{g}^2 + o(\bar{g}^4), \tag{3.14}$$

which leads to the differential equation

$$b_1 \frac{d\varrho}{d\ln\bar{g}^2} = c_1\varrho^2 + (c_2 - b_1)\varrho + c_3,$$

with the solutions

$$\varrho = \frac{b_1 - c_2}{2c_1} - \frac{b_1}{c_1}\frac{1}{\ln\bar{g}^2 + A} \quad \text{or} \quad \varrho = \frac{b_1 - c_2}{2c_1}.$$

The coefficients $d_{nj}(n \geqq 2)$ are uniquely determined by recursion formulae of the form

$$(n-1)b_1 d_{nj} = E_{nj}, \tag{3.15}$$

where E_{nj} is a function of lower order coefficients.

With these results the general solution may be written in the form

$$\bar{\lambda} = \bar{\lambda}_{\pm} - \frac{b_1}{c_1}\frac{\bar{g}^2}{\ln\bar{g}^2 + A} + \sum_{n=2}^{\infty}\sum_{j=1}^{\infty} d_{nj}\bar{g}^{2n}\ln^{-j}\bar{g}^2, \qquad \xi = 0, \tag{3.16}$$

with the power series (3.8) and (3.9) for $\bar{\lambda}_{\pm}$.

It remains to construct the general solution for non-integral ξ. As suggested by the solution (1.14) of the approximate system we include a term

$$d_{11}g^{2k}, \tag{3.17}$$

where $k = \xi + 1$ if $\xi > 0$, $k = 1 - \xi$ if $\xi < 0$. Comparing coefficients in the order $g^{2(k+1)}$ of the differential Eq. (1.11) of $\bar{\lambda}$, one finds

$$(kb_1 - 2a_1c_1 - c_2)d_{11} = 0. \tag{3.18}$$

We have

$$kb_1 - 2a_1c_1 - c_2 = 0 \quad \text{if} \quad \xi > 0, a_1 = \varrho_- \quad \text{or} \quad \xi < 0, a_1 = \varrho_+. \tag{3.19}$$

Hence for $\xi > 0$ a term (3.17) with arbitrary coefficient d_{11} may be included in the expansion starting with $\varrho_-\bar{g}^2$. Similarly for $\xi < 0$ when the expansion starting with $\varrho_+\bar{g}^2$ is used.

When the expansion including the term (3.17) is inserted into the differential Eq. (1.11) terms of the form

$$d_{mn}\bar{g}^{2(m|\xi|+n)}, \qquad m, n = 1, 2, \ldots \tag{3.20}$$

are generated in the expansion of $\bar{\lambda}$. If ξ is rational we write

$$|\xi| = \frac{p}{q} \tag{3.21}$$

as the ratio of relative prime integers p and q. The expansion

$$\bar{\lambda} = \sum_{n=1}^{\infty} a_n \bar{g}^{2n} + \sum_{m=1}^{q-1} \sum_{n=1}^{\infty} d_{mn} \bar{g}^{2(m|\xi|+n)} \quad \text{if } \xi \text{ is rational} \tag{3.22}$$

or

$$\bar{\lambda} = \sum_{n=1}^{\infty} a_n \bar{g}^{2n} + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} d_{mn} \bar{g}^{2(m|\xi|+n)} \quad \text{if } \xi \text{ is irrational} \tag{3.23}$$

solves the differential equation provided the coefficients satisfy

$$a_1 \begin{cases} = \varrho_- & \text{if} \quad \xi > 0\,, \\ = \varrho_+ & \text{if} \quad \xi < 0\,, \end{cases} \tag{3.24}$$

$$(b_1(|\xi|+1) - 2a_1 c_1 - c_2) d_{11} = 0\,, \tag{3.25}$$

and recursion formulae of the type

$$(b_1 n - 2a_1 c_1 - c_2) a_n = E_n\,, \tag{3.26}$$

$$(b_1(m|\xi|+n) - 2a_1 c_1 - c_2) d_{mn} = E_{mn}\,. \tag{3.27}$$

The inhomogeneous terms E_n and E_{mn} only depend on lower order coefficients. The coefficients a_n are uniquely determined by (3.27). Since $b_1(m|\xi|+n) - 2a_1 c_1 - c_2 = 0$ only for $m=n=1$, the coefficients d_{mn} are also uniquely determined once the arbitrary value of d_{11} is given. For rational ξ the coefficients a_n in general involve d_{11}. If ξ is irrational the coefficients a_n are not affected by the value of d_{11}. Then the first sum in (3.23) represents the power series solution (3.8) or (3.9) respectively.

For the discussion of the leading asymptotic behavior of $\bar{\lambda}$ we assume for simplicity that the roots $\varrho_\pm$ are non-vanishing. The supersymmetric case for which $\varrho_- = 0$ will be treated separately in the following section. By (3.8), (3.9), (3.11), (3.16), and (3.23) the asymptotic behavior of the solutions obtained is

$$\bar{\lambda}_+ \simeq \varrho_+ \bar{g}^2\,, \quad \bar{\lambda}_- \simeq \varrho_- \bar{g}^2\,, \quad \bar{\lambda} \simeq \varrho_- \bar{g}^2 \text{ if } \xi \geqq 0\,, \quad \bar{\lambda} \simeq \varrho_+ \bar{g}^2 \text{ if } \xi \leqq 0\,, \tag{3.28}$$

where $\bar{\lambda}$ denotes the general solution. A particular solution $\bar{\lambda}_0$ is called stable if for almost all solutions

$$\lim_{g\to 0} \frac{\bar{\lambda}}{\bar{\lambda}_0} = 1\,. \tag{3.29}$$

According to (3.24) $\bar{\lambda}_-$ is stable for $\xi \geqq 0$ and $\bar{\lambda}_+$ is stable for $\xi \leqq 0$. $\bar{\lambda}_+$ is unstable for $\xi > 0$ and $\bar{\lambda}_-$ unstable for $\xi < 0$.

In conclusion we discuss the question of asymptotic freedom. Inserting any of the expansions of $\bar{\lambda}$ back into the differential Eq. (1.3) of $\bar{g}$ with respect to u we obtain

$$u \frac{\partial \bar{g}^2}{\partial u} = \beta_1(\bar{g}^2, \bar{\lambda}(\bar{g}^2)) \begin{cases} < 0 & \text{if} \quad b_1 < 0 \\ > 0 & \text{if} \quad b_1 > 0 \end{cases}$$

for sufficiently small values of $\bar{g}^2$. Hence $\Delta \geqq 0$ and $b_1 < 0$ are necessary conditions for $\lim_{u\to\infty} \bar{g} = 0$ and $\lim_{u\to\infty} \bar{\lambda} = 0$. On the other hand if b_1 is negative, the solution

$$u = \exp \int_{g^2}^{\bar{g}^2} \frac{dx}{\beta_1(x, \bar{\lambda}(x))} \quad (g^2, \bar{g}^2 \text{ sufficiently small})$$

implies $\bar{g} \to 0$ for $u \to \infty$. Hence

$$\lim_{u\to\infty} \bar{g} = 0 \quad \text{if} \quad \Delta \geqq 0 \quad \text{and} \quad b_1 < 0 .$$

With this also

$$\lim_{u\to\infty} \bar{\lambda} = 0$$

follows for all expansions. This result seems to indicate that the conditions

$$\Delta \geqq 0, b_1 < 0 \tag{3.30}$$

are not only necessary but also sufficient for asymptotic freedom. However, for models involving two independent coupling parameters the full range in both variables is not covered by the asymptotic behavior for large Euclidean momenta. For instance, if the roots $\varrho_\pm$ do not vanish the ratio $\bar{\lambda}/\bar{g}^2$ approaches a non-vanishing value. Hence only for values of λ and g^2 in a sufficient neighborhood of the line $\lambda = \varrho_\pm g^2$ asymptotic freedom is guaranted. Though an initial domain of coupling parameters can be enlarged by the equivalence transformations of the renormalization group non-perturbative effects may restrict the ratio λ/g^2 so that asymptotic freedom does not hold. A most obvious restriction of this kind was first found by Browne, O'Raifeartaigh, and Sherry for supersymmetric interactions [6]. We will now give a generalization of this restriction to the class of models considered here. In order to simplify the following discussion we assume $c_1 > 0$ as is usually the case. Then the necessary condition $b_1 < 0$ implies $\xi \geqq 0$, according to the definition (1.15). A model with $\lambda > 0$ cannot be asymptotically free if $\varrho_- \leqq \varrho_+ < 0$. If $\varrho_- < 0$, but $\varrho_+ \geqq 0$, only the unstable mode of the model corresponding to the solution $\bar{\lambda} = \bar{\lambda}_+$ can be asymptotically free.

4. Supersymmetric Case[2]

For supersymmetric interactions β_2 vanishes if $\lambda = 0$. Therefore $\bar{\lambda} \equiv 0$ is always a solution of the differential equation (1.11). This trivial solution corresponds to the case $\lambda = 0$ of no interaction. For the following discussion we exclude the non-interacting case and require $\lambda = h^2 > 0$. $c_1 > 0$ is assumed throughout this section. Since $c_3 = 0$ the roots of (1.16) are

$$\varrho_\pm = 0, \frac{b_1 - c_2}{c_1}$$

and $\Delta \geqq 0$ is always satisfied. If the root $(b_1 - c_2)/c_1$ were negative the general solution (3.11) would become negative for large Euclidean momenta in contradiction to $\lambda > 0$. Hence

$$\begin{gathered} \frac{b_1 - c_2}{c_1} \geqq 0, \quad \varrho_+ = \frac{b_1 - c_2}{c_1}, \quad \varrho_- = 0, \\ \xi = -\frac{c_1}{b_1}(\varrho_+ - \varrho_-) = \frac{c_2}{b_1} - 1 . \end{gathered} \tag{4.1}$$

2 We consider the class of models studied in [6]

By a similar argument we can exclude $\xi=0$, which is equivalent to $\Delta=0$ or $c_2=b_1$. Then the leading term of the general solution (3.16) takes the form

$$\bar{\lambda}=-\frac{b_1}{c_1}\frac{\bar{g}^2}{\ln\bar{g}^2+A}+o(\bar{g}^4). \tag{4.2}$$

Since $b_1<0$ and $c_1>0$, this becomes negative for sufficiently large $|k^2|$ if asymptotic freedom holds with $\bar{g}\to 0$ for $|k^2|\to\infty$. The stable solution (3.8) and (3.9) as the only power series expansion possible for $\xi=0$ must reduce to the trivial solution $\bar{\lambda}=0$. Hence for the interacting case $\lambda>0$ a model can only be asymptotically free if

$$b_1<0 \quad \text{and} \quad \xi>0 \quad \text{or equivalently} \quad c_2<b_1<0. \tag{4.3}$$

We will now simplify the general solutions (3.11), (3.22), and (3.23) for the supersymmetric case. We begin with the case $\xi=1,2,\ldots$ and show that logarithms are absent in (3.11), as well as terms of order less than $g^{2(\xi+1)}$. If $a_1=\varrho_-=0$. Assuming $a_1=\ldots=a_{j-1}=0$, it follows $(c_2-jb_1)a_j=0$. Hence $a_j=0$ if $j\leqq\xi$ and $a_{\xi+1}$ arbitrary. Thus (3.11) reduces to the power series

$$\bar{\lambda}=\sum_{j=\xi+1}^{\infty}a_j g^{2j}, \quad \xi=1,2,\ldots; \quad a_{\xi+1} \text{ arbitrary}. \tag{4.4}$$

If ξ is irrational the expansion (3.23) becomes

$$\bar{\lambda}=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}d_{mn}\bar{g}^{2(m\xi+n)}, \quad \xi \text{ irrational}, \quad d_{11} \text{ arbitrary}, \tag{4.5}$$

since the coefficients a_n are the same as for the solution $\bar{\lambda}_-$ which vanishes in this case.

For rational ξ,

$$\xi=\frac{p}{q}>0,$$

with p and q relative prime integers. The first integral power generated in the expansion of $\bar{\lambda}$ is $\bar{g}^{2p+2}$ so that (3.22) becomes

$$\bar{\lambda}=\sum_{n=p+1}^{\infty}a_n\bar{g}^{2n}+\sum_{m=1}^{a-1}\sum_{n=1}^{\infty}d_{mn}\bar{g}^{2(m\xi+n)}, \quad d_{11} \text{ arbitrary}. \tag{4.6}$$

In all cases the leading behavior of the general solution is given by $\bar{\lambda}\simeq a\bar{g}^{2(\xi+1)}$ with arbitrary a. In contradistinction the unstable solution (3.8) is of order

$$\bar{\lambda}_+\simeq\varrho_+\bar{g}^2, \quad \varrho_+=\frac{b_1-c_2}{c_1}>0.$$

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2.3 Renormalization group equations with vanishing lowest order of the primary β-function

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Comment (*Klaus Sibold*)
Whereas in subsections 2.1, 2.2 the general method of reduction of couplings has been exposed, in the present paper a class of theories is envisaged which represents a special case only, but nevertheless is of quite some importance for all applications to follow: it is assumed that the lowest order of the primary β-function vanishes. This is of interest in supersymmetric theories in particular. The study has been performed in massless models with two couplings and it follows the pattern which had been suggested by QCD: the primary coupling is asymptotically free and one supposes a secondary coupling to be given whose behavior is investigated as dictated by its β-function. Here it only assumed that not all coefficients of sixth order in the primary β-function vanish. Then asymptotic behavior and stability of the solutions of the evolution equations are derived.
The asymptotic behavior is studied under the assumption that the secondary coupling considered as a function of the primary vanishes when the primary tends to zero. As one of the results for supersymmetric Yang-Mills theories with one Yukawa coupling constant for the interaction of chiral superfields it turns out that they are unstable if they are UV-asymptotically free. Here, as said above, the conclusion holds for the embedding into a theory with two couplings.

RENORMALIZATION GROUP EQUATIONS WITH VANISHING LOWEST ORDER OF THE PRIMARY β-FUNCTION

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The evolution equations for the effective couplings are solved for massless models with two couplings and vanishing lowest order contribution of the primary β-function. The stability and asymptotic freedom of the solutions are analyzed. A specialization to supersymmetric theories with nonvanishing Yukawa couplings of superfields shows instability for asymptotically free models and for possible cases with identically vanishing β-functions.

1. Introduction. In a recent paper [1] asymptotic properties of effective couplings have been studied for massless models which are asymptotically free and involve two coupling constants. In the present note we continue this study for a case not treated in [1] which is of considerable interest, in particular for supersymmetric theories. Let us be specific:

$$\bar{g}=\bar{g}(t, g, \lambda), \quad \bar{\lambda}=\bar{\lambda}(t, g, \lambda) \tag{1.1,2}$$

denote effective couplings depending on the scale variable t and the couplings g, λ. They satisfy the evolution equations

$$\mathrm{d}\bar{g}/\mathrm{d}t=\beta_1(\bar{g},\bar{\lambda}), \quad \mathrm{d}\bar{\lambda}/\mathrm{d}t=\beta_2(\bar{g},\bar{\lambda}), \tag{1.3,4}$$

where we assume the β-functions to have the form

$$\beta_1=b_0 g^4+b_1\lambda^2 g^2+b_2\lambda g^4+b_3 g^6 + \sum_{n=4}^{\infty}\sum_{m=0}^{n-1} b_{n-m,m}g^{2(n-m)}\lambda^m, \tag{1.5}$$

$$\beta_2=c_1\lambda^2+c_2\lambda g^2+c_3 g^4 + \sum_{n=3}^{\infty}\sum_{m=0}^{n} c_{n-m,m}\lambda^{n-m}g^{2m}, \tag{1.6}$$

and, in particular, b_0 to vanish:

$$b_0=0. \tag{1.7}$$

The coefficients b, c are independent of t. It is assumed that $c_1 \neq 0$ and that not all sixth order coefficients b_1, b_2 and b_3 of β_1 vanish. The constraint (1.7) is the case excluded in ref. [1]. Special restrictions for supersymmetric theories will be mentioned later.

We will study the asymptotic behavior of solutions $\bar{\lambda}(g^2(t))$, $\bar{g}^2(t)$ of (1.3), (1.4) in the weak coupling limit

$$\bar{\lambda}(\bar{g}^2)\to 0, \quad \bar{g}^2\to 0. \tag{1.8}$$

The function $\bar{\lambda}(\bar{g}^2)$ is assumed to be defined in an interval $0\le \bar{g}^2\le\eta$ with continuous derivative. If the limit (1.8) holds for $t\to\infty$, the system is

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asymptotically free. For $d\bar{g}/dt \not\equiv 0$, the scale variable t can be eliminated from (1.1), (1.2) so that (1.3) and (1.4) lead to the condition

$$\beta_1 \, d\bar{\lambda}/d\bar{g}^2 = \beta_2 . \tag{1.9}$$

For functions $\bar{\lambda}(\bar{g})$ with $\beta_1 = \beta_2 = 0$ (or equivalently $d\bar{g}/dt \equiv 0$ and $d\bar{\lambda}/dt \equiv 0$) the condition (1.9) is trivially satisfied. The integration of (1.9) thus yields all candidates for the solution of (1.3) and (1.4) in the required asymptotic region. In particular, all possibilities of $\bar{\lambda}$ as a function of $\bar{g}$, consistent with renormalization group properties, are governed by (1.9).

$N = 1$ supersymmetric models described by two coupling parameters g and λ can be obtained by writing the Yukawa coupling coefficients of the superfields in the form

$$d_{ijk} = \sqrt{\lambda}\, C_{ijk} .$$

The factor C_{ijk} is symmetric in all indices and depends only upon the group and the representation content. It satisfies the standard relations. The interest in the case $b_0 = 0$ (eq. (1.7)) lies in the fact that, for appropriate coefficients C_{ijk} and ρ_0, the limit $\lambda = \rho_0 g^2$ gives rise to models with β-functions which vanish at least up to two loops ($N = 1$) [2] or three loops ($N = 2, 4$) [3]. In the latter case there are arguments for the vanishing of the β-function in every order of the perturbation expansion [4].

A solution $\lambda^{(0)}(g^2)$ of (1.8), (1.9) is called stable if in a sufficiently small neighborhood of $\lambda^{(0)}$ almost all solutions $\lambda(g^2)$ satisfy [‡1]

$$\lim_{g^2 \to +0} \lambda(g^2)/\lambda^{(0)}(g^2) = 1 . \tag{1.10}$$

In ref. [1] it was shown that in general no regular reparametrization of g and λ is possible which renders β_1 and β_2 polynomial. Lowest order approximations may therefore be misleading, and it is recommendable to take into account the complete β-functions for the study of (1.9). To be independent of renormalization scheme and chosen normalization conditions, the asymptotic behavior must be based on quantities which are invariant under reparametrizations expressed by eqs. (2.1), (2.2) of ref. [1], with arbitrary coefficients a_{ij}, d_{ij}. Since e.g. the coefficient b_0 does not change under reparametrizations, the constraint (1.7) is invariant.

For the actual solution of (1.9) it is convenient to introduce a function ρ by putting

$$\lambda = g^2 \rho , \tag{1.11}$$

considered as a function of either g^2 or g^{-2} satisfying

$$\varphi g^4 \, d\rho/dg^2 - \psi = 0 , \tag{1.12}$$

or

$$\varphi \, d\rho/dg^{-2} + \psi = 0 , \tag{1.13}$$

where

$$\varphi = \beta_1/g^6 , \quad \psi = \beta_2/g^4 - (\beta_1/g^4)\rho . \tag{1.14}$$

Our aim is now to find all solutions λ satisfying (1.8), (1.9). According to ref. [5], a necessary condition for the existence of such solutions is that the discriminant of the quadratic equation

$$c_1 x^2 + c_2 x + c_3 = 0 \tag{1.15}$$

is non-negative,

$$\Delta = c_2^2 - 4 c_1 c_3 \geq 0 . \tag{1.16}$$

In this note we restrict ourselves to the case $\Delta > 0$ in which the roots ρ_+ and ρ_- of (1.15) are different. ρ_+ denotes the larger root: $\rho_+ > \rho_-$.

2. Solutions in lowest order approximation. We first discuss the system where the functions φ and ψ are approximated by their lowest order

$$\varphi = b_1 \rho^2 + b_2 \rho + b_3 ,$$
$$\psi = c_1 \rho^2 + c_2 \rho + c_3 . \tag{2.1}$$

In this case (1.9) can be integrated yielding special

[‡1] In the following equations the notation $\bar{\lambda}, \bar{g}^2$ will be replaced by λ, g^2.

solutions

$$\rho = \rho_+, \quad \rho = \rho_-, \tag{2.2}$$

and the general solution

$$g^{-2} = -(b_1/c_1)\rho - [b_+/c_1(\rho_+ - \rho_-)] \lg|\rho - \rho_+| + [b_-/c_1(\rho_+ - \rho_-)] \lg|\rho - \rho_-| + C, \tag{2.3}$$

where

$$b_\pm = b_1\rho_\pm^2 + b_2\rho_\pm + b_3, \tag{2.4}$$

with an arbitrary integration constant C. $b_\pm$ is invariant under reparametrizations. The inversion $\rho(g^2)$ of (2.3) contains two branches near $g^2 = 0$, namely

$$\rho = \rho^{(+)}(g^2) \quad \text{with } \rho \to \rho_+ \quad \text{for } g^2 \to 0, \tag{2.5}$$

and

$$\rho = \rho^{(-)}(g^2) \quad \text{with } \rho \to \rho_- \quad \text{for } g^2 \to 0, \tag{2.6}$$

which imply ‡2

$$\lambda = g^2\rho \to 0 \quad \text{as } g^2 \to 0. \tag{2.7}$$

Acceptable are only those branches for which the limit $g^2 \to +0$ is approached from positive values of g^2. These branches also represent solutions which are stable in the sense of (1.10). Since usually $c_1 > 0$, we have necessary condition

$$b_+ > 0 \quad \text{for } \lambda^{(+)} = g^2\rho^{(+)} \to 0, \tag{2.8}$$

$$b_- < 0 \quad \text{for } \lambda^{(-)} = g^2\rho^{(-)} \to 0, \tag{2.9}$$

in the limit $g^2 \to +0$. We note that for $b_+ > 0$ and $b_- < 0$ two different branches of the function $\lambda(g^2)$ exist which vanish for $g^2 \to +0$. In this case $\rho = \rho_+$ as well as $\rho = \rho_-$ are stable.

3. Exact solutions. We now discuss the solutions of (1.9) without approximating the β-functions. If $\Delta > 0$ (see (1.16)), two (formal) power series solutions can be constructed starting with either one of the roots

$$\lambda^{(+)} = g^2\rho^{(+)} = \rho_+ g^2 + \rho_{+1} g^4 + \rho_{+2} g^6 + \cdots,$$
$$\lambda^{(-)} = g^2\rho^{(-)} = \rho_- g^2 + \rho_{-1} g^4 + \rho_{-2} g^6 + \cdots. \tag{3.1}$$

The coefficients of these two expansions are unique although in the stable case, the solutions represented by them are not (see below).

We first show that any given solution $\lambda^{(+)}$ or $\lambda^{(-)}$ with an expansion (3.1) is stable if β_1 is positive or negative respectively

$$\lambda^{(+)} \text{ is stable if } \beta_1(\lambda^{(+)}, g^2) > 0,$$
$$\lambda^{(-)} \text{ is stable if } \beta_1(\lambda^{(-)}, g^2) < 0, \tag{3.2}$$

for sufficiently small $g^2 > 0$. $c_1 > 0$ is assumed as usual. For the proof of (3.2) a general solution λ around the given solution $\lambda^{(\pm)}$ will be constructed which satisfies the stability condition (1.10). To this end we set

$$\rho = \rho^{(\pm)} + \exp\left(-\int_{g_0^{-2}}^{g^{-2}} \mathrm{d}x\, a(x)\right)\chi,$$

$$a(g^{-2}) = \varphi_0^{-1}[\psi_0' + (\mathrm{d}\rho^{(\pm)}/\mathrm{d}g^{-2})\varphi_0'], \tag{3.3}$$

$$\varphi_0 = \varphi(\rho^{(\pm)}, g^2),$$

$$\varphi_0' = \partial\varphi/\partial\rho|_{\rho=\rho^{(\pm)}}, \quad \psi_0' = \partial\psi/\partial\rho|_{\rho=\rho^{(\pm)}}, \tag{3.4}$$

and write the differential equation in terms of the function χ

$$\mathrm{d}\chi/\mathrm{d}g^2 = F(\chi, g^2). \tag{3.5}$$

With (3.2), a Lipschitz condition can be shown to hold for F in a neighborhood of $\chi = 0$ and an interval $0 \le g^2 \le \eta$, provided certain differentiability assumptions on the β-functions are made. This implies the existence of a solution χ uniquely determined by its value at $g^2 = 0$ ‡3. Hence a

‡2 For $g^2 \to 0$ and ρ unbounded, the function λ approaches a non-vanishing constant c_1/b_1 if $b_1 = 0$, or increases exponentially if $b_1 = 0$.

‡3 Since no information for $g^2 < 0$ is available, the right-sided version of the Picard–Lindelöf theorem should be used here. See ref. [6], p. 8.

general solution (3.3) exists where the value χ at $g^2 = 0$ can be chosen arbitrarily provided $|\chi|$ is not too large. The condition (1.10) is obviously satisfied since the factor of χ in (3.3) is exponentially decreasing for $g^2 \to +0$.

The solution (3.3) can be constructed by an expansion of the form

$$\rho = \rho^{(\pm)} + \sum_{n=1}^{\infty} r_n, \tag{3.6}$$

with the expansion terms of the order

$$r_n = \mathrm{O}\left(\exp\left(-n\int^{g^{-2}} \mathrm{d}x\, a\right)\right). \tag{3.7}$$

r_n is determined as a solution of the differential equation

$$\mathrm{d}r_n/\mathrm{d}g^{-2} + a r_n + b_n = 0. \tag{3.8}$$

The coefficient a is given by (3.4). The coefficients b_i are

$$\begin{aligned} &b_1 = 0,\\ &\varphi_0 b_n = \sum_{\alpha=1}^{n-1}\left(r_\alpha \varphi_0' + \sum_{m=2}^{\alpha}\sum_{i_1\ldots i_m}^{\alpha} \frac{r_{i_1}\cdots r_{i_m}}{m!}\varphi_0^{(m)}\right)\frac{dr_{n-\alpha}}{dg^{-2}}\\ &\quad + \sum_{m=2}^{n}\sum_{i_1\ldots i_m}^{n} \frac{r_{i_1}\cdots r_{i_m}}{m!}\\ &\quad \times\left[\psi_0^{(m)} + (\mathrm{d}\rho^{(\pm)}/\mathrm{d}g^{-2})\varphi_0^{(m)}\right], \end{aligned} \tag{3.9}$$

with

$$\varphi_0 = \varphi(\rho^{(\pm)}, g^2), \quad \psi_0 = \psi(\rho^{(\pm)}, g^2), \tag{3.10}$$

$$\varphi_0^{(m)} = \partial^m\varphi/\partial\rho^m|_{\rho=\rho^{(\pm)}}, \quad \psi_0^{(m)} = \partial^m\psi/\partial\rho^m|_{\rho=\rho^{(\pm)}}. \tag{3.11}$$

The sum $\Sigma^n_{i_1\ldots i_m}$ extends over all $i_1, \ldots, i_m$ with $i_\alpha \geq 1$, $\Sigma^n_{\alpha=1} i_\alpha = n$.

The differential equation (3.8) can be integrated explicitly. The first term r_1 is

$$r_1 = A\exp\left(-\int_{g_0^{-2}}^{g^{-2}} \mathrm{d}x\, a\right). \tag{3.12}$$

The arbitrary integration constant A (correlated to g_0) determines the value of χ at $g^2 = 0$ in (3.3). The higher order terms r_n are uniquely determined by the required asymptotic behavior (3.7). It is

$$r_n = \exp\left(-\int_{g_0^{-2}}^{g^{-2}} \mathrm{d}x\, a\right)\int_{\infty}^{g^{-2}} \mathrm{d}x'\, b_n(x') \times\exp\left(\int_{g_0^{-2}}^{x'} \mathrm{d}x''\, a\right). \tag{3.13}$$

Each term r_n is exponentially decreasing for $g^2 \to +0$. Moreover, all derivatives of the general solution ρ with respect to g^2 are identical at $g^2 = 0$:

$$\mathrm{d}^n\rho/\mathrm{d}g^{2n} = \mathrm{d}^n\rho^{(\pm)}/\mathrm{d}g^{2n}, \quad g^2 \to +0. \tag{3.14}$$

Thus, all solutions ρ have the same power series expansion (3.1) as $\rho^{(\pm)}$.

The leading behavior of the difference $\rho - \rho^{(\pm)}$ for $g^2 \to +0$ is given by the first term r_1 of the expansion (3.6):

$$\rho - \rho^{(\pm)} \approx r_1. \tag{3.15}$$

The asymptotic form of r_1 will be worked out in some detail. We begin with the case where

$$b_\pm = b_1\rho_\pm^2 + b_2\rho_\pm + b_3 \neq 0. \tag{3.16}$$

Then r_1 is of the form

$$r_1 = Cg^{2q}\exp\left(-\frac{p}{g^2} + \sum_{j=1}^{\infty} f_j g^{2j}\right). \tag{3.17}$$

The exponents p and q are

$$\begin{aligned} p &= \pm(c_1/b_\pm)(\rho_+ - \rho_-),\\ q &= b_\pm^{-1}\big[2c_1\rho_1 + 3(c_{30} - b_1)\rho_\pm^2\\ &\quad + 2(c_{21} - b_2)\rho_\pm\\ &\quad + c_{12} - b_3 - p(2b_1\rho_1\rho_\pm + b_2\rho_\pm + d)\big], \end{aligned} \tag{3.18}$$

ρ_1 is the coefficient of the g^2 term in (3.1) with the value

$$\rho_1 = -(2c_1\rho_\pm + c_2)^{-1}[(c_{30} - b_1)\rho_\pm^3 + (c_{21} - b_2)\rho_\pm^2 + (c_{12} - b_3)\rho_\pm + c_{03}],$$
$$d = b_{13}\rho_\pm^3 + b_{22}\rho_\pm^2 + b_{31}\rho_\pm + b_{40}. \qquad (3.19)$$

As a consequence of the stability condition (3.2), i.e. $b_+ > 0$ or $b_- < 0$ respectively, the exponent p is positive. We also note that p and q are invariant under reparametrizations.

If some, but not all orders of β_1 vanish:

$$g^{-6}\beta_1(\lambda^{(\pm)}, g^2) = a_0^\pm g^{2n} + \mathrm{O}(g^{2n+2}), \qquad (3.20)$$

we have stability according to (3.2) if $a_0^+ > 0$ or $a_0^- < 0$. The term r_1 then takes the form

$$r_1 = cg^{-2f_0} \exp\left(-\frac{f_{-(n+1)}}{g^{2(n+1)}} - \frac{f_{-n}}{g^{2n}} - \dots - \frac{f_{-1}}{g^2} - \sum_{j=1}^{\infty} f_j g^{2j} \right), \qquad (3.21)$$

with positive leading exponent

$$f_{-(n+1)} = \pm[c_1/(n+1)a_0^\pm](\rho_+ - \rho_-) > 0. \qquad (3.22)$$

The stability criterion (3.2) also applies to the interesting case where β_1 vanishes to all orders of perturbation theory, but does not vanish as an exact expression due to an exponentially decreasing behavior.

In all cases where the condition (3.2) does not hold, instability can be shown.

We begin proving that

$$\lambda^{(+)} \text{ is unstable if } \beta_1(\lambda^{(+)}, g^2) < 0,$$
$$\lambda^{(-)} \text{ is unstable if } \beta_1(\lambda^{(-)}, g^2) > 0, \quad g^2 \text{ small}. \qquad (3.23)$$

The defining equation (1.10) of stability implies that a function $\delta = \rho - \rho^{(\pm)}$ exists which satisfies the relation

$$\delta^{-1}\mathrm{d}\delta/\mathrm{d}g^2 = (g^4\varphi)^{-1}\{(\psi - \psi_0)/\delta - [(\varphi - \varphi_0)/\delta]g^4\mathrm{d}\rho^{(\pm)}/\mathrm{d}g^2\}, \qquad (3.24)$$

and vanishes in the limit $g^2 \to +0$:

$$\lim_{g^2 \to +0} \delta = 0, \qquad (3.25)$$

(3.24), (3.25) are necessary conditions for stability. Since

$$\lim_{g^2 \to +0} \left(\partial\psi/\partial\rho|_{\rho=\rho^{(\pm)}} - \partial\varphi/\partial\rho|_{\rho=\rho^{(\pm)}} \times g^4\mathrm{d}\rho^{(\pm)}/\mathrm{d}g^2 \right) = \pm c_1(\rho_+ - \rho_-), \qquad (3.26)$$

the condition (3.23) implies that the right-hand side of (3.24) becomes negative for sufficiently small δ and g^2:

$$\delta^{-1}\mathrm{d}\delta/\mathrm{d}g^2 < 0. \qquad (3.27)$$

Then δ is monotonically decreasing (increasing) if $\delta > 0$ ($\delta < 0$), which is incompatible with (3.25).

We finally prove instability for the case that the β-functions vanish identically. Since then $\varphi_0 = 0$, we have

$$[(\varphi - \varphi_0)/\delta]g^4\mathrm{d}\delta/\mathrm{d}g^2 = (\psi - \psi_0)/\delta - [(\varphi - \varphi_0)/\delta]g^4\mathrm{d}\rho^{(\pm)}/\mathrm{d}g^2. \qquad (3.28)$$

Here the left-hand side vanishes in the limit $\delta \to 0$, $g^2 \to +0$, while the right-hand side does not (see (3.26)).

The stability condition (3.2) is to be contrasted to the condition for asymptotic freedom. With the momentum dependence (1.3), (1.4), we find that $\lambda^{(\pm)}$ is asymptotically free if, for small g^2,

$$\beta_1(\lambda^{(\pm)}, g^2) < 0. \qquad (3.29)$$

Hence only solutions $\lambda^{(-)}$ starting with the smaller root ρ_- can be stable and asymptotically free. For the solutions $\lambda^{(+)}$ belonging to the larger root, asymptotic freedom and stability are incompatible.

4. Supersymmetric models. Applications of our results to supersymmetric theories deserve a separate exposition. Here we summarize some of the results. We consider theories with Yukawa couplings of superfields satisfying

$$d_{ijk} = C_{ijk}\sqrt{\lambda}\,,$$

$$\sum_{m,n} C_{inm}C_{jnm} = \delta_{ij}.$$

Supersymmetry furthermore implies $c_3 = c_{0m} = 0$. The two roots of eq. (1.15) are then $\rho_+ = -c_2/c_1$ and $\rho_- = 0$. With $b_0 = 0$ and $\lambda = \rho_+ g^2$, the one-loop contributions to β_1 and β_2 vanish and it has been argued that the same is true for two-loop contributions [2]:

$$b_1\rho_+^2 + b_2\rho_+ + b_3 = 0,$$

$$c_{30}\rho_+^2 + c_{21}\rho_+ + c_{12} = 0.$$

On the other hand, for the root $\rho_- = 0$ we get $\lambda \equiv 0$ and $\beta_1 = b_3 g^6 + \cdots$.

Specialization of our general results to the supersymmetric models described above leads to the following conclusions:

(1) Branches with non-vanishing superfield Yukawa coupling based upon the root $\rho_+ = -c_2/c_1$ of eq. (1.15) are unstable if they are UV-asymptotically free [‡4], stable if they are IR-free. This result is also true for possible models with β-functions vanishing exponentially in the limit $g^2 \to +0$.

(2) Theories with identically vanishing β-functions would be unstable.

(3) Branches of the theory with vanishing superfield Yukawa coupling could, in principle, have stability and asymptotic freedom. But in the supersymmetric case with $b_0 = 0$, there are indications that $b_3 > 0$, and hence we expect IR-freedom and instability [8].

Instability here is understood with respect to the two-coupling embeddings.

[‡4] Supersymmetric models with $b_0 \neq 0$ and nonvanishing superfield Yukawa coupling also show an incompatibility of UV-asymptotic freedom and stability, at least in two-coupling versions like those discussed here [1]. For a special UV-free model and embeddings which break supersymmetry, Suzuki has previously found that the symmetry limit is not stable [7].

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2.4 Construction of gauge theories with a single coupling parameter for Yang-Mills and matter fields

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Comment (*Klaus Sibold*)
This paper continues via two examples the application of the reduction method to construct in a neighbourhood of four couplings a gauge theory depending on one coupling, the gauge coupling, only. The respective solutions of the reduction equations are power series in the remaining coupling, hence strictly renormalizable.
The matter field content is chosen such that one of the examples can lead to $N = 2$ supersymmetry in a component formulation, the other one to $N = 4$ supersymmetry. And, indeed the respective values of the matter couplings appear as solutions, hence to all orders of perturbation theory there exist Green functions which depend on one coupling only and whose tree approximation has the respective symmetry. Of course nothing can be derived from this analysis alone, on how the symmetry is realized in higher orders.
In both cases there exists a second solution, also to all orders, which does not show supersymmetry. All of these solutions go to zero with the primary coupling.
A stability analysis along the lines of Lyapunov's theory has been performed. The $N = 2$ example is UV unstable. For the $N = 4$ theory the system is UV-unstable if $\beta \leq 0$ and it is IR-unstable if $\beta \geq 0$ for small coupling. Even after the proof that perturbatively the β-function vanishes identically (cf. subsection 4.2) one cannot exclude terms which vanish exponentially, hence the unequality assumptions are relevant.

CONSTRUCTION OF GAUGE THEORIES WITH A SINGLE COUPLING PARAMETER FOR YANG–MILLS AND MATTER FIELDS

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Massless gauge theories are considered involving matter fields coupled invariantly to a Yang–Mills field. In general renormalization induces additional couplings with independent coefficients λ_j. Consistent descriptions depending only upon the gauge coupling g may be constructed by determining the functions $\lambda_j(g)$ which are independent of the normalization point and allow for an asymptotic power series in g. Two examples with four couplings are shown to result in the pure $N = 2$, $N = 4$ supersymmetric Yang–Mills theories. In addition, one obtains some non-supersymmetric models. Stability properties are discussed using Lyapunov's theory.

In this note matter fields consisting of spinor and (pseudo)scalar fields are studied which interact by minimal gauge invariant couplings to a Yang–Mills field. Apart from gauge parameters the free parameters of the model are mass parameters and the gauge coupling constant (we consider a simple gauge group). Usually such models are not renormalizable in their original form without direct interaction among the matter fields. By adding appropriate direct coupling terms a renormalizable formulation can be obtained which however involves additional free parameters. We want to study the question under which conditions such a formulation depending on several independent coupling parameters can be reduced to a consistent description in terms of the gauge coupling constant only.

We restrict ourselves to the discussion of the massless case in a covariant guage. Then the free parameters are gauge parameters, the renormalized gauge coupling constant g, the renormalized parameters $\lambda_1, ..., \lambda_n$ of the direct coupling terms, and the euclidean normalization mass x^2. By reduction the coupling parameters λ_j become functions

$$\lambda_j = \lambda_j(g), \quad j = 1, ..., n \tag{1}$$

of the gauge coupling constant for which the following conditions will be required [1]:

(i) Renormalization group invariance. The functions $\lambda_j(g)$ should be independent of the normalization point x^2.

(ii) Renormalizability. The functions $\lambda_j(g)$ should possess asymptotic expansions with respect to powers of g.

A formulation satisfying these requirements comes closest to the concept of the original model and may therefore be called the proper renormalized version of the minimal gauge invariant interaction between matter fields and a Yang–Mills field.

Two examples will be discussed briefly leaving detailed derivations for a separate publication. The first example is a system of one Dirac spinor, one scalar, and pseudoscalar field, all transforming according to the adjoint representation of the gauge group SU(2).

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The second example also concerns the gauge group SU(2) with four Majorana spinors, three real scalars, and three real pseudoscalar fields, all in the adjoint representation. For simplicity an internal SU(2) $\times$ SU(2) symmetry will be imposed in addition.

Dirac spinor, scalar and pseudoscalar field transforming according to the adjoint representation of the gauge group SU(2). The most general form of the direct interaction part consisting of terms with dimension four, invariant under Lorentz and gauge transformations, space reflection, and R-transformations is

$$\mathcal{L}_{\mathrm{dir}} = -\mathrm{i}\lambda_1 \epsilon^{abc}\overline{\psi}^a(A^b + B^b\gamma_5)\psi^c - \tfrac{1}{4}\lambda_2(\boldsymbol{A}^2 + \boldsymbol{B}^2)^2 + \tfrac{1}{4}\lambda_3[(\boldsymbol{A}^2)^2 + (\boldsymbol{B}^2)^2 + 2(\boldsymbol{A}\cdot\boldsymbol{B})^2]. \quad (2)$$

The β-functions corresponding to the gauge coupling $\lambda_0 = g^2$ and the direct coupling constants $\lambda_1, \lambda_2, \lambda_3$ respectively are [‡1]

$$\beta_0 = b_0 g^4 + \ldots, \quad b_0 = -16c, \quad c = (32\pi^2)^{-1}, \quad (3)$$

$$\beta_1 = c(32\lambda_1^2 - 48\lambda_0\lambda_1) + \ldots, \quad (4)$$

$$\beta_2 = c(56\lambda_2^2 - 48\lambda_2\lambda_3 + 12\lambda_3^2 - 48\lambda_0\lambda_2 + 32\lambda_1\lambda_2 + 12\lambda_0^2 - 32\lambda_1^2) + \ldots, \quad (5)$$

$$\beta_3 = c(-36\lambda_3^2 + 48\lambda_2\lambda_3 - 48\lambda_0\lambda_3 + 32\lambda_1\lambda_3 - 12\lambda_0^2) + \ldots . \quad (6)$$

Since $\beta_1 \equiv 0$ for $\lambda_1 = 0$ the Yukawa coupling term would not be required for a consistent renormalization scheme with independent coupling parameters g, λ_2 and λ_3. For $\lambda_1 = 0$, however, no renormalizable reduction is possible (see below). The Yukawa coupling should therefore be included for the purpose of constructing a minimal gauge invariant interaction.

The coupling parameters (1) of the reduced model satisfy the differential equations [1,3]

$$\beta_0\, \mathrm{d}\lambda_j/\mathrm{d}g^2 = \beta_j, \quad j = 1,2,3, \quad (7)$$

which imply [1,3,4] the bilinear equations

$$\rho_1^2 - \rho_1 = 0, \quad (8)$$

$$8\rho_1^2 - 14\rho_2^2 - 3\rho_3^2 - 8\rho_1\rho_2 + 12\rho_2\rho_3 + 8\rho_2 - 3 = 0, \quad (9)$$

$$9\rho_3^2 - 8\rho_1\rho_3 - 12\rho_2\rho_3 + 8\rho_3 + 3 = 0, \quad (10)$$

for the lowest order values of the ratios

$$\rho_j = \lim_{g\to 0} \lambda_j/g^2, \quad j = 1,2,3. \quad (11)$$

Eq. (8) implies that $\rho_1 = 0$ or $\rho_1 = 1$. If $\rho_1 = 0$ it can be shown that eqs. (9),(10) have no real solution for ρ_2 and ρ_3. Therefore, it is not possible to construct a renormalizable reduced model in the absence of Yukawa couplings. If $\rho_1 = 1$ there are four solutions:

$$\rho_1 = 1, \quad \rho_2 = 1, \quad \rho_3 = 1, \quad (12)$$

$$\rho_1 = 1, \quad \rho_2 = -1, \quad \rho_3 = -1, \quad (13)$$

$$\rho_1 = 1, \quad \rho_2 = 9/\sqrt{105}, \quad \rho_3 = 7/\sqrt{105}, \quad (14)$$

$$\rho_1 = 1, \quad \rho_2 = -9/\sqrt{105}, \quad \rho_3 = -7/\sqrt{105}. \quad (15)$$

All remaining solutions of eqs. (8)–(10) are non-real. The two solutions (13) and (15) with negative ρ_2 and ρ_3 belong to models with a negative classical potential approaching $-\infty$ in almost all directions. Since the existence of such models is doubtful we will not discuss them further in this note.

The solutions with positive ρ_j lead to two expansions of λ_j with respect to powers of g^2

$$\lambda_j' = g^2 + \sum_{n=2}^{\infty} c_{jn}' g^{2n}, \quad (16)$$

$$\lambda_1'' = g^2 + \sum_{n=2}^{\infty} c_{1n}'' g^{2n},$$

$$\lambda_2'' = (9/\sqrt{105})g^2 + \sum_{n=2}^{\infty} c_{2n}'' g^{2n},$$

$$\lambda_3'' = (7/\sqrt{105})g^2 + \sum_{n=2}^{\infty} c_{3n}'' g^{2n}, \quad (17)$$

which solve (7) to any order in g^2. The coefficients

[‡1] See, for instance, ref. [2].

c'_{jn}, c''_{jn} are uniquely determined by the expansion coefficients of the β-functions with respect to $g^2, \lambda_1, \lambda_2$ and λ_3. In addition, (7) is solved by the following expansions which also involve odd powers of g

$$\lambda_1^{(a)} = g^2 + \sum_{m=4}^{\infty} d_{1m} g^m ,$$

$$\lambda_2^{(a)} = g^2 + ag^3 + \sum_{m=4}^{\infty} d_{2m} g^m ,$$

$$\lambda_3^{(a)} = g^2 + 3ag^3 + \sum_{m=4}^{\infty} d_{3m} g^m . \qquad (18)$$

The parameter a is arbitrary. All other coefficients are uniquely determined for given a. For the positive limits (12), (14) of ρ_j eqs. (16)–(18) constitute all possible solutions of (7) which can be expanded with respect to powers of g.

Inserting the expansions (16), (17) or (18) into β_0 the β-function of the reduced model becomes

$$\beta'(g^2) = \beta_0(g^2, \lambda'_1(g^2), ..., \lambda'_3(g^2)),$$

$$\beta''(g^2) = \beta_0(g^2, \lambda''_1(g^2), ..., \lambda''_3(g^2)),$$

$$\beta^{(a)}(g^2) = \beta_0(g^2, \lambda_1^{(a)}(g^2), ..., \lambda_3^{(a)}(g^2)). \qquad (19)$$

Likewise the Green functions of the reduced models are obtained by inserting (16), (17) or (18) into the Green functions of the embedding theory.

The coupling parameters (16) belong to the model of extended supersymmetry with $N = 2$ and gauge symmetry SU(2) [5,6]. The Green functions of this model are thus uniquely determined to any order of g although the supersymmetric properties have not been proved rigorously in higher orders ‡2.

The expansion (17) determines a non-supersymmetric model which also provides a minimal form of the gauge invariant interaction between the given multiplet of matter fields and the Yang–Mills field.

Expansion (18) provides an example for a hard breaking of supersymmetry which is still consistent with the renormalization procedure. However, if we require invariance under the gauge reflection symmetry $A_\mu \to -A_\mu, g \to -g$ it follows that $a = 0$ so that the solution with expansion (16) becomes unique.

The models with the couplings (16)–(18) are all asymptotically free. Due to the negative sign of β_0 in lowest order the effective coupling $\bar{g}$ vanishes,

$$\bar{g} \to 0 \quad \text{for} \quad k^2 \to -\infty,$$

in the large euclidean limit, while the ratios

$$\bar{\lambda}_j/\bar{g}^2 \to \rho_j$$

approach their lowest order values (12) or (14).

By a suitable redefinition of the coupling parameters the lowest order ratios (11) can be made exact so that

$$\lambda'_1 = \lambda'_2 = \lambda'_3 = g^2, \qquad (20)$$

$$\lambda''_1 = g^2, \quad \lambda''_2 = (9/\sqrt{105})g^2, \quad \lambda''_3 = (7/\sqrt{105})g^2 \quad (21)$$

for all orders of perturbation theory.

Suzuki found that the $N = 2$ extended supersymmetric model is ultraviolet unstable within its embedding theory [8] ‡3. For models of this type a systematic and rigorous study of stability properties can be provided by Lyapunov's theory [10]. Without approximating the β-functions, solutions

$$\bar{g}^2 + \delta\bar{g}^2, \quad \bar{\lambda}_j + \delta\bar{\lambda}_j$$

of the evolution equations are considered which lie in a neighborhood of a given solution $\bar{g}^2, \bar{\lambda}_j$ for the reduced model. The system is called ultraviolet stable if the variations $\delta\bar{g}^2$ and $\delta(\bar{\lambda}_j/\bar{g}^2)$ can be made arbitrarily small for large euclidean momenta k provided the initial variations at some fixed momentum are chosen small enough. By applying Lyapunov's theory we found

$$b_0 < 0, \quad \mathrm{Re}\ \kappa < 0 \qquad (22)$$

for all eigenvalues κ of the characteristic matrix

$$\Gamma = \| \partial\beta_k^{(0)}/\partial\rho^l - b_0\delta_{kl} \|, \quad \beta_k^{(0)} = \lim_{g\to 0} \beta_k/g^4, \qquad (23)$$

as sufficient criterion for ultraviolet stability. On the other hand, if Re $\kappa > 0$ for at least one eigenvalue κ the system is necessarily unstable. Although the conditions refer to the lowest order approximation the

‡2 It was first proved by Maison in ref. [7] that the β-functions for super Yang–Mills theories are uniquely determined by the β-functions of the embedding theory.

‡3 Similarly instability was found for some other supersymmetric models, see refs. [3,9].

theorems apply to solutions of the evolution equations with the exact β-functions.

For the supersymmetric case (11) the eigenvalues of Γ are

$$\kappa_1 = -2b_0 > 0, \quad \kappa_2 = -3b_0 > 0, \quad \kappa_3 = \tfrac{1}{2}b_0 < 0. \tag{24}$$

The solutions with the expansions (16) or (18) are therefore ultraviolet unstable against independent variations of the coupling terms. Since one of the eigenvalues is negative one expects a one-parametric solution of the differential equations (7) with the asymptotic behavior (11), (12) for $g \to 0$. This one-parameter family is represented by the expansion (18) with the arbitrary parameter a.

For the non-supersymmetric values (14) the characteristic matrix has only positive eigenvalues. Therefore also this solution is ultraviolet unstable. Moreover no further solutions should be expected with the same asymptotic behavior as (17).

All other solutions of (7) which do not approach the values (11)–(15) for $g \to 0$ are driven away into a domain which in lowest order is controlled by non-real roots of (8)–(10). In this region no asymptotic properties of the β-functions can be established so that lowest order calculations are meaningless [11]. Hence the only solutions which are asymptotically free are those with the limit (11)–(15), i.e. the expansions (16)–(18), and expansions corresponding to (13) or (15).

Model of matter fields invariant under SU(2) × SU(2) and transforming according to the adjoint representation of the gauge group SU(2). The model consists of Majorana spinors ψ_K^a, real scalars A_i^a and real pseudoscalars B_i^a ($a = 1,2,3; K = 1,...,4; i = 1,2,3$). The subscripts refer to the additional symmetry group SU(2) × SU(2). Under the right symmetry group SU(2) the fields transform as

$$\delta_i^{\rm r} A_j = -\epsilon_{ijk} A_k, \quad \delta_i^{\rm r} B_j = 0, \quad \delta_i^{\rm r}\psi_K = \tfrac{1}{2}\alpha_{iKL}\,\psi_L. \tag{25}$$

For the left symmetry group SU(2) the transformations are

$$\delta_i^{\ell} A_j = 0, \quad \delta_i^{\ell} B_j = -\epsilon_{ijk} B_k, \quad \delta_i^{\ell}\psi_K = \tfrac{1}{2}\beta_{iKL}\,\psi_L. \tag{26}$$

The 4 × 4 matrices α_i, β_i satisfy the commutation relations

$$\{\alpha^i,\alpha^j\} = \{\beta^i,\beta^j\} = -2\delta^{ij}, \quad [\alpha^i,\beta^j] = 0,$$

$$[\alpha_i,\alpha_j] = 2\epsilon_{ijk}\alpha_k, \quad [\beta_i,\beta_j] = 2\epsilon_{ijk}\beta_k. \tag{27}$$

The most general form of the direct interaction part consisting of terms with dimension four, invariant under Lorentz and gauge transformations, space reflection, R-transformations and the SU(2) × SU(2)-symmetry is

$$\mathcal{L}_{\rm dir} = -\tfrac{1}{2}\lambda_1\epsilon^{abc}\bar{\psi}_K^a(\alpha_{iKL}A_i^b + {\rm i}\gamma_5\beta_{iKL}B_i^b)\psi_L^c - \tfrac{1}{4}\lambda_2(A_i^aA_i^a + B_i^aB_i^a)^2 + \tfrac{1}{4}\lambda_3(A_i^aA_i^b + B_i^aB_i^b)^2. \tag{28}$$

The β-functions corresponding to the gauge coupling $\lambda_0 = g^2$ and the direct coupling constants $\lambda_1, \lambda_2, \lambda_3$ respectively are ‡4

$$\beta_0 = o(g^6), \quad c = (32\pi^2)^{-1}, \tag{29}$$

$$\beta_1 = c(-48\lambda_0\lambda_1 + 48\lambda_1^2) + ..., \tag{30}$$

$$\beta_2 = c(12\lambda_0^2 - 48\lambda_0\lambda_2 - 64\lambda_1^2 + 64\lambda_1\lambda_2 + 104\lambda_2^2 - 80\lambda_2\lambda_3 + 12\lambda_3^2) + ..., \tag{31}$$

$$\beta_3 = c(-12\lambda_0^2 - 48\lambda_0\lambda_3 + 64\lambda_1\lambda_3 + 48\lambda_2\lambda_3 - 52\lambda_3^2) + ... \tag{32}$$

The limits (11) of the ratios λ_j/g^2 satisfy the following system of bilinear equations [1,3,4]

$$\rho_1^2 - \rho_1 = 0, \tag{33}$$

$$16\rho_1^2 - 26\rho_2^2 - 3\rho_3^2 - 16\rho_1\rho_2 + 20\rho_2\rho_3 + 12\rho_2 - 3 = 0, \tag{34}$$

$$13\rho_3^2 - 16\rho_1\rho_3 - 12\rho_2\rho_3 + 12\rho_3 + 3 = 0. \tag{35}$$

For $\rho_1 = 0$ it can be shown that no real solutions for ρ_2 and ρ_3 exist. Hence $\rho_1 = 1$ follows implying the existence of Yukawa couplings. There are only two sets of real solutions

$$\rho_1' = 1, \quad \rho_2' = 1, \quad \rho_3' = 1, \tag{36}$$

$$\rho_1'' = 1, \quad \rho_2'' = 0.757944 ..., \quad \rho_3'' = 0.352305 .... \tag{37}$$

‡4 See, for instance, ref. [2].

Both solutions can be completed to power series expansions

$$\lambda_j' = g^2 + \sum_{n=2}^{\infty} c_{jn}' g^{2n}, \tag{38}$$

$$\lambda_j'' = \rho_j'' g^2 + \sum_{n=2}^{\infty} c_{jn}'' g^{2n}. \tag{39}$$

The coefficients of the expansions are uniquely determined. (36) corresponds to the model of extended supersymmetry with $N = 4$ and gauge symmetry SU(2) [6]. Thus the Green functions of this model follow uniquely in any order of g although the supersymmetric properties of the model have not been established in higher order [7]. The values (37) do not seem to be related to a symmetry.

By a suitable redefinition of the coupling parameters the lowest order relations among the couplings become exact:

$$\lambda_1' = \lambda_2' = \lambda_3' = g^2, \tag{40}$$

$$\lambda_1'' = g^2, \quad \lambda_2'' = \rho_2'' g^2, \quad \lambda_3'' = \rho_3'' g^2. \tag{41}$$

For both models the β-function vanishes at least in the order g^4. In the supersymmetric case arguments have been given that the β-function vanishes in all orders of perturbation theory. For $\beta \leqslant 0$ in the small coupling region the system is ultraviolet unstable if Re $\kappa > 0$ for at least one eigenvalue κ of the characteristic matrix Γ given by (23) with $b_0 = 0$. Similarly the system is infrared unstable if $\beta \geqslant 0$ for small enough g and if Re $\kappa < 0$ for at least one eigenvalue.

For the values (36) the characteristic matrix Γ has the eigenvalues

$$\kappa_1 = 24c, \quad \kappa_2 = -12c, \quad \kappa_3 = 64c, \quad c = (32\pi^2)^{-1}.$$

Since there are eigenvalues of opposite sign the system must be ultraviolet unstable if $\beta \leqslant 0$ and infrared unstable if $\beta \geqslant 0$ for small couplings. For the case (37) the eigenvalues are

$$\kappa_1 = 24c, \quad \kappa_2 = 69.2 \ldots c, \quad \kappa_3 = 11.4 \ldots c.$$

If $\beta \leqslant 0$ for small couplings the system is ultraviolet unstable. If $\beta \geqslant 0$ for small enough g it can be shown to be infrared stable.

All solutions of (7) with the asymptotic behavior (36) or (37) respectively have the same power series expansion with uniquely determined coefficients. The reason is that for $b_0 = 0$ two solutions with the same asymptotic behavior differ only by terms which decrease exponentially for $g \to 0$.

In conclusion we remark that the ultraviolet instability of the solutions found seems less disturbing if the conjecture should be correct that only those models exist which are asymptotically free or have vanishing β-function. Further excluding models with unstable classical potential only (16)–(18) remain as expansions of possibly existing reduced models in the first example. Within this subset stability holds for the solution (16) of extended supersymmetry with $N = 2$. For the solution (17) the stability question cannot be settled on the basis of the asymptotic expansions. The corresponding discussion of the second example is more involved depending whether or not $\beta \equiv 0$ for the supersymmetric solution.

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2.5 Additional Remarks to Section 2

Klaus Sibold
We first mention the review papers [3], [4] where many examples and some general discussion of the method have been presented.
Next we draw the readers attention to papers [5], [6], [7]. As contribution to a systematic application of the reduction principle they pose and answer the question how gauge theories live in a gauge-nonivariant surrounding. The free theory can, of course be analyzed and understood as consisting of a gauge invariant and gauge fixing part leading to the well-known factor space structure of the physical Hilbert space. When tackling the interacting theory by reduction of gauge-noninvariant couplings non-linear gauge fixing has to be singled out which indeed can be achieved as suggested by the gauge fixing parameter dependence of the free theory. The abelian case can be mastered in full generality, whereas the non-abelian one requires some additional assumption, either on the gauge fixing parameters or on the complete model. E.g. demanding rigid gauge invariance suffices in the important example of $SU(N)$ to find as unique solution of the reduction equations the BRS invariant gauge theory with one coupling and a β-function which is gauge parameter independent.
Most interesting is the result of the stability analysis (following Lyapunov's theory). The eigenvalues of the stability matrix around the BRS-symmetric solution are complex and change their (UV-, IR-) behavior depending on the value of the gauge fixing parameter. Together with the results for the other examples examined in the present section the following pattern for eigenvalues and general solutions arises:

- for gauge theories: BRS-invariant theory embedded in non-invariant surrounding: eigenvalues complex.
  Supersymmetric gauge theory embedded in non-supersymmetric surrounding: eigenvalues real; general solutions exist which are not supersymmetric but still are power series with integer exponents of the primary coupling. Asymptotic behavior fixed.
  SYM with vanishing first order β-function of the gauge coupling, embedded in non-supersymmetric surrounding: eigenvalues real; general solutions exist which vanish exponentially for small coupling.

- Models with spin $0, 1/2$ only: Field content not compatible with $N = 1$ supersymmetry: eigenvalues real; general solution with irrational exponents of the primary coupling.
  Field content compatible with $N = 1$ supersymmetry: eigenvalues real; general solutions with power series of integer powers of the primary coupling.

These regularities have not yet found any deeper understanding. In any case they underline that for characterizing a specific solution of the renormalization group equations one may either demand a symmetry or a power series in a primary coupling. One may very rarely rely on an "automatic" realization via renormalization group flow. This fact supports constructions of asymptotically vanishing solutions by "partial reduction" as used below in the standard model (subsection 3.3) and in its minimally supersymmetric extension (subsection 5.10) .

3 Reduction of couplings in the standard model

Comment (*Klaus Sibold*)
The following remarks form a general introduction to the above section
Even today, almost twenty years after our first paper on reduction of couplings in the standard model the original motivation for applying this method to this model has not become obsolete, neither by time nor by new insight. The theoretical predictions originating from the standard model are in extremely good agreement with experiment. Actually the most precisely measured physical quantities, the anomalous magnetic moment of electron and myon agree within 3 parts per 10^{-9} with their prediction by theory. Two decades of precision measurement and precision calculation yielded essentially on all available observables a truely astonishing coincidence [1]. And, yet there is no convincing explanation why the number of families is three; why the mass scales – the Planck mass and the electroweak breaking scale – differ so much in magnitude, why the Higgs mass is small compared with the Planck scale. And, quite generally, there is also no explanation for the mixing of the families.
Reduction of couplings offers a way to understand at least to some degree masses and mixings of charged leptons and quarks and the mass of the Higgs particle. It extends the well known case of closed renormalization orbits due to symmetry to other, more general ones. Which structure these orbits have had to be learned, i.e. deduced from the relevant renormalization group equation in the specific model. In particular, one had to take into account the different behavior of abelian versus non-abelian gauge groups and of the Higgs self-coupling, say in the ultraviolet region. If asymptotic expansions should make sense in the transition from a non-perturbative theory to a perturbative version it should be possible to rely on common ultraviolet asymptotic freedom. One also has to respect gross features coming from phenomenology. In mathematical terms this is the problem of integrating partial differential equations by imposing suitable boundary conditions (originating from physical requirements): partial reduction.
And, indeed this is how we proceeded historically. In subsection 3.1 mixing of families has been neglected and the structure in the space of running gauge, Higgs and Yukawa couplings has been found, when asking for common ultraviolet behavior. In subsection 3.2 quark family mixing has been analyzed, in subsection 3.3 the method of partial reduction has been introduced. (Actually, in subsection 5.2 this concept has been extended to couplings carrying dimensions.) In subsection 3.4 as an other, additional ingredient we imposed the condition that quadratical divergencies be absent. This requirement makes sense in the context of the standard model, because these divergencies refer to a gauge invariant quantity. Remarkably enough, it turned out that this postulate is indeed consistent with reduction. Subsection 3.5 concludes these earliest investigations in the standard model with an update as of 1991. It yields as values for Higgs and top mass roughly 65, respectively 100 GeV.
Perhaps the most important and not obvious result of the entire analysis is the fact that reduction of couplings (even the version of "partial reduction") is extremely sensitive to the model. If one accepts the integration "paths" as derived in the papers of this section the ordinary standard model can neither afford a mass of the top quark nor of the Higgs particle as large as they have been found experimentally. The mismatch of the fact that the experimental findings are in very good agreement with calculations and the fact that the reduction paths of integration rule out the SM is only apparent: renormalization group improvement of the theoretical predictions concerns essentially the QCD sector, where it

is taken into account in the reduction. Whereas the differences originating from the other couplings turn out to be negligibly small.
Hence it became clear that other model classes are to be studied and further constraining principles had to be found. This will be the subject of sections four and five.

These earliest papers on reduction of couplings have been reviewed e.g. in
[3], [4].

3.1 Higgs and top mass from reduction of couplings

Title: Higgs and top mass from reduction of couplings
Authors: J. Kubo, K. Sibold, W. Zimmermann
Journal: Nucl. Phys. **B259** (1985) 331-350

Comment (*Klaus Sibold*)
In the context of the standard model with one Higgs doublet and n families the principle of reduction of couplings is applied. For simplicity mixing of the families is assumed to be absent: the Yukawa couplings are diagonal and real. For the massless model reduction solutions can be found to all orders of perturbation theory as power series in the "primary" coupling, thus superseding fixed point considerations based on one-loop approximations. Due to the different asymptotic behaviour of the $SU(3), SU(2)$ and $U(1)$ couplings the space of solutions is clearly structured and permits reduction in very distinct ways only. Since reducing the gauge couplings relative to each other is either inconsistent or phenomenologically not acceptable, α_S (the largest coupling) has been chosen as the expansion parameter – the primary coupling – and thus UV-asymptotic freedom as the relevant regime. This allows to neglect in the lowest order approximation the other gauge couplings and to take their effect into account as corrections.
In the matter sector (leptons, quarks, Higgs) discrete solutions emerge for the reduced couplings which permit essentially only the Higgs self-coupling and the Yukawa coupling to the top quark to be non-vanishing.
Stability considerations (Liapunov's theory) show how the power series solutions are embedded in the set of the general solutions. The free parameters in the general solution represent the the integration constants over which one had disposed in the power series, i.e. perturbative reduction solution.
Couplings of the massless model are converted into masses in the tree approximation of the spontaneously broken model. For three generations one finds $m_H = 61$ GeV, $m_{top} = 81$ GeV with an error of about 10-15%.

Nuclear Physics B259 (1985) 331–350

HIGGS AND TOP MASS FROM REDUCTION OF COUPLINGS

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We reduce the couplings in the standard model with one Higgs doublet and n generations and obtain for three generations 61 GeV and 81 GeV for the mass of the Higgs particle and the top quark respectively. The error is estimated to be about 10–15%.

1. Introduction

The standard model for the electroweak and strong interactions is phenomenologically very successful, the price for this success being the relatively large number of free parameters. In the gauge field sector there are the three gauge couplings associated with SU(3), SU(2) and U(1) respectively, in the matter sector there are Higgs and Yukawa couplings [1,2]. The main aim of model building is to reduce this number of free parameters and not to lose the good agreement with experiment. Grand unified theories permit to replace the three gauge couplings by one, but do not substantially help in the matter sector [3]. Supersymmetric theories may relate Higgs and Yukawa couplings (even to gauge couplings), but they introduce many new particles which are not yet observed [4]. Composite model building, as the third possibility, has not yet led to a realistic alternative [5]. In a less ambitious attempt one may therefore look for other relations amongst couplings within a given model. Estimates of the masses of heavy quarks and/or higgs(es) fall into this category since within a class of models the values of the masses reflect the values of the couplings via the Higgs effect. The main idea put forward thus far is the use of renormalization group equations. In [6] e.g. one exploits a fixed point structure of the Yukawa couplings in first order perturbation theory, in [7] one argues with consistency limits for effective couplings again in first order. Since the reliability of first order approximations is doubtful we apply in the present paper a concept which leads to

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all possible relations amongst couplings to *all* orders [8,9]. If g and $\lambda_1, \ldots, \lambda_n$ are arbitrary coupling constants of a theory, then *in general* relations of the type

$$\lambda_i = \lambda_i(g), \qquad i = 1, \ldots, n \tag{1.1}$$

with

$$\lambda_i \to 0 \qquad \text{for} \quad g \to 0$$

will not be compatible with renormalization group invariance: if such relations hold at one value t_1 of the scale parameter t, they will not hold at any other value t_2 of it. Or else: the infinities associated with the full set of couplings will in general not be removed for the reduced set. Relations (1.1) can hold [8] only if they are solutions of

$$\beta_g \frac{\mathrm{d}\lambda_i}{\mathrm{d}g} = \beta_{\lambda_i}, \qquad i = 1, \ldots, n. \tag{1.2}$$

By establishing and solving (1.2) in a given model one checks therefore in an exhaustive manner whether relations amongst couplings do or do not exist. Amongst the solutions one finds in particular all those relations corresponding to a symmetry, but there may be others, not attributable to any known symmetry (see [8,10] for examples). This method therefore provides the means to search for relations amongst couplings where the commitment to a symmetry or even *any* specific mechanism causing them is not desirable.

In sect. 2 we describe the (simplified) model which we treat and collect all information needed for the subsequent discussion. In sect. 3 we study systematically the possible reduction to *one* coupling constant. In sect. 4 we study the general solution of the reduction equations. In sect. 5 we collect and discuss the results of our analysis.

2. The model

Let us first of all describe the model we shall treat afterwards and in particular list all simplifications leading to it. We consider the gauge theory of the group SU(3) × SU(2) × U(1) with one Higgs doublet and n generations of quarks and leptons put into the usual family structure.

$$\mathcal{L} = \mathcal{L}_{\mathrm{YM}} + \mathcal{L}_{\mathrm{F}} + \mathcal{L}_{\mathrm{Yuk}} + \mathcal{L}_{\mathrm{H}}, \tag{2.1}$$

$$\mathcal{L}_{\mathrm{YM}} = -\tfrac{1}{4} F^i_{\mu\nu} F^{i\mu\nu} - \tfrac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu},$$

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g_3 f^{ijk} A^j_\mu A^k_\nu,$$

$$F^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g \varepsilon^{abc} W^b_\mu W^c_\nu,$$

$$F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu\,,$$

$$\mathcal{L}_F = \sum_{j=1}^{n}\left(i\bar{L}_j \not{D} L_j + i\bar{E}_j \not{D} E_j + i\bar{Q}_j \not{D} Q_j + i\bar{U}_j \not{D} U_j + i\bar{D}_j \not{D} D_j\right),$$

$$\not{D} = \gamma^\mu\left(\partial_\mu - ig_3 T^i A^i_\mu - igT^a W^a_\mu + iYg'B_\mu\right),$$

$$\mathcal{L}_H = |\left(\partial_\mu - \tfrac{1}{2}ig\tau^a W^a_\mu + \tfrac{1}{2}ig'B_\mu\right)\Phi|^2 + \tfrac{1}{2}\mu^2\Phi^+\Phi - \tfrac{1}{4}\lambda'(\Phi^+\Phi)^2,$$

$$\mathcal{L}_{\text{Yuk}} = -\sum_{j=1}^{n}\left\{G_j^{(l)}\left(\bar{L}_j\Phi L_j + \text{h.c.}\right) + G_j^{(m)}\left(\bar{Q}_j i\tau^2\Phi^* U_j + \text{h.c.}\right)\right.$$

$$\left. + G_j^{(d)}\left(\bar{Q}_j\Phi D_j + \text{h.c.}\right)\right\},$$

$$L_j = \frac{1-\gamma_5}{2}\begin{pmatrix}\nu_j\\ l_j\end{pmatrix},$$

$$E_j = \frac{1+\gamma_5}{2} l_j\,,$$

$$Q_j = \frac{1-\gamma_5}{2}\begin{pmatrix}p_j\\ n_j\end{pmatrix},$$

$$U_j = \frac{1+\gamma_5}{2} p_j\,,$$

$$D_j = \frac{1+\gamma_5}{2} n_j\,,$$

$$\Phi = \begin{pmatrix}\phi^+\\ \phi^\bullet\end{pmatrix},$$

$$\phi^0 = \sqrt{\tfrac{1}{2}}\,(v + \varphi + i\chi)\,.$$

For simplicity we have restricted ourselves to diagonal (hence real) Yukawa couplings, i.e. we impose family number conservation. We thereby give up the possibility to predict the Kobayashi-Maskawa mixing angles by the method of reduction.

In the tree approximation we have the following expressions for the physical masses:

$$M_W = \tfrac{1}{2}gv\,, \qquad m_H^2 = \tfrac{1}{2}\lambda' v^2\,, \qquad m_j^{(\ell)} = \sqrt{\tfrac{1}{2}}\,G_j^{(\ell)} v\,, \qquad m_j^{(q)} = \sqrt{\tfrac{1}{2}}\,G_j^{(q)} v\,. \quad (2.2)$$

The electric charge e is related to g and g' via the Weinberg angle:

$$e = g\sin\theta_{\mathrm{w}} = g'\cos\theta_{\mathrm{w}}. \tag{2.3}$$

We shall use a normalization for the U(1) coupling which is suggested by grand unified theories:

$$g_1 := \sqrt{\tfrac{5}{3}}\, g' \tag{2.4}$$

and the notation $g_2 := g$. The completely massless version of (2.1) ($\mu = 0$, hence $v = 0$) is presumably renormalizable, the proof going along the lines of [11]. Then the vertex functions of the model satisfy a renormalization group equation

$$\left(\kappa\partial_\kappa + \beta_{g_i}\partial_{g_i} + \beta_{\lambda'}\partial_{\lambda'} + \beta_{G_j}\partial_{G_j} - \sum_a \gamma_a N_a\right)\Gamma = 0 \tag{2.5}$$

and the reduction of couplings according to [8] can be applied. We shall indeed perform reduction to all orders of perturbation theory in this theory and then enter into the *tree* approximation of (2.1) i.e. into (2.2) – the massive theory – with the values of the couplings obtained by reduction. This yields mass values neglecting radiative corrections in the massive theory.

It may very well be that there exists a renormalization prescription for the massive (and spontaneously broken) theory which has in all orders the *same* β-functions as the massless model. In this case our reduction results would be automatically the same as those of the massive theory. The only change would occur in the relation equivalent to (2.2) which had to express the physical mass in terms of the (unphysical) shift v and the (also unphysical) couplings g_i.

The β-functions read in the one-loop approximation [12]

$$\beta_{g_1} = \frac{1}{16\pi^2}\left(\frac{1}{10} + \tfrac{4}{3}n\right)g_1^3 + \cdots,$$

$$\beta_{g_2} = \frac{1}{16\pi^2}\left(-\tfrac{43}{6} + \tfrac{4}{3}n\right)g_2^3 + \cdots,$$

$$\beta_{g_3} = \frac{1}{16\pi^2}\left(-11 + \tfrac{4}{3}n\right)g_3^3 + \cdots,$$

$$\beta_{G_i^{(\ell)}} = \frac{1}{16\pi^2}G_i^{(\ell)}\left(\tfrac{3}{2}G_i^{(\ell)2} + \sum_j\left(G_j^{(\ell)2} + 3G_j^{(\mathrm{u})2} + 3G_j^{(\mathrm{d})2}\right)\right.$$
$$\left. - \tfrac{9}{4}g_1^2 - \tfrac{9}{4}g_2^2\right) + \cdots,$$

$$\beta_{G_i^{(\mathrm{u})}} = \frac{1}{16\pi^2} G_i^{(\mathrm{u})} \Big(\tfrac{3}{2} G_i^{(\mathrm{u})2} - \tfrac{3}{2} G_i^{(\mathrm{d})2} + \sum_j \left(G_j^{(\ell)2} + 3G_j^{(\mathrm{u})2} + 3G_j^{(\mathrm{d})2} \right)$$

$$- \tfrac{17}{20} g_1^2 - \tfrac{9}{4} g_2^2 - 8 g_3^2 \Big) + \cdots ,$$

$$\beta_{G_i^{(\mathrm{d})}} = \frac{1}{16\pi^2} G_i^{(\mathrm{d})} \Big(\tfrac{3}{2} G_i^{(\mathrm{d})2} - \tfrac{3}{2} G_i^{(\mathrm{u})2} + \sum_j \left(G_j^{(\ell)2} + 3G_j^{(\mathrm{u})2} + 3G_j^{(\mathrm{d})2} \right)$$

$$- \tfrac{1}{4} g_1^2 - \tfrac{9}{4} g_2^2 - 8 g_3^2 \Big) + \cdots ,$$

$$\beta_{\lambda'} = \frac{1}{16\pi^2} \Big(6\lambda'^2 + 4\lambda' \sum_j \left(G_j^{(\ell)2} + 3G_j^{(\mathrm{u})2} + 3G_j^{(\mathrm{d})2} \right)$$

$$- \tfrac{9}{5}\lambda' g_1^2 - 9\lambda' g_2^2 + \tfrac{27}{50} g_1^4 + \tfrac{9}{5} g_1^2 g_2^2 + \tfrac{9}{2} g_2^4$$

$$- 8 \sum_j \left(G_j^{(\ell)4} + 3G_j^{(\mathrm{u})4} + 3G_j^{(\mathrm{d})4} \right) \Big) + \cdots . \tag{2.5}$$

(The dots stand for higher order contributions.) Our aim is to solve reduction equations like (1.2). It is then convenient to introduce the following variables

$$x := \frac{g_3^2}{4\pi}, \qquad y := \frac{g_2^2}{4\pi}, \qquad z := \frac{g_1^2}{4\pi},$$

$$\lambda := \frac{\lambda'}{4\pi}, \qquad u_i := \frac{G_i^2}{4\pi}, \qquad i = \text{leptons, quarks}. \tag{2.6}$$

With these variables a reduction equation

$$\beta_{g_3} \frac{\mathrm{d}G_i}{\mathrm{d}g_3} = \beta_{G_i} \tag{2.7}$$

goes over into

$$\beta_x \frac{\mathrm{d}u_i}{\mathrm{d}x} = \beta_{u_i} \tag{2.8}$$

with

$$\beta_x = \frac{2g_3}{4\pi} \beta_{g_3},$$

$$\beta_{u_i} = \frac{2G_i}{4\pi} \beta_{G_i}. \tag{2.9}$$

The values of the couplings are chosen at the scale of the W-mass and read [13]

$$x_0 = \left.\frac{g_3^2}{4\pi}\right|_{t \hat{=} M_W} \equiv \alpha_s = 0.1\,,$$

$$y_0 = \left.\frac{g_2^2}{4\pi}\right|_{t \hat{=} M_W} = \frac{\alpha}{\sin^2\theta_w} = 0.037\,,$$

$$z_0 = \left.\frac{g_1^2}{4\pi}\right|_{t \hat{=} M_W} = \frac{5}{3}\frac{\alpha}{\cos^2\theta_w} = 0.016\,, \tag{2.10}$$

($\sin^2\theta_w(M_W) = 0.21$; $\alpha(M_W) = 1/128$). Using $M_W = 81$ GeV as known input we may thus evaluate all masses once couplings are given in terms of x, y or z.

3. Complete reduction

Suppose

$$u_i(x) = x\left(u_i^{(0)} + u_i^{(1)}x + u_i^{(2)}x^2 + \cdots\right) \tag{3.1}$$

is a solution of eq. (2.8). Since it is a *power series* in the coupling constant x it corresponds to ordinary perturbation theory. The coupling u_i is expressed by the coupling x compatible with renormalization. This we shall call *complete* reduction. An ansatz like (3.1) fixes, of course, an integration constant and picks a special solution out of the general one. The general solution containing such an integration constant will therefore not be a reduced one. With it we have just traded an integration constant for an ordinary renormalized coupling. If for a certain coupling such a general solution has to be used, e.g. for phenomenological reasons, the reduction is incomplete.

We start now a systematic search as to which couplings can be reduced completely to others.

3.1. GAUGE COUPLINGS

Let us first try to reduce the couplings y and z to x. This means that we have to solve

$$\beta_x \frac{\mathrm{d}y}{\mathrm{d}x} = \beta_y\,, \tag{3.2}$$

$$\beta_x \frac{\mathrm{d}z}{\mathrm{d}x} = \beta_z\,, \tag{3.3}$$

with functions

$$y = x\left(y^{(0)} + y^{(1)}x + \cdots\right), \tag{3.4}$$

$$z = x\left(z^{(0)} + z^{(1)}x + \cdots\right). \tag{3.5}$$

At the order x^2 we find (for 3 families)

$$-14y^{(0)} = -\tfrac{19}{3}y^{(0)2}, \tag{3.6}$$

$$-14z^{(0)} = \tfrac{41}{5}z^{(0)2}, \tag{3.7}$$

i.e. the solutions

$$y_-^{(0)} = 0, \qquad y_+^{(0)} = \tfrac{42}{19} > 1, \tag{3.8}$$

$$z_+^{(0)} = 0, \qquad z_-^{(0)} = -\tfrac{70}{41}. \tag{3.9}$$

Since vanishing gauge coupling is definitely not acceptable we are forced to the choice $y_+^{(0)}$ and $z_-^{(0)}$. But $z_-^{(0)}$ is negative, hence g_1^2 imaginary i.e. excluded. $y_+^{(0)}$ being greater than 1 implies that the SU(2) coupling g_2 would be larger than the SU(3) coupling g_3 – which is phenomenologically unacceptable. Hence we conclude that this reduction is either theoretically inconsistent or phenomenologically bad*. We are therefore not able to predict the Weinberg angle by reduction. This seemingly negative result had in fact to be expected: Grand unified theories give reasonable values for g_3, g_2, g_1 in terms of the coupling g of the grand gauge group G, i.e. for the Weinberg angle. But the vector fields making up G from $SU(3) \times SU(2) \times U(1)$ are missing in our theory hence reduction *must* fail. Reduction could yield a correspondingly good result only if all of those fields were included, i.e. were contributing to the respective β-functions.

Analogously the reduction of y to z is strictly excluded. Generalizing we may summarize by stating that gauge couplings with the same asymptotic behaviour can be reduced (here $x \leftrightarrow y$) whereas those of different asymptotic behaviour (here $x \leftrightarrow z$, $y \leftrightarrow z$) cannot. The respective magnitudes of reduced couplings can only be those of an embedding grand unified theory when *all* fields are included.

We shall have to use the *general* solutions of (3.2), (3.3). They read

$$y = \frac{42}{19}\frac{x}{1 + c_y x}, \tag{3.10}$$

$$z = \frac{70}{41}\frac{x}{c_z x - 1}, \tag{3.11}$$

where c_y, c_z are the integration constants to be fixed by experiment. For the effective couplings at the W-mass we have (cf. sect. 2)

$$x_0 = 0.1, \qquad y_0 = 0.037, \qquad z_0 = 0.016,$$
$$c_y = 49.7, \qquad c_z = 116.7. \tag{3.12}$$

* One can check that the two-loop contributions do not ameliorate the situation.

The use of (3.10), (3.11) is, of course, based on the hope that the numerical values (3.12) obtained in the one-loop approximation are reliable in the range of physically accessible values of x even if one includes higher order corrections. This is a much weaker assumption than to rely on very specific *properties* of the one-loop β-functions like having a fixed point or using the existence of a pole in the corresponding effective coupling.

3.2. REDUCTION OF MATTER COUPLINGS

Since the gauge couplings cannot be reduced we shall put two of them equal to zero, reduce the matter couplings with respect to the third one and then take into account the effect of the others by a switching on procedure. Let us start by putting

$$g_1 = g_2 = 0 \tag{3.13}$$

and reducing to g_3.

The reduction equations

$$\beta_x \frac{\mathrm{d}u_i}{\mathrm{d}x} = \beta_{u_i} \tag{3.14}$$

lead for the power series (3.1) at order x^2 to the following quadratic equations:

$$u_\ell^{(0)}\left(3u_\ell^{(0)} + 2\sum_{\ell'} u_{\ell'}^{(0)} + 6\sum_{\mathrm{q}} u_{\mathrm{q}}^{(0)} + 14\right) = 0\,, \tag{3.15}$$

ℓ: leptons, q: quarks;

$$u_i^{(0)}\left(3u_i^{(0)} - 3u_{\mathrm{d}(i)}^{(0)} + 2\sum_{\ell} u_\ell^{(0)} + 6\sum_{\mathrm{q}} u_{\mathrm{q}}^{(0)} - 2\right) = 0\,, \tag{3.16}$$

i: up-quarks; d(i) down quark associated with ith up quark;

$$u_i^{(0)}\left(3u_i^{(0)} - 3u_{\mathrm{u}(i)}^{(0)} + 2\sum_{\ell} u_\ell^{(0)} + 6\sum_{\mathrm{q}} u_{\mathrm{q}}^{(0)} - 2\right) = 0\,, \tag{3.17}$$

i: down-quarks; u(i) up quark associated with ith down-quark.

The solutions of (3.15) are

(i) $u_\ell^{(0)} = 0$ (3.18)

(ii) If $u_{\ell_1}^{(0)} \neq 0$ and $u_{\ell_2}^{(0)} \neq 0$:

$$u_{\ell_1}^{(0)} = u_{\ell_2}^{(0)} < 0\,. \tag{3.19}$$

Since the variables u have to be non-negative, cf. (2.6), only solutions (3.18) can be chosen. Taking them into account for the solution of (3.16), (3.17) we first find*

(i) $u_{\mathrm{q}}^{(0)}=0,$ (3.20)

(ii) If $u_i \neq 0$ and $u_{\mathrm{d}(i)} \neq 0$ there follows

$$u_i^{(0)}=u_{\mathrm{d}(i)}^{(0)}, \qquad i\text{: up-quark}. \tag{3.21}$$

Now relation (3.21) is certainly a bad approximation for the heaviest quark doublet, hence

$$u_{\mathrm{d}(i)}^{(0)}=0 \tag{3.22}$$

for the down quark of the heaviest doublet. If that mass is neglected one has of course to neglect also the other masses. Demanding gross agreement with phenomenology we thus arrive finally at the solution

$$u_{\mathrm{q}}^{(0)}=0 \quad \text{for q} \neq \text{t} \tag{3.23}$$

and at

$$u_{\mathrm{t}}^{(0)}=\tfrac{2}{9} \quad \text{for top quark}. \tag{3.24}$$

The numerical value of $u_{\mathrm{t}}^{(0)}$ is given for $n=3$ generations. For $n=4$ it is $\frac{14}{27}$ ($\hat{=} m_{\mathrm{t}'} = 136$ GeV), $n=5$ it is $\frac{22}{27}$ ($\hat{=} m_{\mathrm{t}'} = 170$ GeV).

Starting with the values (3.23), (3.24) it can be seen that the respective power series (3.1) are uniquely determined by (3.14):

$$u_{\mathrm{q}} \equiv 0, \qquad \mathrm{q} \neq \mathrm{t} \tag{3.25}$$

$$u_{\mathrm{t}}=\tfrac{2}{9}x+\cdots. \tag{3.26}$$

The dots in (3.26) stand for higher order terms whose coefficients are uniquely given by the higher order terms in the β-functions (2.5). Similarly for the leptons:

$$u_{\ell} \equiv 0 \quad \ell\text{: lepton}. \tag{3.27}$$

We now proceed to determine the Higgs coupling

$$\lambda=x(\lambda^{(0)}+\lambda^{(1)}x+\cdots) \tag{3.27}$$

by solving the reduction equation

$$\beta_x \frac{\mathrm{d}\lambda}{\mathrm{d}x}=\beta_\lambda. \tag{3.28}$$

In β_λ we insert the solution (3.25), (3.26), (3.27) for the Yukawa couplings and

* The trivial solution $u_{\mathrm{q}}^{(0)}=0$ for all quarks seems to be a bad starting point for the generation of masses since it would lead to $u_{\mathrm{q}} \equiv 0$ to all orders. But the general solution surrounding it allows for adjusting all quark masses independently with the Higgs mass determined. (See forthcoming paper, ref. [15].)

obtain from order x^2 the quadratic equation

$$6\lambda^{(0)2} + \left(14 + 12u_t^{(0)}\right)\lambda^{(0)} - 24u_t^{(0)2} = 0. \qquad (3.29)$$

It has the solutions

$$\lambda_{\pm}^{(0)} = \tfrac{1}{18}\left(-25 \pm \sqrt{689}\right) \approx \begin{cases} 0.0694 \\ -2.85. \end{cases}$$

Only the positive root is physically acceptable. It gives also rise to a unique power series

$$\lambda = x\left(\lambda_{+}^{(0)} + \cdots\right). \qquad (3.30)$$

The corresponding numbers read for

$$\begin{aligned} &n = 4, \qquad \lambda_{+}^{(0)} \approx 0.33, \qquad m_{\mathrm{H}} \approx 108\ \mathrm{GeV}, \\ &n = 5, \qquad \lambda_{+}^{(0)} \approx 0.703, \qquad m_{\mathrm{H}} \approx 158\ \mathrm{GeV}. \end{aligned} \qquad (3.31)$$

This concludes the reduction to g_3. It remains to discuss the other possibilities of reduction.

The case $g_1 = g_3 = 0$, $g_2 \neq 0$ and reduction to g_2 is easily treated. It leads only for $n \geqslant 4$ to real Yukawa couplings, hence to no reduction for $n \leqslant 3$.

The case $g_2 = g_3 = 0$, $g_1 \neq 0$ on the other hand is quite analogous to the reduction to g_3. It permits very similar results, at least from a mathematical point of view. As far as physics is concerned it is totally different from reduction to g_3. We understand the massless theory as an approximation to the massive one and are thus considering the ultraviolet limit of the couplings. But in this limit the theory with $g_2 = g_3 = 0$ cannot be adequately described by perturbation theory i.e. by our approximation of the β-functions since this U(1) theory is *infrared* free. Assuming that asymptotic (UV-) freedom is relevant for the physical theory we may therefore exclude this possibility for reduction.

The case $g_1 = g_2 = g_3 = 0$ also permits reduction, namely to $u_i = u_i(\lambda)$. But the physically acceptable solutions ($\lambda > 0$) go into physically unacceptable solutions ($\lambda < 0$) if g_3 is switched on.

Let us summarize this subsection. The reduction of matter couplings is (for $n \geqslant 3$) possible with respect to SU(3) and U(1), the latter being of academic interest only, if for the physical theory asymptotic freedom is important. The reduced couplings are given by (3.25), (3.26), (3.27), (3.30). Inserting the lowest order approximation into (2.2) with (2.10) we find for the masses ($n = 3$)

$$\begin{aligned} &m_{\ell} = 0 &&\ell\text{: leptons} \\ &m_{\mathrm{q}} = 0 &&\mathrm{q} \neq \mathrm{t}\text{: quarks} \\ &m_{\mathrm{t}} = 90\ \mathrm{GeV} &&\text{top quark} \\ &m_{\mathrm{H}} = 50\ \mathrm{GeV} &&\text{Higgs particle}. \end{aligned} \qquad (3.32)$$

Although these are not yet the final predictions for the physical masses – we still have to switch on g_2 and g_1 – it is reassuring that their values are not completely absurd.

4. General solution

4.1. STABILITY OF THE REDUCED SOLUTION

Complete reduction of couplings is closely related to asymptotic expansions for small (or large!) values of these couplings. Let us explain this in the context of the general solutions for the system (3.14), (3.28) in the one-loop approximation.

The reduction equation for the Yukawa coupling of the top quark $u \equiv u_t$ reads

$$-14x^2\frac{du}{dx} = 9u^2 - 16xu \tag{4.1}$$

(for $u_\ell = u_q = 0$, $q \neq t$, $y = z = 0$). Its general solution is given by

$$u = \frac{1}{v} = \frac{2x^{8/7}}{C + 9x^{1/7}}. \tag{4.2}$$

The initial value $C = 0$ (which means x large compared to C) leads to

$$u = \tfrac{2}{9}x \equiv u_+ , \tag{4.3}$$

i.e. to the reduced solution corresponding to the larger root (3.24) of (3.16). The initial condition $C = \infty$ (which means "x small compared to C") leads to

$$u = 0 \equiv u_- , \tag{4.4}$$

i.e. to the reduced solution corresponding to the smaller root (3.23) of (3.16). For eq. (4.1) the reduction solution u_+ is the asymptotically stable one for *large* x, u_- the one for *small* x. (Cf. the stability discussion in [9].)

Precisely the same situation is realized for the Higgs coupling. The reduction equation reads in the one-loop approximation

$$-14x^2\frac{d\lambda}{dx} = 6\lambda^2 + 12\lambda u_t - 24u_t^2. \tag{4.5}$$

For $u_t = \frac{2}{9}x$, $u_\ell = u_q = 0$ ($q \neq t$), $y = z = 0$ we find the general solution in the form

$$\lambda = \rho x + \nu,$$

$$\nu = \frac{x^{-\delta}}{C - \dfrac{3}{7}\dfrac{1}{\delta+1}x^{-\delta-1}}, \tag{4.6}$$

where

$$\delta = \tfrac{1}{21}(18\rho + 4), \tag{4.7}$$

$$\rho = \lambda_{\pm}^{(0)} = \tfrac{1}{18}(-25 \pm \sqrt{689}). \tag{4.8}$$

If $C = 0$ the solution $\lambda(\lambda_{+}^{(0)})$ goes into $\lambda_{-} = \lambda_{-}^{(0)}x$ for *small* x, since $-\delta_{+} - 1$ is negative.

If $C = \infty$, then the solution $\lambda(\lambda_{+}^{(0)})$ goes into $\lambda_{+} = \lambda_{+}^{(0)}x$ for *large* x. (It is easily checked that

$C = 0$ leads from $\lambda(\lambda_{-}^{(0)})$ to λ_{+} for large x,

$C = \infty$ leads from $\lambda(\lambda_{-}^{(0)})$ to λ_{-} for small x.)

For the sake of completeness and in order to prepare for the considerations to follow let us still discuss in the one-loop approximation the complete solution of the system with bottom and top coupling being non-vanishing. We have the reduction equations

$$-14x^2 \frac{du_b}{dx} = 9u_b^2 + 3u_b u_t - 16u_b x, \tag{4.9}$$

$$-14x^2 \frac{du_t}{dx} = 9u_t^2 + 3u_b u_t - 16u_t x, \tag{4.10}$$

which we transform into

$$-\tfrac{4}{3}v \frac{d\rho_b}{dv} = 9\rho_b^2 + 3\rho_b \rho_t - 2\rho_b, \tag{4.11}$$

$$-\tfrac{4}{3}v \frac{d\rho_t}{dv} = 9\rho_t^2 + 3\rho_b \rho_t - 2\rho_t, \tag{4.12}$$

by introducing the functions

$$\rho_b = \frac{u_b}{x}, \qquad \rho_t = \frac{u_t}{x} \tag{4.13}$$

and the new variable

$$v = x^{2/21}. \tag{4.14}$$

(The exponent $\frac{2}{21}$ will – like the exponent $\frac{1}{7}, \delta + 1$ of eqs. (4.2), (4.7) – find its explanation shortly.) By eliminating v we go over to the differential equation

$$\frac{d\rho_t}{d\rho_b} = \frac{9\rho_t^2 + 3\rho_b\rho_t - 2\rho_t}{9\rho_b^2 + 3\rho_b\rho_t - 2\rho_b}. \tag{4.15}$$

All asymptotic solutions for $\rho_b \to 0$ for it are given by the convergent power series

$$\rho_t = \tfrac{2}{9} - \tfrac{1}{5}\rho_b - \tfrac{54}{175}\rho_b^2 + \cdots \tag{4.16}$$

$$\rho_t = c\rho_b + c_2(c)\rho_b^2 + \cdots . \tag{4.17}$$

Here c is an arbitrary integration constant, but all other coefficients of the power series are uniquely determined. Inserting (4.16) into (4.11) we find in terms of x the solutions

$$u_b = ax^{1+2/21} + \cdots , \tag{4.18}$$

$$u_t = \tfrac{2}{9}x - \tfrac{1}{5}ax^{1+2/21} + \cdots , \tag{4.19}$$

where a is an arbitrary integration constant. The function (4.17) yields a two-parametric solution of (4.11), namely

$$u_b = ax^{1+1/7} + \cdots , \tag{4.20}$$

$$u_t = a'x^{1+1/7} + \cdots . \tag{4.21}$$

The physically relevant solution is given by (4.18), (4.19) since it leads to a non-vanishing top mass for vanishing ρ_b. The corrections to ρ_t provided by the terms in ρ_b (see eq. (4.16)) are very small indeed. With the help of (2.2) and (2.10) one finds

$$\rho_b = 7 \cdot 10^{-4} \tag{4.22}$$

(at $x = x_0$).

These asymptotic properties of the one-loop approximation can be extended to all orders. In fact, for β-functions like the present ones

$$\begin{aligned} \beta_x &= b_0 x^2 + \cdots , \\ \beta_{u_i} &= c_{ikj}u_k u_j + c_{ik}u_k x + c_i x^2 + \cdots , \\ \beta_\lambda &= c_\lambda \lambda^2 + c_{\lambda x}\lambda x + c_{\lambda xx}x^2 + c_{\lambda k}\lambda u_k + c_{\lambda kj}u_k u_j + \cdots , \end{aligned} \tag{4.23}$$

where the dots stand for higher orders, it has been shown that Liapunov's theory is applicable [14]. The matrix

$$S_{ij} = \left(\frac{\partial \beta_{i0}}{\partial \rho_{j0}} - b_0 \delta_{ij} \right), \tag{4.24}$$

where

$$\beta_{i0} = \sum_{k,l} c_{ikl}\rho_k \rho_l + \sum_k c_{ik}\rho_k + c_i \tag{4.25}$$

and the ρ_{j0} are a real solution of

$$\beta_{i0} - b_0 \rho_{i0} = 0 \tag{4.26}$$

governs the stability properties of the *full* solutions (neither β-functions nor solutions are approximated). In particular, it follows that general asymptotic solutions exist belonging to positive eigenvalues of S_{ij}/b_0 ($b_0 \neq 0$). In our case the matrix S_{ij} reads for the reduced solution (3.25)–(3.27), (3.30)

$$S = \left[\begin{array}{cccccccccc}
\mathrm{e} & \mu & \tau & \mathrm{d} & \mathrm{s} & \mathrm{b} & \mathrm{u} & \mathrm{c} & \mathrm{t} & \mathrm{H} \\
\hline
\frac{46}{3} & & & & & & & & & \\
 & \frac{46}{3} & & & & & & & & \\
 & & \frac{46}{3} & & & & 0 & & & \\
 & & & -\frac{2}{3} & & & & & & \\
 & & & & -\frac{2}{3} & & & & & \\
 & 0 & & & & -\frac{4}{3} & & & & \\
 & & & & & & -\frac{2}{3} & & & \\
 & & & & & & & -\frac{2}{3} & & \\
\frac{4}{9} & \frac{4}{9} & \frac{4}{9} & \frac{4}{3} & \frac{4}{3} & \frac{2}{3} & \frac{4}{3} & \frac{4}{3} & 2 & \\
4\rho & 4\rho & 4\rho & 12\rho & 12\rho & 12\rho & 12\rho & 12\rho & a & b
\end{array}\right] \tag{4.27}$$

$$\rho = \tfrac{1}{18}(-25 + \sqrt{689}), \qquad a = \tfrac{1}{3}(-82 + 2\sqrt{689}), \qquad b = \tfrac{2}{3}\sqrt{689}.$$

The eigenvalues of S/b_0 are given by

$$\begin{array}{ll}
\xi_\ell = -\frac{23}{21} & \ell = 1,2,3 \quad \ell\text{: lepton} \\[1ex]
\xi^{\mathrm{d}}_{1,2} = \xi^{\mathrm{u}}_{1,2} = \frac{1}{21} & \text{d: down quark, u: up quark} \\[1ex]
\xi_{\mathrm{b}} = \frac{2}{21} & \text{b: bottom quark} \\[1ex]
\xi_{\mathrm{t}} = -\frac{1}{7} & \text{t: top quark} \\[1ex]
\xi_{\mathrm{H}} = -\dfrac{b}{14} < 0 & \text{H: Higgs particle.}
\end{array} \tag{4.28}$$

Hence lepton-, top quark-, Higgs-coupling are unstable: no general solution is approaching their reduced values for $x \to 0$. The other quark-couplings are stable: there are general solutions

$$u_{\mathrm{q}} \sim c_{\mathrm{q}} x^{1+\xi_{\mathrm{q}}} + \cdots, \qquad \mathrm{q} = \mathrm{u}_{1,2}; \mathrm{d}_{1,2}; \mathrm{b} \tag{4.29}$$

with arbitrary coefficients c_{q} approaching for $x \to 0$ the reduced solution (namely 0).

Mathematically the different asymptotic behaviour comes about like in the two coupling situation [9]: for leptons, top and Higgs coupling we have chosen the larger root of the quadratic equations (3.15)–(3.17), (3.29); for the other quarks the smaller roots. As far as physics is concerned we have the interesting consequence that leptons should stay massless in this limit ($g_1 = g_2 = 0$), whereas the quarks and Higgs are massive, but out of the seven mass parameters only five can be chosen at will, the other two are determined. In other words, each unstable solution fixes one coupling, i.e. one seemingly free integration constant. Only the precisely reduced solution goes to zero with the primary coupling (here u_ℓ, u_t, λ are the reduced couplings and x is the primary coupling). As far as dependence of u_t on the other quark couplings is concerned we recall that for the largest one of these couplings – the bottom – the above control in the one-loop approximation shows that it induces very small corrections on the top coupling. Hence all the others can be expected to correct even less. Similarly the effect of the non-vanishing bottom coupling on the Higgs coupling is completely negligible.

4.2. SWITCHING ON SU(2) × U(1)

We have discussed the asymptotic properties of our system in such detail not only in its own right, but also since it is needed for the nontrivial switching-on procedure of g_2 and g_1. Complete reduction corresponds to asymptotic expansions for small x since only there one may trust perturbation theory which provided us with the β-functions. But the discussion of subsect. 3.1 taught us that the physical values of the effective couplings x, y, z cannot go simultaneously to zero since z has the opposite asymptotic behaviour of y for $x \to 0$. We therefore need a more intrinsic characterization of "reduction" which makes a priori no reference to small or large coupling.

The reduction equation for the top quark coupling u is in the one-loop approximation

$$-14x^2 \frac{\mathrm{d}u}{\mathrm{d}x} = 9u^2 + uf(x),$$

$$f(x) = -16x - \tfrac{9}{2}k_y y - \tfrac{17}{10}k_z z. \tag{4.30}$$

Here $y(x), z(x)$ are given by (3.10), (3.11) and k_y, k_z are real constants varying between 0 and 1, introduced for the purpose of switching on/off the functions y, z. (The limits $y \to 0$, $z \to 0$ are not appropriate for this.) The general solution of (4.30) is given by

$$u(x) = \frac{\exp\left[-\int_{x_0}^{x} \mathrm{d}\kappa \frac{f(\kappa)}{14\kappa^2}\right]}{v_0 + \int_{x_0}^{x} \mathrm{d}\kappa \frac{9}{14x^2} \exp\left[-\int_{x_0}^{x} \mathrm{d}\kappa' \frac{f(\kappa')}{14\kappa'^2}\right]} \tag{4.31}$$

with initial values

$$u(x_0) = \frac{1}{v_0}. \tag{4.32}$$

For the special $f(x)$ of (4.30) one integration can be performed, hence

$$u(x) = \frac{x^{8/7}\left(\frac{x}{x+a}\right)^{A/a}\left(\frac{x-b}{x}\right)^{B/b}}{v_0 x_0^{8/7}\left(\frac{x_0}{x_0+a}\right)^{A/a}\left(\frac{x_0-b}{x_0}\right)^{B/b} + \frac{9}{14}\int_{x_0}^{x} \mathrm{d}\kappa\,\kappa^{-6/7}\left(\frac{\kappa}{\kappa+a}\right)^{A/a}\left(\frac{\kappa-b}{\kappa}\right)^{B/b}} \tag{4.33}$$

with

$$A/a = \tfrac{27}{38}k_y, \qquad B/b = \tfrac{17}{82}k_z,$$

$$a = 0.02, \qquad b = 0.0086.$$

It may be seen that one can rewrite the denominator of (4.33) as

$$C + \tfrac{2}{9}x^{1/7} + \tfrac{9}{14}\int_{\kappa}^{\infty} \mathrm{d}\kappa\,\kappa^{-6/7}\left(1 - \left(\frac{\kappa}{\kappa+A}\right)^{A/a}\left(\frac{\kappa-b}{\kappa}\right)^{B/b}\right) \tag{4.34}$$

hence for $C = 0$ (4.33) tends to $\frac{2}{9}x$ for $k_{y,z} \to 0$. This initial value prescription is thus the intrinsic definition of "reduction" in the present case as suggested by the previous two examples (given at the beginning of subsect. 4.1).

We observe that imposing (4.34) puts one on an asymptotic solution for large x i.e. we may expand numerator and denominator of (4.33) in powers of $1/x$. This is tantamount to

$$\hat{u} = u - \tfrac{2}{9}x \tag{4.35}$$

in (4.30) and to expand it in powers of $1/x$. It is to be noted that the effective expansion is in powers of $a/x \approx 0.2$ and $b/x \approx 0.088$ respectively, for the physical value $x = x_0 \approx 0.1$. Up to third order in $1/x$ we find

$$\hat{u} = c_1 + c_2/x + c_3/x^2. \tag{4.36}$$

Hence up to this order in $1/x$

$$u_0 = u(x_0) = 0.018658. \tag{4.37}$$

For the Higgs coupling we proceed in a completely analogous manner and find – also to third order in $1/x$ –

$$\lambda(x) = \lambda_+^{(0)} x + \hat{\lambda} = \lambda_+^{(0)} x + b_1 + b_2/x + b_3/x^2, \tag{4.38}$$

i.e.

$$\lambda(x_0) = 0.010554. \tag{4.39}$$

These values for the top and the Higgs coupling yield inserted into (2.2) with (2.10) for the masses the values

$$m_t = 81 \text{ GeV}, \tag{4.40}$$

$$m_H = 61 \text{ GeV}. \tag{4.41}$$

The reduced solution predicts therefore (4.40), (4.41) and zero for all other quark and lepton masses. (In the case of $n = 3$ generations.)

We now have to check the effect of switching on SU(2)×U(1) on the general solution i.e. $u_\ell \equiv 0$, $u_t = u_{t\ \mathrm{red}} +$ correction, $\lambda = \lambda_{\mathrm{red}} +$ correction and u_q given by (4.13). Clearly any vanishing Yukawa coupling remains a solution and also on the general quark coupling the effect will not be larger than that on the reduced solution. But we should like to see how "far" the vanishing *lepton* coupling solution is from the physical value. The equations to be solved read

$$\beta_x \frac{\mathrm{d}u_\ell}{\mathrm{d}x} = \beta_{u_\ell},$$

$$-14x^2 \frac{\mathrm{d}u_\ell}{\mathrm{d}x} = 3u_\ell^2 + 2u_\ell\Big(\sum_{\ell'} u_{\ell'} + 3\sum_q u_q\Big) - \tfrac{9}{2}u_\ell(k_y y + k_z z). \tag{4.42}$$

We first note that they are symmetric in ℓ, hence for a qualitative understanding it will be sufficient to study a simplified equation for $u_1 = u_2 = u_3 \equiv u$ i.e.

$$-14x^2 \frac{\mathrm{d}u}{\mathrm{d}x} = 9u^2 + uf(x) \tag{4.43}$$

with

$$f(x) = 6\sum_q u_q - \tfrac{9}{2}(k_y y + k_z z).$$

The general solution is given by

$$u(x) = \frac{\exp\left[-\int_{x_0}^{x} \mathrm{d}\kappa \frac{f(\kappa)}{14\kappa^2}\right]}{v_0 + \int_{x_0}^{x} \mathrm{d}\kappa \frac{9}{14\kappa^2} \exp\left[-\int_{x_0}^{x} \mathrm{d}\kappa' \frac{f(\kappa')}{14\kappa'^2}\right]}. \tag{4.44}$$

The gross behaviour of u is determined from presence or absence of a zero of the denominator. A zero can occur only if $f(x)$ contains a term $14\alpha x$ ($\alpha > 0$) which gives rise to $\ln(x/x_0)^\alpha$ in $\int f(x)/14x^2$ and to a function u of the type

$$u = \frac{x}{Cx^{\alpha+1} - \frac{9}{14}\frac{1}{\alpha+1}}. \tag{4.45}$$

Studying terms in $f(x)$ of the type $x^\beta, x^\gamma \ln x, 1/x^\delta$ one may convince oneself that they do not modify the gross features of u. Consequently, the completely reduced case $u_t = \frac{2}{9}x$, $k_y = k_z = 0$ is already a good representative for the general case. The stability discussion of subsect. 4.1 gave us already the approach of u to zero for large x:

$$u \sim xx^{-23/21} = x^{-2/21}. \tag{4.46}$$

Leptons can therefore be massive via the general solution going roughly like $x^{-2/21}$ for large x. Hence it is the mere presence of the $SU(2) \times U(1)$ subgroup which permits their masses, whereas the overall behaviour (the power law) is still dictated by SU(3).

The order of magnitude of the coefficients needed in the general solution suggests that these deviations from the reduction solution might be caused by (and thus computable as) radiative corrections. Whether this is true or not depends on the renormalizability properties of the mixed massive/massless model. If in the mixed case too a vanishing Yukawa coupling stays strictly vanishing then radiative corrections do not lift a zero mass to a finite value. But there does not seem to exist a rigorous treatment of this problem.

5. Discussion and conclusions

We have studied reduction of coupling parameters [8] in the context of the standard model with one Higgs doublet and $n = 3, 4, 5$ generations of fermions. Simplifying assumptions were
—absence of matter mixing angles (i.e. family number conservation)
—complete masslessness (i.e. no symmetry breakdown).
With the values of the couplings obtained this way we entered the classical approximation of the massive model and determined the corresponding masses of the matter fields. The results are as follows:
—In the gauge coupling sector reduction is either inconsistent or phenomenologically unacceptable.
—In the matter sector reduction is possible with respect to U(1) and SU(3), but only the latter can be expected to yield physical results, due to asymptotic freedom.

It yields (for $n = 3$):

$$m_\ell = 0 \qquad \ell\text{: lepton}$$

$$m_q = 0 \qquad q \neq t,\ q\text{: quarks}$$

$$m_t = 81\ \text{GeV} \qquad \text{top quark}$$

$$m_H = 61\ \text{GeV} \qquad \text{Higgs particle} \tag{5.1}$$

This reduction solution is embedded in the general solution where all quarks can be massive, even for switched off SU(2) × U(1), and where – due to the existence of the SU(2) × U(1) subgroup – the leptons too may be massive.

The systematic error of the prediction (5.1) is negligibly small as far as the other masses are concerned; it originates from neglecting family mixing and from the phenomenological determination of the values x_0, y_0, z_0 taken from the literature which includes two-loop corrections and is *not* based on the reduction solution. The uncertainty of the value of $\sin^2\theta_w$ and of α_s is each about 10%; since they enter via square roots they contribute each about 5% uncertainty for the masses. Altogether this may sum up to an error of the order of 10–15%.

What happens if experimentally (5.1) is not verified? In close analogy to the discussion in the gauge coupling sector (cf. subsect. 3.1) this would mean that in the standard model (with $n = 3$) *all* couplings are independent solutions of the reduction equations.

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3.2 Quark family mixing and reduction of couplings

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Comment (*Klaus Sibold*)
After having laid the groundwork for reduction in the standard model in the paper of the previous subsection we continue this analysis by admitting the full-fledged Yukawa coupling matrices. In the case which has been treated three families are being considered hence there appears a complex 3×3 matrix G^d for the down quarks and a similar matrix G^u for the up quarks. Together with the Higgs coupling λ they are understood as functions of α_s which is the primary coupling following the results of the previous paper. Hence we search for solutions of the reduction equations which go to zero with α_s, i.e. we impose asymptotic freedom in the UV region.
The diagonal solutions of the non-trivial reduction which implied non-vanishing masses for the top quark and the Higgs clearly also govern the solution pattern for the mixing. For the trivial reduction case arbitrary masses for the charged leptons and the quarks are permitted. (Neutrinos are by assumption massless.) For the non-trivial reduction, where the Higgs and top quark masses are determined it is found that the Cabibbo angle is arbitrary, mixing between the third and the first two families is however excluded. This result is interesting indeed because the observed parameters in the Kobayashi-Maskawa matrix which express mixing between the third and the first two families are very small. (Warning: The second equation of (6) in the paper contains a misprint. The formula should read $c_{-} \neq 0$.)

QUARK FAMILY MIXING AND REDUCTION OF COUPLINGS

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The principle of reduction is applied to quark mixing in the standard model with three families. In case of the non-trivial reduction for which the top quark and the Higgs mass are determined it is found that the Cabibbo angle is arbitrary, while mixing between the third and the first two families is excluded.

In ref. [1] the reduction method was used for deriving constraints on the Higgs and quark masses in the standard model with three families. By the reduction principle the coupling parameters of a model are required to be functions of a single coupling under the condition that all couplings vanish simultaneously in the weak coupling limit. In this way the original model depending on several coupling parameters is reduced to a description in terms of a single coupling. Invariance under the renormalization group of the original as well as the reduced model implies the reduction equations: a set of ordinary differential equations for the coupling parameters [2]. The same set of equations holds for the effective couplings of the original model after elimination of the scale variable [3]. The asymptotic behavior for the solutions of the reduction equations can be studied systematically by asymptotic expansions valid for small couplings [4]. If the β-function of the reduced model is negative the principle of reduction is equivalent to imposing asymptotic freedom on the original model. For this reason the reduction method cannot be implemented for the three gauge couplings of the standard model due to opposite signs of the β-functions [1]. But for the system of strong interactions as defined by setting $g=g'=0$ in the standard model reasonable constraints among the coupling parameters are obtained which express asymptotic freedom for quantum chromodynamics supplemented by the quark Yukawa and the Higgs couplings.

In this note the reduction method is applied to the family mixing among quarks. After setting $g=g'=0$ the model involves the 3×3 Yukawa coupling matrix G^{d} of the down quarks, the matrix G^{u} of the up quarks and the Higgs coupling λ, all considered as functions of the strong gauge coupling parameter α_s,

$$G^{\mathrm{d}}(\alpha_s),\quad G^{\mathrm{u}}(\alpha_s),\quad \lambda(\alpha_s)\,.$$

The Kobayashi–Maskawa mixing matrix is then given by [5]

$$U=A_{\mathrm{L}}^{\mathrm{u}*}A_{\mathrm{L}}^{\mathrm{d}}\,, \tag{1}$$

where the unitary transformations $A_{\mathrm{L}}^{\mathrm{d}}$, $A_{\mathrm{L}}^{\mathrm{u}}$ acting on the left-handed quark fields diagonalize the matrices $G^{\mathrm{d}*}G^{\mathrm{d}}$ or $G^{\mathrm{u}*}G^{\mathrm{u}}$, respectively. The eigenvalues of $G^{\mathrm{d}*}G^{\mathrm{d}}$ and $G^{\mathrm{u}*}G^{\mathrm{u}}$ determine the quark masses. The reduction equations for the matrices

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$$\rho_+ = \frac{G^{\mathrm{u}*}G^{\mathrm{u}}/4\pi}{\alpha_s}, \quad \rho_- = \frac{G^{\mathrm{d}*}G^{\mathrm{d}}/4\pi}{\alpha_s} \tag{2}$$

are [‡1]

$$(\beta_{\alpha_s}/\alpha_s)\,\mathrm{d}\rho_\pm/\mathrm{d}\alpha_s = 3\rho_\pm^2 - \tfrac{3}{2}\{\rho_+\rho_-\} + 6\rho_\pm\,\mathrm{tr}(\rho_+ + \rho_-) - 2\rho_\pm + \ldots, \quad \beta_\alpha/\alpha_s = -14\alpha_s + \ldots \tag{3}$$

In the limit $\alpha_s \doteq 0$ the equations

$$3\rho_\pm^2 - \tfrac{3}{2}\{\rho_+\rho_-\} + 6\rho_\pm\,\mathrm{tr}(\rho_+ + \rho_-) - 2\rho_\pm = 0 \tag{4}$$

follow. All solutions $\rho_\pm$ with positive eigenvalues can be made diagonal by the same matrix $A_{\mathrm{L}}^{\mathrm{d}} = A_{\mathrm{L}}^{\mathrm{u}}$. Hence

$$U = 1\,,$$

so that there is no family mixing in lowest order.

The diagonal solutions of (3) have been determined by using asymptotic expansions [1]. Among them only two cases are realistic in view of the observed mass spectrum. These are the non-trivial reduction with

$$\lim_{\alpha_s \to 0} \rho_+ = c_+, \quad \lim_{\alpha_s \to 0} \rho_- = c_-\,, \tag{5}$$

$$c_+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad c_- = 0 \tag{6}$$

(with suitable labeling of the quark fields) and the trivial reduction with

$$\lim_{\alpha_s \to 0} \rho_\pm = 0\,. \tag{7}$$

The non-trivial reduction (5), (6) determines the masses of the top quark and the Higgs particle:

$$m_{\mathrm{top}} \approx 81\ \mathrm{GeV}\,, \tag{8}$$

$$m_{\mathrm{Higgs}} \approx 63\ \mathrm{GeV}\,. \tag{9}$$

These values include electroweak corrections and are approximately independent of the other quark masses. The trivial reduction (7) allows for arbitrary top masses bounded by (8) with the Higgs mass being a function of the top mass bounded by (9).

In this note the case (5), (6) of the non-trivial reduction is extended to non-diagonal solutions of (3). All asymptotic expansions solving (3) are determined with the limit (5) or any non-diagonal solution of (4) in the neighborhood of (6). A convenient parametrization for solutions of (4) representing a neighborhood of (6) is given by

$$\rho^{(0)}_{+11} = \tfrac{9}{2}\,|\rho^{(0)}_{+13}|^2 G(u)\,,$$

$$\rho^{(0)}_{+22} = \tfrac{9}{2}\,|\rho^{(0)}_{+23}|^2 G(u)\,,$$

$$\rho^{(0)}_{+12} = \tfrac{9}{2}\rho^{(0)}_{+13}\rho^{(0)*}_{+23} G(u)\,,$$

$$\rho^{(0)}_{+33} = \tfrac{9}{2} - \rho^{(0)}_{+11} - \rho^{(0)}_{+22}\,, \tag{10}$$

[‡1] For the one-loop coefficients of the β-functions used here see ref. [6].

$$\rho_{-}^{(0)}=0\,,$$

$$G=2(1-\sqrt{1-u})/u,\quad u=81(|\rho_{+13}^{(0)}|^2+|\rho_{+23}^{(0)}|^2)\,, \tag{10 cont'd}$$

with arbitrary complex parameters $\rho_{+13}^{(0)}$ and $\rho_{+23}^{(0)}$ restricted by

$$0\leqslant|\rho_{+13}^{(0)}|^2+|\rho_{+23}^{(0)}|^2\leqslant\tfrac{1}{81}\,.$$

All matrices satisfying (10) are unitarily equivalent to (6)

$$\rho_{\pm}^{(0)}=V^{-1}c_{\pm}V\,. \tag{11}$$

Hence applying the constant unitary transformation V to (3) the discussion of the solutions may be restricted to those with the diagonal limit (5), (6).

Neglecting radiative corrections the asymptotic expansions solving (3) involve powers

$$\alpha_s^{p_1\xi_1+\ldots+p_l\xi_l+q} \tag{12}$$

with non-negative integers

$$p_1,\ldots,p_l,\,q\geqslant0\,.$$

$\xi_1,\ldots,\xi_l$ are the non-negative eigenvalues of the 18×18 exponent matrix

$$\Xi=\left\|-\frac{1}{14}\frac{\partial\beta_j^{(0)}}{\partial\rho_k^{(0)}}-\delta_{jk}\right\|,\qquad \beta_j^{(0)}=\lim_{\alpha_s\to0}\frac{\beta_j}{\alpha_s},\quad j,k=1,\ldots,18\,. \tag{13}$$

Here $\rho_1,\ldots,\rho_{18}$ denote the entries of the matrices $\rho_{\pm}$ with the limits $\rho_j^{(0)}$ for $\alpha_s\to0$. β_j denotes the corresponding β-functions. By suitably labeling the ρ_j the matrix Ξ assumes a triangular form, so that its eigenvalues are given by the diagonal elements. Only one eigenvalue of Ξ is negative (belonging to the top quark field), all others are real and non-negative. There are four vanishing eigenvalues, one eigenvalue $\frac{2}{21}$, eight eigenvalues $\frac{1}{21}$ and four eigenvalues $\frac{1}{14}$. The asymptotic expansion of $\rho_{\pm}$ contains as many free real parameters as there are non-negative eigenvalues of Ξ. Accordingly there are 17 free real parameters. The four real parameters belonging to the eigenvalues 0 are given by the complex parameters $\rho_{+13}^{(0)},\rho_{+23}^{(0)}$ of (10) which occur in the constant unitary equivalence transformation (11) and may be disregarded. The four real parameters associated with the eigenvalue $\frac{1}{14}$ occur in the hermitean coefficients of the power $\alpha_s^{1/14}$,

$$d^{\pm}\alpha_s^{1/14}\,,$$

which have the form

$$d^{-}=\begin{pmatrix}0&0&a_{-13}\\0&0&a_{-23}\\a_{-31}&a_{-32}&0\end{pmatrix},\quad d^{+}=\tfrac{1}{3}d^{-}\,, \tag{14}$$

with arbitrary a_{-13}, a_{-23}. It can be shown that

$$a_{-13}=a_{-23}=0\,, \tag{15}$$

if all quark masses should be positive. This excludes exponents $\xi=\frac{1}{14}$ in (12). Only the parameters $\frac{1}{21}$ and $\frac{2}{21}$ remain, so that the solutions of (3) with the limit (5), (6) can be expanded as

$$\rho_{\pm}=c_{\pm}+\sum_{n=1}^{\infty}c_n^{\pm}\alpha_s^{n/21} \tag{16}$$

with the lowest order $c_{\pm}$ given by (6). Due to the absence of powers $\alpha_s^{1/14}$ it follows recursively that the coefficients have the block form

$$c_n^{\pm} = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{$*$}} & 0 \\ & & 0 \\ \hline 0 & 0 & * \end{array}\right) . \tag{17}$$

Accordingly there is no mixing between the first two families and the third family. The coefficients $c_1^{\pm}$ and c_2^{-} have the form

$$c_1^{+} = \begin{pmatrix} a_{+11} & a_{+12} & 0 \\ a_{+21} & a_{+22} & 0 \\ 0 & 0 & -\frac{1}{2}(a_{+11}+a_{+22}+a_{-11}+a_{-22}) \end{pmatrix} ,$$

$$c_1^{-} = \begin{pmatrix} a_{-11} & a_{-12} & 0 \\ a_{-21} & a_{-22} & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad c_2^{-} = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{$*$}} & 0 \\ & & 0 \\ \hline 0 & 0 & a_{-33} \end{array}\right) . \tag{18}$$

The matrix elements $a_{\pm ik}$ $(i, k=1, 2)$, a_{-33} are arbitrary and represent the remaining nine free real parameters of the expansion. As a consequence the Cabibbo angle is not restricted, and the bottom quark mass as well as all masses, of the quarks in the first two families can be chosen independently. The other matrix elements of c_2^{-} and all other coefficients $c_j^{\pm}$ are uniquely determined.

In particular, the top Yukawa coupling and the Higgs mass are determined as functions of the five lower quark masses and the Cabibbo angle. In ref. [1] it was shown that the influence of the bottom quark mass on the top and Higgs mass can already be neglected. This should also be expected in the non-diagonal case regarding all parameters so that the approximate values (8) and (9) are not changed.

For the non-trivial reduction we thus have the result that the Cabibbo angle is arbitrary while mixing with the third family is not allowed. This is of interest since the observed elements of the Kobayashi–Maskawa matrix which express mixing between the third and the first two families are very small [7].

Electroweak corrections have not been included yet. For the trivial reduction we have found that sufficiently small mixing angles are arbitrary. But upper bounds for the mixing parameters of the third family should be expected which vanish in the limit of the non-trivial reduction.

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3.3 New results in the reduction of the standard model

Title: New results in the reduction of the standard model
Authors: J. Kubo, K. Sibold, W. Zimmermann
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Comment (*Klaus Sibold*)
Reduction of couplings is based on the requirement that all reduced couplings vanish simultaneously with the reducing – the primary – coupling. This is clearly only possible if the couplings considered have the same asymptotic behavior or have vanishing β-functions. Hence in the standard model, based on $SU(3) \times SU(2) \times U(1)$ straightforward reduction cannot be realized. Since however the strong coupling α_s is, say at the W-mass, considerably larger than the weak and electromagnetic coupling one may put those equal to zero, reduce within the system of quantum chromodynamics including the Higgs and the Yukawa couplings and subsequently take into account electroweak corrections as a kind of perturbation. This is called "partial reduction". In the present paper a new perturbation method has been developed and then applied with the updated experimental values of the strong coupling and the Weinberg angle.
If β functions are non-vanishing they usually go to zero with some power of the couplings involved. Thus, reduction equations are singular for vanishing coupling and require a case by case study at this singular point. In particular this is true for the reduction equations of Yukawa and Higgs couplings when reducing to α_s. It is shown in the paper that for the non-trivial reduction solution (i.e. only the top Yukawa coupling and the Higgs coupling do not vanish) one can de-singularize the system by a variable transformation and thereafter go over to a partial differential equation which is easier to solve than the ordinary differential equations one started with. The reduction solutions of the perturbed system are then in one-to-one correspondence with the unperturbed one's.
In terms of mass values the non-trivial reduction yields $m_t = 91.3$ GeV, $m_H = 64.3$ GeV. These mass values are at the same time the upper bound for the trivial reduction, where the Higgs mass is a function of the top mass. Here we used as definition for "trivial" that the ratios of top-Yukawa coupling, respectively Higgs coupling to α_s go to zero for the weak coupling limit α_s going to zero.

NEW RESULTS IN THE REDUCTION OF THE STANDARD MODEL

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The reduction of couplings in the standard model is ameliorated and updated: a new method of calculating the electroweak corrections is developed, two-loop effects are estimated and recent experimental values of the strong coupling and the Weinberg angle are incorporated in the explicit calculation of the mass relations.

In a recent paper predictions for the masses of the top and Higgs particles were obtained by applying the reduction method to the standard model [1]. The reduction principle may be applied to any model of quantum field theory involving several independent coupling parameters [2]. The main hypothesis is that all couplings are functions of a single coupling parameter satisfying some general conditions. In the weak coupling limit the couplings are required to vanish simultaneously. Combined with the renormalization group invariance of the original and the reduced model as well, these assumptions imply constraints on the coupling parameters.

Reduced couplings are asymptotically free (in the ultraviolet or infrared region respectively) or have vanishing β-functions. Accordingly, there are no reduction solutions for the standard model as such, since the opposite signs of the electroweak gauge couplings preclude the possibility of asymptotic freedom. But the reduction principle may successfully be applied to the system of strong interactions as defined by setting the electroweak couplings in the standard model equal to zero. This is equivalent to extending the requirement of asymptotic freedom from quantum chromodynamics to the enlarged system including the Higgs and Yukawa interactions. Electroweak corrections to the reduction solutions obtained are computed afterwards using the full set of renormalizations group equations with appropriate boundary conditions.

For the reduction of the strong interactions it is convenient to use α_s as the parameter on which all other couplings should depend. The reduction solutions may be classified according to the behaviour of the ratios

$$\rho_q = \frac{G_q^2/4\pi}{\alpha_s}, \quad \rho_H = \frac{\lambda/4\pi}{\alpha_s}, \tag{1}$$

in the weak coupling limit $\alpha_s \to 0$. G_q denotes the Yukawa coupling of the quark q, λ the Higgs coupling. Reductions for which all ratios (1) vanish for $\alpha_s \to 0$ are called trivial. Among the non-trivial reduction for which at least one ratio does not vanish in the limit only the case

$$\lim \rho_q = 0, \quad q \neq t,$$
$$\lim \rho_t = \tfrac{2}{9}, \quad \rho_H = \tfrac{1}{18}(\sqrt{689} - 25), \tag{2}$$

is compatible with known particle masses. The subscript t refers to the top quark. In this case the values

$$m_t^{NTr} = 81 \text{ GeV}, \quad m_H^{NTr} = 61 \text{ GeV} \tag{3}$$

(including electroweak corrections) were found in ref. [1] for $\alpha_s = 0.1$ and $\sin^2\theta = 0.21$ at the normalization point $M_W = 81$ GeV.

For the corresponding solutions in the trivial reduction the top mass is not fixed, rather the Higgs mass is a function of the top mass. Both masses are

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bounded from above by their values in the non-trivial reduction:

$$m_t < m_t^{\mathrm{NTr}}, \quad m_{\mathrm{H}}(m_t) < m_{\mathrm{H}}^{\mathrm{NTr}} . \tag{4}$$

The trivial reduction does not look very promising at first since in the weak coupling limit of the strong interactions all masses vanish which makes the generation of large quark masses unlikely. However, recent investigations show that quadratic divergencies in the Higgs self-energy cancel in the trivial reduction for specific mass values which come surprisingly close to those of the non-trivial reduction. Thus combining the principle of reduction with Veltman's naturalness requirement that the quadratic divergencies of the Higgs field should cancel lead to massive top and Higgs particles with uniquely determined mass values [3].

In this note we report on a systematic method of computing the electroweak corrections. The numerical values of the top and Higgs mass are given in higher order of the electroweak couplings in dependence on α_s and θ_W in the range

$$0.103 \leqslant \alpha_s \leqslant 0.123 ,$$

$$0.223 \leqslant \sin^2\theta_W \leqslant 0.233 , \tag{5}$$

as suggested by present experimental data.

We begin by discussing the electroweak corrections for the top mass. In ref. [1] it was shown that the influence of the five lower quark masses is negligible. Accordingly we set the corresponding Yukawa couplings equal to zero. Taking the lowest order for the β-functions the top coupling ρ_t expressed as a function of α_s satisfies the differential equation

$$-14\alpha_s \frac{\mathrm{d}\rho_t}{\mathrm{d}\alpha_s} = 9\rho_t^2 - 2\rho_t - \tfrac{9}{2}u\rho_t - \tfrac{17}{10}v\rho_t , \tag{6}$$

with

$$u = \frac{g^2}{4\pi\alpha_s} = \frac{\alpha}{\alpha_s \sin^2\theta_W} ,$$

$$v = \frac{5}{3}\frac{g'^2}{4\pi\alpha_s} = \frac{5}{3}\frac{\alpha}{\alpha_s \cos^2\theta_W} . \tag{7}$$

Neglecting the electroweak interactions by setting $u = v = 0$ the general solution is

$$\rho_t^{(0)} = \tfrac{2}{9}\frac{\alpha_s^{1/7}}{c + \alpha_s^{1/7}} \quad \text{or} \quad \rho_t \equiv 0 . \tag{8}$$

The nontrivial reduction is given by the only solution

$$\rho_t^{(0)} \equiv \tfrac{2}{9} \quad (c=0) \tag{9}$$

with a non-vanishing limit for $\alpha_s \to 0$. All other solutions represent trivial reductions with the limit

$$\lim_{\alpha_s \to 0} \rho_t^{(0)} = 0 . \tag{10}$$

The explicit solutions of the differential equations

$$-14\alpha_s \frac{\mathrm{d}u}{\mathrm{d}\alpha_s} = 14u - \tfrac{19}{3}u^2 ,$$

$$-14\alpha_s \frac{\mathrm{d}v}{\mathrm{d}\alpha_s} = 14v + \tfrac{41}{5}v^2 . \tag{11}$$

for u and v as functions of α_s in lowest order for the β-functions are

$$u = \frac{42}{19}\frac{a}{\alpha_s + a}, \quad v = \frac{70}{41}\frac{b}{\alpha_s - b} , \tag{12}$$

with a and b as constants of integration. With (12) the differential equation (6) was solved in closed form in ref. [1]. More convenient is an approximation method based on a partial differential equation which will be sketched in this note.

First we specify the boundary conditions for solving the ordinary differential equation (6). The task is to find the correct connections between undisturbed solutions $\rho_t^{(0)}$ for $u = v = 0$ and solutions ρ_t of the complete differential equation. The electroweak perturbation in (6) is of the form

$$\tfrac{9}{2}u + \tfrac{17}{10}v$$
$$= \frac{a}{\alpha_s}\left(\frac{14p}{1 + a/\alpha_s} + \frac{b}{a}\frac{14q}{1 - (b/a)a/\alpha_s}\right),$$
$$p = \tfrac{27}{38}, \quad q = \tfrac{17}{82} . \tag{13}$$

It is justified to treat this term as perturbation since the actual values of the variables a/α_s and b/α_s are small. For instance,

$$a/\alpha_s \approx 0.16, \quad b/\alpha_s \approx 0.08$$

for $\alpha_s = 0.113$ and $\sin^2\theta_W = 0.228$. Since the electroweak perturbation vanishes

$$\tfrac{9}{2}u + \tfrac{17}{10}v \to 0 \quad \text{for } a/\alpha_s \to 0 ,$$

we require that the solutions ρ_t of (6) asymptotically approach corresponding undisturbed solutions $\rho_t^{(0)}$

$\rho_t \approx \rho_t^{(0)}$ for $a/\alpha_s \to 0$.

This is the boundary condition which will be used in solving the differential equation (6). In order to discuss the asymptotic behaviour in this limit we introduce

$$w=(a/\alpha_s)^{1/7}$$

as new variable in (6). It is not sufficient to demand $\rho_t \to \rho_t^{(0)}$ for $w \to 0$ since infinitely many solutions have the same value at $w=0$ which is a singular point of the differential equation. Therefore, we transform the differential equation by

$$\rho_t = \tfrac{2}{9} + w\tau . \qquad (14)$$

(6) then takes the form

$$2\frac{d\tau}{dw} = 9\tau^2 - \tfrac{2}{9}\Sigma w^5 - \Sigma w^6 \tau , \qquad (15)$$

with

$$\Sigma = 14\left(\frac{p}{1+w^2} + \frac{b}{a}\frac{q}{1-(b/a)w^2}\right).$$

For (15) $w=0$ is a regular point so that the condition

$$\tau = \tau^{(0)} \quad \text{at } w=0 \qquad (16)$$

uniquely determines the solution τ for given $\tau^{(0)}$. This provides a one-to-one correspondence between the solutions ρ_t of (6) and the solutions $\rho_t^{(0)}$ of the undisturbed form of (6) with $u=v=0$. It can be proved that this correspondence is independent of the chosen regularity transformation.

In order to compute the solution ρ_t which corresponds to the non-trivial reduction (9) it is convenient to employ a partial differential equation. To this end we consider ρ_t as a functional of u and v. Any solution of

$$(14u - \tfrac{19}{3}u^2)\frac{\partial \rho_t}{\partial u} + (14v + \tfrac{41}{5}v^2)\frac{\partial \rho_t}{\partial v}$$
$$= 9\rho_t^2 - 2\rho_t - \tfrac{9}{2}u\rho_t - \tfrac{17}{10}v\rho_t , \qquad (17)$$

satisfies the ordinary differential equation (6) if functions u and v of α_s are inserted which are solutions of (11). A power-series solution

$$\rho_t = \sum_{p,q=0}^{\infty} c_{pq} u^p v^q$$
$$= \tfrac{2}{9} - \tfrac{1}{12}u - \tfrac{17}{540}v - \tfrac{1}{288}u^2 + \tfrac{119}{9360}uv + \tfrac{799}{64800}v^2 + \dots \qquad (18)$$

of (17) can be constructed with uniquely determined coefficients c_{pq} if $c_{00} = \tfrac{2}{9}$ is taken as lowest order solution. By this choice the asymptotic requirement (16) is satisfied with the undisturbed coupling $\rho_t^{(0)} \equiv \tfrac{2}{9}$ of the non-trivial reduction. According to (7) and

$$\rho_t = \tfrac{1}{2}(m_t^2/M_W^2)u , \qquad (19)$$

the expansion (17) expresses the ratio m_t^2/M_W^2 as a functional of α_s, α and θ_W.

For comparison with the result (3) of ref. [1] we first take the same parameter values as in ref. [1], namely

$$\alpha_s(M_W)=0.1, \quad \alpha(M_W)=\tfrac{1}{128} ,$$
$$\sin^2\theta_W(M_W)=0.21, \quad M_W=81 \text{ GeV} . \qquad (20)$$

Then the top mass computed from (18) including the fifth order in u and v becomes

$$m_t^{(0)}=88.5 \text{ GeV}, \quad m_t^{\mathrm{I}}=81.0 \text{ GeV} ,$$
$$m_t^{\mathrm{II}}=81.1 \text{ GeV}, \dots , \quad m_t^{\mathrm{V}}=81.1 \text{ GeV} , \qquad (21)$$

in agreement with ref. [1]. The fifth-order contribution is about 2×10^{-5} GeV.

Present values of α_s and $\sin^2\theta_W$ are considerably higher than (20). In table 1 the fifth order values of m_t^2/M_W^2 are listed for some parameter values from the intervals (5). The corresponding top masses range between 85 GeV and 97 GeV. For instance,

$$m_t^{\mathrm{V}}=91.3 \text{ GeV} \qquad (22)$$

Table 1
m_t^2/M_W^2 in the non-trivial reduction as function of $\alpha_s(M_W)$ and $\sin^2\theta_W(M_W)$ in fifth order of α.

α_s	$\sin^2\theta_W$		
	0.223	0.228	0.233
0.103	1.113	1.142	1.170
0.113	1.240	1.271	1.302
0.123	1.366	1.401	1.435

if

$$\alpha_s(M_W)=0.113,\quad \alpha(M_W)=\tfrac{1}{128}\,,$$
$$\sin^2\theta_W(M_W)=0.228,\quad M_W=81\text{ GeV}\,, \tag{23}$$

The differential equation for the Higgs coupling ρ_H as a function of α_s is

$$-14\alpha_s\frac{d\rho_H}{d\alpha_s}=6\rho_H^2+14\rho_H+12\rho_t\rho_H-24\rho_t^2$$
$$+\tfrac{9}{2}u^2+\tfrac{9}{5}uv+\tfrac{27}{50}v^2-9u\rho_H-\tfrac{9}{5}v\rho_H\,. \tag{24}$$

The electroweak perturbation in this equation vanishes for $a/\alpha_s\to 0$. Accordingly we introduce a one-to-one correspondence between the solutions ρ_H and the solutions $\rho_H^{(0)}$ of the undisturbed equation (with $u=v=0$) by imposing the boundary condition

$$\rho_H\simeq\rho_H^{(0)}\quad\text{for } a/\alpha_s\to 0\,. \tag{25}$$

This condition can be made precise by transforming (24) into a regular form. Instead of solving (23) directly it is more convenient to solve the partial differential equation

$$(14u-\tfrac{19}{3}u^2)\,\frac{\partial\rho_H}{\partial u}+(14v+\tfrac{41}{5}v^2)\,\frac{\partial\rho_H}{\partial v}$$
$$=6\rho_H^2+14\rho_H+12\rho_t\rho_H-24\rho_t^2$$
$$+\tfrac{9}{2}u^2+\tfrac{9}{5}uv+\tfrac{27}{50}v^2-9u\rho_H-\tfrac{9}{5}v\rho_H \tag{26}$$

for ρ_H considered as a functional of u and v. Any solution of (26) becomes a solution of (24) if functions u, v and ρ_t of α_s are inserted which satisfy (11) and (6). After inserting the expansion (18) for ρ_t a power-series solution

$$\rho_H=\sum_{p,q=0}^{\infty}a_{pq}u^pv^q$$
$$=a+\frac{10a-\frac{8}{9}}{12a+\frac{8}{3}}u+\frac{\frac{98}{45}a-\frac{136}{405}}{12a+\frac{8}{3}}v+\dots,$$
$$a=\tfrac{1}{18}(\sqrt{689}-25)\,, \tag{27}$$

is obtained. With $a_{00}=a$ all higher order coefficients a_{pq} are unique. The boundary condition (23) is satisfied for the undisturbed coupling $\rho_H^{(0)}\equiv a$ of the non-trivial reduction. According to (7) and

$$\rho_H=\tfrac{1}{2}(m_H^2/m_W^2)u \tag{28}$$

the expansion (27) expresses the ratio m_H^2/M_W^2 as a functional of α_s, α and θ_W.

With the parameter values (20) of ref. [1] the Higgs mass up to and including the fifth order of (27) becomes

$$m_H^{(0)}=49.0\text{ GeV},\quad m_H^{I}=31.5\text{ GeV}\,,$$
$$m_H^{II}=64.0\text{ GeV},\quad m_H^{III}=63.5\text{ GeV}\,,$$
$$m_H^{IV}=63.8\text{ GeV},\quad m_H^{V}=63.8\text{ GeV}\,. \tag{29}$$

This is somewhat higher than the value (3) obtained in ref. [1]. The fifth-order contribution is about 10^{-3} GeV. In contradistinction to the top mass the Higgs mass depends only slightly on α_s and θ_W. For the intervals (5) it ranges between 63.9 GeV and 65.3 GeV. The fifth-order values of m_H^2/M_W^2 are listed in table 2 for some values of α_s and θ_W. As example we give the value

$$m_H^{V}=64.4\text{ GeV} \tag{30}$$

of the Higgs mass for the parameter values (23).

In the case of the trivial reduction any non-negative solution ρ_t of (6) with $\rho_t\to 0$ for $\alpha_s\to +0$ is admissible. For given ρ_t a non-negative solution ρ_H of (17) is uniquely determined with $\rho_H\to 0$ for $\alpha_s\to +0$. In order to compute ρ_H it is convenient to consider it as a function of u, v and ρ_t. Any such functional satisfying the partial differential equation

$$(14u-\tfrac{19}{3}u^2)\,\frac{\partial\rho_H}{\partial u}+(14v+\tfrac{41}{5}v^2)\,\frac{\partial\rho_H}{\partial v}$$
$$+(9\rho_t^2-2\rho_t-\tfrac{9}{2}u\rho_t-\tfrac{17}{10}v\rho_t)\,\frac{\partial\rho_H}{\partial\rho_t}$$
$$=6\rho_H^2+14\rho_H+12\rho_t\rho_H-24\rho_t^2$$
$$-9u\rho_H-\tfrac{9}{5}v\rho_H+\tfrac{9}{2}u^2+\tfrac{9}{5}uv+\tfrac{27}{50}v^2\,, \tag{31}$$

Table 2
m_H^2/M_W^2 in the non-trivial reduction as function of $\alpha_s(M_W)$ and $\sin^2\theta_W(M_W)$ in fifth order of α.

α_s	$\sin^2\theta_W$		
	0.223	0.228	0.233
0.103	0.623	0.624	0.626
0.113	0.629	0.632	0.635
0.123	0.640	0.645	0.650

becomes a solution of (17) if functions u, v and ρ_t of α_s are inserted which are solutions of (11) and (6). Eq. (31) is solved by an expansion of the form

$$\rho_H = \tfrac{4}{3}\rho_t^2 + \tfrac{9}{28}u^2(1+\rho_t) + \tfrac{9}{70}uv(1+\rho_t) + \tfrac{27}{700}v^2(1+\rho_t) + \tfrac{2}{5}\rho_t^3 + \tfrac{33}{784}u^3 - \tfrac{351}{14000}v^3 - \tfrac{129}{3920}u^2v - \tfrac{1143}{19600}uv^2 - \tfrac{8}{15}v\rho_t^2 + \dots, \tag{32}$$

which fulfils the asymptotic requirement (25). Here all terms up to and including the third order in u, v and ρ_t are listed. In high orders also logarithmic terms occur. According to (7), (19) and (28) the expansion (13) expresses m_H^2/M_W^2 as a functional of m_t^2/M_W^2, α_s, α and θ_W. Table 3 lists the Higgs mass for some values of the top mass with the parameter values (23).

Next we discuss the two-loop correction to the reduction solutions.For a rough estimate of the correction we need to consider only the undisturbed system in the non-trivial reduction. Taking the two-loop β-functions [4], the differential equation for ρ_t (6) becomes

$$[-14\alpha_s - (\alpha_s^2/\pi)(13+\rho_t)]\frac{d\rho_t}{d\alpha_s} = 9\rho_t^2 - 2\rho_t + (\alpha_s/\pi)(-6\rho_t^3 + 19\rho_t^2 - \tfrac{245}{6}\rho_t + \tfrac{3}{16}\rho_H^2\rho_t - \tfrac{3}{2}\rho_H\rho_t^2). \tag{33}$$

Table 3
m_H as function of m_t in the trivial reduction including the fifth order in α (in GeV). The following parameter values were used: $\alpha_s(M_W)=0.113$, $\alpha(M_W)=\frac{1}{128}$, $\sin^2\theta_W(M_W)=0.228$, $M_W=81$ GeV.

m_t	m_H
0	39.9
40	41.7
50	43.4
60	46.3
70	50.4
80	56.0
90	63.0
91.3 [a]	64.3 [a]

[a] Upper bounds from the non-trivial reduction.

The solution which corresponds to (2) at the one-loop level can be uniquely obtained to be

$$\rho_t = \tfrac{2}{9}(1+c_t\alpha_s/\pi), \quad c_t \approx 2.3\,. \tag{34}$$

This would increase the one-loop result for the top mass by about 4%.

Similarly, the differential equation for ρ_H (24) becomes at the two-loop level

$$[-14\alpha_s - (\alpha_s^2/\pi)(13+\rho_t)]\frac{d\rho_H}{d\alpha_s} = 6\rho_H^2 + 12\rho_t\rho_H - 24\rho_t^2 + 14\rho_H + (\alpha_s/\pi)(-\tfrac{39}{8}\rho_H^3 + \rho_H^2 + 13\rho_H - \tfrac{63}{2}\rho_t^2 + 20\rho_H\rho_t + 30\rho_t^3 - 9\rho_H^2\rho_t - \tfrac{3}{4}\rho_H\rho_t^2)\,. \tag{35}$$

The two-loop solution for ρ_H is

$$\rho_H = a(1+c_H\alpha_s/\pi), \quad c_H \approx 2.3\,, \tag{36}$$

where a is given in (27). Again, the two-loop effect increases the one-loop result for m_H by about 4%.

For the disturbed system we expect some additional corrections to (33) and (35), like $(1/\pi)\alpha_s u$. This could slightly change our rough estimate of the two-loop effect. As for the trivial reduction, we may expect the same order of magnitude for the two-loop correction.

There is yet another correction to our predictions which comes from the fact that the mass value obtained from the reduction is not physical, i.e. the pole of the corresponding propagator. To find the magnitude of the correction, we notice that the values for the coupling constants quoted in (5) correspond to those in the modified minimal subtraction scheme with the renormalization scale at M_W. A precise estimate of the correction may depend on m_t and m_H, but it was found to be at most 0.5% [5], which is negligibly small compared to the two-loop correction.

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3.4 Cancellation of divergencies and reduction of couplings

Title: Cancellation of divergencies and reduction of couplings in the standard model
Authors: J. Kubo, K. Sibold, W. Zimmermann
Journal: Phys. Letts. **B220** (1989) 191-194

Comment (*Klaus Sibold*)
Although the standard model describes the experimental situation very well it has (at least) two shortcomings which raise doubts that it can be considered as a fundamental theory as opposed to an effective one. First, due to the quadratical divergencies in the Higgs self-mass there is the problem of "naturalness", also called hierarchy problem. Second, the masses of quarks and leptons as well as the mixing angles enter as free parameters which have to be taken from experiment – these are unaesthetically many.
Reduction of couplings as described in the previous subsections indeed constrains the parameters of the model. In the present paper it has been analyzed whether it is possible to require in addition the absence of quadratical divergencies. If so, then the version with three families would indeed become strengthened as to be fundamental.
In order to proceed it has been shown first that postulating absence of quadratical divergencies is a gauge and renormalization group invariant statement. And, indeed the resulting constraint is compatible with reduction, at least with the trivial one. The non-trivial reduction solution is however off by the uncertainties of the measurement of α_{em}/α_s and $sin^2\theta_W$.
Below, in section 5, the absence of quadratical divergencies will be implemented by relying on supersymmetry and/or by soft breaking of susy which maintains their absence. Hence this requirement and its interplay with reduction of couplings remained substantial.

CANCELLATION OF DIVERGENCIES AND REDUCTION OF COUPLINGS IN THE STANDARD MODEL

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Cancellation of quadratical divergencies in the Higgs propagator is shown to be compatible with renormalization group invariance and the reduction of couplings. Requiring both – cancellation and reduction – fixes Higgs and top mass as a function of the strong coupling and the Weinberg angle.

The standard model is experimentally in good shape [1]. Although it is thought of as an effective theory only, it works better than one had any reason to expect. Thus every result is welcome which is obtained within the model and which reduces its essential theoretical or aesthetic shortcomings:
- due to quadratical divergencies in the Higgs self-mass there is the problem of "naturalness" [2,3];
- the masses of quarks and leptons and the mixing angles are free parameters – these are unaesthetically many.

In earlier papers [4–6] we have shown that the method of reduction of couplings [7] serves to constrain the parameters of the standard model. These results were obtained for three generations and one Higgs doublet. The presence of the full gauge group $SU(3)_C \times SU(2)_L \times U(1)$ was essential.

There are two realistic cases of coupling reductions for the standard model. In case of the nontrivial reduction the top and Higgs mass are determined as functions of the gauge couplings and the other parameters. For the trivial reduction only the Higgs mass is determined with the top mass constrained by an upper bound. All other masses are essentially free in both cases.

In the present note we address ourselves to the divergence problem and relate it to the reduction method. The idea is very simple: Following a suggestion by Veltman we require the quadratical divergencies of the Higgs mass to cancel [3] [#1]. We check that Veltman's cancellation condition is compatible with the reduction principle. Both requirements combined lead to further constraints on the parameters of the model. For the trivial reduction it will be seen that the top and Higgs mass become determined by imposing the cancellation of quadratic divergencies. The numerical values obtained come surprisingly close to those of the non-trivial reduction. This agrees with the observation made by Gérard that the couplings of the non-trivial reduction approximately satisfy the cancellation condition [9].

We first discuss a definition of the Higgs self-mass which is gauge invariant and invariant under the renormalization group. In terms of unrenormalized quantities the Higgs part of the lagrangian is given by

$$\mathscr{L}_{\text{Higgs}} = \partial_\mu \Phi_0^* \partial^\mu \Phi_0 + \mu_0^2 \Phi_0^* \Phi_0 - \tfrac{1}{4}\lambda_0 (\Phi_0^* \Phi_0)^2 \qquad (1)$$

with the doublet

$$\Phi_0 = \begin{pmatrix} \varphi_{00} \\ (\nu_0 + \varphi_0 + i\chi_0)\sqrt{2} \end{pmatrix} . \qquad (2)$$

The parameters of the model are constrained by im-

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[#1] For a possible relation of this cancellation condition to symmetries of the system see ref. [8]. In this paper the top and Higgs mass are determined by requiring the cancellation of quadratic and logarithmic divergencies as well.

posing the condition that the field φ_0 has vanishing vacuum expectation value. In lowest order the pole of the Higgs propagator $G_{H0}(p^2)$ is located at $p^2=\frac{1}{2}\lambda_0 v_0^2=2\mu_0^2$. Accordingly, we define the unrenormalized Higgs mass m_{H0} by the gauge invariant quantity [#2]

$$m_{H0}^2=2\mu_0^2\,. \tag{3}$$

Let φ_{i0}, ..., φ_{i0} denote the unrenormalized fields of the model. Renormalized fields

$$\varphi_i=\bar{Z}_i^{1/2}\varphi_{i0} \tag{4}$$

are introduced by imposing suitable normalization conditions on their propagators. The renormalization group is the group of all transformations

$$\varphi_i'=Z_i^{1/2}\varphi_i,\quad Z_i>0\,, \tag{5}$$

which relate differently normalized finite field operators.

The physical mass and width of the Higgs particle determine a pole of the Higgs propagator which is reached by analytic continuation across the cut. We define the renormalized Higgs mass m through the real part m^2 of this pole. The self-mass

$$\delta m_H^2=m_H^2-m_{H0}^2 \tag{6}$$

of the Higgs particle is then gauge invariant and invariant under the renormalization group. Its quadratically divergent part can be isolated in a scheme independent manner and has in the one-loop approximation the coefficient [10]

$$\delta m_H^2\,|_{\substack{\text{quadr.}\\ \text{div.}}}\sim\tfrac{3}{2}\lambda+\tfrac{3}{2}g^2+\frac{3}{4\cos^2\theta_W}g^2-6G_t^2\,. \tag{7}$$

In (7) we have neglected the contributions coming from light fermions and the mixing angles, g denotes the SU(2) gauge coupling, θ_W the Weinberg angle, G_t the top quark Yukawa coupling. In terms of the masses

$$m_H^2=\tfrac{1}{2}\lambda v^2\,,$$

$$m_t^2=\tfrac{1}{2}G_t^2 v^2\,,$$

$$M_W^2=\tfrac{1}{4}g^2v^2\,, \tag{8}$$

[#2] At first sight it may seem natural to define m_0^2 by the coefficient $\frac{3}{4}\lambda_0 v_0^2-\mu_0^2$ of $\frac{1}{2}\varphi_0^2$ in the lagrangian. But this expression is not gauge invariant.

$$M_Z^2=\frac{1}{\cos^2\theta_W}M_W^2=\frac{g^2v^2}{4\cos\theta_W}\,, \tag{8 cont'd}$$

the quadratically divergent part of the Higgs self-energy reads

$$\delta m_H^2\,|_{\substack{\text{quadr.}\\ \text{div.}}}\sim 3m_H^2+6M_W^2+3M_Z^2-12m_t^2\,. \tag{9}$$

Pursuing an old suggestion by Veltman [3] we now postulate the quadratical divergence in the Higgs selfmass to be absent

$$\tfrac{3}{2}\lambda+\tfrac{3}{2}g^2+\frac{3}{4\cos^2\theta_W}g^2-6G_t^2=0\,. \tag{10}$$

In this order this requirement is obviously invariant under renormalization group transformations since the couplings are those of the *tree* approximation. In higher orders individual terms will be scheme dependent but according to the arguments given above the entire sum will again be invariant.

We now use the results of the reduction of couplings. For simplicity we neglect the mixing angles and all fermion masses but the top quark mass. According to ref. [4] we have to distinguish two cases:

(i) *Non-trivial reduction.* The top and Higgs coupling, hence their masses turn out to be uniquely determined functions of α_s and $\sin^2\theta_W$ (which themselves cannot be fixed by reduction within the standard model).

$$\frac{1}{4\pi}G_t^2=\alpha_s\rho_t=\alpha_s\sum c_{mn}u^m v^n\,, \tag{11}$$

$$\lambda=\alpha_s\rho-\alpha_s\sum a_{mn}u^m v^n\,, \tag{12}$$

$$u\equiv\frac{\alpha_{em}}{\alpha_s}\frac{1}{\sin^2\theta_W},\quad v\equiv\frac{\alpha_{em}}{\alpha_s}\frac{5}{3\cos^2\theta_W}\,, \tag{13}$$

$$\frac{1}{2}\frac{m_t^2}{M_W^2}=\sum c_{mn}\left(\frac{\alpha_{em}}{\alpha_s}\right)^{m+n-1}\times\left(\frac{1}{\sin^2\theta_W}\right)^{m-1}(\cos^2\theta_W)^n\,,$$

$$\frac{1}{2}\frac{m_t^2}{M_W^2}=\sum a_{mn}\left(\frac{\alpha_{em}}{\alpha_s}\right)^{m+n-1}\times\left(\frac{1}{\sin^2\theta_W}\right)^{m-1}(\cos^2\theta_W)^n\,. \tag{14}$$

(The list of numerical coefficients c_{mn}, a_{mn} is too long to be reproduced here, see ref. [11].)

Thus any additional relation, like (10), implies a functional relationship between α_s and $\sin^2\theta_W$. A convenient way of plotting the latter is found to be the ratio α_{em}/α_s versus $\sin^2\theta_W$ (see fig. 1), with all values taken at the scale M_W. The result is almost a straight line and it is seen that the previous values [12] for α_s and $\sin^2\theta_W$

$$\alpha_s = 0.1 \pm 0.015, \quad \sin^2\theta_W = 0.21 \pm 0.01 ,$$

$$\alpha_{em}/\alpha_s = 0.0625 ,$$

are very close to the line, whereas the recent values [13]

$$\alpha_s = 0.125 \pm 0.015, \quad \sin^2\theta_W = 0.23 \pm 0.01 ,$$

$$\alpha_{em}/\alpha_s = 0.0625 ,$$

are clearly off the line. Thus the Weinberg angle is determined in the non-trivial reduction by the cancellation of quadratic divergencies, but comes out too small for current values of α_s.

(ii) *Trivial reduction.* Here the top mass is another free parameter bounded from above by its value for the non-trivial reduction. The Higgs mass is given as a function of α_s, $\sin^2\theta_W$ and m_t. The requirement (10) yields then the top mass for any given value of α_s and $\sin^2\theta_W$. In order to minimize the error introduced by

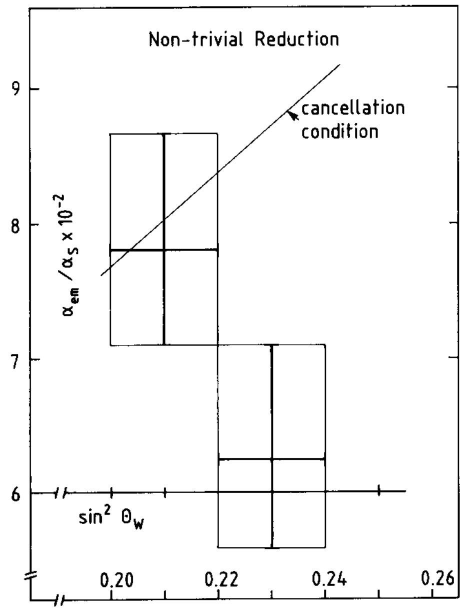

Fig. 1. Previous and recent experimental values and uncertainties of $\sin^2\theta_W$, α_{em}/α_s compared with the cancellation condition.

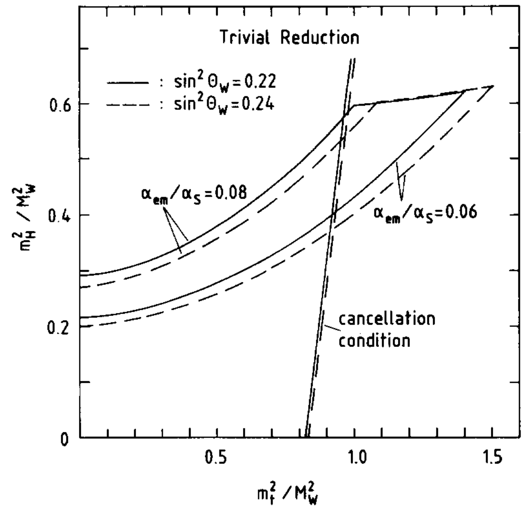

Fig. 2. Range of m_H^2/M_W^2 versus m_t^2/M_W^2 for the trivial reduction.

Table 1
Top mass m_t (in GeV) as function of $\alpha_s(M_W)$ and $\sin^2\theta_W(M_W)$ in the trivial reduction with cancellation of quadratic divergencies

α_s	$\sin^2\theta_W$		
	0.223	0.228	0.233
0.103	79.14	79.10	79.06
0.113	78.52	78.50	78.48
0.123	78.02	78.01	78.01

their experimental uncertainty it is best to go over to mass ratios m_H^2/M_W^2, m_t^2/M_W^2. The result is shown graphically in fig. 2, where these ratios are plotted as given by reduction and then intersected with the straight line indicating cancellation of the quadratical divergence. It is remarkable that reduction and cancellation are compatible.

Conceptually it is important to note that the quantities ρ, ρ_t, α_s, α_{em} refer to the *effective* couplings, hence depend on the scale (already in the order we are calculating). Thus "intersecting" (14) with (10) makes sense only for a given, fixed value of the scale. (Here taken to be the W mass.)

In tables 1 and 2 the masses of the top quark and Higgs particles obtained by combining the trivial reduction with the cancellation of quadratic divergencies are listed for some values of α_s and $\sin^2\theta_W$ (at M_W). The electroweak corrections of the reduction solutions are computed up to and including the fifth

Table 2
Higgs mass m_H (in GeV) as function of $\alpha_s(M_W)$ and $\sin^2\theta_W(M_W)$ in the trivial reduction with cancellation of quadratic divergencies

α_s	$\sin^2\theta_W$		
	0.223	0.228	0.233
0.103	59.06	58.35	57.65
0.113	55.65	55.00	54.39
0.123	52.78	52.19	51.63

order in α. For the W mass the value $M_W=81$ GeV was used.

Demanding reduction of couplings means requiring asymptotic freedom – one desideratum for a model to exist non-perturbatively [#3]. Absence of quadratical divergencies presumably also points towards existence of the corresponding theory and solves the problem of "naturalness" in a way similar to supersymmetry. In the present note we have shown that both requirements are compatible in the standard model with three generations and one Higgs doublet. They are satisfied for very specific values of the top and the Higgs mass, see fig. 2, and leave room for all other masses. It is thus suggestive to speculate that the standard model exists non-perturbatively and that these mass values are realized in nature.

[#3] Strictly speaking, asymptotic freedom only holds for the strong interaction part of the system if reduction is applied. The electroweak couplings should then be regarded as small perturbations of an asymptotically free system.

We are indebted to J.H. Kühn for very helpful discussions on the present values of α_s and $\sin^2\theta_W$.

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3.5 Precise determination of the top quark and Higgs masses

Title: Precise determination of the top quark and Higgs masses in the reduced standard theory for electroweak and strong interactions
Author: J. Kubo
Journal: Phys. Letts. **B262** (1991) 472-476

Comment (*Jisuke Kubo*)
The top quark and Higgs mass, m_t and m_h, can be predicted within the standard model (SM) when reduction of coupling constants (s. subsection 2.1) is applied. At the one-loop order we obtained (s. subsection 3.1)

$$m_t \simeq 81 \text{ GeV} , m_h \simeq 61 \text{ GeV} .$$

There are corrections to these values:

1. The above mass values depend on the SM parameters, in particular the strong coupling constant α_3 and $\sin\theta_W$. Since the values of α_3 and $\sin\theta_W$ have been updated, the above predictions need to be updated, too.

2. Two-loop corrections may be important.

3. In subsection 3.1 the difference of the physical mass (pole mass) and the mass defined in the $\overline{\text{MS}}$ scheme has been ignored.

In the present article all these corrections are included. We find that the correction coming from the $\overline{\text{MS}}$ to the pole mass transition increases m_t by about 4 %, while m_h is increased by about 1 %. The two-loop effect is non-negligible especially for m_t: +2 % for m_t and 0.2 % for m_h. Taking into account all these corrections we obtain

$$m_t = 98.6 \pm 9.2 \text{ GeV}, m_h = 64.5 \pm 1.5 \text{ GeV},$$

where the 1991 values of $M_Z, \alpha_3(M_Z), sin^2\theta_W(M_Z)$ and $\alpha_{em}(M_Z)$ are used.

If we use their 2013 values given in [2], we find that the change of the prediction is negligible. Obviously, this prediction is inconsistent with the experimental observations. This may be seen as a good news, because we know that the SM has to be extended to explain the recent experimental observations such as the non-zero neutrino mass. Even a simplest extension to include a dark matter candidate will change the 1991-prediction (which coincides essentially with a 2013-prediction).

Precise determination of the top quark and Higgs masses in the reduced standard theory for electroweak and strong interactions

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Using the latest experimental data, we recalculate the top quark and Higgs masses, m_t and m_h, on the basis of the reduction of coupling constants in the standard theory for electroweak and strong interactions. The reduced standard theory predicts $m_t = 99.2 \pm 5.7$ GeV and $m_h = 64.6 \pm 0.9$ GeV, where the uncertainty mostly originates from that of the QCD coupling constant.

Six years ago, we applied [1] the idea of reduction of coupling constants [2] to the standard theory for electroweak and strong interactions. We found that within this scheme the top quark and Higgs masses, m_t and m_h, are strongly constrained, and obtained [1]

$$m_t \simeq 81 \text{ GeV}, \quad m_h \simeq 61 \text{ GeV} \tag{1}$$

for the standard theory parameters used at that time. Since then, those parameters have slightly changed according to the improvements in experiments, and, moreover, the recent experimental data imply that $m_t \geqslant 89$ GeV [3]. Taking into account those changes of the standard theory parameters and also corrections which should still be included in (1), we shall recalculate m_t and m_h in this paper. We will find that the corrected mass values are consistent with the present experimental data. But a top quark mass $\geqslant 111$ GeV would definitely exclude the realization of our idea in the standard theory, unless it is somehow modified.

Detailed discussions on how to implement the reduction method in the standard theory are given in refs. [1,4,5]. Here we would like to briefly outline our idea. There are 13 coupling constants in the theory if one neglects the Kobayashi–Maskawa angles. Except the Higgs self-coupling, $\alpha_h \equiv \lambda/4\pi$, and the Yukawa coupling for the top quark, $\alpha_t \equiv G_t^2/4\pi$, the values of other couplings are experimentally known, some of them precisely and the others less precisely [#1]. As has been well known for a long time, the QCD coupling, α_3, is the largest in the hierarchy of those 11 known couplings. One finds that

$$\tilde{\alpha}_i \equiv \alpha_i/\alpha_3 \leqslant 0.35, \quad i \neq \text{t, h and } 3. \tag{2}$$

Of course, the hierarchy depends absolutely on the energy scale where the couplings are defined. In (2) we considered the energy scale at $\mu = M_Z$ [#2]. Observing that hierarchy of couplings, we were led to the assumption that the $\tilde{\alpha}$'s can be used as formal expansion parameters in the standard theory (at least at the present energies), and investigated whether this makes sense theoretically. We thus started with the unperturbed system which is defined as containing only α_3, α_t and α_h as the non-vanishing couplings. In order to perform rigorous, theoreti-

[1] Permanent address.
[#1] We use coupling constants defined in the $\overline{\text{MS}}$ scheme.
[#2] In ref. [1], we actually considered the energy scale at $\mu = M_W$. Today it is more convenient to define the couplings at $\mu = M_Z$ for obvious reasons.

cal investigations for our purpose, the requirement of asymptotic freedom is indispensable. We found there is a unique possibility that satisfies the asymptotic freedom requirement and that α_t and α_h appear in the same order as α_3 in the formal perturbation expansions mentioned above. And we called that solution the non-trivial reduction solution in ref. [1]. The unperturbed system which satisfies our requirements contains only one coupling constant, α_3, while the others, α_t and α_h, are power series of α_3. To two-loop order, for instance, one finds

$$\alpha_t = \tfrac{2}{9}\alpha_3 + \frac{31\,359 + 41\sqrt{689}}{62\,208}\frac{\alpha_3^2}{\pi} + \ldots, \quad \alpha_h = \frac{-25+\sqrt{689}}{18}\alpha_3 + \frac{14\,701\,515 - 535\,843\sqrt{689}}{3\,856\,896}\frac{\alpha_3^2}{\pi} + \ldots, \tag{3}$$

where the expansion coefficients in higher orders can be uniquely computed [4,6] in perturbation theory if the β-functions (which we assume are polynomials in perturbation theory) are given. The solution (3) satisfies the reduction equations [2]

$$\beta_t^0 = \beta_3^0 \frac{d\alpha_t}{d\alpha_3}, \quad \beta_h^0 = \beta_3^0 \frac{d\alpha_h}{d\alpha_3}, \tag{4}$$

where the β^0's are the β-functions for α_t, α_h and α_3, respectively, in the unperturbed system and given by, to two-loop order [7],

$$4\pi\beta_t^0 = \alpha_t(9\alpha_t - 16\alpha_3) + \frac{\alpha_t}{\pi}\left(-6\alpha_t^2 + \tfrac{3}{16}\alpha_h^2 - \tfrac{3}{2}\alpha_h\alpha_t - 54\alpha_3^2 + 18\alpha_t\alpha_3\right),$$

$$4\pi\beta_h^0 = 6\alpha_h^2 + 12\alpha_h\alpha_t - 24\alpha_t^2 + \frac{1}{\pi}\left(-\tfrac{39}{8}\alpha_h^3 - 32\alpha_3\alpha_t^2 + 20\alpha_t\alpha_h\alpha_3 - 8\alpha_h^2\alpha_t - \tfrac{3}{4}\alpha_h\alpha_t^2 + 30\alpha_t^3\right),$$

$$4\pi\beta_3^0 = -14\alpha_3^2 + \frac{\alpha_3^2}{\pi}(\alpha_t - 13\alpha_3). \tag{5}$$

So, the zeroth order system is an asymptotically free system which contains quarks that are strongly interacting and the self-interacting Higgs that feels the strong force via Yukawa coupling for the top quark. Perturbations caused by the non-vanishing $\tilde{\alpha}$'s break the asymptotic freedom property of the unperturbed, reduced system. Therefore, the whole system – the reduced standard theory – may be regarded as asymptotically free in a restricted sense [#3].

Next we come to corrections. Let us first discuss the corrections coming from the perturbations caused by the non-vanishing $\tilde{\alpha}$'s. In ref. [4], it has been shown that the perturbations can be incorporated into (3) by solving a set of partial differential equations:

$$\left(\tilde{\beta}_3 \frac{\partial}{\partial\alpha_3} + \sum_{i\neq t,h,3} \tilde{\beta}_i \frac{\partial}{\partial\tilde{\alpha}_i}\right)\rho_t = \tilde{\beta}_t, \quad \left(\tilde{\beta}_3 \frac{\partial}{\partial\alpha_3} + \sum_{i\neq t,h,3} \tilde{\beta}_i \frac{\partial}{\partial\tilde{\alpha}_i}\right)\rho_h = \tilde{\beta}_h, \tag{6}$$

where $\rho_t \equiv \alpha_t/\alpha_3$ and $\rho_h \equiv \alpha_h/\alpha_3$. The $\tilde{\beta}$-functions are defined as

$$\tilde{\beta}_3 \equiv \beta_3/\alpha_3, \quad \tilde{\beta}_t \equiv \beta_t/\alpha_3^2 - (\tilde{\beta}_3/\alpha_3)\rho_t, \quad \tilde{\beta}_h \equiv \beta_h/\alpha_3^2 - (\tilde{\beta}_3/\alpha_3)\rho_h, \quad \tilde{\beta}_i \equiv \beta_i/\alpha_3^2 - (\tilde{\beta}_3/\alpha_3)\tilde{\alpha}_i, \tag{7}$$

where [7]

$$4\pi\beta_t = 4\pi\beta_t^0 + \alpha_t\left(-\tfrac{9}{2}\alpha_2 - \tfrac{17}{10}\alpha_1 + 3\alpha_b\right)$$
$$+ \frac{\alpha_t}{\pi}\left(\tfrac{1187}{1200}\alpha_1^2 - \tfrac{9}{40}\alpha_1\alpha_2 + \tfrac{19}{30}\alpha_1\alpha_3 - \tfrac{23}{8}\alpha_2^2 + \tfrac{9}{2}\alpha_2\alpha_3 + \tfrac{393}{160}\alpha_1\alpha_t + \tfrac{225}{32}\alpha_t\alpha_2\right) + \ldots,$$

[#3] There is an alternative way to define an unperturbed system by treating α_3 as the only non-vanishing coupling in zeroth order. The resultant solution is called the trivial reduction solution in refs. [1,4]. This solution can be combined with the cancellation of quadratic divergences in the theory [8], leading to $m_t \simeq 80$ GeV and $m_h \simeq 55$ GeV [9].

$$
\begin{aligned}
4\pi\beta_{\mathrm{h}} &= 4\pi\beta^0_{\mathrm{h}} - \tfrac{9}{5}\alpha_{\mathrm{h}}\alpha_1 - 9\alpha_{\mathrm{h}}\alpha_2 + \tfrac{27}{50}\alpha_1^2 + \tfrac{9}{2}\alpha_2^2 + \tfrac{9}{5}\alpha_1\alpha_2 + 12\alpha_{\mathrm{h}}\alpha_{\mathrm{b}} - 24\alpha_{\mathrm{b}}^2 \\
&\quad + \frac{1}{\pi}\,[\tfrac{27}{20}\alpha_{\mathrm{h}}^2\alpha_1 + \tfrac{27}{4}\alpha_{\mathrm{h}}^2\alpha_2 - \tfrac{1119}{800}\alpha_{\mathrm{h}}\alpha_1^2 - \tfrac{73}{32}\alpha_{\mathrm{h}}\alpha_2^2 - \tfrac{117}{80}\alpha_{\mathrm{h}}\alpha_1\alpha_2 - \tfrac{3411}{2000}\alpha_1^3 - \tfrac{1677}{400}\alpha_1^2\alpha_2 - \tfrac{289}{80}\alpha_1\alpha_2^2 + \tfrac{305}{16}\alpha_2^3 \\
&\quad - (\tfrac{8}{5}\alpha_1 + \tfrac{9}{4}\alpha_2)\alpha_{\mathrm{t}}^2 + (\tfrac{17}{8}\alpha_1 + \tfrac{45}{8}\alpha_2)\alpha_{\mathrm{h}}\alpha_{\mathrm{t}} + (-\tfrac{171}{100}\alpha_1 + \tfrac{63}{10}\alpha_2)\alpha_{\mathrm{t}}\alpha_1] + \dots, \\
4\pi\beta_3 &= 4\pi\beta^0_3 + \frac{\alpha_3^2}{\pi}\,(\tfrac{11}{20}\alpha_1 + \tfrac{9}{4}\alpha_2) + \dots, \\
4\pi\beta_1 &= \tfrac{41}{5}\alpha_1^2 + \frac{\alpha_1^2}{\pi}\,(\tfrac{44}{10}\alpha_3 + \tfrac{17}{20}\alpha_{\mathrm{t}} + \tfrac{199}{100}\alpha_1 + \tfrac{27}{20}\alpha_2) + \dots, \\
4\pi\beta_2 &= -\tfrac{19}{3}\alpha_2^2 + \frac{\alpha_2^2}{\pi}\,(6\alpha_3 + \tfrac{3}{4}\alpha_{\mathrm{t}} + \tfrac{9}{20}\alpha_1 + \tfrac{35}{12}\alpha_2) + \dots, \\
4\pi\beta_{\mathrm{b}} &= \alpha_{\mathrm{b}}(-16\alpha_3 + 3\alpha_{\mathrm{t}} + 9\alpha_{\mathrm{b}} - \tfrac{9}{2}\alpha_2 - \tfrac{1}{2}\alpha_1) + \dots, \\
&\vdots\,.
\end{aligned} \tag{8}
$$

In (8) we have suppressed terms indicated by ... that are irrelevant for a numerical study on m_{t} and m_{h}, and the β^0's are given in (5).

With the boundary condition that in the vanishing $\tilde{\alpha}$'s the solution of (6) reduces to that of the unperturbed system, i.e. (3), ρ_{t} and ρ_{h} are *unique to all orders* in perturbation theory [4]. It has been also shown [4] that, for small $\tilde{\alpha}$'s, ρ_{t} and ρ_{h} can be expressed as power series of $\tilde{\alpha}$'s and α_3 with unique expansion coefficients. We find

$$
\begin{aligned}
\rho_{\mathrm{t}} &= \tfrac{2}{9} - \tfrac{17}{540}\tilde{\alpha}_1 - \tfrac{1}{12}\tilde{\alpha}_2 - \tfrac{1}{5}\tilde{\alpha}_{\mathrm{b}} + \tfrac{799}{64800}\tilde{\alpha}_1^2 - \tfrac{1}{288}\tilde{\alpha}_2^2 + \tfrac{119}{9360}\tilde{\alpha}_2\tilde{\alpha}_1 + \tfrac{9}{400}\tilde{\alpha}_{\mathrm{b}}\tilde{\alpha}_1 - \tfrac{54}{175}\tilde{\alpha}_{\mathrm{b}}^2 - \tfrac{5593}{972000}\tilde{\alpha}_1^3 - \tfrac{1}{1728}\tilde{\alpha}_2^3 - \tfrac{323}{62400}\tilde{\alpha}_1^2\tilde{\alpha}_2 \\
&\quad + \tfrac{17}{56160}\tilde{\alpha}_1\tilde{\alpha}_2^2 - 0.001\ldots\tilde{\alpha}_1\tilde{\alpha}_2\tilde{\alpha}_{\mathrm{b}} - 0.009\ldots\tilde{\alpha}_{\mathrm{b}}\tilde{\alpha}_1^2 + 0.0029\ldots\tilde{\alpha}_1^4 + 0.0025\ldots\tilde{\alpha}_1^3\tilde{\alpha}_2 - 0.000\,08\ldots\tilde{\alpha}_1^2\tilde{\alpha}_2^2 \\
&\quad + 0.000\,05\ldots\tilde{\alpha}_1\tilde{\alpha}_2^3 - 0.000\,14\ldots\tilde{\alpha}_2^4 + \ldots + \frac{\alpha_3}{\pi}\left(\frac{31\,359 + 41\sqrt{689}}{62\,208} - 0.2231\ldots\tilde{\alpha}_1 - 0.8262\ldots\tilde{\alpha}_2\right. \\
&\quad \left. + 0.1690\ldots\tilde{\alpha}_1^2 - 0.0664\ldots\tilde{\alpha}_2^2 + 0.1824\ldots\tilde{\alpha}_1\tilde{\alpha}_2 + \ldots\right) + \ldots,
\end{aligned} \tag{9}
$$

and

$$
\begin{aligned}
\rho_{\mathrm{h}} &= \frac{-25 + \sqrt{689}}{18} + \frac{1295 - 83\sqrt{689}}{16\,740}\,\tilde{\alpha}_1 + \frac{163 - 7\sqrt{689}}{372}\,\tilde{\alpha}_2 - 0.148\,645\ldots\tilde{\alpha}_{\mathrm{b}} + 0.091\,372\,6\ldots\tilde{\alpha}_1^2 \\
&\quad + 0.437\,165\ldots\tilde{\alpha}_2^2 + 0.212\,713\ldots\tilde{\alpha}_1\tilde{\alpha}_2 + 0.145\ldots\tilde{\alpha}_1\tilde{\alpha}_{\mathrm{b}} + 0.094\ldots\tilde{\alpha}_2\tilde{\alpha}_{\mathrm{b}} + 1.090\ldots\tilde{\alpha}_{\mathrm{b}}^2 - 0.068\,89\ldots\tilde{\alpha}_1^3 \\
&\quad - 0.131\,18\ldots\tilde{\alpha}_1^2\tilde{\alpha}_2 - 0.086\,32\ldots\tilde{\alpha}_1\tilde{\alpha}_2^2 + 0.035\,40\ldots\tilde{\alpha}_2^3 - 0.0639\ldots\tilde{\alpha}_{\mathrm{b}}\tilde{\alpha}_1^2 + 0.3739\ldots\tilde{\alpha}_{\mathrm{b}}\tilde{\alpha}_2^2 + 0.0858\ldots\tilde{\alpha}_{\mathrm{b}}\tilde{\alpha}_1\tilde{\alpha}_2 \\
&\quad + 0.0497\ldots\tilde{\alpha}_1^4 + 0.0903\ldots\tilde{\alpha}_1^3\tilde{\alpha}_2 + 0.0609\ldots\tilde{\alpha}_1^2\tilde{\alpha}_2^2 + 0.0235\ldots\tilde{\alpha}_1\tilde{\alpha}_2^3 + 0.0369\ldots\tilde{\alpha}_2^4 + \ldots \\
&\quad + \frac{\alpha_3}{\pi}\left(\frac{14\,701\,515 - 535\,843\sqrt{689}}{3\,856\,896} - 0.1235\ldots\tilde{\alpha}_1 - 0.4820\ldots\tilde{\alpha}_2 + 0.3340\ldots\tilde{\alpha}_1^2\right. \\
&\quad \left. - 0.0459\ldots\tilde{\alpha}_1\tilde{\alpha}_2 - 0.0033\ldots\tilde{\alpha}_2^2 + \ldots\right) + \ldots\,.
\end{aligned} \tag{10}
$$

Our next concern is to relate ρ_{t} and ρ_{h} to m_{t} and m_{h}. At the tree level, we have $m_{\mathrm{t}}^2/M_{\mathrm{Z}}^2 = 2\cos^2\theta_{\mathrm{w}}\,\alpha_{\mathrm{t}}/\alpha_2$ and $m_{\mathrm{h}}^2/M_{\mathrm{Z}}^2 = 2\cos^2\theta_{\mathrm{w}}\,\alpha_{\mathrm{h}}/\alpha_2$. In higher orders in perturbation theory, these relations are modified in general. But in the $\overline{\mathrm{MS}}$ scheme the same relations among renormalized parameters hold:

$$\frac{\bar{m}_t^2}{\bar{M}_Z^2}=2\frac{\rho_t}{\tilde{\alpha}_2}\cos^2\bar{\theta}_w\,,\quad \frac{\bar{m}_h^2}{\bar{M}_Z^2}=2\frac{\rho_h}{\tilde{\alpha}_2}\cos^2\bar{\theta}_w\,. \tag{11}$$

The $\bar{m}$'s are the $\overline{\mathrm{MS}}$ masses, and differ from the physical masses, m_t and m_h, by a finite renormalization. This is a correction which should be taken into account in (1). So, we need to determine the physical masses in the $\overline{\mathrm{MS}}$ scheme. To this end, we have to calculate the corresponding self-energy diagrams [10,11] and look for poles in the propagator. In this way, one finds

$$m_t=(1+\Delta_t)\bar{m}_t(M_Z)\,,\quad m_h=(1+\Delta_h)\bar{m}_h(M_Z)\,,\quad \overline{M_Z}(M_Z)=(1+\Delta_Z)M_Z\,, \tag{12}$$

with

$$\Delta_t\simeq\left(\frac{4}{3}+\ln\frac{M_Z^2}{m_t^2}\right)\frac{\alpha_3(M_Z)}{\pi}-0.04\frac{\alpha_2(M_Z)}{\pi}\,,\quad \Delta_h\simeq0.45\frac{\alpha_2(M_Z)}{\pi}\,,\quad \Delta_Z\simeq0.45\frac{\alpha_2(M_Z)}{\pi}\,. \tag{13}$$

The Δ's depend on the standard theory parameters, especially on m_t and m_h. In (13) we have used $m_t=100$ GeV, $m_h=65$ GeV, $\sin^2\theta_w=0.23$ and $M_Z=91.2$ GeV.

Inserting (12) into (11) we finally obtain

$$\frac{m_t}{M_Z}=(1+\Delta_t)(1+\Delta_Z)\cos\bar{\theta}_w(M_Z)\sqrt{2\frac{\alpha_3(M_Z)}{\alpha_2(M_Z)}\rho_t(M_Z)}\,, \tag{14}$$

$$\frac{m_h}{M_Z}=(1+\Delta_h)(1+\Delta_Z)\cos\bar{\theta}_w(M_Z)\sqrt{2\frac{\alpha_3(M_Z)}{\alpha_2(M_Z)}\rho_h(M_Z)}\,, \tag{15}$$

where ρ_t and ρ_h are given in (9) and (10). We are now in the position to give numerical values for m_t and m_h, and use [11,12] [#4]

$$\sin^2\bar{\theta}_w(M_Z)=0.2333\pm0.0002\,,\quad \alpha_3(M_Z)=0.116\pm0.010\,,$$
$$\alpha_{em}(M_Z)=\sin^2\bar{\theta}_w(M_Z)\,\alpha_2(M_Z)=\tfrac{3}{5}\cos^2\bar{\theta}_w(M_Z)\,\alpha_1(M_Z)=(127.8\pm0.1)^{-1}\,,$$
$$M_Z=91.177\pm0.021\text{ GeV}\,,\quad m_b=5\text{ GeV}\,. \tag{16}$$

Inserting (16) into (14) and (15), we find

$$m_t=99.2\pm5.7\text{ GeV}\,,\quad m_h=64.6\pm0.9\text{ GeV}\,, \tag{17}$$

which are consistent with the present knowledge of the standard theory.

Near the central values of (16), (14) and (15) can be approximately written as

$$\frac{m_t}{M_Z}=1.088+0.716\left(\frac{\alpha_3(M_Z)}{0.116}-1\right)+0.452\left(\frac{\sin^2\bar{\theta}_w(M_Z)}{0.2333}-1\right)-0.637[127.8\alpha_{em}(M_Z)-1]\,, \tag{18}$$

$$\frac{m_h}{M_Z}=0.708+0.107\left(\frac{\alpha_3(M_Z)}{0.116}-1\right)+0.002\left(\frac{\sin^2\bar{\theta}_w(M_Z)}{0.2333}-1\right)-0.099[127.8\alpha_{em}(M_Z)-1]\,. \tag{19}$$

To obtain (1) we used in ref. [1], $\alpha_3=0.1$, $\sin^2\bar{\theta}_w=0.21$, and $M_W=81$ GeV, and ignored the Δ's in (1) and also the two-loop effects and $O(\tilde{\alpha}^4)$ contributions to ρ_t and ρ_h in (9) and (10). The QCD contribution to Δ_t which is absent in Δ_h and Δ_Z at the one-loop level increases m_t by about 4%, and the two-loop effects in ρ_t (ρ_h) shift up m_t (m_h) by about 2% (0.2%). As can be seen from (18) and (19), m_t is more sensitive than m_h against the change of the standard theory parameters, especially against the change of α_3 which has the largest experimental

[#4] The value of $\sin^2\bar{\theta}_w(M_Z)$ quoted in (16) is determined for $m_t=100$ GeV and $m_h=65$ GeV. I thank W. Hollik for the determination.

uncertainty. So, the large uncertainty in our prediction for m_t mostly originates from the uncertainty in α_3, and a precise measurement of m_h therefore would provide a clear experimental test of the reduced standard theory.

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4 Abstract interludium

Comment (*Klaus Sibold*)
In the third section we presented the principle of reduction of couplings and its application to the standard model. These investigations took place, roughly, during the years 1983 until 1991. In parallel to them a program of renormalizing supersymmetric theories was carried out which culminated for models with one supersymmetric generator, $N = 1$ in short, in a fairly complete understanding of its maximal symmetry content: superconformal symmetry. It turned out that in all $N = 1$ models the anomalies of the superconformal tranformations lie in some susy multiplet and are provided by the supercurrent and its moments in superspace. Next, it is crucial that a specific $U(1)$ axial transformation, called R, forms part of the superconformal algebra. For, axial transformations may lead to non-renormalization theorems, which then affect the (non-)renormalization behavior of the anomalies of the other transformations.
In the usual setup of perturbative quantum field theories ultraviolet divergencies occur and have to be taken care of in such a way that the fundamental postulates – Lorentz covariance, unitarity and causality – are not violated. In supersymmetric theories, as a rule, fewer divergencies show up than in ordinary models of spin zero, one-half and one. The non-abelian gauge theory with $N = 4$ supersymmetries has only one coupling, the gauge coupling. Its respective $\beta-$function automatically vanishes; this theory has been called "finite". In the more general case of $N = 1$ supersymmetry one can now search if this can take place by reducing the matter couplings to the gauge coupling, follow the effect of reduction and combining the result with relations provided by the superconformal symmetry. The non-renormalization theorems of axial current anomalies yield then very interesting results. This refers to subsections 4.1 and 4.2. (A somewhat non-technical report on the outcome of these investigations is provided by [8].)
In section 5 models will be considered which are based on supersymmetry and finiteness, i.e. the proliferation of free parameters introduced by "supersymmetrizing" a phenomenologically viable theory, say in order to suppress naturally quadratical divergencies, is counterbalanced by restricting matter couplings via reduction and asking for finiteness in the sense of having vanishing β-functions. This application justifies the inclusion of the respective papers in the present section.

In subsection 4.3 a first step has been made towards incorporating masses and gauge parameters when performing reduction of couplings: it is shown that reduction of dimensionless couplings is possible in the presence of such parameters.
These considerations are extended in subsection 4.4 to refer to the notion of reduction itself by formulating the method also for "couplings" carrying dimension; this includes mass parameters. These investigations provide the basis for the exploration and application of soft susy breaking in the papers presented in section 5. Obviously nature is not supersymmetric, but mechanisms for breaking supersymmetry are rare. Dynamical mass generation is not easy to implement, spontaneous breaking of susy does not lead very far, hence soft breaking which maintains the benefits of susy is the most suitable tool. In practice it has been found (s. section 5) that there exist also on the level of soft terms closed renormalization orbits. Those can be systematically searched for by reduction. It is then a matter of detailed analysis to relate (running) mass parameters to physical masses and to clarify the different renormalization effects. Most important is the identification of renormalization scheme independent quantities and resulting calculational rules.

4.1 Vanishing $\beta-$functions in $N = 1$ supersymmetric gauge theories

Title: Vanishing $\beta-$functions in supersymmetric gauge theories
Authors: Lucchesi, O. Piguet, K. Sibold
Journal: Helv. Physica Acta **61** (1988) 321-344

Comment (*Olivier Piguet*)
This paper presents a non-renormalization theorem for the vanishing, at all orders of perturbation theory, of the Callan-Symanzik β-functions for a class of $N = 1$ supersymmetric non-abelian gauge theories where the gauge group is simple. The matter content of the theory is assumed to be such that the anomaly in the Slavnov-Taylor identity is absent, hence the gauge theory is consistent. The necessary and sufficient conditions for the theorem to hold are:
(i) the β-function of the gauge coupling vanishes in one-loop order;
(ii) the anomalous dimensions of the matter superfields vanish in one-loop order;
(iii) the Yukawa couplings of the matter supermultiplets solve as power series in the gauge coupling the Oehme-Zimmermann reduction equations (see Section 1).
The proof exploits the supersymmetric correspondence of the conformal anomaly with a certain axial current anomaly through the supercurrent multiplet. The theorem allows the formulation of a simple criterion, involving only one-loop order quantities. The outcome is a class of $N = 1$ supersymmetric theories with a single coupling constant which are "finite", i.e., whose β-function vanish to all orders of perturbation theory. An example based on the unitary group $SU(6)$ is worked out, showing that this class of finite theories is not empty and contains theories without extended supersymmetry.

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Vanishing β-functions in $N = 1$ supersymmetric gauge theories

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(13. II. 1987)

Abstract. Necessary and sufficient conditions for the all-order vanishing of the β-functions in $N = 1$ supersymmetric gauge theories with simple gauge group are given. They contain well-known one-loop conditions and require the Yukawa coupling constants to be power series in the gauge coupling constant solving the reduction equations of Oehme and Zimmermann. A simple criterion for vanishing β-functions involving only one-loop quantities is then proposed.

1. Introduction

Many attempts have been made during the last years to obtain finite quantum field theories in four-dimensional space-time. For general theories, such a search has hardly gone beyond the one-loop approximation [1]. There is a strong indication that only supersymmetric gauge theories (SYM) can eventually be completely free of ultra-violet divergences [1], although examples of non-supersymmetric models with vanishing one-loop β-functions, i.e., without coupling constant renormalization, are known [2]. Much work [3–10] has been dedicated to the investigation of the SYM theories. The authors of Refs. [9] and [10], in particular, deal with this problem at all orders for $N = 1$ SYM theories. They demand the all order vanishing of the anomalous dimensions for all fields; this ensures the vanishing of the β-functions too, hence the complete finiteness of the theory. For this purpose, they require the Yukawa coupling constants λ (self-interaction of the matter fields) to be power series in the gauge coupling

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constant g: these functions $\lambda(g)$ have to solve the condition of vanishing matter field anomalous dimensions. The authors must, however, impose some restrictions; in particular, they cannot treat theories where the number of independent anomalous dimensions exceeds that of Yukawa coupling constants. Their proof also heavily relies on the dimensional regularization which is known [11] to face difficulties in preserving supersymmetry in higher orders.

The present paper is an extension of a previous work [12] in which sufficient conditions for 'finiteness' were presented. By 'finiteness' we mean the vanishing of the β-functions – the physically relevant objects – to all orders but not necessarily of all anomalous dimensions to any order: this allows us to abandon any a priori restriction on the number of fields and couplings. The functions $\lambda(g)$ are now solutions of the reduction equations of Oehme and Zimmermann [13, 14], a necessary condition for the consistency of the theory. In order to avoid any problem with regularization, the theory is assumed to be renormalized by using the superspace renormalization scheme of Ref. [15], where it is also shown [16] that BRS invariance can be maintained at all orders of perturbation theory, provided the usual gauge anomaly is absent.

The criterion of 'finiteness' here gains precision with respect to that of Ref. [12]. Our first main result (Theorem 5.2) is that the conditions of Jones, Mezincescn, Parkes and West [4] for the one-loop and two-loop finiteness of $N = 1$ SYM theories – namely the vanishing of the gauge β-function and of the matter field anomalous dimensions at one-loop – are actually necessary and sufficient in order to have β-functions vanishing to all orders, if one completes them with the requirement that the reduction equations possess a power series solution $\lambda = \lambda(g)$. Our second main result is a set of sufficient 'finiteness' conditions relying only on one-loop quantities (Theorem 5.3): it consists of adding to the conditions of Ref. [4] a condition which ensures the existence of all-order solutions to the reduction equations.

We further show that the vanishing of the anomalies associated with all the chiral symmetries the model may have is necessary and sufficient for ensuring the compatibility of the vanishing conditions for all the one-loop anomalous dimensions of the matter fields.

Hence the 'finite' SYM theories are completely free of anomalies, of the conformal ones, i.e., the β-functions, as well as of the chiral ones. The strategy of our proof is a rigorous extension of an old formal argument [17] proposed for showing the finiteness of the $N = 4$ SYM theory. Our approach depends on the detailed structure of the supercurrent multiplet anomaly [18, 19, 15] and in particular on an explicit relation we derive, combining the β-functions, the anomalous dimensions and the axial anomalies [equation (4.14)]. The usefulness of this relation relies on the non-renormalization theorem we prove for the latter anomalies. Note that a recent paper [20] gives a 'proof' of the theorem for the anomaly of the axial current (R-current) related to the supercurrent multiplet. It uses, however, the regularization by dimensional reduction*).

*) The authors of Ref. [20] in fact claim to have a successful demonstration using this regularization, otherwise criticized [11] as being inconsistent.

In order to render the present paper self-contained and also to fill a loophole found in Ref. [12], we shall repeat part of the material presented there. Section 2 reviews general features of $N=1$ SYM models. Section 3 deals with the one-loop approximation and in particular with the relationship between vanishing axial anomalies and anomalous dimensions (Lemma 3.1). The general structure of the supercurrent anomalies and their relation with the axial anomalies are explained in Section 4. The main results mentioned in this Introduction are derived in Section 5. We apply them to an example in Section 6 and draw some general conclusions in Section 7. Appendix A gives the corrected statement and the proof of the supersymmetric non-renormalization theorem, which was formulated under too weak hypotheses in Ref. [12]. Finally, a one-loop condition for the existence of power series solutions to the reduction equations is given in Appendix B.

2. The model and its invariances

The physical field content of a general $N=1$ SYM theory [15] consists of a real gauge superfield of dimension 0, $\phi = \phi_i \tau^i$ (τ^i the generators of the gauge group G, assumed to be *simple*), and of chiral matter superfields A^R of dimension one. The upper index R labels both the field itself and the irreducible representation (irrep.) of G it belongs to. The complex conjugate field $\bar{A}_R$ transforms in the representation conjugate to R. We shall also use the multi-index notation [4]

$$A^r \equiv A^{(R,\rho)}, \tag{2.1}$$

where ρ labels the components within the irrep. R

The BRS transformations read

$$\begin{aligned} se^{\phi} &= e^{\phi}c_+ - \bar{c}_+ e^{\phi}, \\ sA^{(R,\rho)} &= -c_{+i}(T^i_R)^{\rho}_{\sigma}A^{(R,\sigma)}, \\ sc_+ &= -\tfrac{1}{2}\{c_+, c_+\}, \end{aligned} \tag{2.2}$$

and are nilpotent:

$$s^2 = 0. \tag{2.3}$$

Here $c_+ = c_{+i}\tau^i$ is the (chiral) Faddeev–Popov ghost. The Hermitian matrices T^i_R are the generators of G in the irrep. R. We omit the Lagrange multiplier and antighost fields involved in the gauge fixing of the theory.

A more general BRS transformation law preserving the nilpotency property is obtained by performing a generalized field amplitude renormalization [15], i.e., by replacing ϕ in the first line of (2.2) by

$$\mathscr{F}(\phi) = \phi + \sum_{k=2}^{\infty} a_k \phi^k, \tag{2.4}$$

where the infinite set of parameters a_k can be shown [15, 21] to be non-physical.

The most general gauge-invariant classical action is [15]*)

$$\Gamma_{inv} = -\frac{1}{128g^2}\operatorname{Tr}\int dSF^{\alpha}F_{\alpha} + \frac{1}{16}\int dV \sum_R \bar{A}_R \exp(\phi_i T^i_R)A^R + \int dSU(A) + \int d\bar{S}\bar{U}(\bar{A}), \tag{2.5}$$

with the SYM field strength F_α given in terms of the 'chiral connection' φ_α by

$$F_\alpha = \overline{DD}\varphi_\alpha, \qquad \varphi_\alpha = e^{-\phi}D_\alpha e^{\phi} \tag{2.6}$$

where ϕ is replaced by (2.4) in the general case. g is the gauge coupling constant. The two last terms in (2.5) describe the self-interaction of the matter fields in term of the chiral superpotential. With the use of notation (2.1), these terms read:

$$\begin{aligned} U(A) &= \tfrac{1}{6}\lambda_{(rst)}A^r A^s A^t, \\ \bar{U}(\bar{A}) &= \tfrac{1}{6}\bar{\lambda}^{(rst)}\bar{A}_r\bar{A}_s\bar{A}_t, \end{aligned} \tag{2.7}$$

the complex "Yukawa" coupling constants λ_{rst} being invariant symmetric tensors of G.

Beyond supersymmetry and BRS invariance, the massless action (2.5) is invariant under the R-transformations [22, 15]

$$\delta_R\psi = i(n_\psi + \theta^\alpha\partial_{\theta^\alpha} - \bar{\theta}^{\dot\alpha}\partial_{\bar\theta^{\dot\alpha}})\psi, \tag{2.8}$$

with the R-weights being respectively

$$n_\psi = 0,\ -\tfrac{2}{3}(\tfrac{2}{3}),\ 0(0) \quad \text{for} \quad \psi = \phi,\ A(\bar{A}),\ c_+(\bar{c}_+).$$

The theory is, in general, also invariant under a (possibly empty) set of chiral transformations.

$$\begin{aligned} &\delta_a\phi = \delta_a c_+ = 0, \\ &\delta_a A^R = ie_{aS}^{\ R}A^S, \qquad \delta_a\bar{A}_R = -i\bar{A}_S e_{aR}^{\ S}, \end{aligned} \tag{2.9}$$

where the chiral charge matrices e_a are Hermitian. These transformations commute with the BRS transformations (hence $e_{aS}^{\ R} = 0$ if irrep. $R \neq$ irrep. S), and with supersymmetry.

The classical action (2.5) is invariant under (2.9) if and only if the Yukawa coupling constants obey the constraints

$$\forall_a\colon\ \lambda_{rsu}e_{at}^{\ u} + \text{cycl. perm. } (r, s, t) = 0, \tag{2.10}$$

with the notation

$$M_t^u = \delta_\tau^\mu M_T^U. \tag{2.11}$$

The quantum theory in loop expansion, described by the vertex functional

*) $dV = d^4x d^4\theta = d^4x\, DD\bar{D}\bar{D}$, $dS = d^4x d^2\theta = d^4x DD$.

$\Gamma(\phi, A, \ldots) = \Gamma_{\text{class}} + O(\hbar)$, can be shown to preserve all the invariances listed above, up to soft breakings induced by supersymmetric masses (which we add to the action (2.5) in order to avoid infra-red difficulties [15]). Supersymmetry is explicit (and exact) due to the use of a superspace subtraction scheme [15]. BRS invariance is expressed by the Slavnov identity*)

$$\mathscr{S}(\Gamma) \sim 0 \tag{2.12}$$

which holds (up to soft breakings: this is the meaning of the symbol $\sim$) provided the representation of the matter fields A^R is chosen to be anomaly free [15, 16]:

$$a \equiv \sum_R a(R) = 0, \tag{2.13}$$

$a(R)$ being the 'anomaly index' of the irrep. R; these indices are tabulated, e.g., in Ref. [23]. R-invariance (2.8) and the chiral invariances (2.9) are expressed by the Ward identities

$$W_R\Gamma \equiv -i \sum_{\psi=\phi,A,c_+} \int \delta_R\psi \frac{\delta\Gamma}{\delta\psi} \sim 0, \tag{2.14}$$

$$\begin{aligned} W_a\Gamma &\equiv -i \sum_{\psi} \int \delta_a\psi \frac{\delta\Gamma}{\delta\psi} \\ &\equiv \sum_{R,S} e_{aS}^{\ R}\left[\int dSA^S \frac{\delta}{\delta A^R} - \int d\bar{S}\bar{A}_R \frac{\delta}{\delta \bar{A}_S}\right]\Gamma \sim 0 \end{aligned} \tag{2.15}$$

holding at all orders [15], up to soft breakings, too. The operators W_a generate the Lie algebra $\mathscr{C}$ associated to the infinitesimal chiral transformations (2.9), with the commutation relations

$$[W_a, W_b] = W_c, \tag{2.16}$$

W_c having the charge matrix $e_c = -[e_a, e_b]$. We shall denote by W_{0a} a basis of the centre $\mathscr{C}_0$ of the algebra $\mathscr{C}$:

$$[W_{0a}, W_b] = 0 \quad \text{for any} \quad W_b \in \mathscr{C}, \tag{2.17}$$

and by e_{0a} the corresponding charge matrices.

Let us close this section by recalling the Callan–Symanzik equation [15] fulfilled by the vertex functional Γ, up to soft mass insertions:

$$\begin{aligned} C\Gamma \equiv [&m\,\partial_m + \beta_g\,\partial_g + \beta_{rst}\partial_{\lambda_{rst}} \\ &+ \bar{\beta}^{rst}\,\partial_{\bar{\lambda}^{rst}} - \gamma_\phi \mathscr{N}_\phi - \gamma_R^S \mathscr{N}_S^R - \gamma_k\,\partial_{a_k}]\Gamma \sim 0. \end{aligned} \tag{2.18}$$

where $m\,\partial_m$ (with summation over all mass parameters of the theory) is the

*) We shall not give the explicit form of the (non-linear) Slavnov functional operator $\mathscr{S}$; it involves external superfields coupled to the BRS variations of the different fields of the theory (see for instance Ref. [12]).

scaling operator. The counting operators $\mathcal{N}$ are*)

$$\mathcal{N}_\phi = \mathrm{Tr}\int dV\phi\delta_\phi, \tag{2.19}$$

$$\mathcal{N}_S^R = \int dSA^R\delta_{A^S} + \int d\bar{S}\bar{A}_S\delta_{\bar{A}_R}. \tag{2.20}$$

Due to the reality of Γ, the gauge beta-function β_g, the anomalous dimension γ_ϕ and the coefficients γ_k – which describe the generalized amplitude renormalization (2.4) – are real. The Yukawa beta-functions β_λ and $\beta_{\bar{\lambda}}$ are the complex conjugates of each other, and the matrix γ_S^R of matter field anomalous dimensions is Hermitian. Note the absence of an anomalous dimension term for the ghost c_+: we are using a particular renormalization scheme with the effect that its anomalous dimension vanishes [12].

In fact, due to the chiral invariances (2.9) and (2.15), only combinations of the counting operators (2.20) which commute with the Ward identity operators W_a can occur in the Callan–Symanzik equation. They have the form

$$\mathcal{N} = g_S^R\mathcal{N}_R^S, \tag{2.21}$$

where the Hermitian matrix g commutes with all matrices e_a of (2.9). A convenient choice for a basis of such counting operators is realized by

$$\mathcal{N}_{0a} = e_{0a}{}_S^R\mathcal{N}_R^S, \tag{2.22}$$

$$\mathcal{N}_{1K} = f_{1K}{}_S^R\mathcal{N}_R^S. \tag{2.23}$$

Here the matrices e_{0a} are the charge matrices of the centre of the algebra of chiral transformations W_a [see equations (2.9), (2.15)–(2.17)]. The operators $\mathcal{N}_{1K}$, with f_{1K} Hermitian and commuting with all e_a complete the basis. Let us note for later use that the $\mathcal{N}_{0a}$ form a basis for the counting operators commuting with all chiral symmetries W_a and annihilating the superpotential (2.7):

$$\mathcal{N}_{0a}U(A) = 0. \tag{2.24}$$

It follows that the chiral field polynomials

$$\mathcal{N}_{1K}U(A) = 3\lambda_{rsu}f_{1K}{}_t^u A^rA^sA^t, \tag{2.25}$$

[with the notation (2.11) for $M = f_{1K}$] are linearly independent, and the invariant symmetric tensors

$$T_{(rst)}^K \equiv \lambda_{rsu}f_{1K}{}_t^u + \text{cycl. perm. } (r, s, t) \tag{2.26}$$

are therefore independent.

In the basis (2.22) and (2.23) the Callan–Symanzik equation now reads

$$\begin{aligned} C\Gamma = [&m\,\partial_m + \beta_g\,\partial_g + \beta_{rst}\,\partial_{\lambda_{rst}} + \bar{\beta}^{rst}\,\partial_{\bar{\lambda}^{rst}} \\ &- \gamma_\phi\mathcal{N}_\phi - \gamma_{0a}\mathcal{N}_{0a} - \gamma_{1K}\mathcal{N}_{1K} - \gamma_k\,\partial_{a_k}]\Gamma \sim 0. \end{aligned} \tag{2.27}$$

*) We neglect contributions from the external fields, antighost fields, etc., cf. Refs. [15, 12].

3. The one-loop problem

The one-loop β-functions and anomalous dimensions of the matter fields are [4]

$$\beta_g^{(1)} = \frac{g^3}{4(4\pi)^2}\left[\sum_R T(R) - 3C_2(G)\right], \tag{3.1}$$

$$\beta_{rst}^{(1)} = \lambda_{rsu}\gamma^{(1)u}{}_t + \text{cycl. perm. } (r, s, t), \tag{3.2}$$

$$\begin{aligned} \gamma^{(1)r}{}_s &\equiv \gamma^{(1)R}{}_S\delta^\rho_\sigma \\ &= \frac{1}{2\pi^2}[\bar{\lambda}^{ruv}\lambda_{suv} - \tfrac{1}{16}g^2C_2(R)\delta^r_s] \\ &\equiv K[\bar{\lambda}^{ruv}\lambda_{suv} - \alpha C_2(R)\delta^r_s]. \end{aligned} \tag{3.3}$$

where the Dynkin index $T(R)$ and the Casimir eigenvalue $C_2(R)$ of the irrep. R are defined by

$$\begin{aligned} &\mathrm{Tr}\,(T^i_R T^j_R) = \delta^{ij}T(R), \\ &(T^i_R T^i_R)^\rho_\sigma = \delta^\rho_\sigma C_2(R), \\ &C_2(G) = C_2(\text{adj.}) = T(\text{adj.}), \end{aligned} \tag{3.4}$$

and are related by the identity

$$d(G)T(R) = d(R)C_2(R), \tag{3.5}$$

$d(G)$ and $d(R)$ being the dimensions of the gauge group and of the irrep. R, respectively.

We shall see in Section 5 that the vanishing of the β-functions to all orders requires that the one-loop anomalous dimensions (3.3) vanish too. This last condition, however, is in general stronger than the vanishing of the β-functions (3.2), since there may be more γ's than β's. Thus, the equations*)

$$\gamma^{(1)R}{}_S(\lambda, g) = 0 \tag{3.6}$$

may overdetermine the solution $\lambda = \lambda(g)$. Let us look for conditions ensuring the compatibility of these equations. They are provided by the following

Lemma 3.1. *The equations* (3.6) *are compatible if and only if the conditions*

$$x_{0a} \equiv \sum_R e_{0a}{}^R_R T(R) = 0 \tag{3.7}$$

hold. The charge matrices e_{0a} here correspond to the Ward identity operators W_{0a} generating the centre of the algebra (2.16) *of chiral symmetries.*

Remark. The quantities x_{0a} are the coefficients of the anomalies of the

*) These, together with the condition $\beta_g^{(1)} = 0$, are the one-loop conditions of Ref. [4].

(classically conserved) axial currents associated to the symmetries W_{0a}. They will be later shown (Appendix A) to be not renormalized. It will also be proved in Section 5, Lemma 5.1, that the conditions (3.6) are necessary for having vanishing β-functions at all orders.

Proof. Let us begin by proving the sufficiency: we show that, under condition (3.7), the equations $\beta^{(1)}_{rst} = 0$ – which are compatible since their number equals the number of unknowns λ_{rst} – imply the vanishing of all $\gamma^{(1)R}{}_S$. Thus, let us assume $\beta^{(1)}_{rst}$ to be zero. Multiplying (3.2) with $\bar{\lambda}^{rst}$ and using the expressions (3.3) for the one-loop anomalous dimensions yields

$$\begin{aligned} 0 &= [\gamma^{(1)r}{}_u + \alpha\delta^r_u C_2(R)]\gamma^{(1)u}{}_r \\ &= \sum_{R,U} d(R)\gamma^{(1)R}{}_U\gamma^{(1)U}{}_R + \alpha\sum_R d(R)C_2(R)\gamma^{(1)R}{}_R \\ &= \sum_{R,U} d(R)\,|\gamma^{(1)R}{}_U|^2 + \alpha d(G)\sum_R T(R)\gamma^{(1)R}{}_R, \end{aligned} \tag{3.8}$$

where use has been made of the Hermiticity of $\gamma^{(1)R}{}_S$ and of the relation (3.5). On the other hand, let us insert in $\beta^{(1)}_{rst}$ (3.2) the expression

$$\gamma^{(1)R}{}_S = \gamma^{(1)}_{0a} e_{0a}{}^R{}_S + \gamma^{(1)}_{1K} f_{1K}{}^R{}_S \tag{3.9}$$

deduced by comparing the two forms (2.18) and (2.27) of the Callan–Symanzik equation. The contributions of the $\gamma^{(1)}_{0a}$ drop out because of the chiral invariance conditions (2.10) for the Yukawa coupling constants and we are left with

$$0 = T^K_{rst}\gamma^{(1)}_{1K}, \tag{3.10}$$

where the tensors T^K, given by (2.26), are independent. Thus,

$$\begin{aligned} &\gamma^{(1)}_{1K} = 0, \\ &\gamma^{(1)R}{}_S = \gamma^{(1)}_{0a} e_{0a}{}^R{}_S, \end{aligned} \tag{3.11}$$

and we get

$$\sum_R T(R)\gamma^{(1)R}{}_R = \sum_a \gamma^{(1)}_{0a} \sum_R e_{0a}{}^R{}_R T(R). \tag{3.12}$$

Here the right-hand side vanishes due to (3.7), hence equation (3.8) reduces to

$$\sum_{R,U} d(R)\,|\gamma^{(1)R}{}_U|^2 = 0, \tag{3.13}$$

which means the vanishing of all $\gamma^{(1)R}{}_S$ and ends the proof of the sufficiency of conditions (3.7).

In order to show their necessity, let us multiply the chiral invariance conditions (2.10) by $\bar{\lambda}^{rst}$. Using the expression (3.3) for $\gamma^{(1)}$, we get in the same way as we obtained equation (3.8),

$$\sum_{R,U} d(R)\gamma^{(1)R}{}_U e_a{}^U{}_R + \alpha d(G)\sum_R T(R)e_a{}^R{}_R = 0. \tag{3.14}$$

The compatibility of equations (3.6) then implies

$$\sum_R T(R)e_{aR}^{\ R} = 0. \tag{3.15}$$

For the special case $e_a = e_{0a}$ these are conditions (3.7).

From (3.14) follows also the

Corollary 3.2. *The vanishing of the one-loop anomalous dimensions of the matter fields implies the conditions* (3.7) *of Lemma* 3.1.

4. The supercurrent anomaly

The supercurrent [18, 15, 19] is a *BRS* invariant supermultiplet containing the conserved spinor current and energy momentum tensor associated with supersymmetry and translation invariance, together with the anomalous axial current associated with R-invariance (2.14). The anomalies of the R-axial current, of the spinor current 'trace' and of the energy-momentum tensor trace belong to a chiral supermultiplet whose superfield representation is denoted by S [15, 19]. This chiral insertion*) S has dimension 3, R-weight -2 [see (2.8)] and is invariant under *BRS,* as well as under the chiral transformations (2.9). It can be expanded as [15, 19, 12]

$$\begin{aligned} S &= \beta_g L_g + \beta_{rst} L^{rst} - \gamma_\Phi L_\Phi - \gamma_k L_k - \gamma_S^R L_R^S \\ &= \beta_g L_g + \beta_{rst} L^{rst} - \gamma_\Phi L_\Phi - \gamma_k L_k - \gamma_{0a} L_{0a} - \gamma_{1K} L_{1K}. \end{aligned} \tag{4.1}$$

The coefficients β and γ are those of the Callan–Symanzik equation, either in the form (2.18) or in the form (2.27). Each set of insertions L appearing in the two expressions above forms a basis for the chiral insertions which have the dimension, R-weight and invariances of S. They are defined through the quantum action principle [24, 15] by

$$\nabla_i \Gamma \sim \int dS L_i + \int d\bar{S} \bar{L}_i, \tag{4.2}$$

for

$$\nabla_i = \partial_g,\ \partial_{\lambda_{rst}},\ \mathcal{N}_\Phi,\ \partial_{a_k},\ \mathcal{N}_R^S,\ \mathcal{N}_{0a},\ \mathcal{N}_{1K}.$$

In particular,

$$\begin{aligned} &L_R^S = A^S \delta_{A^R}, \\ &L_{0a} = e_{0a}{}_S^R L_R^S, \qquad L_{1K} = f_{1K}{}_S^R L_R^S, \end{aligned} \tag{4.3}$$

the Hermitian matrices e_{0a} and f_{1K} being defined in equations (2.22) and (2.23).

*) An 'insertion' I is the generating functional of the (one-particle-irreducible) Green functions with the composite field operator I inserted in.

One can show [12] that

$$L_\Phi = \overline{DD}\mathscr{L}_\Phi, \tag{4.4}$$

where $\mathscr{L}_\Phi$ is BRS invariant and real. It has been proved [12] that any BRS invariant chiral insertion T of dimension 3 and R-weight -2 admits the representation

$$T \sim \overline{DD}(r_T K^0 + J_T^{inv}) + T^c, \tag{4.5}$$

where K^0 is the 'supersymmetric Chern–Simons insertion' defined in Appendix A and related to the finite insertion Trc_+^3 through the quantum extension of the classical descent equations (A.2). The coefficient r_T is gauge independent and uniquely defined. J_T^{inv} is BRS invariant and T^c, BRS invariant as well, is a 'genuinely chiral' insertion, i.e., it cannot be written as a double derivative $\overline{DD}(\ldots)$. The basis of genuinely chiral insertions with the appropriate dimension and R-weight is a quantum extension of the independent field polynomials constituting the superpotential U (2.7). One can choose the basis*)

$$\{L_{1K}, U_{0L}\} \tag{4.6}$$

with L_{1K} given by (4.3) – the L_{1K} are independent, see the remark following equation (2.25) – and with some insertions U_{0L} for completing the basis if necessary.

Let us use the representation (4.5) for the supercurrent anomaly S and for each of the L_i appearing in both right-hand sides of (4.1):

$$\begin{aligned}
&S \sim \overline{DD}[rK^0 + J^{inv}],\\
&L_g \sim \overline{DD}\left[\left(\frac{1}{128g^3} + r_g\right)K^0 + J_g^{inv}\right] + L_g^c,\\
&L^{rst} \sim \overline{DD}[r^{rst}K^0 + J^{rst,inv}] + L^{rst,c},\\
&L_k \sim \overline{DD}[r_k K^0 + J_k^{inv}] + L_k^c,\\
&L_S^R \sim \overline{DD}[r_S^R K^0 + J_S^{R\,inv}] + L_S^{R,c},\\
&L_{0a} \sim \overline{DD}[r_{0a}K^0 + J_{0a}^{inv}].
\end{aligned} \tag{4.7}$$

We have not written the corresponding representations of L_Φ and L_{1K} which are trivial due to (4.4) and the choice of (4.6) for the basis of genuinely chiral insertions. Note the absence of genuinely chiral terms for S and L_{0a}. This is due to the R-invariance (2.14) and to the identity [15, 19]

$$i\int dSS - i\int d\bar{S}\bar{S} \sim W_R\Gamma, \tag{4.8}$$

*) There is in fact also a term involving the ghost c_+: $c_+\delta_{c_+}\Gamma$ which, however, does not contribute to the Green function without external ghost lines and which is irrelevant for the present discussion.

and, for L_{0a}, to the Ward identities (2.15) which read

$$\int dSL_{0a} - \int d\bar{S}\bar{L}_{0a} \sim 0. \tag{4.9}$$

All coefficients r, r_g, etc., in (4.7) are of order $\hbar$ at least – we have explicitly displayed the zeroth order in L_g.

It is shown in Appendix A that r – the anomaly of the R-axial current – and r_{0a} – the anomalies of the axial currents associated to the chiral symmetries W_{0a}, i.e., with the centre of the algebra of all chiral symmetries – are not renormalized: they are exactly given by their one-loop contributions. The coefficients r and r_{0a} turn out [12] to be proportional respectively to the one-loop gauge β-function (3.1) and to the expression (3.7):

$$r = \frac{1}{128g^3}\beta_g^{(1)}, \tag{4.10}$$

$$r_{0a} = -\frac{1}{256(4\pi)^2}x_{0a}. \tag{4.11}$$

We also show in Appendix A that the coefficients r_k in the representation (4.7) for L_k (although renormalized contrary to the claim in Ref. [12]) are of order $\hbar^2$ at least and governed by the non-renormalized coefficients r_{0a}:

$$r_k = \sum_a t_{ka}r_{0a}, \tag{4.12}$$

where t_{ka} is of order $\hbar$ at least.

Let us come back to equations (4.7), insert them in each of the two equations (4.1) and identify the coefficients of the K^0 term. We thus get two relations:

$$r = \beta_g\left(\frac{1}{128g^3} + r_g\right) + \beta_{rst}r^{rst} - \gamma_k r_k - \gamma_S^R r_R^S, \tag{4.13}$$

$$r = \beta_g\left(\frac{1}{128g^3} + r_g\right) + \beta_{rst}r^{rst} - \gamma_k r_k - \gamma_{0a}r_{0a}. \tag{4.14}$$

The first of these equations will be useful for proving Lemma 5.1 in Section 5, whereas the second one will be crucial for proving the vanishing of the β-functions to all orders (Theorem 5.2, Section 5), due to the non-renormalization properties of r and r_{0a}.

An identity similar to equation (4.13) was proposed in Ref. [25] where the terms with coefficients r^{rst} and r_k are absent. Moreover r_g and r_R^S are claimed to be strictly one-loop. We remark that our less spectacular result takes rigorously into account all the possible renormalization effects.

5. The criteria for vanishing β-functions

Before stating the main theorems (Theorems 5.2 and 5.3), let us prove a result which yields necessary conditions for the vanishing of the β-functions to all orders:

Lemma 5.1. *Let us assume that the gauge β-function vanishes up to the two-loop order and the Yukawa β-functions at the one-loop order, i.e.,*

$$\beta_g = O(\hbar^3), \qquad \beta_{rst} = O(\hbar^2). \tag{5.1}$$

Then the following three conditions are necessarily fulfilled:

1) *The axial current of R-invariance is anomaly free:*

$$r = \frac{1}{128 g^3} \beta_g^{(1)} = 0. \tag{5.2}$$

2) *The one-loop anomalous dimensions* (3.3) *of the matter fields vanish:*

$$\gamma^{(1)R}_{\ S} = 0. \tag{5.3}$$

3) *The axial currents of the symmetries W_{0a} belonging to the centre of the algebra of chiral symmetries* (2.16) *are anomaly free:*

$$r_{0a} = 0. \tag{5.4}$$

Remark. The anomaly coefficients and the one-loop anomalous dimensions above are given in (4.10), (4.11) and (3.3). The third condition ensures the compatibility of the system of equations (5.3) – the second condition – due to Lemma 3.1 and relation (4.11).

Proof. The first condition is obvious and the third one follows from the second according to Corollary 3.2. Let us show the necessity of the second condition. In view of the last equality (3.8) used in the proof of Lemma 3.1, it is enough to check that

$$\sum_R T(R) \gamma^{(1)R}_{\ R} = 0. \tag{5.5}$$

But the latter follows from the identity (4.13) and the hypotheses (5.1), if we recall that in (4.13) the coefficients r_g and r^{rst} are of order $\hbar$ and r_k of order $\hbar^2$ [see (4.12)], and if we note that

$$r_R^S = -\frac{1}{256(4\pi)^2} \delta_R^S T(R) + O(\hbar^2), \tag{5.6}$$

as it results from a one-loop computation.

The present Lemma shows that the Yukawa and gauge coupling constants are not independent. The former must be functions of the latter,

$$\lambda_{rst} = \lambda_{rst}(g), \tag{5.7}$$

solutions of equation (5.3). These functions also solve the equations

$$\beta^{(1)}_{rst} = 0 \tag{5.8}$$

in view of (3.2).

So far so good for the one loop approximation, where the functions (5.7) are proportional to g [see (3.3)]. We now have to extend such a relationship to all orders, the functions (5.7) being formal power series in g. It is well known [13, 14] that these functions must then be solutions of the 'reduction equations'

$$\beta_{rst} = \beta_g \frac{d\lambda_{rst}}{dg}, \tag{5.9}$$

in order for the resulting theory, depending on the single coupling constant g, to be consistent. We note that the equations (5.8) are just the reduction equations at the one-loop order. But we also know from Lemma 5.1 that the stronger condition (5.3) of vanishing anomalous dimensions must in fact hold at this order. Let us thus state and prove the following.

Theorem 5.2. *The three conditions hereafter are necessary and sufficient for the β-functions of the gauge and Yukawa couplings to vanish to all orders of perturbation theory:*

(1) $\beta^{(1)}_g = 0$,

(2) $\gamma^{(1)R}{}_S = 0$,

(3) *The reduction equations* (5.9) *admit a formal power series solution which, in its lowest order, also has to be a solution of the condition* (2).

Remark. These conditions are in fact those of Ref. [4] [conditions (1) and (2)], but supplemented by a consistency requirement [condition (3)].

Proof. The necessity follows from Lemma 5.1 and from the discussion above. Let us show the sufficiency. The starting point is the identity (4.14). From condition (1) it follows that the R-current axial anomaly r vanishes [see (4.10)]. Condition (2) implies through Lemma 3.1 and its Corollary 3.2 that the quantities x_{0a} (3.7), hence the axial anomalies r_{0a} (4.11), vanish. This, in turn, ensures the vanishing of the coefficients r_k (4.12). At this stage the identity (4.14) becomes homogeneous in the β-functions. Condition (3) allows us to substitute for β_{rst} the right-hand side of the reduction equations (5.9), and we get

$$0 = \beta_g \left(\frac{1}{128 g^3} + r_g + \frac{d\lambda_{rst}}{dg} r^{rst} \right). \tag{5.10}$$

The term in brackets being invertible in the perturbative sense, it results from this equation and from the reduction equations (5.9) that

$$\beta_g = 0, \qquad \beta_{rst} = 0 \tag{5.11}$$

at all orders. This concludes the proof.

The first two conditions of Theorem 5.2 are simple one-loop criteria. On the other hand, the last condition demands that the reduction equations be solvable at all orders. It is shown in Appendix B that a solution exists at all orders (and is unique) if the lowest-order solution, i.e., the solution of (5.8), is isolated and non-degenerate. We can thus state the following criterion:

Theorem 5.3 (criterion for vanishing β-functions). *Let us assume that a* SYM *gauge theory with simple gauge group obeys the following four conditions:*

(1) *It is free of gauge anomalies* [*equation* (2.13)];

(2) *The one-loop gauge β-function* (3.1) *vanishes,*

$$\beta_g^{(1)} = 0. \tag{5.12}$$

(3) *There exist solutions of the form*

$$\lambda_{rst} = \rho_{rst}\, g, \qquad \rho_{rst} \text{ complex number}, \tag{5.13}$$

to the condition of vanishing one-loop matter field anomalous dimensions (3.3),

$$\gamma^{(1)R}{}_S = 0. \tag{5.14}$$

(4) *The solutions* (5.13) *of* (5.14) *are isolated and non-degenerate when considered as solutions to the condition of vanishing one-loop Yukawa β-functions,*

$$\beta_{rst}^{(1)} = 0. \tag{5.15}$$

Then each of the solutions (5.13) *can be uniquely extended to a formal power series of g, giving a theory which depends on a single coupling constant – the gauge coupling g – with a β-function vanishing to all orders.*

The last theorem provides us with a simple criterion for vanishing β-functions which involves only standard one-loop computations. It can, in principle, be checked explicitly for every model at hand. However, the last condition can cause problems: the solutions of (5.14) are generally far from being isolated and non-degenerate. But it may happen that an extension of the given group of chiral symmetries W_a (2.15) yields enough supplementary constraints on the Yukawa coupling constants in order to lift the degeneracy. The use of a special renormalization scheme, based on the non-renormalization of chiral vertices, may also help to reach this goal. The example treated in the next section will show clearly how all this works in practice.

6. An $SU(6)$ model with vanishing β-functions

We consider here one of the 'two-loop finite' models of Ref. [5]. We shall show by checking the criterion given in Theorem 5.3 that it can be made 'all-loop finite' in the sense of vanishing β-functions.

This model has $SU(6)$ gauge invariance and its chiral matter fields belong to a complex representation of $SU(6)$, as one can read off from the Table. The present representation is free of the gauge anomaly (2.13) and makes the one-loop gauge β-function vanish: conditions (1) and (2) of Theorem 5.3 are fulfilled. The most general gauge invariant superpotential (2.7) is

Fields	ψ_i^α $(i = 1, \ldots, 8)$	ϕ_α^a $(a = 1, \ldots, 16)$	Λ_M	H^A
Representations	6	$\bar{6}$	$\overline{15}$	21
C_2	35/12	35/12	14/3	20/3
T	1/2	1/2	2	4

Chiral matter field representations, Casimir eigenvalues C_2 and Dynkin indices T [according to the definitions (3.4)]. The letters $\alpha = 1, \ldots, 6$; $M = 1, \ldots, 15$; $A = 1, \ldots, 21$ are representation indices. i and a are 'flavour' indices.

$$\begin{aligned}
&U = U^1 + U^2,\\
&U^1 = \lambda_{(ab)} t_A^{(\alpha\beta)} \phi_\alpha^a \phi_\beta^b H^A, \\
&U^2 = \lambda^{[ij]} u_{[\alpha\beta]}^M \psi_i^\alpha \psi_j^\beta \Lambda_M + \lambda_3 v^{(MNP)} \Lambda_M \Lambda_N \Lambda_P,
\end{aligned} \tag{6.1}$$

where t, u and v are $SU(6)$ invariant tensors normalized by

$$\begin{aligned}
&t_A^{\alpha\beta} \bar{t}_{\alpha\beta}^B = 2\delta_A^B,\\
&u_{\alpha\beta}^M \bar{u}_N^{\alpha\beta} = 2\delta_N^M, \\
&v^{MNP} \bar{v}_{MNQ} = \tfrac{1}{18}\delta_Q^P.
\end{aligned} \tag{6.2}$$

The superpotential is invariant under the chiral transformations [see (2.9)]

$$\begin{aligned}
&\delta_1 \phi^a = i\phi^a, \qquad \delta_1 H = -2iH,\\
&\delta_1 \psi_i = 0, \qquad \delta_1 \Lambda = 0,
\end{aligned} \tag{6.3}$$

with vanishing anomaly $r_{01} = 0$ [see (4.11) and (3.7)]. Hence from Lemma 3.1, the one-loop matter field anomalous dimensions can consistently be set to zero, thus the third condition of Theorem 5.3 is satisfied. The equations are

$$\begin{aligned}
&\gamma_{\phi\ b}^{(1)a} = 28x(L_b^a - \alpha\delta_b^a) = 0,\\
&\gamma_H^{(1)} = 4x(L_a^a - 16\alpha) = 0,\\
&\gamma_{\psi\ i}^{(1)j} = 5x(4K_i^j - 7\alpha\delta_i^j) = 0,\\
&\gamma_\Lambda^{(1)} = 2x(2K_i^i + 9\,|\lambda_3|^2 - 28\alpha) = 0,
\end{aligned} \tag{6.4}$$

where

$$L_b^a = \bar{\lambda}^{ac}\lambda_{cb}, \qquad K_i^j = \bar{\lambda}_{ik}\lambda^{kj}, \tag{6.5}$$

α is proportional to the square of the gauge coupling constant g and x is a numerical constant. The solutions are

$$\lambda_{(ab)} = \sqrt{\alpha}\, l_{(ab)}, \qquad \lambda^{[ij]} = \sqrt{\frac{7\alpha}{4}}\, k^{[ij]}, \qquad \lambda_3 = 0, \tag{6.6}$$

where (l_{ab}, k^{ij}) is any solution of

$$L^a_b = \delta^a_b, \qquad K^j_i = \delta^j_i. \tag{6.7}$$

We see that the last condition of Theorem 5.3 is *not* fulfilled, and this for two reasons. First, the solutions are not isolated: they form a continuous family parametrized by the complex numbers l_{ab}, k^{ij} solutions of (6.7). Second, the value $\lambda_3 = 0$ is a double, hence degenerate, root of the equations [see (3.2)]

$$\begin{aligned} &\beta^{(1)}_{\lambda_3} = 3\lambda_3\gamma^{(1)}_\Lambda = 0, \\ &\beta^{(1)ij} = \lambda^{ik}\gamma^{(1)j}_{\psi\ k} - \lambda^{jk}\gamma^{(1)i}_{\psi\ k} + \lambda^{ij}\gamma^{(1)}_\Lambda = 0. \end{aligned} \tag{6.8}$$

But there is a way out. Let us pick out an element of the family (6.6) by choosing an arbitrary solution (l_{ab}, k^{ij}) of (6.7). Then the superpotential

$$U_{(l,k)} = U^1_{(l)} + U^2_{(k)}, \tag{6.9}$$

obtained by replacing in (6.1) λ_{ab}, λ^{ij} and λ_3 by l_{ab}, k^{ij} and 0, is invariant under the three chiral symmetries.

$$\delta_2\psi_i = i\psi_i, \qquad \delta_2\Lambda = -2i\Lambda, \qquad (\phi,\ H \text{ invariant}) \tag{6.10}$$

$$\delta_{(l)}\phi^a = ie_{(l)}{}^a_b\phi^b, \qquad (H,\ \psi,\ \Lambda \text{ invariant}) \tag{6.11}$$

$$\delta_{(k)}\psi_i = ie_{(k)}{}^j_i\psi_j, \qquad (\phi,\ H,\ \Lambda \text{ invariant}) \tag{6.12}$$

provided the matrices $e_{(l)}$ and $e_{(k)}$ are constrained by [see (2.10)]

$$\begin{aligned} &l_{ac}e_{(l)}{}^c_b + l_{bc}e_{(l)}{}^c_a = 0, \\ &k^{il}e_{(k)}{}^j_l - k^{jl}e_{(k)}{}^i_l = 0. \end{aligned} \tag{6.13}$$

Conversely, keeping the choice of (l, k) as a solution of (6.7), we find that these chiral symmetries fix the superpotential up to two complex coupling constants:

$$U = \lambda_1 U^1_{(l)} + \lambda_2 U^2_{(k)}, \tag{6.14}$$

i.e.,

$$\lambda_{ab} = \lambda_1 l_{ab}, \qquad \lambda^{ij} = \lambda_2 k^{ij}, \qquad \lambda_3 = 0. \tag{6.15}$$

The system of equations for vanishing anomalous dimensions is still compatible*) and one finds

$$\begin{aligned} &\lambda_I = \rho_I e^{i\varphi_I}, \qquad \varphi_I \text{ arbitrary}, \quad (I = 1, 2), \\ &\rho_1^2 = \alpha, \qquad \rho_2^2 = \tfrac{7}{4}\alpha. \end{aligned} \tag{6.16}$$

Unluckily, this is again a continuous family of solutions parametrized by the phases left undetermined in (6.16): the last condition of Theorem 5.3 is still not satisfied. One can, however, fix by hand these phases to be zero if the

*) One can check that the anomalies of the chiral symmetries (6.10)–(6.12) are zero.

corresponding β-functions identically vanish:

$$\beta_{\varphi_I} = \mathrm{Im}\left(\frac{\beta_{\lambda_I}}{\lambda_I}\right) = 0. \tag{6.17}$$

It is easy to see that (6.17) is achieved if the renormalization scheme used to define the theory – prior to reduction – preserves for all orders the one-loop relations (3.2) between the matter β- and γ-functions**). In the present case – with chiral symmetries (6.10)–(6.12) and superpotential (6.14) – these relations read

$$\frac{\beta_{\lambda_1}}{\lambda_1} = 2\gamma_\Phi + \gamma_H, \qquad \frac{\beta_{\lambda_2}}{\lambda_2} = 2\gamma_\Psi + \gamma_\Lambda. \tag{6.18}$$

We used the fact that the chiral symmetries imply

$$\gamma_{\Phi}{}^a_b = \gamma_\Phi \delta^a_b, \qquad \gamma_{\Psi}{}^j_i = \gamma_\Psi \delta^j_i, \tag{6.19}$$

and substituted this in (6.4). Equations (6.17) are now seen to hold due to the reality of the expressions (6.18).

After having set to zero by hand the phases φ_I in (6.16), we get a unique solution of the one-loop problem: the last condition of Theorem 5.3 is now satisfied and its conclusion then follows.

7. Conclusions and outlook

i) The criterion given in Theorem 5.3 for all-order 'finiteness', i.e., for vanishing β-functions, is specially simple since it only involves standard one-loop quantities. Its conditions are sufficient. They are also necessary, condition (4) excepted. This last condition – existence of isolated and non-degenerate solutions to the one-loop problem – ensures the existence of power series solutions to all orders. If condition (4) is not met, this is not guaranteed but still possible, and such solutions are then to be characterized by additional requirements. Section 6 actually shows that such a solution exists for the model considered there, although condition (4) is violated when one starts with the most general interaction: this is the solution we got after reducing the dimension of the coupling constant space through the imposition of additional chiral invariances and the use of a particular renormalization scheme.

In general, one can expect the procedure for getting 'finite' theories from theories obeying the first three conditions of Theorem 5.3 to have two steps. Reduce first the number of independent Yukawa coupling constants by means of new symmetries and/or the use of a particular renormalization scheme, until the fourth condition is met. Then solve iteratively the reduction equations (5.9),

**) This scheme consists of replacing the normalization conditions on the vertex functions defining the Yukawa coupling constants, by the requirement of the absence of counterterms cubic in the chiral fields. This is consistent since the corresponding vertex graphs are ultra-violet finite [26, 15].

starting with a lowest-order solution for which the matter field anomalous dimensions vanish.

The models with complex representations obeying the first three conditions are listed in Ref. [5]. Those with real representations may be found in Ref. [8]. That all or part of them lead to 'finite' theories is under investigation.

ii) All anomalies vanish. Indeed the β-functions are set to zero: there is no conformal anomaly. Moreover all chiral anomalies vanish too, according to Lemma 5.1. One may ask whether at least a subclass of these theories are completely finite, i.e., whether the anomalous dimensions, which are in general gauge dependent, may all vanish as well. For the gauge field anomalous dimension, this may be the case in a suitable gauge, e.g., in the background gauge [27] where the gauge field anomalous dimension and the gauge β-function are not independent. The question is anyway more relevant for the matter field anomalous dimensions due to their relation with the Yukawa β-functions. For instance, in the $N = 4$ SYM theory written in terms of $N = 1$ superfields, which fulfils our criterion [12], there is one independent anomalous dimension and one β-function in the 'matter field' sector, thus both have to vanish. But in a generic case with more anomalous dimensions than β-functions – such cases are in fact excluded in Refs. [8–10] – we do not see any way for these anomalous dimensions to vanish altogether, although they have to do so in the one-loop approximation. Let us, however, mention Ref. [7], which suggests the possibility of a renormalization scheme where this vanishing holds at all orders.

iii) We have introduced masses in order to avoid the complications of the off-shell infra-red problem [15, 28]. These masses have been taken to be supersymmetric so that the finiteness of chiral insertions, used in the proof of the non-renormalization theorem for axial anomalies, holds. But they break softly the *BRS* invariance. There exists [15, 28], however, an infra-red cut-off procedure which preserves *BRS* invariance but softly breaks supersymmetry. The cut-off is shown to be a gauge parameter, hence unphysical. One has to extend our results to these truly gauge invariant theories. An argument will be presented elsewhere [29].

iv) In the present work we have restricted ourselves to the case of simple gauge groups. For semi-simple groups, the non-renormalization theorem for axial anomalies certainly holds (see Ref. [30] for usual gauge theories). In this case there is more than one gauge coupling constant and one will presumably have to reduce them too, so that all Yukawa and gauge coupling constants will be functions of a single one. The case of a gauge with $U(1)$ factors is excluded since the corresponding gauge β-functions can never be set to zero unless the $U(1)$ coupling constants vanish.

Appendix A. The supersymmetric non-renormalization theorem for the axial anomalies

We present here a corrected proof of the non-renormalization theorem of Ref. [12]. The hypotheses are now a little stronger but this is of no concern in view of the applications we discuss at the end of this Appendix and which are needed in the text.

Let us first introduce the 'supersymmetric Chern–Simons insertions' [12] K^q, $q = 0, \ldots, 3$. Their classical approximations are the following superfield 'polynomials'

$$\begin{aligned} k^0 &= \mathrm{Tr}\,(\varphi^\alpha \overline{DD}\varphi_\alpha), \\ k^{1\dot\alpha} &= -\mathrm{Tr}\,(D^\alpha c_+ \bar D^{\dot\alpha}\varphi_\alpha + \bar D^{\dot\alpha} D^\alpha c_+ \varphi_\alpha), \\ k^2_\alpha &= \mathrm{Tr}\,(c_+ D_\alpha c_+), \\ k^3 &= \tfrac{1}{3}\,\mathrm{Tr}\, c_+^3, \end{aligned} \tag{A.1}$$

where φ_α is the chiral superconnection (2.6). The K^q are solutions of the quantum extension of the classical descent equations*)

$$\begin{aligned} sk^0 &= \bar D_{\dot\alpha} k^{1\dot\alpha}, \\ sk^{1\dot\alpha} &= (\bar D^{\dot\alpha} D^\alpha + 2D^\alpha \bar D^{\dot\alpha})k^2_\alpha, \\ sk^2_\alpha &= D_\alpha k^3, \\ sk^3 &= 0, \end{aligned} \tag{A.2}$$

where s is the *BRS* operator (2.2). K^3 is uniquely defined as the insertion of k^3 which is finite due to the non-renormalization of chiral vertices [26, 15]. Then one can show that K^0 is uniquely defined up to a *BRS* invariant insertion and a total derivative $\bar D(\ldots)$.

We can now state and prove the general theorem:

Theorem A.1. *Let T be a BRS invariant chiral superfield insertion of dimension 3 and R-weight**) −2. Moreover let its chiral superspace integral fulfil the Callan–Symanzik equation (2.27) without anomalous dimension, i.e.*

$$C \int dST \sim 0. \tag{A.3}$$

Then:

1) *T admits the representation*

$$T \sim \overline{DD}(rK^0 + J^{inv}) + T^c, \tag{A.4}$$

where J^{inv} and T^c are BRS invariant, T^c is genuinely chiral [i.e., $T^c \neq \overline{DD}(\ldots)$], and the coefficient r of the Chern–Simons insertion K^0 is gauge independent and uniquely defined.

2) *The coefficient r is not renormalized, i.e., only one-loop graphs contribute to it.*

Proof. The first conclusion does not depend on the hypothesis (A.3). It was

*) See Ref. [12] for more details.
**) See equation (2.8).

proved in Ref. [12] (Proposition 3). In order to prove the second conclusion, let us begin by showing *) that the condition (A.3) of the theorem implies

$$CT \sim \overline{DD}X^{inv}, \tag{A.5}$$

where X^{inv} is BRS invariant. The proof of (A.5) at all orders being iterative, it suffices to discuss the classical problem, i.e., to show that

$$\int dSU = 0 \Rightarrow U = \overline{DD}X^{inv}, \tag{A.6}$$

where U and X^{inv} are classical insertions. U admits a representation analogous to (A.4)

$$U = \overline{DD}(xk^0 + X^{inv}), \tag{A.7}$$

without a genuinely chiral term since its chiral integral vanishes by assumption. Then

$$\int dV(xk^0 + X^{inv}) = 0, \tag{A.8}$$

and the integrand must be a total superspace derivative:

$$xk^0 + X^{inv} = D^\alpha A_\alpha + \bar{D}_{\dot\alpha}B^{\dot\alpha}. \tag{A.9}$$

Applying the BRS operator to this equation and using the descent equations (A.2) yields

$$x\bar{D}_{\dot\alpha}k^{1\dot\alpha} = D^\alpha sA_\alpha + \bar{D}_{\dot\alpha}sB^{\dot\alpha}. \tag{A.10}$$

A detailed superspace analysis then shows the existence of classical insertions G^1 and $\hat{G}^1$ such that

$$\begin{aligned} sA_\alpha &= -\overline{DD}G^1_\alpha + (D\bar{D} + 2\bar{D}D)_{\alpha\dot\alpha}\hat{G}^{1\dot\alpha}, \\ xk^{1\dot\alpha} - sB^{\dot\alpha} &= -DD\hat{G}^{1\dot\alpha} + (\bar{D}D + 2D\bar{D})^{\dot\alpha\alpha}G^1_\alpha. \end{aligned} \tag{A.11}$$

Applying s again gives the equations

$$\begin{aligned} (D\bar{D} + 2\bar{D}D)_{\alpha\dot\alpha}s\hat{G}^{1\dot\alpha} &= \overline{DD}sG^{1\alpha}, \\ (\bar{D}D + 2D\bar{D})^{\dot\alpha\alpha}(xk^2_\alpha - sG^1_\alpha) &= -DDs\hat{G}^{1\dot\alpha}, \end{aligned} \tag{A.12}$$

which can be solved by

$$s\hat{G}^{1\dot\alpha} = \bar{D}^{\dot\alpha}I^2, \qquad sG^1_\alpha = -D_\alpha I^2 + xk^2_\alpha, \tag{A.13}$$

where I^2 has dimension 0.

A last application of s and of the descent equations yields

$$\bar{D}^{\dot\alpha}sI^2 = 0, \qquad -D_\alpha sI^2 + xD_\alpha k^3 = 0. \tag{A.14}$$

*) It is just here that the present proof differs from the one given in Ref. [12], the condition (A.3) here being stronger than the corresponding one there.

The first equation means that sI^2 is chiral. Being of dimension 0, it must be proportional to k^3 (A.1) which, however, is not an s-variation. Therefore, $sI^2 = 0$ and the second equation (A.14) implies the vanishing of x. Equation (A.7) then yields the desired result (A.6).

We now insert the representation (A.4) of T in the relation (A.5) we have just proved and thus get

$$\overline{DD}[C(rK^0) + CJ^{inv} - X^{inv}] \sim 0, \qquad CT^c \sim 0, \tag{A.15}$$

the genuinely chiral part CT^c dropping out. The term in brackets must be a total $\bar{D}$ derivative:

$$C(rK^0) + CJ^{inv} - X^{inv} \sim \bar{D}_{\dot{\alpha}} L^{\dot{\alpha}}. \tag{A.16}$$

A sequence of BRS variations and of integrations with respect to superspace differential operators, combined with the quantum descent equations, finally yields [12]

$$C(rK^3) \sim 0. \tag{A.17}$$

Then, since K^3 is finite, $CK^3 \sim 0$, and

$$O = Cr = (\beta_g\, \partial_g + \beta_{rst}\, \partial_{\lambda_{rst}} + \bar{\beta}^{rst}\, \partial_{\bar{\lambda}^{rst}})r. \tag{A.18}$$

The second equality results from r being dimensionless and gauge independent. The non-renormalization of r then follows [12] from equation (A.18).

Corollary A.2. *The coefficients r and r_{0a} of S and L_{0a} respectively in equations* (4.7) *are not renormalized. Their values are given in the text* [*equations* (4.10) *and* (4.11)].

Proof. R-invariance implies [see Eq. (4.8)]

$$\int dSS - \int d\bar{S}\bar{S} \sim 0. \tag{A.19}$$

On the other hand, the equation

$$\int dSS + \int d\bar{S}\bar{S} \sim (C - m\,\partial_m)\Gamma \tag{A.20}$$

follows [15, 19] from the relation of the Callan–Symanzik equation with the broken dilatation invariance. Hence the hypothesis (A.3) holds for S, as one can see by applying the Callan–Symanzik operator C to both equations (A.19) and (A.20), and r is not renormalized. For r_{0a} we note that [see equations (4.3) and (2.15)]

$$\begin{aligned} &L_{0a} = D_{0a}\Gamma, \qquad D_{0a} = e_{0a}{}^R_S A^S \delta_{A^R}, \\ &W_{0a} = \int dS D_{0a} - \int d\bar{S}\bar{D}_{0a}. \end{aligned} \tag{A.21}$$

Hence the differential operator D_{0a} commutes with the Callan–Symanzik operator C since W_{0a} is a symmetry. It follows that the hypothesis (A.3) holds for L_{0a}; r_{0a} is thus not renormalized.

Remarks. The coefficient r in the representation (4.7) for S is the anomaly of the axial current associated with R-invariance [12]. The coefficients r_{0a} are the anomalies of the axial currents associated with the invariances W_{0a}: the representation (4.7) for L_{0a} is nothing other than the anomalous Ward identity for the associated current which is a component of the superfield J^{inv}_{0a} (the left-hand side L_{0a} is a contact term) [31]. We have formulated the Corollary above for the anomalies r_{0a} corresponding to the centre of the algebra of chiral symmetries W_a. This is what we need in the text; in particular just these r_{0a} participate in equation (4.14) and have to vanish. This Corollary is the supersymmetric extension of the well-known Alder–Bardeen theorem for the $U(1)$ anomalies [32, 30].

The coefficients r_k in the representation (4.7) for L_k are renormalized, but they are governed by the anomalies r_{0a}. Let us recall the definition (4.2) of L_k:

$$\partial_{a_k}\Gamma \sim \int dSL_k + \int d\bar{S}\bar{L}_k. \tag{A.22}$$

a_k is a gauge parameter [15, 21], i.e.,

$$\partial_{a_k}\Gamma \sim \mathscr{B}\Delta_k, \tag{A.23}$$

where $\mathscr{B}$ denotes the quantum extension of the BRS operator s [15, 12] and Δ_k is an insertion of dimension 4 and ghost number -1. The most general form for Δ is

$$\Delta_k = \int dV\hat{\mathscr{L}}_k + t_{kS}^{\ R} \int dSY_RA^S + \text{conj.}, \tag{A.24}$$

where Y_R is the chiral external field coupled with the BRS variation of A^R [12, 15]. The chiral invariances W_a imply that the matrices t_k can be expanded in the matrices e_{0a} and f_{1K}, defined by equations (2.22) and (2.23):

$$t_{kS}^{\ R} = t_{ka}e_{0aS}^{\ \ R} + t'_{kK}f_{1KS}^{\ \ R}. \tag{A.25}$$

Moreover [12, 15]

$$\mathscr{B}(Y_RA^S) = L_R^S \tag{A.26}$$

hence we can choose, for L_k, in agreement with the definition (A.22),

$$L_k = \overline{DD}\mathscr{L}_k^{inv} + t_{ka}L_{0a} + t'_{kK}L_{1K}, \tag{A.27}$$

where $\mathscr{L}_k^{inv} = \mathscr{B}\mathscr{L}_k$. Inserting here the representation (4.7) of L_{0a} and comparing the result with the representation (4.7) of L_k, keeping in mind that L_{1K} belongs to the basis of genuinely chiral insertions, we get the result we looked for:

$$r_k = \sum_a t_{ka}r_{0a}. \tag{A.28}$$

The coefficients t are of order $\hbar$, hence r_k is of order $\hbar^2$. Moreover r_k vanishes if the axial anomalies r_{0a} vanish.

Appendix B. Reduction of coupling constants for SYM theories

We want to show that the reduction equations (5.9) admit a power series solution $\lambda_{rst}(g)$ if there is a lowest-order solution which is isolated and non-degenerate. If the gauge β-function is zero at the one-loop order – the case of interest here – the lowest-order equations are

$$\beta_\lambda^{(1)}(\lambda, g) = 0 \tag{B.1}$$

for all Yukawa coupling constants λ.

By separating the complex coupling constants λ_{rst} into their real and imaginary parts – we consider only the set of independent ones – we can assume all Yukawa coupling constants to be real and denote them by λ_i. The reduction equations read

$$\beta_i = \beta_g \frac{d\lambda_i}{dg}. \tag{B.2}$$

We shall follow Ref. [14], specializing to the structure of SYM gauge theories, for which the power series expansion of the β-functions has the form

$$\begin{aligned}\beta_i &= \sum_{n=1}^{\infty} \sum_{a=0}^{n} \sum_{k} C_i^{(n)k_1\cdots k_{2a+1}} g^{2n-2a} \lambda_{k_1} \cdots \lambda_{k_{2a+1}} \\ &= C_i^{(1)k} g^2 \lambda_k + C_i^{(1)klm} \lambda_k \lambda_l \lambda_m + O(\hbar^2), \\ \beta_g &= g^3 \sum_{n=2}^{\infty} \sum_{a=0}^{n-1} \sum_{k} B^{(n)k_1\cdots k_{2a}} g^{2n-2-2a} \lambda_{k_1} \cdots \lambda_{k_{2a}} \\ &= O(\hbar^2).\end{aligned} \tag{B.3}$$

The index n denotes the loop order. We have assumed β_g to vanish at order 1. Let us look for a solution of (B.2) of the form

$$\lambda_i(g) = \sum_{n=0}^{\infty} \rho_i^{(n)} g^{2n+1}. \tag{B.4}$$

At the lowest order, one finds that $\rho_i^{(0)}$ must be a solution of the equations

$$F_i(\rho^{(0)}) \equiv C_i^{(1)k} \rho_k^{(0)} + C_i^{(1)klm} \rho_k^{(0)} \rho_l^{(0)} \rho_m^{(0)} = 0, \tag{B.5}$$

which are just equations (B.1).

In higher orders we get the recurrence equations

$$M_i^k \rho_k^{(n)} = f_i, \qquad n \geqslant 1, \tag{B.6}$$

where the right-hand side depends only on the $\rho^{(p)}$ for $p < n$. The matrix M

depends on $\rho^{(0)}$ only:

$$M_i^k = \partial F_i(\rho^{(0)})/\partial\rho_k^{(0)}. \tag{B.7}$$

If this matrix is non-singular, i.e., if and only if the solution $\rho^{(0)}$ of equation (B.5) is isolated and not degenerate, then (B.6) determines the higher coefficients of (B.4) in terms of $\rho^{(0)}$.

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4.2 Necessary and sufficient conditions for all order vanishing $\beta-$functions in supersymmetric Yang-Mills theories

Title: Necessary and sufficient conditions for all order vanishing $\beta-$functions in supersymmetric Yang-Mills theories
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Comment (*Klaus Sibold*)
Based on the theorems of the preceding subsection one-loop criteria are given which are necessary and sufficient for the vanishing of β-functions to all orders of perturbation theory. They are operative in the fairly general setting of consistent $N = 1$ supersymmetric Yang-Mills theories. The following three conditions have to be satisfied:
(i) the β-function of the gauge coupling vanishes in one-loop order;
(ii) the anomalous dimensions of the matter superfields vanish in the one-loop order;
(iii) the Yukawa couplings solve the reduction equations (and satisfy (ii)) in such a way that the solution is isolated and non-degenerate.
Isolation and non-degeneracy can usually be established (if not automatically true) by imposing additional chiral symmetries or fixing arbitrary phases by hand: the non-renormalization theorem for chiral vertices guarantees that they are not affected by higher orders.
The second – physicswise very interesting – result of this paper is that it contains an interpretation of what "finiteness" means. Vanishing β-functions say, of course, that dilatations and special (super-)conformal symmetry are unbroken. Clearly also R-invariance is maintained. But all other chiral symmetries which act as outer automorphisms on susy are also unbroken: that their one-loop anomaly coefficients vanish guarantees the compatibility of the equations used in condition (ii). Hence one has a model which is free of all possible anomalies: those related to geometry and those related to internal symmetries.
In section 5 the preceding criteria will be extensively used for finding finite theories which are phenomenologically acceptable.
Another immediate application is possible in investigations of anomalies via local coupling (with or without supergravity background). Based on calculations in components within SYM with local gauge coupling [9], [10] an anomaly had been found and attributed to supersymmetry. For a manifestly supersymmetric gauge in the analogous study by [11] it was realized that this anomaly could be shifted into a renormalization of the θ-angle. Remarkably enough, in a finite SYM theory this anomaly is absent and thus the θ-angle is not renormalized.
It is then tempting to speculate that amongst such finite $N = 1$ models there is (at least) one which permits to cancel the Weyl anomaly in conformal supergravity theory. That, in turn might permit to construct power counting renormalizable theories containing quantized gravity. (As a guide to the rich literature one may consult [12].)

NECESSARY AND SUFFICIENT CONDITIONS FOR ALL ORDER VANISHING β-FUNCTIONS IN SUPERSYMMETRIC YANG–MILLS THEORIES

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If the matter self-couplings in SYM theories are certain uniquely determined power series of the gauge coupling then it is necessary and sufficient, for the vanishing of all β-functions to all orders, that the gauge β-function and the anomalous dimensions of the matter fields vanish at the one-loop order.

1. The first quantum field theories in four-dimensional space–time which were argued to be ultraviolet finite were gauge theories with extended supersymmetry [1]. It was later shown for $N=1$ supersymmetric gauge theories (SYM) that finiteness in the one-loop approximation

$$\beta_g^{(1)}=0\,,\quad \gamma_{\text{matter}}^{(1)}=0 \tag{1}$$

(gauge coupling β-function, anomalous dimensions of the matter fields) implies finiteness at two loops [2]. A table of models fulfilling these conditions is given in ref. [3]. For the special case that the number of independent anomalous dimensions does not exceed the number of "Yukawa" couplings (matter self-interactions) an extension of this result to al orders has been proposed [4]. Unfortunately dimensional regularization, whose validity is doubtful [5], has been used there. A corresponding class of models has been constructed [6]. In the realm of general theories the search for completely finite models [7] seems to point to the necessity of supersymmetry [8].

In this letter [#1] we shall not demand complete finiteness but only that all β-functions vanish, to all orders. It will turn out as a result that the conditions of Parkes and West, eq. (1), are neccessary and sufficient for this to happen, if in addition the Yukawa coupling constants λ_{rst} (see below) are uniquely determined power series solutions $\lambda=\lambda(g)$ of the reduction equations [10,11]

$$\beta_{\lambda_{rst}}=\beta_g\,\mathrm{d}\lambda_{rst}/\mathrm{d}g\,. \tag{2}$$

Thereby we improve an earlier version [12] by showing the necessity of these conditions and also by giving a physical interpretation of them.

2. The gauge invariant lagrangian of a general $N=1$ SYM theory theory with a simple gauge group is [13]

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[#1] A more detailed account will be presented elsewhere [9].

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$$\mathscr{L}_{\mathrm{inv}} = -\frac{1}{128g^2}\int \mathrm{d}^2\theta\,\mathrm{Tr}[\overline{\mathrm{DD}}(\mathrm{e}^{-\phi}\mathrm{D}\,\mathrm{e}^{\phi})]^2$$
$$+\frac{1}{16}\int \mathrm{d}^4\theta \sum_R \bar{A}_R \mathrm{e}^{\phi} A^R$$
$$+\frac{1}{6}\left(\int \mathrm{d}^2\theta\, \lambda_{rst}A^r A^s A^t + \mathrm{conj.}\right). \tag{3}$$

Here ϕ is the matrix of gauge superfields. The chiral superfields A^R describe matter: R labels both the field and the irreducible representation (irrep). of the gauge group it belongs to. The last term of the lagrangian uses the multiindex notation $r=(R,\rho)$, ρ labeling the components in the irrep. R.

The generators of the gauge group in the irrep. R are hermitian matrices $(T^i_R)^\rho_\sigma$, $i=1, ..., \dim \mathrm{G}$. The quadratic Casimir eigenvalue C_2 and the Dynkin index T of the irrep. R are defined by

$$\sum_i T^i_R T^i_R = C_2(R)\cdot \mathbb{1},$$
$$\mathrm{Tr}(T^i_R T^j_R) = T(R)\delta^{ij}. \tag{4}$$

The model may also be invariant under a set of chiral transformations ($a=1,2,...$)

$$\delta_a\phi = 0,$$
$$\delta_a A^R = \mathrm{i}e_a{}^R{}_S A^S, \quad \delta_a\bar{A}_R = -\mathrm{i}\bar{A}_S e_a{}^S{}_R, \tag{5}$$

the matrices e_a being hermitian. These infinitesimal transformations form a Lie algebra and we shall denote by δ_{0a}, e_{0a} the elements of its center. The lagrangian is invariant under the transformations (5) if the Yukawa coupling constants obey the constraints

$$\lambda_{rsu}e_a{}^u{}_t + \text{cycl. perm. } (r,s,t) = 0, \tag{6}$$

where

$$e_a{}^r{}_s \equiv e_a{}^R{}_S \delta^\rho{}_\sigma. \tag{7}$$

The theory is known to be renormalizable if the matter field representation is anomaly free [13]. We assume the presence of supersymmetric masses for all fields in order to avoid the off-shell infrared problem of SYM theories [13]. The gauge invariance – more precisely, BRS invariance – and the chiral symmetries (5) will hold up to soft (mass) breakings. The massless limit will be discussed elsewhere [14].

The Callan–Symanzik (CS) equation reads (Γ denoting the generating functional of one-particle-irreducible Green functions)

$$\Big(\sum_{\substack{\text{all}\\ \text{masses}}} m\partial_m + \beta_g\partial_g + \beta_{rst}\partial_{\lambda_{rst}} + \bar{\beta}^{rst}\partial_{\bar{\lambda}^{rst}}$$
$$-\gamma_\phi N_\phi - \gamma^R{}_S N^S{}_R - \gamma_k\partial_{a_k}\Big)\Gamma \sim 0. \tag{8}$$

The sign $\sim$ means "up to soft terms". The (unphysical) parameters a_k define the generalized field amplitude renormalization [13]

$$\phi \to \phi + \sum_{k\geq 2} a_k\phi^k. \tag{9}$$

The counting operators N are defined by

$$N_\phi = \int \phi\delta_\phi,$$
$$N^R{}_S = \int A^R\delta_{A^S} + \int \bar{A}_S\delta_{\bar{A}_R}. \tag{10}$$

The chiral invariances (5) impose that the hermitian matrix of matter field anomalous dimensions $\gamma^R{}_S$ commutes with all chiral charge matrices e_a. One can thus expand γ as

$$\gamma^R{}_S = \gamma_{0a}e_{0a}{}^R{}_S + \gamma_{1K}f_{1K}{}^R{}_S, \tag{11}$$

where the linearly independent matrices e_{0a} were defined to generate the center of the algebra of chiral transformations (5) and the f_{1K} complete the basis for γ.

3. The CS equation is directly related to the anomalies of the supercurrent multiplet, in particular to the trace anomaly and to the anomaly of the axial current associated to R-invariance [3]. A detailed study of these anomalies [12,9] leads to the following two identities:

$$r = \beta_g(1/g^3 + x_g) + \beta_{rst}x^{rst}$$
$$+\gamma^R{}_S x^S{}_R + \gamma_k x^k, \tag{12}$$
$$r = \beta_g(1/g^3 + y_g) + \beta_{rst}y^{rst}$$
$$+\gamma_{0a}r_{0a} + \gamma_k y^k. \tag{13}$$

In the second one the coefficients γ_{0a} are those of the expansion (11) for the matter field anomalous dimensions. Note the absence here of the coefficients

γ_{1K}: they are effectively reabsorbed in the β_{rst}-term. The coefficients x, y in (12), (13) are of order $\hbar$ at least. In particular:

$$x^S{}_R = [1/2(4\pi)^2]T(R)\delta^S{}_R + O(\hbar^2)\,, \tag{14}$$

$$x^k = O(\hbar^2)\,, \quad y^k = O(\hbar^2)\,. \tag{15}$$

On the other hand the coefficients r and r_{0a} in (12), (13) are the anomalies of the axial currents associated to R-invariance and to the basis elements of the center of the algebra of chiral invariances (5), respectively. They are exactly given by their one-loop values:

$$r = \frac{1}{4(4\pi)^2}\left(\sum_R T(R) - 3C_2(G)\right), \tag{16}$$

$$r_{0a} = \frac{1}{2(4\pi)^2}\sum_R e_{0a}{}^R{}_R T(R)\,. \tag{17}$$

The correct statement and the proof of this non-renormalization theorem is to be found in ref. [9], appendix A. Moreover the coefficients y^k turn out to vanish if all anomalies r_{0a} vanish [9]:

$$y^k = O(\hbar r_{0a})\,. \tag{18}$$

4. We are now going to show, first that the conditions (1) are *necessary* for the β-functions to vanish at all orders, and second that the fulfillment of these conditions implies the vanishing of the anomalies (16) and (17). The main tool is the identity (12).

The first of the conditions (1) is obvious. Then eq. (12) in the one-loop approximation implies the vanishing of the R-anomaly (16).

At two loops, eq. (12) gives (with the help of (14), (15))

$$\sum_R T(R)\gamma^{(1)R}{}_R = 0\,. \tag{19}$$

The one-loop β-functions are linear combinations of the γ-functions [13]. In the notation of (6):

$$\beta^{(1)}_{rst} = \lambda_{rsu}\gamma^{(1)u}{}_t + \text{cycl.perm.}(r,s,t)\,. \tag{20}$$

Multiplying this identity by $\bar\lambda^{rst}$, and using the explicit expression [2]

$$\gamma^{(1)r}{}_s = (1/2\pi^2)(\bar\lambda^{ruv}\lambda_{suv} - \tfrac{1}{16}g^2C_2(R)\delta^r{}_s)\,, \tag{21}$$

we get, for $\beta_{rst}=0$,

$$0 = \sum_{R,S} d(R)\,|\gamma^{(1)R}{}_S|^2 + \tfrac{1}{16}g^2 d(G)\sum_R T(R)\gamma^{(1)R}{}_R\,, \tag{22}$$

where we have used the identity

$$d(G)T(R) = d(R)C_2(R)\,, \tag{23}$$

with $d(G)$ and $d(R)$ the dimensions of the gauge group and of the irrep. R, respectively. Combining eqs. (19) and (22) yields the second set of the conditions (1):

$$\gamma^{(1)R}{}_S = 0\,. \tag{24}$$

The same treatment applied to the chiral symmetry conditions (6) in place of (20) yields the vanishing of the chiral anomalies (17), if eq. (24) holds. This last result also shows that the absence of chiral anomalies *is* the compatibility condition for solving the system of equations (24), and thus gives physical justification for them.

5. The preceeding discussion has shown that the Yukawa coupling constants must be functions $\lambda_{rst}(g)$, solutions of (24) in the lowest order. Consistency of the theory in higher orders implies [10,11] that these functions must be solutions of the reduction equations (2).

We can now prove that the conditions (1), together with the requirement that the Yukawa coupling constants be functions of g solving the reduction equations (2), are also *sufficient* for the all-order vanishing of the β-functions. We used the identity (13). Since the anomalies r and r_{0a} vanish (hence the coefficients y^k (18), too), as we have shown above, and since the reduction equations hold, the identity (13) reads

$$0 = \beta_g[1/g^3 + y_g + (d\lambda_{rst}/dg)y_{rst}]\,. \tag{25}$$

The bracket being perturbatively invertible, β_g vanishes. The same conclusion holds for the Yukawa β-functions, which are related to β_g through the reduction equations.

6. For the use of the above results it is clearly crucial to have a convenient (say one-loop) criterion guaranteeing that a solution of the reduction equations exists to all orders. As in ref. [11] one can show the following: any solution of the lowest order reduction equations

$$\beta^{(1)}_{rst} = 0 , \tag{26}$$

– simultaneously solving (24) – which is isolated and non-degenerate can be uniquely continued to a formal power series

$$\lambda_{rst} = \sum_{n=0}^{\infty} \rho^{(n)}_{rst} g^{2n+1} , \tag{27}$$

i.e. the solution exists to all orders.

For a given model with anomaly free representation of the matter fields, but general matter self-interaction, it will often turn out that eqs. (24), (26) have solutions $\lambda(g)$ depending on a parameter, i.e. forming a family and thus not satisfying the present sufficiency condition. In this case one might still proceed by imposing additional chiral symmetries of type (5) until a unique solution is singled out. For complex representations one can fix undetermined phases by hand and control their renormalization with the non-renormalization theorem of chiral vertices [13]. This procedure was sucessfully applied to an SU(6) model [9]. The set of all one- and two-loop finite models, listed in refs. [3,15], is under investigation.

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4.3 Reduction of couplings in the presence of parameters

Title: Reduction of couplings in the presence of parameters
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Comment (*Klaus Sibold, Wolfhart Zimmermann*)
In the papers on reduction and its application in the above sections two and three reduction had been performed for massless theories. It is however obvious that reduction is of considerable interest also in massive theories and in particular reduction of couplings carrying dimension is a very important issue (s. section 5). In the present paper this problem has been addressed in its simplest version: in a gauge theory mass parameters m_a and a gauge fixing parameter α are permitted, where masses are fixed on-shell and matter couplings are fixed by α-independent normalization conditions. It is also necessary to introduce a special value α_0 for the gauge parameter α in addition to the standard normalization point parameter κ.
Due to the presence of mass parameters one has now to distinguish between renormalization group and Callan-Symanzik equations. All β-functions can be rendered α-independent to all orders, independent of α_0 to one-loop order and the Callan-Symanzik β-functions mass independent to one-loop. The β-functions of the renormalization group equations will in general depend on mass ratios already in one loop.
When setting up reduction equations for the dimensionless coupling parameters those for the renormalization group equations turn out to involve partial derivatives with respect to mass values. But for the Callan-Symanzik equation, fortunately, they take the form of ordinary differential equations quite similar to the massless case with only parametric dependence on the mass and gauge fixing values. The problem of consistency between potentially different solutions originating from either renormalization group respectively Callan-Symanzik equation can be solved by employing the consistency of the original differential equations referring to the original parameters: one can show that the reduced couplings satisfy the required differential equations (namely variations with respect to α, α_0, κ) for power series solutions of the reduction equations. Hence these reduced theories can be considered as renormalizable field theories. Furthermore the mass dependence of the RG β-functions in order $n-1$ determines the mass dependence of the reduction solution in order n. For more general solutions this is unlikely to happen.
The general case will be presented in the next subsection.

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REDUCTION OF COUPLINGS IN THE PRESENCE OF PARAMETERS

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We show that reduction of couplings and parameter dependence are consistent for reduction solutions which are uniquely determined power series in the primary coupling.

1. Introduction and statement of the problem

Suppose a power counting renormalizable theory is given with fields of spins 0, $\frac{1}{2}$, 1 interacting with strength characterized by couplings $g, \lambda_1, ..., \lambda_n$. Then the "reduction of couplings" [1] deals with the question under which conditions the couplings $\lambda_1, ..., \lambda_n$ can be functions of the "primary" coupling g. The answer [1] is simple:

$$\lambda_i = \lambda_i(g), \quad i=1, ..., n, \tag{1}$$

is consistent with renormalization if and only if these functions are solutions of the "reduction equations"

$$\beta_g \frac{d\lambda_i}{dg} = \beta_{\lambda_i}, \quad i=1, ..., n, \tag{2}$$

where the functions β originate from the renormalization group equation. If one has found the general solution of (2) containing n integration constants, then one can understand reduction of couplings as given by (1) as a mere transformation of variables. But as soon as one demands that e.g. the λ_i vanish asymptotically with g or that they be a power series in g, one selects in a non-trivial way amongst different models. Power series solutions correspond to perturbatively renormalized theories and enumerate in particular all symmetries, but there are often solutions not belonging to any known symmetry. The examples considered thus far (see refs. [2,3] for a review) are restricted to mass-independent β-functions and in the case of gauge models to the Landau gauge.

In the present paper the notion of reduction is studied in theories with physical normalization of the mass and in general gauges, hence besides the coupling parameters g, λ_i and κ, a parameter characterizing the normalization point, physical mass parameters m_a ($a=1, ..., A$) are present as well as α_0, a parameter serving as zero point for the gauge fixing parameter α (s.b. and ref. [4]). Instead of (1) we shall have

$$\lambda_i = \lambda_i(g, m/\kappa, \alpha_0) \tag{3}$$

for the reduction of dimensionless couplings and the question arises which β-functions occur in the reduction equations. Since every one of the parameters κ, m_a, α_0 gives rise to a partial differential equation [#1]

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[#1] The terms $\gamma^\kappa \mathcal{N}$, etc. are a somewhat symbolic notation for the sum over all counting operators, symmetric with respect to rigid symmetries and BRS invariance. In gauge theories they contain the derivative with respect to the gauge parameter α (cf. ref. [4]).

$$\text{(RG)}\quad \Big(\kappa\partial_\kappa+\beta_g^\kappa\partial_g+\sum_i\beta_{\lambda_i}^\kappa\partial_{\lambda_i}-\gamma^\kappa\mathcal{N}\Big)\Gamma$$
$$=0\,, \tag{4}$$

$$(m_a)\quad \Big(m_a\partial m_a+\beta_g^a\partial_g+\sum_i\beta_{\lambda_i}^a\partial_{\lambda_i}-\gamma^a\mathcal{N}\Big)\Gamma$$
$$=\alpha_{m_a}\Delta_{m_a}\cdot\Gamma\,, \tag{5}$$

$$(\alpha_0)\quad \Big(\alpha_0\partial_{\alpha_0}+\beta_g^0\partial_g+\sum_i\beta_{\lambda_i}^0\partial_{\lambda_i}-\gamma^0\mathcal{N}\Big)\Gamma$$
$$=0\,, \tag{6}$$

$$\text{(CS)}\quad \Big(\mathrm{D}+\beta_g\partial_g+\sum_i\beta_{\lambda_i}\partial_{\lambda_i}-\gamma\mathcal{N}\Big)\Gamma$$
$$=\alpha_m\Delta_m\cdot\Gamma\,, \tag{7}$$

$$\mathrm{D}\equiv\kappa\partial_\kappa+\sum_a m_a\partial_{m_a},\quad \gamma\equiv\gamma^\kappa+\sum_a\gamma^a\,,$$
$$\beta_g\equiv\beta_g^\kappa+\sum_a\beta_g^a,\quad \beta_{\lambda_i}\equiv\beta_{\lambda_i}^\kappa+\sum_a\beta_{\lambda_i}^a\,,$$
$$\alpha_m\Delta_m\equiv\sum_a\alpha_{m_a}\Delta_{m_a}\,,$$

this question is non-trivial. In writing down (4)–(7) we have assumed matter mass to be normalized on the mass shell, normalization of the matter interaction to be independent of α and the gauge coupling to be defined at a zero-point of the gauge parameter:

$$\Gamma_{A_\mu A_\nu}=\Big(g_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\Big)\gamma_{AA}^{\mathrm{T}}+\frac{p_\mu p_\nu}{p^2}\gamma_{AA}^{\mathrm{L}}\,, \tag{8}$$

$$\frac{\partial}{\partial p^2}\gamma_{AA}^{\mathrm{T}}\Big|_{\substack{p^2=-\kappa^2\\ \alpha=\alpha_0}}=-\frac{1}{g^2}\,. \tag{9}$$

The definition (9), i.e. the introduction of a special value α_0 of the usual gauge parameter α, is necessary, since the amplitude of $\mathrm{Tr}\int F_{\mu\nu}F^{\mu\nu}$ is α-independent, but the vertex function $\Gamma_{A_\mu A_\nu}$ is not. In this respect α_0 resembles κ which is also introduced for the purpose of normalization only. By enlarging the ordinary BRS invariance to include variations of the gauge parameter [4] one can show that all β-functions are independent of α to all orders [#2] independent of α_0 to one-loop order and that the β-functions occurring in the Callan–Symanzik equation (7) are mass-independent in one loop. Those of the renormalization-group equation (4) will in general depend on the mass ratios already in one loop.

Let us now derive the reduction equations. The usual assumption [1] is that after imposing (3) again partial differential equations hold

$$(x\partial_x+\hat\beta_g^x\partial_g-\hat\gamma^x\mathcal{N})\hat\Gamma=\delta_{xm_a}\hat\alpha_{m_a}\Delta_{m_a}\cdot\hat\Gamma,$$
$$x=\kappa,\ m_a,\ \alpha_0\,, \tag{10}$$

where

$$\hat\Gamma=\hat\Gamma(g)=\Gamma\Big(g,\lambda\Big(g,\frac{m}{\kappa},\alpha_0\Big)\Big)\,. \tag{11}$$

With this definition it follows first that

$$\hat\beta_g^x=\beta_g^x\Big(g,\lambda\Big(g,\frac{m}{\kappa},a_0\Big)\Big),\quad x=\kappa,\ m_a,\ \alpha_0\,, \tag{12}$$

and then that

$$x\partial_x\lambda_i+\beta_g^x\frac{\partial\lambda_i}{\partial g}=\beta_{\lambda_i}^x\,,$$
$$x=\kappa,\ m_a,\ \alpha_0,\quad i=1,\dots,n\,, \tag{13}$$

are conditions to be satisfied by the functions λ_i of (3). Since the couplings λ_i are dimensionless

$$\mathrm{D}\lambda_i\equiv\kappa\partial_\kappa\lambda_i+\sum_a m_a\partial_{m_a}\lambda_i=0,\quad i=1,\dots,n\,, \tag{14}$$

it is suggestive to form the sum which yields the β-functions of the CS equation. We find

$$\beta_g\frac{\partial\lambda_i}{\partial g}=\beta_{\lambda_i},\quad i=1,\dots,n\,, \tag{15}$$

an equation most similar to (2) with α_0 and the masses m_a being purely parametric. Assuming now that (15) is solved by

$$\lambda_i=g(\rho_i^{(0)}+\rho_i^{(1)}g+\rho_i^{(2)}g^2+\dots) \tag{16}$$

the problem consists in showing that (13) is solved at the same time.

[#2] In theories with BRS invariance it is thus reasonable to suppose that no α-dependence appears in the reducing relation (3). In more general models α-dependence occurs and can be handled analogously [5].

2. Consistency condition

We wish to prove that

$$E_i^x \equiv x\partial_x\lambda_i + \beta_g^x \frac{\partial\lambda_i}{\partial g} - \beta_{\lambda_i}^x ,$$
$$x=\kappa, m_a, \alpha_0, \quad i=1, ..., n , \qquad (17)$$

vanishes identically as a formal power series in g. In order to do so we study its derivative with respect to g and use again (15)

$$\frac{dE_i^x}{dg} = x\frac{d}{dx}\left(\frac{\beta_{\lambda_i}}{\beta_g}\right) + \frac{d}{dg}\left(\beta_g^x\frac{\beta_{\lambda_i}}{\beta_g}\right) - \frac{d}{dg}\beta_{\lambda_i}^x , \qquad (18)$$

here we consider the β-functions as functions of g and λ $(g, m/k, \alpha_0)$. It follows that

$$\frac{dE_i^x}{dg} = x\partial_x\left(\frac{\beta_{\lambda_i}}{\beta_g}\right) + x\partial_x\lambda_j\frac{\partial}{\partial\lambda_j}\left(\frac{\beta_{\lambda_i}}{\beta_g}\right)$$
$$+\frac{d}{dg}\left(\beta_g^x\frac{\beta_{\lambda_i}}{\beta_g}\right) - \frac{d}{dg}\beta_{\lambda_i}^x . \qquad (19)$$

With the definition of E_i^x (17) we can rewrite (19)

$$\frac{dE_i^x}{dg} - E_j^x\frac{\partial}{\partial\lambda_j}\left(\frac{\beta_{\lambda_i}}{\beta_g}\right)$$
$$= x\partial_x\left(\frac{\beta_{\lambda_i}}{\beta_g}\right) + \beta_{\lambda_j}^x\partial_{\lambda_j}\left(\frac{\beta_{\lambda_i}}{\beta_g}\right) + \frac{d\beta_g^x}{dg}\frac{\beta_{\lambda_i}}{\beta_g}$$
$$+\beta_g^x\frac{\partial}{\partial g}\left(\frac{\beta_{\lambda_i}}{\beta_g}\right) - \frac{d\beta_{\lambda_i}^x}{dg} . \qquad (20)$$

Further information is now provided by the fact that eqs. (4–6) are consistent; this means that integrability conditions hold

$$y\partial_y\beta_g^x + \beta_g^y\partial_g\beta_g^x + \beta_{\lambda_j}^y\partial_{\lambda_j}\beta_g^x$$
$$= x\partial_x\beta_g^y + \beta_g^x\partial_g\beta_g^y + \beta_{\lambda_j}^x\partial_{\lambda_j}\beta_g^y , \qquad (21)$$
$$y\partial_y\beta_{\lambda_i}^x + \beta_g^y\partial_g\beta_{\lambda_i}^x + \beta_{\lambda_j}^y\partial_{\lambda_j}\beta_{\lambda_i}^x$$
$$= x\partial_x\beta_{\lambda_i}^y + \beta_g^x\partial_g\beta_{\lambda_i}^y + \beta_{\lambda_j}^x\partial_{\lambda_j}\beta_{\lambda_i}^y , \qquad (22)$$

for all $x, y \in \{\kappa, m_a, \alpha_0\}$. For the special choice $y \equiv D$, see eq. (7), we have

$$y\partial_y\beta_g^x = 0 = y\partial_y\beta_{\lambda_i}^x , \qquad (23)$$

and (21), (22) may be combined to yield

$$x\partial_x\left(\frac{\beta_{\lambda_i}}{\beta_g}\right) = \frac{d}{dg}\beta_{\lambda_i}^x - \frac{\beta_{\lambda_i}}{\beta_g}\frac{d\beta_g^x}{dg}$$
$$-\beta_g^x\partial_g\left(\frac{\beta_{\lambda_i}}{\beta_g}\right) - \beta_{\lambda_j}^x\partial_{\lambda_j}\left(\frac{\beta_{\lambda_i}}{\beta_g}\right) . \qquad (24)$$

As a consequence of (20) and (24) we arrive finally at

$$\left.\frac{dE_i^x}{dg} - E_j^x\frac{\partial}{\partial\lambda_j}\left(\frac{\beta_{\lambda_i}}{\beta_g}\right)\right|_{\lambda_i=\lambda_i(g,m/\kappa,\alpha_0)} = 0 ,$$
$$x=\kappa, m_a, \alpha_0, \quad i=1, ..., n . \qquad (25)$$

In order to solve (25) we multiply it by β_g

$$\left.\left(\delta_{ij}\beta_g\frac{d}{dg} - \partial_{\lambda_j}\beta_{\lambda_i} + \frac{\beta_{\lambda_i}}{\beta_g}\partial_{\lambda_j}\beta_g\right)E_j^x\right|_{\lambda=\lambda(g, ...)}$$
$$=0 \qquad (26)$$

and insert the series expansion for the respective functions[#3]

$$\beta_g = b_0g^3 + b_{1ij}\lambda_i\lambda_jg^3 + b_2g^5 + ... , \qquad (27)$$
$$\beta_{\lambda_i} = c_{1iklm}\lambda_k\lambda_l\lambda_m + c_{2ik}\lambda_kg^2 + ... , \qquad (28)$$
$$\beta_g = ag^2 + a_ig\lambda_i + a_{(ij)}\lambda_{(i}\lambda_{j)} + ... , \qquad (27')$$
$$\beta_{\lambda_i} = c_{i(jk)}\lambda_{(j}\lambda_{k)} + c_{ij}g\lambda_j + c_ig^2 + ... , \qquad (28')$$
$$E_j^x = e_{j2}^xg^2 + e_{j3}^xg^3 + ... , \qquad (29)$$
$$\frac{\beta_{\lambda_i}}{\beta_g} = \frac{\partial\lambda_i}{\partial g} = \rho_i^{(0)} + 2\rho_i^{(1)}g + ... \qquad (30)$$

The choice (27), (28) covers for instance gauge theories (with g being the gauge, λ being Yukawa couplings). The form of (27′), (28′) is e.g. relevant for self-interacting scalar fields.

As coefficient of the power g^{N+2} (g^{N+1}) we thus obtain

$$S_{ij}e_{jN}^x = \text{l.o.} , \qquad (31)$$
$$S_{ij} = \delta_{ij}b_0N - (3c_{1ijlm}\rho_l^{(0)}\rho_m^{(0)} + c_{2ij})$$
$$\text{for (27), (28)} , \qquad (32)$$
$$S'_{ij} \equiv \delta_{ij}N(a + a_l\rho_l^{(0)} + a_{(lk)}\rho_{(l}^{(0)}\rho_{k)}^{(0)})$$
$$+ a_i\rho_j^{(0)} + 2a_{(ij)}\rho_j^{(0)} - 2c_{i(jk)}\rho_k^{(0)} - c_{ji}$$
$$\text{for (27′), (28′)} , \qquad (32')$$

[#3] E_j^x starts with power g^2 at least since $\rho_j^{(0)}$, see eq. (16), does not depend on x.

and l.o. standing for terms involving only coefficients e^x_{jk} with k strictly less than N. But the matrices S and S' are precisely those which are also responsible for the solvability of the reduction equations (15)! If det S (det S') is non-vanishing for all N, then the reduction solution (16) is uniquely determined to all orders, i.e. the coefficients $\rho_i^{(N)}$ are uniquely given by (15). Hence

$$E^x_i=0, \quad x=\kappa, m_a, \alpha_0, \quad i=1, ..., n \tag{33}$$

for all unique power series solutions (16), i.e. all other reduction equations (13) are satisfied as well. Let us note that the case $b_0=0$ and even the case with identically vanishing β-functions $\beta_g=\beta_g(g, \lambda(g, ...))$ = $\beta_{\lambda_i}(g, \lambda(g, ...))=0$ has been checked to be included in the above derivation.

3. An explicit example

In order to test the above abstract considerations and to gain some insight into the consistency mechanism at work let us consider the simple example of two massive scalar fields A, B as described by the classical action

$$\Gamma_{\mathrm{cl}}=\int\bigg(\tfrac{1}{2}\partial A\partial A-\tfrac{1}{2}m_A^2A^2+\tfrac{1}{2}\partial B\partial B-\tfrac{1}{2}m_B^2B^2 - \frac{1}{4!}(\lambda_A A^4+6gA^2B^2+\lambda_B B^4)\bigg). \tag{34}$$

For simplicity we have imposed the discrete symmetries $A\to -A$, $B\to -B$ and $A\to A$, $B\to -B$ which exclude the couplings with an odd number of fields.

The masses are fixed by physical normalization conditions, the couplings by normalizing at a symmetric normalization point with characteristic value κ^2. We then have the CS equation

$$(\kappa^2\partial_{\kappa^2}+m_A^2\partial_{m_A^2}+m_B^2\partial_{m_B^2}+\beta_g\partial_g+\beta_{\lambda_A}\partial_{\lambda_A}+\beta_{\lambda_B}\partial_{\lambda_B} -\gamma_{ab}\mathcal{N}_{ab})\Gamma=(\alpha_A m_A^2\Delta_{m_A}+\alpha_B m_B^2\Delta_{m_B})\cdot\Gamma \tag{35}$$

and the RG equation

$$(\kappa^2\partial_{\kappa^2}+\beta_g^\kappa\partial_g+\beta^\kappa_{\lambda_A}\partial_{\lambda_A}+\beta^\kappa_{\lambda_B}\partial_{\lambda_B}-\gamma^\kappa_{ab}\mathcal{N}_{ab})\Gamma=0 \tag{36}$$

with β-functions in one-loop order given by

$$\beta_g=\tfrac{1}{2}(\lambda_1+\lambda_2)g+2g^2,$$
$$\beta_{\lambda_A}=\tfrac{3}{2}(\lambda_A^2+g^2), \quad \beta_{\lambda_B}=\tfrac{3}{2}(g^2+\lambda_B^2), \tag{37}$$
$$\beta^\kappa_g=\tfrac{1}{2}(\lambda_A X_A+\lambda_B X_B)g+2g^2X_{AB},$$
$$\beta^\kappa_{\lambda_A}=\tfrac{3}{2}(\lambda_A^2X_A+g^2X_B), \quad \beta^\kappa_{\lambda_B}=\tfrac{3}{2}(g^2X_A+\lambda_B^2X_B),$$

$$X\equiv\frac{1}{8\pi^2}\left(1-\frac{3m^2/\kappa^2}{2\sqrt{1+3m^2/\kappa^2}}\ln\frac{\sqrt{1+3m^2/\kappa^2}+1}{\sqrt{1+3m^2/\kappa^2}-1}\right),$$

$$X_{AB}\equiv\frac{1}{8\pi^2}\bigg[1+\frac{3}{8}\frac{m_A^2-m_B^2}{\kappa^2}\ln\frac{m_A^2}{m_B^2} -\frac{1}{2}\sqrt{\left(1+\frac{M^2}{\frac{4}{3}\kappa^2}\right)^2-\frac{4m_A^2m_B^2}{\frac{4}{3}\kappa^2\frac{4}{3}\kappa^2}}\ln\frac{\surd+1+M^2/\frac{4}{3}\kappa^2}{\surd-1-M^2/\frac{4}{3}\kappa^2} -\frac{1}{2}\frac{1+M^2/\frac{4}{3}\kappa^2}{\surd}\ln\frac{\surd+1+M^2/\frac{4}{3}\kappa^2}{\surd-1-M^2/\frac{4}{3}\kappa^2}\bigg],$$

$$M^2\equiv m_A^2+m_B^2,$$
$$\surd\equiv\sqrt{\left(1+\frac{M^2}{\frac{4}{3}\kappa^2}\right)^2-\frac{4m_A^2m_B^2}{\frac{4}{3}\kappa^2\frac{4}{3}\kappa^2}}. \tag{38}$$

The reduction equations read for the CS β-functions

$$\beta_g\frac{\partial\lambda_i}{\partial g}=\beta_{\lambda_i}, \quad i=A, B; \tag{39}$$

they have in one-loop the solutions

$$\text{(I)} \quad \rho_A^{(0)}=\rho_B^{(0)}=3, \tag{40}$$

$$\text{(II)} \quad \rho_A^{(0)}=\rho_B^{(0)}=1 \tag{41}$$

for

$$\lambda_i=g(\rho_i^{(0)}+\rho_i^{(1)}g+...), \tag{42}$$

and the matrices S'

$$S'_{\mathrm{I}}=\begin{pmatrix} 5n-\frac{9}{2} & \frac{3}{2} \\ \frac{3}{2} & 5n-\frac{9}{2}\end{pmatrix}, \quad \det S'_{\mathrm{I}}\neq 0, \tag{43}$$

$$S'_{\mathrm{II}}=\begin{pmatrix} -n-\frac{5}{2} & \frac{1}{2} \\ \frac{1}{2} & -n-\frac{5}{2}\end{pmatrix}, \quad \det S'_{\mathrm{II}}\neq 0. \tag{44}$$

Hence both solutions are uniquely determined to all orders, solution I corresponding to the O(2)-symmetric theory.

The reduction equations for the RG equation

$$\kappa^2\partial_{\kappa^2}\lambda_i+\beta^\kappa_g\frac{\partial\lambda_i}{\partial g}=\beta^\kappa_{\lambda_i}, \quad i=A, B, \tag{45}$$

are to the lowest order in g the simple statement

$$\kappa^2\partial_{\kappa^2}\rho_i^{(0)}=0, \quad i=A,B, \tag{46}$$

corresponding to

$$\kappa^2\partial_{\kappa^2}\beta_g=\kappa^2\partial_{\kappa^2}\beta_{\lambda_i}=0, \quad i=A,B, \tag{47}$$

of the consistency (21), (22) in one loop. In the order g^2 the reduction equations read

$$\text{(I)} \quad \kappa^2\partial_{\kappa^2}\rho_A^{(1)}=+9X_A-3X_B-6X_{AB}, \tag{48}$$

$$\text{(I)} \quad \kappa^2\partial_{\kappa^2}\rho_B^{(1)}=-3X_A+9X_B-6X_{AB}, \tag{49}$$

$$\text{(II)} \quad \kappa^2\partial_{\kappa^2}\rho_i^{(1)}=+X_A+X_B-2X_{AB}, \quad i=A,B. \tag{50}$$

We thus have the remarkable result that the one-loop β-functions of the RG equation determine the mass dependence of the reduction solution of the CS equation at *two* loops.

One can even go a step further and integrate (48)–(50) with respect to κ^2:

$$\rho_i^{(0)} = \int_{\ln\kappa_0^2}^{\ln\kappa^2} \frac{1}{g^2}\left(\beta_g^\kappa(u)\frac{\partial\lambda_i}{\partial g}-\beta_{\lambda_i}^\kappa(u)\right)\bigg|_{\lambda_i=\rho_i^{(0)}g} \mathrm{d}u + \bar{\rho}_i^{(1)}(g), \tag{51}$$

where $\bar{\rho}_i^{(1)}$ does not depend on the masses. Hence

$$\lambda_i^{(1)}=\rho_i^{(0)}g + \left[-\int_{\ln\kappa_0^2}^{\ln\kappa^2} \frac{1}{g^2}\left(\beta_g^\kappa(u)\frac{\partial\lambda_i}{\partial g}-\beta_{\lambda_i}^\kappa(u)\right)\bigg|_{\lambda_i=\rho_i^{(0)}g} \mathrm{d}u + \bar{\rho}_i^{(1)}(g)\right]g^2+\dots. \tag{52}$$

At $m_A=m_B=0$ we find

$$\lambda_i=\rho_i^{(0)}g+\bar{\rho}_i^{(1)}(g)g^2, \tag{53}$$

which identifies $\bar{\rho}_i^{(1)}(g)$ as the two-loop value of the CS reduction in the massless limit of the model. Hence in order to know the complete mass dependence of the reduced coupling λ_i in two loops it is sufficient to calculate the mass-dependent β-functions of the RG equations in *one* loop and the β-functions of the CS equation in two loops only for the massless theory – quite a simplification. Clearly the same interplay occurs in higher orders. It furthermore suggests another way of solving (13): if one considers together with the massive theory its massless version (for which $\beta^\kappa=\beta!$), (13) determines in one order of g (at $m\neq 0$) the mass-dependent part of the reduction solution, in the following order (at $m=0$) the mass-independent part[#4].

4. Discussion

In theories with physical normalization of the mass and in gauge theories formulated in a general linear gauge with gauge parameter α, reduction of couplings should be performed via (15) i.e. with the β-functions of the Callan–Symanzik equation. Inserting the solutions (16) into all other partial differential equations which govern the parameter dependence of the theory – namely (4)–(6) – will be consistent only if eqs. (13) are satisfied. They control the dependence of $\lambda(g, m/\kappa, \alpha_0)$ on the mass and gauge zero point parameters. The above analysis shows that these partial differential equations are indeed satisfied for all unique power series solutions (16) of the reduction equations (15). Hence reduction and parametric dependence are consistent. There is furthermore an intriguing interplay of different orders of perturbation theory: the mass dependence of the RG β-function in the order $n-1$ determines the mass dependence of the reduction solution in the order n.

One might wonder whether this consistency extends itself also to more general solutions of (15). The answer seems to be clearly no. Any general solution contains integration "constants" – which are in this context arbitrary functions of m_a/κ and α_0. Introducing them by integrating (15) is an ad hoc procedure seen from eqs. (13) hence consistency can neither be guaranteed not expected to hold a priori.

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4.4 Scheme independence of the reduction principle and asymptotic freedom in several couplings

Title: Scheme independence of the reduction principle and asymptotic freedom in several couplings
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Comment (*Wolfhart Zimmermann*)
For renormalizable models of quantum field theory there is considerable arbitrariness in setting up schemes of renormalization. But different schemes should be equivalent in the sense that Green's functions – apart from normalization factors of the fields – become identical after an appropriate transformation of the coupling parameters. For the reduction principle to be a meaningful concept it must be invariant under such scheme changing transformations. The freedom of choosing a convenient renormalization scheme may be used to simplify the form of conditions for the reduction principle to hold.
In the first part of the present work the scheme independence of the reduction principle is proved. Apart from dimensionless couplings, pole masses and gauge parameters the model may also involve coupling parameters carrying a dimension and variable masses. Pole masses refer to the lowest propagator singularities, variable masses are defined by propagators at the normalization point and treated like couplings with dimension. Since relevant for some applications also partial reductions are included. Accordingly, some of the couplings are selected as primary couplings on which the remaining reduced couplings depend. The reduction principle states that Green's functions expressed in terms of the primary couplings satisfy the corresponding renormalization group equations. In addition, it is required that all couplings simultaneously vanish in the weak coupling limit and allow for power series expansions in the primary couplings. All these requirements are shown to be invariant under scheme changing transformations thus establishing the scheme independence of the reduction principle.
As an application massive models of quantum field theory are treated with several dimensionless couplings. One of them is selected as primary coupling on which the other couplings depend according to the reduction principle. A transformation of the coupling parameters is constructed for defining an equivalent renormalization scheme in which the original β-functions are replaced by their massless limits. Due to the scheme independence the reductions equations also hold in the new renormalization scheme with mass independent β-functions as coefficients. Their final form is a set of ordinary differential equations with only parametric dependence on the masses.

The last part of this work concerns the property of asymptotic freedom for models involving several couplings. Renormalizable models of quantum field theory are studied with positive dimensionless coupling parameters. Effective couplings are introduced by appropriate vertex functions. Their momentum dependence is controlled by the evolution equations, a system of ordinary differential equations in the momentum variable with the β-functions as coefficients. Asymptotic freedom states that all effective couplings simultaneously vanish in the high momentum limit. As a consequence all β-functions are negative in the domain considered. For models with only one coupling the negative sign of the β-function is also a sufficient condition for asymptotic freedom. In case of several couplings asymptotic freedom is not a property of the model as such, but selects particular solutions of the system by placing constraints on the coupling parameters. These are obtained by eliminating the momentum variable in the evolution equations. To this end the momentum variable is replaced by one of the effective couplings, called the primary coupling, as independent variable. With this substitution the evolution equations take the form of reduction equations for the other effective couplings (the reduced couplings) as functions of the primary coupling. The momentum dependence is then regulated by the remaining evolution equation of the primary coupling with negative β-function. For asymptotic freedom to hold the reduced couplings must vanish with the primary coupling in the weak coupling limit (or high momentum limit) in accordance with the reduction principle.

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Scheme Independence of the Reduction Principle and Asymptotic Freedom in Several Couplings

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Dedicated to the memory of Harry Lehmann

Abstract: It is proved that reduction in the number of coupling and mass parameters is a scheme independent concept. This result justifies to use special renormalization schemes suitable for applications of the reduction method. Scheme changing transformations are discussed with the aim of removing gauge and mass parameters in the reduction equations. Necessary and sufficient conditions for asymptotic freedom in models with several couplings are stated.

1. Introduction

The method of reducing the number of couplings was originally proposed for renormalizable models of quantum field theory with dimensionless couplings $\lambda_0, \lambda_1, \ldots, \lambda_n$ and a normalization mass κ as the only parameters [1]. Since the reduction method is exclusively based on the form of the β functions it may as well be applied to other models in formulations for which the β functions are massless and independent of gauge parameters. To this end the Landau gauge is used for gauge theories and a scheme of renormalization like dimensional renormalization in which β functions are mass independent [2,3]. Then the β functions depend on the dimensionless couplings only

$$\beta_j = \beta_j(\lambda_0, \lambda_1, \ldots, \lambda_n), \quad j = 0, 1, \ldots, \lambda_n. \tag{1.1}$$

By the principle of reduction all couplings λ_j are required to be functions of a single one denoted by λ_0,

$$\lambda_j = \lambda_j(\lambda_0) \quad (j = 1, \ldots, n), \tag{1.2}$$

in a way which is compatible with invariance under the renormalization group [4–6]. Substituting the functions (1.2) for the couplings λ_j of the original model one obtains a formulation of a reduced model involving a single coupling parameter λ_0 only. As a consequence of the renormalization group invariance of the original and the reduced model as well one finds a system of ordinary differential equations

$$\beta_0 \frac{d\lambda_j}{d\lambda_0} = \beta_j \tag{1.3}$$

to be satisfied by the functions (1.2). For the solutions to be meaningful it is required that all couplings simultaneously vanish in the weak coupling limit

$$\lambda_j \to 0 \quad \text{for} \quad \lambda_0 \to 0. \tag{1.4}$$

In many cases it is natural to impose further that all couplings allow for power series expansions with respect to a suitably selected primary coupling λ_0,

$$\lambda_j = \sum c_{jl} \lambda_0^l. \tag{1.5}$$

In this case the correlation functions of the reduced model have formal expansions with respect to powers of λ_0, thus resembling a renormalizable theory with a single coupling λ_0. For some applications it is useful to consider partial reductions, where several parameters remain independent. It may also be of interest to require – instead of (1.4) – that all couplings simultaneously approach a non-trivial zero of the β functions.

Coupling relations (1.2) which follow from the invariance of a model under a symmetry group satisfy the conditions (1.3)–(1.5) provided the symmetry can be implemented to all orders of perturbation theory. The reduction method may thus be considered as a generalization of this particular aspect of symmetry[1].

The reduction method was extended by Piguet and Sibold to formulations of models with β functions depending on mass and gauge parameters [20]. In that case the reduction equations become a system of partial differential equations including derivatives with respect to the normalization mass and gauge parameters. Due to these partial derivatives it is difficult to study the solutions of the reduction equations in the general case. However, Piguet and Sibold found the remarkable result that on the basis of the Callan–Symanzik equations [21, 22] the reduction equations have the form of ordinary differential equations with parametric dependence on the mass and gauge parameters. Since in general the renormalization group equations [23] and the Callan–Symanzik equations are independent, the question of consistency between the two types of reduction equations comes up. For solutions which are uniquely determined power series in the primary coupling Piguet and Sibold proved the consistency. For general solutions the issue is more involved. But transforming to a scheme with massless β functions for which renormalization group equations and Callan–Symanzik equations coincide should furnish a resolution of this problem in general.

Another important development concerns the combined reduction of couplings and masses in supersymmetric grand unified theories [24]. In this work Kubo, Mondragón and Zoupanos reduced the coefficients of the soft supersymmetry breaking terms in order to minimize the number of independent parameters. The scheme of dimensional renormalization was used with mass parameters introduced similarly to couplings. Then the differential equations of the renormalization group also involve derivatives with respect to the masses. It is characteristic for dimensional renormalization that those β functions which carry a dimension are linear or quadratic forms in the dimensional

[1] For reviews see, for instance, refs. [7–14]. Refs. [15–19] contain earlier work related to the reduction of couplings.

couplings and masses, while the coefficients of these polynomials depend on the dimensionless couplings only. Since in this approach the mass parameters enter similarly to the couplings, masses are included with the couplings in the reduction process. In this way Kubo, Mondragón and Zoupanos obtained non-trivial constraints on the soft supersymmetry breaking terms which are compatible with renormalization and lead to surprisingly simple sum rules [25].

In the present paper it will be proved that the principle of reduction is invariant under transformations of couplings and masses which change the scheme of renormalization[2]. This scheme independence justifies the use of special schemes of renormalization chosen such that the β functions take a particularly simple form. The proof includes the case of couplings with the dimension of mass and variable masses treated similarly to couplings (Sect. 2).

In Sect. 3 methods of eliminating gauge and mass parameters are discussed. It is referred to the work of Breitenlohner and Maison for a comprehensive treatment [27]. For the purpose of the reduction method an alternative approach is proposed which is exclusively based on the differential equations of the renormalization group. In models with dimensionless couplings and pole masses transformations are constructed which lead to a scheme of renormalization with massless β functions. The proof is based on formal expansions with respect to powers of the coupling and uses the assumption that the massless limit of the β functions exists and is approached smoothly. The formulation obtained should be equivalent to the scheme of dimensional renormalization with appropriate normalization conditions. The generalization to models which also involve dimensional couplings and variable mass parameters is only sketched. In this case mass parameters cannot be eliminated completely from the β functions. Instead a polynomial dependence on dimensional couplings and masses remains. The final form of the reduction equations is in agreement with ref. [24].

A different interpretation of the reduction method is provided by the evolution equations [28]. A systematic discussion of the effective couplings in this respect is given in Sect. 4 for models with dimensionless couplings and pole masses. It is shown how in the reduced model the effective couplings are expressed as functionals of the primary coupling. An evolution equation for the primary coupling alone is derived. Again, particularly simple results are obtained, if a scheme of renormalization is used with massless β functions, as is justified by scheme independence. Then the reduction equations follow in the form

$$\bar{\beta}_0 \frac{d\bar{\lambda}_j}{d\bar{\lambda}_0} = \bar{\beta}_j \quad (j = 1, \dots n), \quad \bar{\beta}_j = \beta_j(\bar{\lambda}_0, \bar{\lambda}_1, \dots, \bar{\lambda}_n), \tag{1.6}$$

for the effective couplings $\bar{\lambda}_j$ by eliminating the momentum variable $|k|$ in the evolution equations. Corresponding to (1.4) the condition

$$\bar{\lambda}_j \to 0 \quad \text{for} \quad \bar{\lambda}_0 \to 0 \tag{1.7}$$

is imposed. In the case of

$$\bar{\lambda}_0 \to 0 \quad \text{for} \quad |k| \to \infty \tag{1.8}$$

the property of asymptotic freedom holds [29,30]: All couplings vanish simultaneously in the high momentum limit,

$$\bar{\lambda}_0 \to 0, \dots, \bar{\lambda}_n \to 0 \quad \text{for} \quad |k| \to \infty. \tag{1.9}$$

2 A preliminary report on this work was given in ref. [26].

Equation (1.8) is implied by the evolution equation for $\bar{\lambda}_0$, if $\bar{\beta}_0$ has the appropriate sign for $\bar{\lambda}_0 \to 0$. For example $\bar{\beta}_0$ should be negative in the case of $\lambda_0 > 0$ in the model considered. In this way necessary and sufficient conditions for asymptotic freedom in several couplings follow[3].

2. Scheme Independence

We consider models of local quantum field theory with renormalizable interactions involving several coupling and mass parameters. Apart from dimensionless coupling parameters and a normalization mass we allow for the possibility of intrinsic masses, coupling parameters of dimension mass and gauge parameters, should gauge fields be present. For the intrinsic masses either pole masses are used defined by the lowest propagator singularities or variable masses suitably defined by propagators at the normalization point. For implementing the concept of reduction some of the parameters are selected as an independent variables with other parameters depending on them. Usually one single parameter is chosen as independent variable. There are interesting applications, however, where a partial reduction with several independent parameters is useful, see ref. [24], for instance. For this reason the case of partial reduction is included. Following is a list of all parameters involved:

- dimensionless couplings $g^0_{01}, \dots, g^0_{0A}, \quad g^1_{01}, \dots, g^1_{0E}$;
- couplings of dimension mass $g^0_{11}, \dots, g^0_{1B}, \quad g^1_{11}, \dots, g^1_{1F}$;
- variable masses $g^0_{21}, \dots, g^0_{2C}, \quad g^1_{21}, \dots, g^1_{2G}$;
- variable mass squares $g^0_{31}, \dots, g^0_{3D}, \quad g^1_{31}, \dots g^1_{3H}$;
- pole masses $m_1, \dots, m_I$;
- gauge parameters $\alpha_1, \dots, \alpha_J$;
- normalization mass κ.

The independent parameters are denoted by g^0_{ij} , the parameters g^1_{ij} will be considered to be functions of them,

$$g^1_{ij} = r_{ij}(g^0_{01}, \dots, g^0_{3D}, m_1, \dots, m_I, \alpha_1, \dots, \alpha_J, \kappa^2) \tag{2.1}$$

or

$$g^1 = r(g^0, m, \alpha, \kappa^2) \tag{2.2}$$

in vector notation

$$\begin{gathered} g^0 = (g^0_{01}, \dots, g^0_{3D}), \quad g^1 = (g^1_{01}, \dots, g^1_{3H}), \quad r = (r_{01}, \dots, r_{3H}), \\ m = (m_1, \dots, m_I), \qquad \alpha = (\alpha_1, \dots, \alpha_J). \end{gathered} \tag{2.3}$$

The distinction between linear and quadratic mass parameters is a matter of convenience relevant for the massless limit. For the time ordered correlation functions

$$\tau = \tau(k, g^0, g^1, m, \alpha, \kappa^2), \tag{2.4}$$

[3] For reduced models with asymptotic freedom see refs. [15–18,31–34], reviews are given in refs. [8,10].

(k denotes the vector of momentum variables) the partial differential equations of the renormalization group are

$$\kappa^2 \frac{\partial\tau}{\partial\kappa^2} + \sum \beta^l_{ij} \frac{\partial\tau}{\partial g^l_{ij}} + \sum \delta_j \frac{\partial\tau}{\partial\alpha_j} + \sum \gamma_j \tau = 0. \tag{2.5}$$

In the original model all variables g^l_{ij} of the correlation functions are independent. By substituting the functions (2.1) for the variables g^1_{ij} in (2.4) the number of independent parameters is decreased. The correlation functions thus obtained,

$$\tau' = \tau'(k, g^0, m, \alpha, \kappa^2) = \tau(k, g^0, r(g^0, m, \alpha, \kappa^2), m, \alpha, \kappa^2), \tag{2.6}$$

define a new model which is called a reduced model with the reducing functions (2.1). By the reduction principle the reduced model is again invariant under the renormalization group. This means that the correlation functions (2.6) should also satisfy partial differential equations of the form

$$\kappa^2 \frac{\partial\tau'}{\partial\kappa^2} + \sum \beta'^0_{ij} \frac{\partial\tau'}{\partial g^0_{ij}} + \sum \delta'_j \frac{\partial\tau'}{\partial\alpha_j} + \sum \gamma'_j \tau' = 0. \tag{2.7}$$

Comparing (2.5) with (2.7) we obtain

$$\beta'^0_{ij} = \beta^0_{ij}, \quad \delta'_j = \delta_j, \quad \gamma'_j = \gamma_j \tag{2.8}$$

with the prime indicating that the functions (2.1) should be inserted for the variables g^1_{ij}. For the reducing functions (2.1) the partial differential equations

$$\kappa^2 \frac{\partial r_{st}}{\partial\kappa^2} + \sum \beta'^0_{ij} \frac{\partial r_{st}}{\partial g^0_{ij}} + \sum \delta'_j \frac{\partial r_{st}}{\partial\alpha_j} = \beta'^1_{st} \tag{2.9}$$

follow. The reduction principle requires further that the couplings vanish simultaneously in the weak coupling limit $g^0 \to 0$,

$$r_{0t} = 0, \quad r_{1u} = 0 \quad \text{at} \quad g^0_{oj} = 0, \quad g^0_{1l} = 0. \tag{2.10}$$

A considerably stronger restriction may be imposed on the reducing functions by demanding that – in addition to (2.10) – formal expansions of the dependent couplings r_{0t}, r_{1u}, and masses r_{2u}, r_{3w} as well, exist with respect to the independent couplings g^0_{0j}, g^0_{1l}. In that case the correlation functions can also be expanded with respect to the independent couplings so that the reduced system resembles a renormalizable model.

If the scheme of renormalization is changed, the couplings and variable masses are transformed like

$$G^l_{ij} = \Gamma^l_{ij}(g^0_{01}, \dots, g^1_{3H}, m, \alpha, \kappa^2) \tag{2.11}$$

or

$$G^l = \Gamma^l(g^0, g^1, m, \alpha, \kappa^2)$$

in vector form. Here G^l and Γ^ldenote the vectors

$$G^l = (G^0_{01}, \dots, G^1_{2F}), \quad \Gamma^l = (\Gamma^0_{01}, \dots, \Gamma^1_{2F}). \tag{2.12}$$

These transformations can be expanded with respect to powers of the couplings g^u_{0t}, g^w_{1v}. In lowest order we have

$$\Gamma^l_{ij} = g^l_{ij} + \text{higher orders in} \quad g^u_{0t}, g^w_{1v}. \tag{2.13}$$

The correlation functions $\hat{\tau}$ in the new scheme are given by

$$\tau(k, g^0, g^1, m, \alpha, \kappa^2) = \hat{\tau}(k, G^0, G^1, m, \alpha, \kappa^2) \tag{2.14}$$

with the transformation (2.11) to be substituted for G^0, G^1. In the new scheme the renormalization group equations are

$$\kappa^2 \frac{\partial \hat{\tau}}{\partial \kappa^2} + \sum \hat{\beta}^u_{st} \frac{\partial \hat{\tau}}{\partial G^u_{st}} + \sum \delta_j \frac{\partial \hat{\tau}}{\partial \alpha_j} + \sum \gamma_j \hat{\tau} = 0 \tag{2.15}$$

with the coefficients

$$\hat{\beta}^u_{st} = \kappa^2 \frac{\partial \Gamma^u_{st}}{\partial \kappa^2} + \sum \beta^l_{ij} \frac{\Gamma^u_{st}}{\partial g^l_{ij}} + \sum \delta_j \frac{\partial \Gamma^u_{st}}{\partial \alpha^j}. \tag{2.16}$$

The functions (2.1) represent a surface S in the space of coordinates g^l_{ij}. By the transformation (2.11) the surface S will be mapped into a surface $\hat{S}$ in the space of coordinates G^l_{ij} which will be described by functions

$$G^1 = R(G^0, m, \alpha, \kappa^2), \qquad R = (R_{01}, \ldots, R_{2F}). \tag{2.17}$$

Inserting these functions into the transformed correlation functions we obtain a reduced system with the correlation functions

$$\hat{\tau}'(k, G^0, m, \alpha, \kappa^2) = \hat{\tau}(k, G^o, R(G^0, m, \alpha, \kappa^2), m, \alpha, \kappa^2). \tag{2.18}$$

In order to prove the scheme independence of the reduction principle we have to show that $\hat{\tau}'$ satisfies a renormalization group equation.

We begin with the construction of the functions (2.17). The surface S is mapped into the surface $\hat{S}$ by

$$G^0 = \Gamma^0(g^0, r(g, m, \alpha, \kappa^2), m, \alpha, \kappa^2) = L^0(g^0, m, \alpha, \kappa^2), \tag{2.19}$$

$$G^1 = \Gamma^1(g^0, r(g^0, m, \alpha, \kappa^2), m, \alpha, \kappa^2) = L^1(g^0, m, \alpha, \kappa^2) \tag{2.20}$$

(see Eqs. (2.1) and (2.11)). At given m, α and κ^2 the coordinates of G^l of $\hat{S}$ are thus expressed as functions of g^0 which we denote by L^l. For constructing the parametrization (2.17) we have to replace g^0 by G^0. To this end we invert (2.19) with respect to g^0,

$$g^0 = f(G^0, m, \alpha, \kappa^2) \quad (\text{inversion of} \quad G^0 = L^0(g^0, m, \alpha, \kappa^2)). \tag{2.21}$$

The inversion is possible for values of g^0 not too large, since

$$\frac{\partial L^0_{ij}}{\partial g^0_{st}} = \frac{\partial \Gamma^0_{ij}}{\partial g^0_{st}} + \sum \frac{\partial \Gamma^0_{ij}}{\partial g^1_{vw}} \frac{\partial g^1_{vw}}{\partial g^0_{st}} = \delta_{is}\delta_{jt} \tag{2.22}$$

$$\text{at} \quad g^0_{1p} = 0, \quad g^0_{1q} = 0$$

(see Eq. (2.13)). Substituting (2.21) for g^0 into (2.20) we obtain

$$G^1 = L^1(f(G^0, m, \alpha, , \kappa^2), m, \alpha, \kappa^2) = R(G^0, m, \alpha, \kappa^2). \tag{2.23}$$

By this we have constructed the parametrization (2.17) of the surface $\hat{S}$.

After this preparation we turn to the proof of the renormalization group equations for the functions $\hat{\tau}'$ defined by (2.18). Into the transformation law (2.14) of the correlation functions we substitute the reducing functions (2.1) and their image (2.17) for the variables g^1 or G^1 resp.,

$$\tau(k, g^0, r(g^0, m, \alpha, \kappa^2), m, \alpha, \kappa^2) = \hat{\tau}(k, G^0, R(G^0, m, \alpha, \kappa^2), m, \alpha, \kappa^2). \tag{2.24}$$

By definition (2.6) and (2.18) of τ' and $\hat{\tau}'$ this represents the transformation law for the correlation functions of the reduced system

$$\tau'(k, g^0, m, \alpha, \kappa^2) = \hat{\tau}'(k, G^0, m, \alpha, \kappa^2) \tag{2.25}$$

with (2.19) expressing the dependence of G^0 on g^0. Differentiating (2.25) with respect to κ^2, g^0_{ij} and α_j we get

$$\frac{\partial \tau'}{\partial \kappa^2} = \frac{\partial \hat{\tau}'}{\partial \kappa^2} + \sum \frac{\partial \hat{\tau}'}{\partial G^0_{st}} \frac{\partial L^0_{st}}{\partial \kappa^2} = \frac{\partial \hat{\tau}'}{\partial \kappa^2} + \sum \frac{\partial \hat{\tau}'}{\partial G^0_{st}} \frac{\partial \Gamma^0_{st}}{\partial \kappa^2} + \sum \frac{\partial \hat{\tau}'}{\partial G^0_{st}} \frac{\partial \Gamma^0_{st}}{\partial g^1_{vw}} \frac{\partial r_{vw}}{\partial \kappa^2}, \tag{2.26}$$

$$\frac{\partial \tau'}{\partial g^0_{ij}} = \sum \frac{\partial \hat{\tau}'}{\partial G^0_{st}} \frac{\partial L^0_{st}}{\partial g^0_{ij}} = \sum \frac{\partial \hat{\tau}'}{\partial G^0_{st}} \frac{\partial \Gamma^0_{st}}{\partial g^0_{ij}} + \sum \frac{\partial \hat{\tau}'}{\partial G^0_{st}} \frac{\partial \Gamma^0_{st}}{\partial g^1_{vw}} \frac{\partial r_{vw}}{\partial g^0_{ij}}, \tag{2.27}$$

$$\frac{\partial \tau'}{\partial \alpha_j} = \frac{\partial \hat{\tau}'}{\partial \alpha_j} + \sum \frac{\partial \hat{\tau}'}{\partial G^0_{st}} \frac{\partial L_{st}}{\partial \alpha_j} = \frac{\partial \hat{\tau}}{\partial \alpha_j} + \sum \frac{\partial \hat{\tau}'}{\partial G^0_{st}} \frac{\partial \Gamma^0_{st}}{\partial \alpha_j} + \sum \frac{\partial \hat{\tau}'}{\partial G^0_{st}} \frac{\partial \Gamma^0_{st}}{\partial g^1_{vw}} \frac{\partial r_{vw}}{\partial \alpha_j}. \tag{2.28}$$

Inserting these expressions into (2.7), (2.8) and using (2.9) first, then (2.16) (for $u = 0$), we obtain

$$\kappa^2 \frac{\partial \hat{\tau}'}{\partial \kappa^2} + \sum \hat{\beta}^0_{ij} \frac{\partial \hat{\tau}'}{\partial G^0_{ij}} + \sum \delta_j \frac{\partial \hat{\tau}'}{\partial \alpha_j} + \sum \gamma_j \hat{\tau}' = 0. \tag{2.29}$$

These are the renormalization group equations of the reduced system in the new scheme. Combining this result with the renormalization group equations (2.5) of the original system in the new scheme we find the differential equations

$$\kappa^2 \frac{\partial R_{st}}{\partial \kappa^2} + \sum \hat{\beta}^0_{ij} \frac{\partial R_{st}}{\partial G^0_{ij}} + \sum \delta_j \frac{\partial R_{st}}{\partial \alpha_j} = \hat{\beta}^1_{st} \tag{2.30}$$

for the reducing functions (2.17). This completes the proof for the scheme independence of the reduction principle.

It is easy to check that condition (2.10) – and the power series requirement as well – are scheme independent. We begin with transforming (2.10). By (2.13)

$$\begin{aligned} &\Gamma^0_{0s} = 0, \quad \Gamma^0_{1t} = 0, \quad \Gamma^1_{0u} = 0, \quad \Gamma^1_{1v} = 0, \\ \text{at} \quad &g^0_{0a} = 0, \quad g^0_{1b} = 0, \quad g^1_{0c} = 0, \quad g^1_{1d} = 0. \end{aligned} \tag{2.31}$$

Setting

$$g^0_{0a} = 0, \quad g^0_{1b} = 0,$$

it follows

$$r_{0c} = 0 \quad \text{and} \quad r_{1d} = 0$$

from (2.10) so that in (2.19), (2.20)

$$L^0_{0s} = 0, \quad L^0_{1t} = 0 \quad \text{at} \quad g^0_{0a} = 0, \quad g^0_{1b} = 0 \tag{2.32}$$

and

$$L^1_{0u} = 0, \quad L^1_{1v} = 0 \quad \text{at} \quad g^0_{0a} = 0, \quad g^0_{1b} = 0 \tag{2.33}$$

using (2.31). Since (2.19) is inverted uniquely by (2.21), (2.32) implies

$$f_{0a} = 0, \quad f_{0b} = 0 \quad \text{at} \quad G^0_{0s} = 0, \quad G^0_{1t} = 0. \tag{2.34}$$

Inserting (2.34) followed by (2.33) into (2.23) the final result

$$R_{0u} = 0, \quad R_{1v} = 0 \quad \text{at} \quad G^0_{0s} = 0, \quad G^0_{1t} = 0 \tag{2.35}$$

is obtained. This is the transformed version of (2.10) in the new scheme.

Similarly the power series requirement can be checked. An expansion of r and the expansion (2.13) implies that L^0 and L^1 as defined by (2.19) or (2.20) resp. can be expanded with respect to powers of g^0_{0a}, g^0_{0b}. The power series of L^0 may be inverted to a power series of f (see Eq. (2.21)) because of (2.22). Inserting the power series of f into (2.23) followed by the expansion of L^1 we find that the reducing functions R in the new scheme can be expanded with respect to powers of G^0_{0s} and G^0_{1t}. This completes the proof of the scheme independence for the condition that all couplings simultaneously approach zero and the additional requirement that the reducing functions can be expanded in the independent couplings.

3. Elimination of Parameters

A comprehensive treatment on the elimination of gauge and mass parameters is given in the work of Breitenlohner and Maison published in this volume [27]. In this section we discuss possibilities of eliminating parameters which are based on the renormalization group alone and should be sufficient for applications to the reduction method. Only minimal assumptions on the dynamics of the system will be needed for that purpose.

The aim is to find parameter transformations which lead to schemes with particularly simple β functions. In the last section the relations (2.16) served to determine the β functions β^u_{st} in a new scheme after applying a given transformation (2.11) to the parameters. A different point of view will be taken now: We consider the β functions $\hat{\beta}^u_{st}$ as given in a suitable form and determine transformations (2.11) as solutions of Eqs. (2.16).

Postponing the removal of masses as a second step we discuss the elimination of gauge parameters first. For this purpose we consider (2.16) with $\hat{\beta}^u_{st}$ taken to be the values of the β functions in the Landau gauge. In this case solutions of (2.16) can be found, but in general they involve additional parameters carrying a dimension or require a positive lower bound for the masses. Thus the correlation functions will either depend on new mass parameters or a final elimination of masses is impossible. But using a few simple consequences of gauge invariance parameter transformations can be constructed as solutions of (2.16) which do not introduce new parameters and apply to a range of mass values including the massless limit. A detailed treatment of this possibility for eliminating the gauge parameters will be given in another publication. For the remainder of this section it will be assumed that the gauge parameters have been removed.

We next turn to the problem of eliminating masses. First we consider models with parameters

$$\lambda_0, \lambda_1, \ldots, \lambda_n; m_1, \ldots, m_I; \zeta. \tag{3.1}$$

The couplings λ_i are all dimensionless. The mass parameters m_j denote pole masses defined by the location of the lowest propagator singularities. The normalization mass κ is replaced by its inverse

$$\zeta = \frac{1}{|\kappa|} \tag{3.2}$$

which is more convenient for the discussion of the massless limit. Opposite signs of the same coupling parameter are interpreted as belonging to different models, unless the square may be used instead of the original coupling parameter in the renormalization group analysis. For a specific model each coupling parameter is defined such that

$$\lambda_j \geq 0 \tag{3.3}$$

by changing sign, if necessary. The renormalization group equations (2.5) simplify to

$$\sum \beta_s \frac{\partial \tau}{\partial \lambda_s} + \sum \gamma_s \tau - \frac{1}{2}\zeta \frac{\partial \tau}{\partial \zeta} = 0 \tag{3.4}$$

with

$$\beta_s = \beta_s(\lambda_0, \lambda_1, \ldots, \lambda_n, m_1\zeta, \ldots, m_I\zeta) \tag{3.5}$$

(similarly for γ_s). In this and the following section it is assumed that the β functions are differentiable and do not vanish in[4]

$$(\lambda_0, \ldots, \lambda_n) \in \mathcal{D}, \quad 0 \leq m_j\zeta < \pi_j, \quad \zeta > 0, \tag{3.6}$$

[4] Instead of differentiability Lipschitz conditions would be sufficient for the existence theorems applied in this paper.

where $\mathcal{D}$ is a bounded domain in the sector $\lambda_j > 0 \quad (j = 1, \ldots, \lambda_n)$ with the origin on the boundary of $\mathcal{D}$. In the simplest case a cube

$$0 < \lambda_j < \omega_j \quad (j = 0, ..., n), \quad \omega_j > 0,$$

may be chosen for $\mathcal{D}$. The interior of a cone section in $\lambda_j > 0 \quad (j = 1, \ldots, n)$ with tip at the origin should be sufficiently general. This assumption excludes the case that the β functions vanish identically and restricts (3.6) by an appropriate boundary such that non-trivial zeroes of the β functions remain outside. Moreover, by (3.5) and (3.6) the massless limit

$$\hat{\beta}_j(\lambda_0, \ldots, \lambda_n) = \beta_j(\lambda_0, \ldots, \lambda_n, 0, \ldots, 0) \tag{3.7}$$

exists independently of the way the limit $m_j \to 0$ is taken.

We want to change the scheme by constructing a transformation (2.11),

$$\Lambda_j = \Gamma_j(\lambda_0, \lambda_1, \ldots, \lambda_n, m, \zeta), \quad m = (m_1, \ldots, m_I), \tag{3.8}$$

which leads to renormalization group equations

$$\sum \hat{\beta}_s \frac{\partial \hat{\tau}}{\partial \Lambda_s} + \sum \gamma_s \hat{\tau} - \frac{1}{2}\zeta \frac{\partial \hat{\tau}}{\partial \zeta} = 0 \tag{3.9}$$

with the massless β functions (3.7),

$$\hat{\beta}_s = \hat{\beta}_s(\Lambda_0, \ldots, \Lambda_n). \tag{3.10}$$

The transformations (3.8) are solutions of the partial differential equations (2.16),

$$\sum \beta_s \frac{\partial \Gamma_j}{\partial \lambda_s} - \frac{1}{2}\zeta \frac{\partial \Gamma_j}{\partial \zeta} = \hat{\beta}_j. \tag{3.11}$$

There are many solutions of (3.11). A unique solution can be constructed, for instance, by adjusting the new couplings to the old ones at a normalization mass $\kappa = \kappa_0$, i.e.

$$\Lambda_j = \lambda_j \quad \text{at} \quad \zeta = \zeta_0 = 1/|\kappa_0| > 0. \tag{3.12}$$

The existence of such a solution will be proved in a region (3.6). For given mass values the functions (3.8) represent an $(n+2)$-dimensional surface S in the $(2n+3)$-dimensional space of coordinates $\lambda_i, \Lambda_j, \zeta$. A solution of (3.11) must be found for which S contains the $(n+l)$-dimensional surface S_0 given by (3.12). The characteristic determinants of the $n+1$ equations (3.11) are identical and have the value $-\frac{1}{2}\zeta_0$ on the surface S_0. Thus the characteristic determinants do not vanish at $\zeta = \zeta_0 > 0$. Therefore, a unique solution of (3.11) exists which satisfies the initial conditions (3.12)[5]. In this way a new scheme of renormalization is defined for which the β functions are those of the massless model. By this construction, however, a new dimensional parameter κ_0^2 is introduced. The β functions of the new scheme do not depend on it, but the transformation (3.8) as well as the correlation functions $\hat{\tau}$ in the new scheme involve this parameter κ_0. Moreover, the dependence on κ_0 is not controlled by the renormalization group equation.

Instead, a satisfactory method of eliminating masses is provided by adjusting the couplings

$$\Lambda_j = \lambda_j \quad \text{at} \quad \zeta = 0. \tag{3.13}$$

[5] See ref. [35], Chapter 2 and ref. [36], Chapter 2.2.

This condition may be interpreted as adjusting the couplings of the old and the new scheme for $|\kappa_0^2| \to \infty$. The procedure should not be confused with trying to normalize coupling parameters at infinite momentum. Even in the case of asymptotic freedom such normalization is not easily possible, since then all effective couplings vanish in the high momentum limit. In contradistinction the issue here is to find solutions of the partial differential equations (3.11) satisfying the initial conditions (3.13) with the β functions (3.5) and (3.10). The choice of boundary conditions (3.13) seems to be particularly natural, since the new β functions $\hat{\beta}_s$ are the limits of the original β functions for vanishing ζ,

$$\hat{\beta}_s = \lim_{\zeta \to 0} \beta_s(\lambda_0, \ldots, \lambda_n, m_1\zeta, \ldots, m_I\zeta). \tag{3.14}$$

For the method to work this limit should exist, of course. But it should be stressed that the massless limit of the correlation functions is not required here.

It will be shown that indeed a power series solution of (3.11) can be constructed uniquely by imposing condition (3.13). An existence and uniqueness proof which is not based on expansions is also possible, but requires the use of Callan–Symanzik equations in addition as in the work of Breitenlohner and Maison [27]. For the construction of the power series expansions a few assumptions concerning the limit (3.14) will be made. In the formal expansions

$$\begin{gathered} \beta_j = \sum \beta_{j\mu} \lambda_0^{\mu_0} \cdots \lambda_n^{\mu_n}, \quad \mu = (\mu_0, \ldots, \mu_n), \\ M = \sum \mu_j \geq 2, \end{gathered} \tag{3.15}$$

the coefficients

$$\begin{aligned} \beta_{j\mu} &= \beta_{j\mu}(m_1, \ldots, m_I; \zeta) \\ &= \beta_{j\mu}(\nu_1, \ldots, \nu_I), \quad \nu_j = m_j\zeta, \end{aligned} \tag{3.16}$$

are assumed to exist in a region including the massless case $\zeta = 0$. The expansions of the β functions in the new scheme are then

$$\hat{\beta}_j = \sum \hat{\beta}_{j\mu} \lambda_0^{\mu_0} \cdots \lambda_n^{\mu_n} \tag{3.17}$$

with the constants $\hat{\beta}_{j\mu}$ given as the values of $\beta_{j\mu}$ at $\zeta = 0$,

$$\hat{\beta}_{j\mu} = \beta_{j\mu}(0, \ldots, 0). \tag{3.18}$$

It is further assumed that the value $\hat{\beta}_{j\mu}$ is approached smoothly by $\beta_{j\mu}$ in the limit $\zeta \to 0$. The condition that

$$\Delta\beta_{j\mu}(m_1\zeta, \ldots, m_I\zeta)| \leq a_{j\mu}\zeta^{\epsilon_{j\mu}}, \quad \text{if} \quad 0 < \zeta < z \tag{3.19}$$

will be sufficient for the deviations

$$\Delta\beta_{j\mu} = \beta_{j\mu} \tag{3.20}$$

from the zero mass values. The numbers $a_{j\mu}$, $\epsilon_{j\mu}$ and z are suitably chosen with

$$a_{j\mu} > 0, \quad 0 < \epsilon_{j\mu} < 1, \quad z > 0.$$

The aim is to solve (3.11) by a formal expansion

$$\begin{aligned}\Lambda_s &= \Gamma_s(\lambda_0, \lambda_1, \dots, \lambda_n, m_1, \dots, m_I, \zeta) \\ &= \sum \Lambda_{s\mu}(m_1, \dots, m_I; \zeta)\lambda_0^{\mu_0} \cdots \lambda_n^{\mu_n}\end{aligned} \tag{3.21}$$

with the initial condition (3.13) imposed. This implies

$$\Lambda_{s\mu}(m_1, \dots, m_I; 0) = 0 \tag{3.22}$$

for all coefficients except

$$\Lambda_{s(s)}(m_1, \dots, m_I; 0) = 1 \tag{3.23}$$

for the coefficient of λ_s.

For the low order terms of the β functions (3.15), (3.17) and the transformation (3.21) we use the simplified notation

$$\beta_j = \frac{1}{2}\sum b_j^{kl}\lambda_k\lambda_l + \cdots, \tag{3.24}$$

$$\hat{\beta}_j = \frac{1}{2}\sum \hat{b}_j^{kl}\Lambda_k\Lambda_l + \cdots, \tag{3.25}$$

$$\Delta b_j^{kl} = b_j^{kl} - \hat{b}_j^{kl}, \tag{3.26}$$

$$\Lambda_s = L_s + \sum L_s^k\lambda_k + \frac{1}{2}\sum L_s^{kl}\lambda_k\lambda_l. \tag{3.27}$$

The differential equations (3.11) imply

$$\begin{aligned}\frac{\partial L_s}{\partial\zeta} &= 0, \quad \frac{\partial L_s^k}{\partial\zeta} = 0, \\ L_s &= 0, \qquad L_s^k = \delta_{sk}\end{aligned}$$

by the conditions (3.22), (3.23). With this the expansion (3.21) takes the form

$$\Lambda_s = \lambda_s + \sum_{M\geq 2} \Lambda_{s\mu}(m_1, \dots, m_I; \zeta)\lambda_0^{\mu_0} \cdots \lambda_n^{\mu_n}. \tag{3.28}$$

In the notation of (3.24)–(3.27) we obtain the differential equations

$$\frac{1}{2}\zeta\frac{\partial L_s^{kl}}{\partial\zeta} = \Delta b_s^{kl} \tag{3.29}$$

for the coefficients of the quadratic terms. The solutions are

$$L_s^{kl} = 2\int_0^{\zeta} \Delta b_s^{kl}(m_1 x, \dots, m_I x)\frac{dx}{x}. \tag{3.30}$$

By (3.19) the integrals converge, additional constants of integration vanish due to the initial condition (3.13).

For treating higher orders we proceed by induction. The hypothesis of induction is: On the basis of the differential equations (3.11) with the initial conditions (3.13) all coefficients

$$\Lambda_{s\mu} = \Lambda_{s\mu}(m_1, ..., m_I; \zeta) \tag{3.31}$$

of the expansion (3.28) with

$$2 \leq M = \sum \mu_j < N \tag{3.32}$$

have been constructed. This construction is unique and it has been shown that the coefficients (3.31) are bounded by

$$|\Lambda_{s\mu}(m_1, \dots m_I; \zeta)| \leq c_{s\mu} \zeta^{\eta_{s\mu}}, \quad \text{if} \quad 0 < \zeta < u_{s\mu}, \tag{3.33}$$

for suitable numbers $c_{s\mu}, \eta_{s\mu}, u_{s\mu}$ with

$$c_{s\mu} > 0, \quad 0 < \eta_{s\mu} < 1, \quad u_{s\mu} > 0.$$

We remark that (3.33) holds for the integral (3.30) as a consequence of (3.19).

It will now be shown that each coefficient

$$\Lambda_{t\nu} = \Lambda_{t\nu}(m_1, \dots, m_I; \zeta), \quad \nu = (\nu_0, \dots, \nu_n), \tag{3.34}$$

with

$$\sum \nu_j = N$$

is also determined uniquely by (3.11), (3.13) and bounded similarly to (3.33). Equation (3.11) implies the differential equation

$$\beta_{t\nu} - \frac{1}{2}\zeta \frac{\partial \Lambda_{s\nu}}{\partial \zeta} + \sum_l E^l_{t\nu} = \hat{\beta}_{t\nu} \tag{3.35}$$

for (3.34). The terms $E^l_{t\nu}$ are determined by lower orders only with $M < N$. They are monomials in the coefficients (3.31) with (3.32) and involve coefficients of the β functions. Therefore, they are bounded similarly to (3.33). Equation (3.35) is solved by

$$\Lambda_{t\nu} = 2\int_0^\zeta \Delta\beta_{t\nu}(m_1 x, \dots, m_I x)\frac{dx}{x} + 2\sum_l \int_0^\zeta E^l_{t\nu}(m_1, \dots, m_I; x)\frac{dx}{x}. \tag{3.36}$$

Due to (3.19) and similar bounds for $E^l_{t\nu}$ all integrals converge and are again bounded like (3.33). Therefore, (3.33) also holds for $\Lambda_{t\nu}$. This completes the proof of induction.

On the basis of formal expansions it is thus possible to construct a scheme of renormalization in which the β functions do not depend on the pole masses m_j nor on the normalization mass κ. This result will now be applied to the reduction of a model involving the parameters (3.1) with λ_0 chosen as primary coupling. For a set of reducing functions

$$\lambda_j = r_j(\lambda_0, m_1\zeta, \dots, m_I\zeta), \tag{3.37}$$

the reduction equations (2.9) take the form

$$\beta_0' \frac{\partial r_j}{\partial \lambda_0} - \frac{1}{2}\zeta \frac{\partial r_j}{\partial \zeta} = \beta_j' \quad (j = 1, \dots, n), \tag{3.38}$$

$$\beta_j' = \beta_j(\lambda_0, r_1, \dots, r_n, m_1\zeta, \dots, m_I\zeta). \tag{3.39}$$

The reducing functions are supposed to satisfy the condition

$$\lim_{\lambda_0 \to 0} r_j = 0 \tag{3.40}$$

or the stronger power series requirement

$$\begin{aligned} r_j &= \sum_{l=1}^{\infty} c_{jl} \lambda_0^l, \\ c_{jl} &= c_{jl}(m_1\zeta, \ldots, m_I\zeta). \end{aligned} \tag{3.41}$$

After transforming to massless β functions (3.37) is mapped into

$$\Lambda_j = R_j(\Lambda_0, m_1\zeta, \ldots, m_I\zeta)$$

satisfying

$$\hat{\beta}'_0 \frac{\partial R_j}{\partial \Lambda_0} - \frac{1}{2}\zeta \frac{\partial R_j}{\partial \zeta} = \hat{\beta}'_j \qquad (j = 1, \ldots, n), \tag{3.42}$$

$$\begin{aligned} \hat{\beta}'_j &= \hat{\beta}'_j(\Lambda_0, R_1, \ldots, R_n) \\ &= \beta_j(\Lambda_0, R_1, \ldots, R_n, 0, \ldots, 0). \end{aligned}$$

Although the β functions do not explicitly depend on m_j or ζ, such dependence cannot be excluded for the solutions r_j. But it will be shown in the following section that any ζ-dependent solution of (3.42) may be replaced by an equivalent solution of the same equations which is independent of ζ. Therefore, we may set

$$\frac{\partial R_j}{\partial \zeta} = 0$$

in (3.42) and solve the ordinary differential equations

$$\hat{\beta}'_0 \frac{dR_j}{d\Lambda_0} = \hat{\beta}'_j \quad (j = 1, \ldots, n) \tag{3.43}$$

by functions

$$\Lambda_j = R_j(\Lambda_0)$$

with the requirements

$$\lim_{\Lambda_0 \to 0} R_j = 0 \tag{3.44}$$

or the stronger power series condition

$$R_j = \sum_{l=1}^{\infty} C_{jl} \Lambda_0^l. \tag{3.45}$$

We conclude this section by making some brief remarks on the elimination of the normalization mass and the reduction method for models involving dimensional couplings and variable mass squares as in ref. [24]. The parameters are denoted by

- dimensionless couplings $\lambda_0, \lambda_1, \ldots, \lambda_n$,
- couplings of dimension mass $\xi_1^0, \ldots, \xi_B^0, \quad \xi_1^1, \ldots, \xi_F^1$,
- variable mass squares $\omega_1^0, \ldots, \omega_C^0, \quad \omega_1^1, \ldots, \omega_G^1$,
- inverse normalization mass $\zeta = 1/|\kappa|$.

The independent parameters are

$$\lambda_0, \xi_1^0, \ldots, \xi_B^0, \omega_1^0, \ldots, \omega_C^0, \tag{3.46}$$

while the parameters

$$\lambda_1, \ldots, \lambda_n, \xi_1^1, \ldots, \xi_F^1, \omega_1^1, \ldots, \omega_G^1 \tag{3.47}$$

are treated as functions depending on (3.46),

$$\begin{aligned} \lambda_t &= r_t(\lambda_0, \xi^0, \omega^0, \zeta) \quad (t = 1, \ldots, n), \\ \xi_t^1 &= r_{1t}(\lambda_0, \xi^0, \omega^0, \zeta) \quad (t = 1, \ldots, F), \\ \omega_t^1 &= r_{2t}(\lambda_0, \xi^0, \omega^0, \zeta) \quad (t = 1, \ldots, G) \end{aligned} \tag{3.48}$$

with the vector notation

$$\xi^0 = (\xi_1^0, \ldots, \xi_B^0), \quad \omega^0 = (\omega_1^0, \ldots, \omega_C^0). \tag{3.49}$$

The renormalization group equations (2.5) are

$$\sum_j \beta_j \frac{\partial \tau}{\partial \lambda_j} + \sum_{jl} \beta_{1j}^l \frac{\partial \tau}{\partial \xi_j^1} + \sum_{jl} \beta_{2j}^l \frac{\partial \tau}{\partial \omega_j^l} + \sum \gamma_j \tau - \frac{1}{2} \zeta \frac{\partial \tau}{\partial \zeta} = 0. \tag{3.50}$$

Taking into account the dimensionality of the β functions we write the representations

$$\begin{aligned} \beta_t &= \beta_t, \\ \beta_{1t}^u &= \sum \beta_{1tk}^{ui} \xi_k^i, \\ \beta_2^u t &= \sum \beta_{2tk}^{ui} \omega_k^i + \sum \beta_{2tkl}^{uij} \xi_k^i \xi_l^j \end{aligned} \tag{3.51}$$

with coefficients

$$F = \beta_t, \beta_{1tk}^{ui}, \beta_{2tk}^{ui}, \beta_{2tkl}^{uij}$$

depending on dimensionless ratios only

$$F = F(\lambda_0, \lambda_1, \ldots, \lambda_n, \zeta\xi^0, \zeta\xi^1, \zeta^2\omega^0, \zeta^2\omega^1). \tag{3.52}$$

Terms involving ζ^{-1} or ζ^{-2} with non-vanishing coefficients for $\zeta \to 0$ should not be expected in realistic models. It is assumed that the limits

$$\hat{\beta}_t' = \lim_{\zeta \to 0} \beta_t, \quad \hat{\beta}_{st}^u = \lim_{\zeta \to 0} \beta_{st}^u \tag{3.53}$$

exist. By (3.51) and (3.52) these limits yield quadratic forms in the dimensional couplings and masses with coefficients depending on the dimensionless couplings. The reduction equations (2.9) take the form

$$\beta_0' \frac{\partial r_t}{\partial \lambda_0} + \sum \beta_{1j}'^0 \frac{\partial r_t}{\partial \xi_j^0} + \sum \beta_{2j}^0 \frac{\partial r_t}{\partial \omega_j^0} - \frac{1}{2}\zeta \frac{\partial r_t}{\partial \zeta} = \beta_t', \tag{3.54}$$

$$\beta_0' \frac{\partial r_{st}}{\partial \lambda_0} + \sum \beta_{1j}'^0 \frac{\partial r_{st}}{\partial \xi_j^0} + \sum \beta_{2j}'^0 \frac{\partial r_{st}}{\partial \omega_j^0} - \frac{1}{2}\zeta \frac{\partial r_{st}}{\partial \zeta} = \beta_{st}' \tag{3.55}$$

with primes indicating the insertion of the reducing functions. On the basis of formal power series expansions a transformation to a scheme can be constructed for which the β functions assume their value at $\zeta = 0$. Details will not be given in this paper. The transformed coupling and mass parameters are denoted by

$$\begin{gathered} \Lambda_0, \Lambda_1, \ldots, \Lambda_n, \\ \Xi_1^0, \ldots, \Xi_B^0, \Xi_1^1, \ldots, \Xi_F^1, \\ \Omega_1^0, \ldots, \Omega_C^0, \Omega_1^1, \ldots, \Omega_G^1. \end{gathered} \tag{3.56}$$

For the transformed reducing functions we write the representations

$$\Lambda_t = R_t(\Lambda_0, \Xi^0, \Omega^0, \zeta) = R_t(\Lambda_0, \zeta\,\Xi^0, \zeta^2\Omega^0), \tag{3.57}$$

$$\Xi_t^1 = R_{1t}(\Lambda_0, \Xi^0, \Omega^0, \zeta) = \sum S_{tk}\,\Xi_k^0 + S_t^0\zeta^{-1}, \tag{3.58}$$

$$\begin{aligned} \Omega_t^1 &= R_{2t}(\Lambda_0, \Xi^0, \Omega^0, \zeta) \\ &= \sum T_{tk}\Omega_k^0 + S_t^0\zeta^{-2} + \sum T_{tkl}\,\Xi_k^0\Xi_l^0 + \sum T_{tk}^0\,\Xi_k^0\zeta^{-1}. \end{aligned} \tag{3.59}$$

Here the coefficients

$$F = S_{tk}, S_k^0, T_{tk}, T_t^0, T_{tkl}, T_{tk}^0$$

depend on dimensionless ratios

$$F = F(\Lambda_0, \zeta\,\Xi^0, \zeta^2\Omega^0). \tag{3.60}$$

In the transformed version of the reduction equations

$$\hat\beta_0' \frac{\partial R_t}{\partial \Lambda_0} + \sum \hat\beta_{1j}'^0 \frac{\partial R_t}{\partial \Xi_j^0} + \sum \hat\beta_{2j}'^0 \frac{\partial R_t}{\partial \Omega_j^0} - \frac{1}{2}\zeta \frac{\partial R_t}{\partial \zeta} = \hat\beta_t', \tag{3.61}$$

$$\hat\beta_0' \frac{\partial R_{st}}{\partial \Lambda_0} + \sum \hat\beta_{1j}'^0 \frac{\partial R_{st}}{\partial \Xi_j^0} + \sum \hat\beta_{2j}'^0 \frac{\partial R_{st}}{\partial \Omega_j^0} - \frac{1}{2} \frac{\partial R_{st}}{\partial \zeta} = \hat\beta_{st}', \tag{3.62}$$

the β functions are ζ-independent. Therefore, it is consistent (and can be justified by an equivalence argument) that the reducing functions (3.57)–(3.59) do not depend on ζ. This excludes terms involving ζ^{-1} or ζ^{-2}. In the remaining terms ζ may be set equal to zero so that the coefficients (3.60) become independent of masses and dimensional couplings. Thus

$$\Lambda_t = R_t(\Lambda_0), \tag{3.63}$$

$$\Xi_t^1 = R_{1t}(\Lambda_0, \Xi^0) = \sum S_{tk}\,\Xi_k^0, \tag{3.64}$$

$$\begin{aligned}\Omega_t^1 &= R_{2t}(\Lambda_0, \Xi^0, \Omega^0) \\ &= \sum T_{tk}\Omega_k^0 + \sum T_{tkl}\xi_k^0\,\Xi_l^0.\end{aligned} \tag{3.65}$$

After insertion of (3.63) the β functions take the form

$$\begin{aligned}\hat{\beta}_t' &= \phi_t(\Lambda_0), \\ \hat{\beta}_{1t}'^{u} &= \sum \chi_{tk}^{ui}(\Lambda_0)\,\Xi_k^i, \\ \hat{\beta}_{2t}'^{u} &= \sum \psi_{tk}^{ui}(\Lambda_0)\Omega_k^i + \sum \psi_{tkl}^{uij}(\Lambda_0)\,\Xi_k^i\,\Xi_l^j.\end{aligned} \tag{3.66}$$

Here (3.64) and (3.65) should be substituted for the variables Ξ_k^1 and Ω_l^1. Eventually the β functions and the reducing functions become expressed as quadratic forms of the independent variables Ξ_k^0 and Ω_l^0. Using

$$\frac{\partial R_t}{\partial \Xi_j^0} = 0, \quad \frac{\partial R_t}{\partial \Omega_j^0} = 0, \quad \frac{\partial R_t}{\partial \zeta} = 0,$$
$$\frac{\partial R_{1t}}{\partial \Omega_j^0} = 0, \quad \frac{\partial R_{1t}}{\partial \zeta} = 0, \quad \frac{\partial R_{2t}}{\partial \zeta} = 0$$

the reduction equations (3.61), (3.62) simplify considerably. With the representations (3.63)–(3.66) a first order system of ordinary differential equations is found for the coefficients

$$R_t,\ S_{tk},\ T_{tk},\ T_{tkl}$$

of the reducing functions (3.63)–(3.65). The final result are Eqs. (2)–(11) of ref. [24].

4. Evolution Equations and Asymptotic Freedom

In this section evolution equations will be studied in connection with asymptotic freedom and reduction for models involving dimensionless couplings and masses defined by propagator singularities. For the notation see (3.1). Effective couplings

$$\bar{\lambda}_j = \bar{\lambda}_j(z, m; \lambda_0, \lambda_1, \dots, \lambda_n, \zeta) \quad (j = 0, \dots, n), \tag{4.1}$$

$$z = \frac{1}{|k|}, \quad \zeta = \frac{1}{|\kappa|}, \quad m = (m_1, \dots, m_I),$$

depending on a momentum square k^2 are introduced by suitable vertex functions with initial values at the normalization point,

$$\bar{\lambda}_j = \lambda_j > 0 \quad \text{at} \quad z = \zeta > 0. \tag{4.2}$$

For the effective couplings evolution equations hold in the form

$$-\frac{1}{2}z\frac{d\bar{\lambda}_j}{dz} = \beta_j(\bar{\lambda}_0, \ldots, \bar{\lambda}_n, m_1 z, \ldots, m_n z) \tag{4.3}$$

with the initial values (4.2). The masses and initial values we restrict by (3.6), likewise z and the values $\bar{\lambda}_j$ assumed by the solutions of (4.3). Then by the Cauchy–Picard theorem a unique solution (4.1) of (4.3) exists with initial values (4.2). Unless the dependence on the initial values λ_j, ζ is relevant, the simplified notation

$$\bar{\lambda}_j = \bar{\lambda}_j(z, m) \tag{4.4}$$

will be used instead of (4.1).

Asymptotic freedom means that all effective couplings vanish in the high momentum limit

$$\lim_{z\to\infty} \bar{\lambda}_j(z, m) = 0 \quad (j = 0, 1, \ldots, n). \tag{4.5}$$

In the case of several couplings this is not a property of the model as such, but selects, if at all possible, particular solutions of the evolution equations, while other solutions are not asymptotically free. By imposing (4.5) the couplings are no longer independent. In fact, it will be seen that (4.5) induces a reduction of couplings.

Since zeroes are absent in the domain (3.6), the evolution equations (4.3) imply that each effective coupling is either monotonically increasing or decreasing. Therefore, condition (4.5) combined with convention (3.3) implies

$$\frac{d\bar{\lambda}_j}{dz} > 0 \tag{4.6}$$

and

$$\beta_j(\bar{\lambda}_0, \ldots, \bar{\lambda}_n, m_1 z, \ldots, m_I z) < 0 \tag{4.7}$$

for asymptotically free couplings $\bar{\lambda}_j$ on the domain (3.6). Thus a negative sign for the β functions is a necessary condition for asymptotic freedom. It is, however, – unlike the case of a single coupling – not sufficient in general. Sufficient conditions will be stated later after elimination of mass parameters in the β functions. In preparation for this, how evolution equations transform under a change of the renormalization scheme will be discussed.

After a scheme changing transformation (3.7) new effective couplings may be defined by

$$\bar{\Lambda}_j = \Gamma_j(\bar{\lambda}_0, \ldots, \bar{\lambda}_n, m, z). \tag{4.8}$$

Through the dependence (4.4) the transformed couplings (4.8) also become functions of z and m with initial values

$$\bar{\Lambda}_j = \Lambda_j \quad \text{at} \quad z = \zeta. \tag{4.9}$$

For these functions the notation

$$\bar{\Lambda}_j = \bar{\Lambda}_j(z, m; \Lambda_0, \ldots, \Lambda_n, \zeta), \tag{4.10}$$

or simpler,

$$\bar{\Lambda}_j = \bar{\Lambda}_j(z, m) \tag{4.11}$$

will be used. With Eq. (3.11) it is easy to check that the new effective couplings again satisfy evolution equations in the form

$$-\frac{1}{2} z \frac{d\bar{\Lambda}_j}{dz} = \hat{\beta}_j(\bar{\Lambda}_0, \ldots, \bar{\Lambda}_n, m_1 z, \ldots, m_I z). \tag{4.12}$$

The condition (4.5) of asymptotic freedom is scheme independent. For a Taylor formula

$$\bar{\Lambda}_j = \bar{\lambda}_j + \sum \bar{\lambda}_s \bar{\lambda}_t R^j_{st} \tag{4.13}$$

with appropriate remainders R^j_{st} holds according to the properties of transformations (2.11) stated in the last section. Thus (4.5) implies the corresponding condition

$$\lim_{z \to 0} \bar{\Lambda}_j(z, m) = 0 \quad (j = 1, \ldots, n) \tag{4.14}$$

in the new scheme.

The scheme independence justifies studying asymptotic freedom in a special scheme, where the β functions are massless. The evolution equations then take the simplified form

$$-\frac{1}{2} z \frac{d\bar{\Lambda}_j}{dz} = \hat{\beta}_j(\bar{\Lambda}_0, \ldots, \bar{\Lambda}_n) \quad (j = 0, \ldots, n) \tag{4.15}$$

with $\hat{\beta}_j$ denoting the massless limit (3.7). For asymptotically free solutions we write (4.6) and (4.7) in transformed form

$$\frac{d\bar{\Lambda}_j}{dz} > 0, \tag{4.16}$$

$$\hat{\beta}_j(\bar{\Lambda}_0, \ldots, \bar{\Lambda}_n) < 0. \tag{4.17}$$

With massless β functions it is possible to treat asymptotic freedom in two separate steps: First, all couplings are reduced to functions of a primary coupling, then the high momentum behavior is determined by a single evolution equation involving the primary coupling only. In order to show this we select $\bar{\Lambda}_0$ as a primary coupling and introduce it in (4.15) as an independent variable instead of z. Because of (4.16) the function

$$\bar{\Lambda}_0 = \bar{\Lambda}_0(z, m) \tag{4.18}$$

may be inverted to

$$z = \bar{\zeta}(\bar{\Lambda}_0, m). \tag{4.19}$$

By this all $\bar{\Lambda}_j$ may be expressed as functionals of $\bar{\Lambda}_0$,

$$\bar{\Lambda}_j = \bar{\Lambda}_j(z, m) = \bar{\Lambda}_j(\bar{\zeta}(\bar{\Lambda}_0, m), m), \tag{4.20}$$

which we denote by

$$\bar{\Lambda}_j = \bar{s}_j(\bar{\Lambda}_0, m), \quad j = 1, \ldots, n. \tag{4.21}$$

Introducing $\bar{\Lambda}_0$ as an independent variable the system (4.15) takes the equivalent form

$$\hat{\beta}_0' \frac{d\bar{\zeta}}{d\bar{\Lambda}_0} = -\frac{1}{2}\bar{\zeta}, \tag{4.22}$$

$$\hat{\beta}_0' \frac{d\bar{s}_j}{d\bar{\Lambda}_0} = \hat{\beta}_j' \tag{4.23}$$

with the notation

$$\hat{\beta}_j' = \beta_j(\bar{\Lambda}_0, \bar{s}_1(\bar{\Lambda}_0, m), \ldots, \bar{s}_n(\bar{\Lambda}_0, m)). \tag{4.24}$$

Equation (4.22) is integrated by

$$\lg \bar{\zeta} = \frac{1}{2}\int_{\bar{\Lambda}_0}^{c} \frac{dx}{\tilde{\beta}_0} + d, \quad c > \bar{\Lambda}_0, \tag{4.25}$$

$$\tilde{\beta}_0 = \beta_0(x, \bar{s}_1(x, m), \ldots, \bar{s}_n(x, m)).$$

Equation (4.14) may be written equivalently as

$$\lim_{\bar{\Lambda}_0 \to 0} \bar{\zeta}(\bar{\Lambda}_0, m) = 0, \tag{4.26}$$

$$\lim_{\bar{\Lambda}_0 \to 0} \bar{s}_j(\bar{\Lambda}_0, m) = 0. \tag{4.27}$$

Equations (4.23) constitute reduction equations for the reducing functions (4.21) of the primary coupling $\bar{\Lambda}_0$ with the condition (4.27) to be imposed. With the solution of the reduction equations (4.21) the evolution of the system becomes a problem in one variable only: Eq. (4.22) or (4.25) controls the momentum dependence of the primary coupling $\bar{\Lambda}_0$ in the high momentum limit. Depending on the sign of $\tilde{\beta}_0$ for small x the divergence of the integral for small couplings implies either $\bar{\zeta} \to 0$ or $\bar{\zeta} \to \infty$ for $\bar{\Lambda}_0 \to 0$. The results of this analysis are summarized by the following necessary and sufficient conditions for asymptotic freedom:

Among the effective couplings a primary coupling $\bar{\Lambda}_0$ is chosen so that the other couplings $\bar{\Lambda}_j$ become functions of $\bar{\Lambda}_0$. These functions should satisfy the reduction equations (4.23) with the requirement (4.27) that the couplings vanish together with $\bar{\Lambda}_0$. The β function of $\bar{\Lambda}_0$ should be negative for sufficiently small couplings after inserting the solution of (4.23).

As a corollary we note that for asymptotically free couplings all β functions simultaneously become negative for small couplings. More generally, as a consequence of (4.27) reduction solutions of (4.23) satisfy

$$\frac{d\bar{s}_j}{d\bar{\Lambda}_0} > 0 \tag{4.28}$$

in (3.6) due to the absence of zeroes of the β functions and the convention (3.3). This means that all β functions have the same sign for small couplings. Negative sign corresponds to asymptotic freedom in the original sense. Positive sign of the β functions can be interpreted as asymptotic freedom in the infrared region. This is relevant for models without intrinsic masses. Not discussed in this paper is the case that β functions vanish identically for some solutions of the evolution equations.

We return to the theory of reduction in general schemes of renormalization. In the last section it was found that the reduction equations still involve the normalization mass after transforming to massless β functions. The resulting reduction equations in the case of asymptotic freedom seem to indicate that such a dependence should not be expected. It will be shown that indeed the normalization mass can be eliminated independently of the scheme by making use of the evolution equations.

We begin by setting up the evolution equations of the reduced model. To this end we combine the reduction equations (3.38) with the original form (4.3) of the evolution equations. As initial values (4.2) for the solutions (4.1) of (4.3) reducing functions r_j will be taken:

$$\bar{\lambda}_0 = \lambda_0, \quad \bar{\lambda}_j = r_j(\lambda_0, m, \zeta) \quad \text{at} \quad z = \zeta \quad (j = 1, \ldots, n). \tag{4.29}$$

The functions r_j are supposed to obey the reduction equations (3.38) with the condition (3.40) or the stronger power series requirement (3.41). By the assumptions stated on the β functions for the domain (3.6) existence and uniqueness of the effective couplings (4.1) is implied.

Corresponding to the primary coupling λ_0 we define an effective coupling $\bar{\lambda}_0$ by (4.1),

$$\bar{\lambda}_0 = \bar{\lambda}_0(z, m), \tag{4.30}$$

using the simplified notation (4.4). For the reduced model an evolution equation for $\bar{\lambda}_0$ alone is expected. As such we propose

$$-\frac{1}{2} z \frac{d\bar{\lambda}'_0}{dz} = \bar{\beta}'_0 \tag{4.31}$$

with the notation

$$\hat{\beta}'_j = \beta_j(\bar{\lambda}'_0, r_1(\bar{\lambda}'_0, m, z), \ldots, r_n(\bar{\lambda}'_0, m, z)), \tag{4.32}$$

and the initial conditions

$$\bar{\lambda}'_0 = \lambda_0 \quad \text{at} \quad z = \zeta \tag{4.33}$$

to be imposed. We have chosen another notation $\bar{\lambda}'_0$ for the effective coupling, since it has yet to be shown that (4.30) indeed solves (4.31). In the domain (3.6),

$$\bar{\lambda}'_0 = \bar{\lambda}'_0(z, m) \tag{4.34}$$

exists as a unique solution of (4.31) with the initial condition (4.33). The other effective couplings $\bar{\lambda}'_j$ are introduced by

$$\bar{\lambda}'_j = \bar{\lambda}'_j(z, m) = r_j(\bar{\lambda}'_0(z, m), m, z) \quad (j = 1, \ldots, n) \tag{4.35}$$

as functionals of $\bar{\lambda}'_0$. It will be seen that the functions (4.34) solving (4.31)–(4.33) combined with the functions (4.35) on the one hand and the function (4.30) solving (4.3), (4.29) on the other hand are identical,

$$\bar{\lambda}'_j \equiv \bar{\lambda}_j \quad (j = 0, \ldots, n). \tag{4.36}$$

For the proof we need only check that the functions (4.34), (4.35) likewise solve the evolution equations (4.3) with the initial conditions (4.2). Identity (4.36) follows by the uniqueness property of these differential equations. For $j = 0$ Eq. (4.3) is satisfied according to the defining equation (4.31) of $\bar{\lambda}'_0$. In order to verify the remaining equations we differentiate (4.35) with respect to z,

$$\begin{aligned} -\frac{1}{2}z\frac{d\bar{\lambda}'_j}{dz} &= -\frac{1}{2}z\frac{\partial r_j}{\partial \zeta} - \frac{1}{2}z\frac{\partial r_j}{\partial \lambda_0}\frac{d\bar{\lambda}'_0}{dz} \\ &= -\frac{1}{2}z\frac{\partial r_j}{\partial \zeta} + \bar{\beta}'_0\frac{\partial r_j}{\partial \lambda_0} = \bar{\beta}'_j. \end{aligned} \tag{4.37}$$

Here $\bar{\lambda}'_0(z, m)$ and z should be substituted for the arguments λ_0 and ζ resp. in the partial derivatives of r_j, similar to (4.35), for the notation $\bar{\beta}'_j$ see Eq. (4.32). Thus we have shown that the functions $\bar{\lambda}'_j$ indeed satisfy the evolution equations (4.3). Since the initial conditions (4.2) are also fulfilled, the proof of (4.36) is completed.

The results may be summarized as follows. The effective coupling (4.30) of the reduced model solves the evolution equations (4.31),

$$-\frac{1}{2}z\frac{d\bar{\lambda}_0}{dz} = \bar{\beta}'_0, \quad \bar{\lambda}_0 = \lambda_0 \quad \text{at} \quad z = \zeta, \tag{4.38}$$

with $\bar{\beta}'_0$ given by (4.32). Defining the other couplings by

$$\bar{\lambda}_j = \bar{\lambda}_j(z, m) = r_j(\bar{\lambda}_0(z, m), m, z), \tag{4.39}$$

a solution of the original evolution equations (4.3) in the form

$$-\frac{1}{2}z\frac{d\bar{\lambda}'_j}{dz} = \bar{\beta}'_j \tag{4.40}$$

is obtained with the initial conditions (4.2).

We next turn to the question to what extent the reduction equations (3.38) contain redundant information and how it can be eliminated. On the basis of the evolution equations a natural constraint on the reducing functions will be found. Obviously, relevant for the interpretation of the reduction method can only be the final functional dependence of the effective couplings $\bar{\lambda}_j$ on the primary coupling $\bar{\lambda}_0$. Accordingly, we call two sets of reducing functions equivalent,

$$r_j^{(1)} \sim r_j^{(2)}, \tag{4.41}$$

if the resulting functional dependence

$$\bar{\lambda}_j(z, m) = \bar{s}_j(\bar{\lambda}_0(z, m), m) \quad (j = 1, \ldots, n) \tag{4.42}$$

is the same. In order to find an appropriate formulation we take the reduced form (4.38) of the evolution equations and invert its solution (4.30) with respect to z,

$$z = \bar{\zeta}(\bar{\lambda}_0, m), \tag{4.43}$$

using that $\bar{\beta}'_0$ does not vanish in the domain considered. Then the effective couplings $\bar{\lambda}_j$ may be expressed as functions of $\bar{\lambda}_0$,

$$\bar{\lambda}_j = r_j(\bar{\lambda}_0, m, \bar{\zeta}(\bar{\lambda}_0, m)) = \bar{s}_j(\bar{\lambda}_0, m). \tag{4.44}$$

Hence $r_j^{(1)}$ and $r_j^{(2)}$ are equivalent, if

$$\begin{gathered} r_j^{(1)}(\bar{\lambda}_0, m, \bar{\zeta}^{(1)}(\bar{\lambda}_0, m)) = r_j^{(2)}(\bar{\lambda}_0, m, \bar{\zeta}^{(2)}(\bar{\lambda}_0, m)) \\ \text{or} \quad \bar{s}_j^{(1)}(\bar{\lambda}_0, m) = \bar{s}_j^{(2)}(\bar{\lambda}_0, m). \end{gathered} \tag{4.45}$$

The $\bar{s}_j$ are not reducing functions per se, but may be viewed as such by admitting a sliding normalization mass. To see this we replace z by $\bar{\lambda}_0$ as an independent variable in (4.40). Similar to the discussion of asymptotic freedom the equivalent set of differential equations

$$\bar{\beta}_0' \frac{d\bar{\zeta}}{d\bar{\lambda}_0} = -\frac{1}{2}\bar{\zeta}, \tag{4.46}$$

$$\bar{\beta}_0' \frac{d\bar{s}_j}{d\bar{\lambda}_0} = \bar{\beta}_j', \tag{4.47}$$

$$\bar{\beta}_j' = \beta_j(\bar{\lambda}_0, \bar{\lambda}_0, \bar{s}_1(\bar{\lambda}_0, m), \ldots, \bar{s}_n(\bar{\lambda}_0, m), m, \bar{\zeta}(\bar{\lambda}_0, m)) \tag{4.48}$$

is obtained. In Eqs. (4.46)–(4.48) we replace $\bar{\lambda}_0$ by its value λ_0 at the normalization point and change the notation $\bar{s}_j, \bar{\zeta}, \bar{\beta}_j'$ to s_j, ζ, β_j' accordingly. Then we have a set of $n + 1$ ordinary differential equations

$$\beta_0' \frac{d\zeta}{d\lambda_0} = -\frac{1}{2}\zeta, \tag{4.49}$$

$$\beta_0' \frac{ds_j}{d\lambda_0} = \beta_j' \quad (j = 1, \ldots, n), \tag{4.50}$$

$$\beta_j' = \beta_j(\lambda_0, s_1(\lambda_0, m), \ldots, s_n(\lambda_0, m), m\zeta(\lambda_0, m)) \quad (j = 0, \ldots, n). \tag{4.51}$$

for the functions

$$\zeta = \zeta(\lambda_0, m), \quad \lambda_j = s_j(\lambda_0, m) \quad (j = 1, \ldots, n). \tag{4.52}$$

The function s_j are related to reducing functions by (4.44):

$$s_j(\lambda_0, m) = r_j(\lambda_0, m, \zeta(\lambda_0, m)). \tag{4.53}$$

Equations (4.50) may thus be interpreted as reduction equations modified by a sliding normalization mass

$$|\kappa| = \frac{1}{\zeta(\lambda_0, m)} \tag{4.54}$$

which satisfies the differential equation (4.49). In general Eqs. (4.49) and (4.50) are coupled by the dependence of the β functions on the normalization mass. But in a scheme with massless β functions the system (4.50) can be solved independently of (4.49). Any set s_j of solutions for (4.50) then also satisfies (3.38) with

$$\frac{\partial s_j}{\partial \zeta} = 0. \tag{4.55}$$

Therefore, in a scheme with massless β functions the functions s_j coincide with reducing functions R_j independent of ζ, thus representing an equivalence class. Hence without loss of information the dependence on the normalization mass may be disregarded so that the reduction equations (3.38) with massless β functions become a set of ordinary differential equations

$$\hat{\beta}_0' \frac{dR_j}{d\Lambda_0} = \hat{\beta}_j' \quad (j = 1, \dots, n) \tag{4.56}$$

for functions

$$\Lambda_j = R_j(\Lambda_0).$$

Equation (4.49) may be integrated to

$$\begin{aligned} \zeta &= c \exp[-\frac{1}{2}\int_a^{\lambda_0} \frac{dx}{\tilde{\beta}_0}], \\ \tilde{\beta}_0 &= \hat{\beta}_0(x, R_1(x), \dots, R_n(x)), \end{aligned} \tag{4.57}$$

so that identity (4.53) becomes

$$\begin{aligned} \Lambda_j &= R_j(\Lambda_0) \\ &= R_j(\Lambda_0, m, c\exp[-\frac{1}{2}\int_a^{\Lambda_0} \frac{dx}{\tilde{\beta}_0}]). \end{aligned} \tag{4.58}$$

Since the constants $m_1, \dots, m_I, a$ and c (correlated to a) do not occur otherwise in (4.56), they may be absorbed by the arbitrary integration constants of the general solution for (4.56).

Thus a set of reducing functions R_j is selected in each equivalence class by the solutions of (4.56). In the original formulation of the model on the basis of mass dependent β functions a corresponding set r_j may be constructed by applying the inverse of (3.8) with (3.13) to R_j. On the reducing functions thus selected the condition (3.40) or the power series requirement (3.41) is imposed.

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5 Phenomenologically viable models; finiteness; top and Higgs mass predictions agreeing with experiment

Comment (*Myriam Mondragón, George Zoupanos*)
Let us first give a general introduction to this section.
In the recent years the theoretical endeavours that attempt to achieve a deeper understanding of Nature have presented a series of successes in developing frameworks such as String Theories and Noncommutativity that aim to describe the fundamental theory at the Planck scale. However, the essence of all theoretical efforts in Elementary Particle Physics (EPP) is to understand the present day free parameters of the Standard Model (SM) in terms of few fundamental ones, i.e. to achieve *reductions of couplings*. Unfortunately, despite the several successes in the above frameworks they do not offer anything in the understanding of the free paramaters of the SM. The pathology of the plethora of free parameters is deeply connected to the presence of *infinities* at the quantum level. The renormalization program can remove the infinities by introducing counterterms, but only at the cost of leaving the corresponding terms as free parameters. To reduce the number of free parameters of a theory, and thus render it more predictive, one is usually led to introduce a symmetry. Grand Unified Theories (GUTs) are very good examples of such a procedure. For instance, in the case of minimal $SU(5)$, because of the (approximate) gauge coupling unification, it was possible to reduce the gauge couplings of the SM to one. In fact, the LEP data suggested that a further symmetry, namely $N = 1$ global supersymmetry should also be required to make the prediction viable. GUTs can also relate the Yukawa couplings among themselves, again $SU(5)$ provided an example of this by predicting the ratio M_τ/M_b in the SM. Unfortunately, requiring more gauge symmetry does not seem to help, since additional complications are introduced due to new degrees of freedom, in the ways and channels of breaking the symmetry, among others. Therefore, the fundamental lesson we have learned from the extensive studies of GUTs was that unification of gauge couplings is a very good idea, which moreover is nicely realized in the minimal supersymmetric version of the Standard Model (MSSM). In addition the use of the renormalization group equations (RGEs) has been established as the basic tool in the corresponding studies.
A natural extension of the GUT idea is to find a way to relate the gauge and Yukawa sectors of a theory, that is to achieve gauge-Yukawa Unification (GYU) that will be presented in the subsections 5.1, 5.2, 5.5. Following the original suggestion for reducing the couplings discussed in the previous sections, within the framework of GUTs we were hunting for renormalization group invariant (RGI) relations holding below the Planck scale, which in turn are preserved down to the GUT scale. It is indeed an impressive observation that one can guarantee the validity of the RGI relations to all-orders in perturbation theory by studying the uniqueness of the resulting relations at one-loop (sect. 2). Even more remarkable is the fact that it is possible to find RGI relations among couplings that guarantee finiteness to all-orders in perturbation theory (sect. 3). The above principles have only been applied in $N = 1$ supersymmetric GUTs for reasons that will be transparent in the following subsections, here we should only note that the use of $N = 1$ supersymmetric GUTs comprises the demand of the cancellation of quadratic divergencies in the SM. The above GYU program applied in the dimensionless couplings of supersymmetric GUTs had already a great success by predicting correctly, among others, the top quark mass in the finite $N = 1$ supersymmetric $SU(5)$ before its discovery [13].

Although supersymmetry seems to be an essential feature for a successful realization of the above program, its breaking has to be understood too, since it has the ambition to supply the SM with predictions for several of its free parameters. Indeed, the search for RGI relations has been extended to the soft supersymmetry breaking sector (SSB) of these theories, which involves parameters of dimension one and two. In addition, there was important progress concerning the renormalization properties of the SSB parameters, based on the powerful supergraph method for studying supersymmetric theories, and it was applied to the softly broken ones by using the "spurion" external space-time independent superfields. According to this method a softly broken supersymmetric gauge theory is considered as a supersymmetric one in which the various parameters, such as couplings and masses, have been promoted to external superfields. Then, relations among the soft term renormalization and that of an unbroken supersymmetric theory have been derived. In particular the β-functions of the parameters of the softly broken theory are expressed in terms of partial differential operators involving the dimensionless parameters of the unbroken theory. The key point in solving the set of coupled differential equations so as to be able to express all parameters in a RGI way, was to transform the partial differential operators involved to total derivative operators. It is indeed possible to do this by choosing a suitable RGI surface.

On the phenomenological side there exist some serious developments too. Previously an appealing "universal" set of soft scalar masses was assumed in the SSB sector of supersymmetric theories, given that apart from economy and simplicity (1) they are part of the constraints that preserve finiteness up to two-loops, (2) they appear in the attractive dilaton dominated supersymmetry breaking superstring scenarios. However, further studies have exhibited a number of problems, all due to the restrictive nature of the "universality" assumption for the soft scalar masses. Therefore, there were attempts to relax this constraint without loosing its attractive features. Indeed an interesting observation on $N = 1$ GYU theories is that there exists a RGI sum rule for the soft scalar masses at lower orders in perturbation theory, which was later extended to all-orders, and that manages to overcome all the unpleasant phenomenological consequences. Armed with the above tools and results we were in a position to study the spectrum of the full finite models in terms of few free parameters, with emphasis on the predictions of supersymmetric particles and the lightest Higgs mass. The result was indeed very impressive since it led to a prediction of the Higgs mass which coincided with the results of the LHC for the Higgs mass, $125.5 \pm 0.2 \pm 0.6$ GeV by ATLAS [14] and $125.7 \pm 0.3 \pm 0.3$ GeV by CMS [15], and predicted a supersymmetric spectrum consistent with the non-observation of coloured supersymmetric particles at the LHC. These successes will be presented in subsections 5.5, 5.8 and 5.9.

Last but certainly not least, the above machinery has been recently applied in the MSSM with impressive results concerning the predictivity of the top, bottom and Higgs masses, being at the same time consistent with the non-observation of supersymmeric particles at the LHC. These results will be presented in subsection 5.10.

5.1 Finite unified models

Title: Finite unified models
Authors: D. Kapetanakis, M. Mondragon, G. Zoupanos
Journal: Z. Phys. **C60** (1993) 181-186
Comment (*Myriam Mondragón, George Zoupanos*)
The principle of finiteness requires perhaps some more motivation to be considered and generally accepted these days than when it was first envisaged. It is however interesting to note that in the old days the general feeling was quite different. Probably the well known Dirac's phrase that "...divergencies are hidden under the carpet" is representative of the views of that time. In recent years we have a more relaxed attitude towards divergencies. Most theorists believe that the divergencies are signals of the existence of a higher scale, where new degrees of freedom are excited. Even accepting this dogma, we are naturally led to the conclusion that beyond the unification scale, i.e. when all interactions have been taken into account in a unified scheme, the theory should be completely finite. In fact, this is one of the main motivations and aims of string, non-commutative geometry, and quantum group theories, which include also gravity in the unification of the interactions. In our work on reduction of couplings and finiteness we restricted ourselves to unifying only the known gauge interactions, based on a lesson of the history of EPP that if a nice idea works in physics, usually it is realised in its simplest form. Finiteness is based on the fact that it is possible to find renormalization group invariant (RGI) relations among couplings that keep finiteness in perturbation theory, even to all orders. Accepting finiteness as a guiding principle in constructing realistic theories of EPP, the first thing that comes to mind is to look for an $N = 4$ supersymmetric unified gauge theory, since these theories are finite to all-orders for any gauge group. However nobody has managed so far to produce realistic models in the framework of $N = 4$ SUSY. In the best of cases one could try to do a drastic truncation of the theory like the orbifold projection of refs. [17, 18], but this is already a different theory than the original one. The next possibility is to consider an $N = 2$ supersymmetric gauge theory, whose beta-function receives corrections only at one-loop. Then it is not hard to select a spectrum to make the theory all-loop finite. However a serious obstacle in these theories is their mirror spectrum, which in the absence of a mechanism to make it heavy, does not permit the construction of realistic models. Therefore, we are naturally led to consider $N = 1$ supersymmetric gauge theories, which can be chiral and in principle realistic.
Before our work the studies on $N = 1$ finite theories were following two directions: (a) construction of finite theories up to two-loops examining various possibilities to make them phenomenologically viable, (b) construction of all-loop finite models without particular emphasis on the phenomenological consequences. The success of our work was that we constructed the first realistic all-loop finite model, based on the theorem presented in the subsection 4.1, realising in this way an old theoretical dream of field theorists. Equally important was the correct prediction of the top quark mass one and half year before the experimental discovery. It was the combination of these two facts that motivated us to continue with the study of $N = 1$ finite theories. It is worth noting that nobody expected at the time such a heavy mass for the top quark. Given that the analysis of the experimental data changes over time, the comparison of our original prediction with the updated analyses will be discussed later, in particular in subsection 5.8.

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Finite unified models

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Abstract. We present phenomenologically viable $SU(5)$ unified models which are finite to all orders before the spontaneous symmetry breaking. In the case of two models with three families the top quark mass is predicted to be 178.8 GeV.

1 Introduction

The apparent success of unified gauge theories describing the observed interactions is restrained by the plethora of arbitrary parameters that one has to introduce by hand. In particular, in the electroweak standard model [1], which is indeed a very successful theory, one has to fit more than twenty parameters if neutrinos are massive or eighteen if they are massless. This is a clear disadvantage as far as the predictivity of the theory is concerned. Grand Unified Theories (GUTs) [2, 3] are doing better in this respect since they can provide predictions for parameters such as $\sin^2\theta_W$ and fermion mass ratios, which are free parameters in the electroweak standard model. In turn, GUTs can be tested and possibly could be ruled out, as for instance is the case of the minimal $SU(5)$ model [4].

There exists another principle that certainly points to the direction of further reduction of the free parameters of a gauge theory, namely, the requirement of finiteness. Moreover, the principle of finiteness goes very deeply to the heart of quantum field theories, supporting strongly the hope that the ultimum theory does not need infinite renormalizations. Although the latter are perfectly legitimate in quantum field theory they still give the feeling that divergences are "hidden under the carpet" [5]. It is not accidental that supersymmetric gauge theories have been so widely explored during the last decade in spite of the lack of any experimental evidence of supersymmetry. The clear motivation for the explosion of interest is due to the absence of quadratic divergences in these theories which guarantees their naturalness.

There have been made many attempts to obtain finite quantum field theories in four dimensions. For general theories such searches are usually limited to one loop approximation [6]. Besides, there is a strong indication that only supersymmetric gauge theories can be completely free from ultraviolet divergences [6]. A very interesting fact is that the one loop finiteness conditions on $N = 1$ supersymmetric theories automatically ensure also two-loop finiteness [7]. Last but not least, there have been given simple criteria [8, 9, 10] which ensure "all orders finiteness" in the sense of vanishing β-functions.

A complete classification of chiral $N = 1$ supersymmetric theories with a simple gauge group that satisfy the one-loop finiteness conditions has been done in refs. [11, 12]. There appear to exist only a few possibilities that have a chance to develop to realistic models. Here we examine to which extent these models can be made realistic, imposing in addition the requirement of all orders finiteness in the sense of [8]. We find interesting solutions to this problem. Furthermore, in the case of the models involving three families a heavy top quark naturally emerges, a feature which seems to be characteristic of this class of models.

2 Finite $N=1$ supersymmetric gauge theories

In order to discuss in detail the finiteness conditions and their implications, let us consider a chiral, anomaly free, globally supersymmetric $N = 1$ gauge theory with gauge group G. The superpotential of such a theory is given by:

$$W = a_i\phi_i + \tfrac{1}{2} m_{ij}\phi_i\phi_j + \tfrac{1}{6} C_{ijk}\phi_i\phi_j\phi_k, \tag{1}$$

where a_i, m_{ij} and C_{ijk} are gauge invariant tensors and the matter fields ϕ_i transform according to an irreducible representation R_i of the gauge group G.

The necessary and sufficient conditions for finiteness at one-loop level are the following:

- One-loop finiteness of the gauge fields self-energy which

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requires:

$$\sum_i \ell(R_i) = 3C_2(G), \tag{2}$$

where $\ell(R_i)$ is the Dynkin index of R_i [13] and $C_2(G)$ is the quadratic Casimir operator of the adjoint representation of the gauge group G.

• One-loop finiteness of the chiral superfields self-energy. In terms of the cubic couplings C_{ijk} appearing in the superpotential given in (1), referred to as Yukawa couplings, this condition requires:

$$C^{ikl}C_{jkl} = 2\delta^i_j g^2 C_2(R_i), \tag{3}$$

where g is the gauge coupling constant, $C_2(R_i)$ is the quadratic Casimir of the representation R_i, and $C^{ijk} = (C_{ijk})^*$. Note that condition 3 forbids the presence of singlets with nonzero coupling. Furthermore, it requires that $C^{ikl}C_{jkl}$ is diagonal in its two free indices.

Therefore, the finiteness conditions given in (2) and (3), which express the vanishing of the one-loop anomalous dimensions of the gauge and matter couplings respectively, restrict considerably the choices of the representations R_is for a given group G as well as their Yukawa couplings appearing in the superpotential, (1). On the other hand due to the non-renormalization theorem [14], which relates the renormalization of a_i, m_{ij} and C_{ijk} to that of the ϕ_i, the finiteness conditions do not restrict the form of a_i and m_{ij}.

An important consequence of the finiteness conditions is that supersymmetry most probably can only be broken by the addition of soft breaking terms. Specifically, due to the exclusion of singlets according to (3) the F-type [15] spontaneous supersymmetry breaking terms are incompatible with finiteness. Also, the D-type [16] spontaneous breaking is ruled out since it requires the existence of a $U(1)$ gauge group which in turn is incompatible with (2). In choosing to break supersymmetry by the addition of soft terms one should be aware of the fact that one-loop finiteness imposes extra conditions on this sector of the theory [17].

A very interesting result proved in [7] is that the one-loop finiteness conditions (2), (3) are necessary and sufficient for finiteness at two-loop level. Even more interesting is the theorem proved in [8]. The theorem states that if a supersymmetric gauge theory with simple gauge group is free from gauge anomalies, obeys (2), and there exist solutions to (3) of the form

$$C_{ijk} = \rho_{ijk} g, \tag{4}$$

where ρ_{ijk} are complex numbers, which are isolated and non-degenerate, then each of these solutions can be uniquely extended to a formal power series of g [18], giving a theory which depends on a single coupling g, with a β-function vanishing to all orders.

3 Finite unified models based on $SU(5)$

An inspection on the Tables of [11, 12] immediately shows the difficulties encountered in constructing phenomenologically viable finite unified theories (FUTs) already at the one- or equivalently two-loop level. In particular, using $SU(5)$ as gauge group there exist only two candidate models which can accommodate three fermion families and they contain the chiral multiplets 5, $\overline{5}$, 10, $\overline{10}$, 24 with multiplicities (6, 9, 4, 1, 0) and (4, 7, 3, 0, 1) respectively. In addition, there exists another model based on $SU(5)$ gauge group which can accommodate five fermion families and contains the same chiral multiplets as the two previous with multiplicities (5, 10, 5, 0, 0). Out of these three models only the second one contains a 24-plet which can be used for the spontaneous symmetry breaking of $SU(5)$ down to the standard model $SU(3)\times SU(2)\times U(1)$. For the other two models one has to incorporate another way such as the Wilson flux breaking mechanism [19] in order to achieve the required superstrong spontaneous symmetry breaking of the $SU(5)$ gauge group.

In the following we will consider in more detail the three family models.

3.A $N = 1$, $SU(5)$ model with three fermion families and without adjoint Higgs

The particle content of this model consists of the following supermultiplets represented by their transformation properties under $SU(5)$: three $(\overline{5} + 10)$, which are identified with the three supermultiplets describing the fermion families, six $(5 + \overline{5})$ which are considered as Higgs supermultiplets, and one $(10 + \overline{10})$ which are considered also as scalar supermultiplets.

The first finiteness condition given in (2) is automatically satisfied in the present model given that this was one of the selection rules for the models appearing in [11, 12]. In order to satisfy the second condition given in (3) we have to consider the superpotential. The most general $SU(5)$ invariant, $N = 1$ cubic superpotential with the above particle content has the form:

$$\begin{aligned}
W = {} & \tfrac{1}{2}\, g_{ija}\, 10_i\, 10_j\, H_a + g_{ia}\, 10_i N\, H_a + \bar{g}_{ija}\, 10_i \overline{5}_j \bar{H}_a \\
& + \tfrac{1}{2}\, g'_{ijk}\, 10_i \overline{5}_j \overline{5}_k + \tfrac{1}{2} f_{ab} N\, \bar{H}_a \bar{H}_b + \tfrac{1}{2} \bar{f}_{ab}\, \bar{N}\, H_a H_b \\
& + \tfrac{1}{2}\, h_a N\, N\, H_a + \tfrac{1}{2}\, \bar{h}_a \bar{N} \bar{N} \bar{H}_a + \tfrac{1}{2}\, q_{iab}\, 10_i\, \bar{H}_a \bar{H}_b \\
& + p_{ia} N \overline{5}_i \bar{H}_a + \tfrac{1}{2}\, t_{ij} N\, \overline{5}_i \overline{5}_j,
\end{aligned} \tag{5}$$

where $i, j, k = 1, \ldots, 3$ and $a, b = 1, \ldots, 6$ and we have suppressed the $SU(5)$ indices. 10_i and $\overline{5}_i$ are the usual three families. The six $(5 + \overline{5})$ Higgses are denoted by H_a, $\bar{H}_a$, while the scalar field belonging to the $(10 + \overline{10})$ representation by $N + \bar{N}$.

Then, (3) imposes the following relations among the Yukawa and gauge couplings:

$$\begin{aligned}
& H{:}\ 3g^{ija}\, g_{ijb} + 6g^{ia}\, g_{ib} + 4f^{ca} f_{cb} + 3h^a\, h_b = \delta^a_b \tfrac{24}{5}\, g^2, \\
& \overline{5}{:}\ 4\bar{g}^{ila}\, \bar{g}_{ima} + 4g'^{ilk}\, g'_{imk} + 4t^{lj}\, t_{mj} + p^{la}\, p_{ma} = \delta^l_m \tfrac{24}{5}\, g^2, \\
& \bar{H}{:}\ 4\bar{g}^{ija}\, \bar{g}_{ijb} + 4f^{ca} f_{cb} + 3\bar{h}^a\, \bar{h}_b + 4q^{ica}\, q_{icb} + 4p^{ia}\, p_{ib} \\
& \qquad = \delta^a_b \tfrac{24}{5}\, g^2, \\
& N{:}\ 3g^{ia}\ \ g_{ia} + f^{cb} f_{cb} + 3h^a\, h_a + 2p^{ia}\, p_{ia} + t^{ij}\, t_{ij} = \tfrac{36}{5}\, g^2, \\
& \bar{N}{:}\ \bar{f}^{ab} \bar{f}_{ab} + 3\bar{h}^a\, \bar{h}_a = \tfrac{36}{5}\, g^2, \\
& 10{:}\ 3g^{lki}\, g_{mki} + 2\bar{g}^{lki}\, \bar{g}_{mki} + 3g^{la}\, g_{ma} + g'^{ljk}\, g'_{mjk} \\
& \qquad + q^{lab}\, q_{mab} = \delta^l_m \tfrac{36}{5}\, g^2.
\end{aligned} \tag{6}$$

As it was already emphasized in sect. 2 the fulfillment of (2) and (3) is necessary and sufficient to guarantee the one-loop as well the two loop finiteness of the theory [7]. Nevertheless, in order to achieve all-loop finiteness one has to do more [8]. Specifically, one has to find a solution of (3) which is isolated and non-degenerate. This is a far from trivial problem given that (3) has infinitely many solutions that can be parametrized by continuous parameters (see for example [20, 21]).

Our strategy to find a unique and phenomenologically interesting solution to (3) is to impose on the model additional symmetries on top of the $SU(5)$ gauge invariance and $N=1$ global supersymmetry. Next recall that the terms of lower dimension such as mass terms are not restricted by the finiteness requirement. We use this freedom to make the model phenomenologically viable. As a result we have found a solution to all-loop finiteness problem with very interesting phenomenological predictions. In particular the top quark mass is predicted. The method can be generalized in a straightforward way in order to take into account all light fermion masses and mixing angles [26]. Specifically, we impose the $Z_7 \times Z_3$ discrete symmetry given in Table 1, together with a multiplicative Q-parity under which the 10_i and $\bar{5}_i$ describing the fermion supermultiplets are odd, while all the other superfields are even. In this way the number of terms that are permitted to appear in the super-potential is severely restricted. Only terms with Yukawa couplings g_{iii}, $\bar{g}_{iii}$, f_{44}, f_{56}, f_{65}, $\bar{f}_{44}$, $\bar{f}_{56}$, $\bar{f}_{65}$, h_4, and $\bar{h}_4$ survive.

We then find the following unique solution to (6),

$$
\begin{aligned}
&g_{111}^2 = g_{222}^2 = g_{333}^2 = \tfrac{8}{5} g^2, \\
&\bar{g}_{111}^2 = \bar{g}_{222}^2 = \bar{g}_{333}^2 = \tfrac{6}{5} g^2, \\
&f_{44}^2 = 0;\ f_{56}^2 = f_{65}^2 = \tfrac{6}{5} g^2, \\
&\bar{f}_{44}^2 = 0;\ \bar{f}_{56}^2 = \bar{f}_{65}^2 = \tfrac{6}{5} g^2, \\
&h_4^2 = \tfrac{8}{5} g^2;\ \bar{h}_4^2 = \tfrac{8}{5} g^2.
\end{aligned} \tag{7}
$$

The uniqueness* of this solution guarantees the all-loop finiteness.

One might wonder if this model could result from some more fundamental theory and, in turn, if there is some justification for its symmetries. It seems that there exist very suggestive hints that the model under consideration belongs to a class of models obtained from superstring compactification over certain Calabi–Yau (CY) manifolds. More specifically, Witten [22] has shown that it is possible to construct stable, irreducible, and holomorphic $SU(5)$ or $SU(4)$ vector bundles over CY manifolds. Then one can start from the heterotic superstring with gauge group $E_8 \times E_8'$ and obtain an $SU(5)$ or $SO(10)$ $N=1$ supersymmetric theory at four dimensions, by embedding the structure group of the bundle ($SU(5)$ or $SU(4)$) in E_8 (E_8' is considered as hidden). It is worth noting that claims that such configurations are generically unstable [23] due to non-perturbative effects appeared unjustified in particular cases. Furthermore the conditions under which a stable configuration emerges are given in [24]. It turns out that the spectrum of a $N=1$, $SU(5)$ gauge theory resulting from a CY compactification is generally of the form $m(10) + n(\bar{5}) + \delta(10 + \overline{10}) + \varepsilon(5 + \bar{5})$, where m, n, δ, and ε are topological numbers of the CY manifold [25]. Therefore, it is not inconceivable to imagine how a model like the one considered here could come from superstring compactification.

* The phase arbitrariness of (7) is not crucial, since it can be removed by using a specific renormalization scheme [8]

Table 1. The charges of the $Z_7 \times Z_3$ symmetry

	10_1	10_2	10_3	$\bar{5}_1$	$\bar{5}_2$	$\bar{5}_3$	H_1	H_2	H_3	H_4	H_5	H_6	N
Z_7	1	2	4	4	1	2	5	3	6	0	0	0	0
Z_3	1	2	0	0	0	0	1	2	0	0	1	2	0

Furthermore, since in the present model we are interested in applying the Wilson flux breaking mechanism, we, naturally, assume that the CY which is going to be used should admit a freely acting discrete group F. Then the light fields will be the ones which are invariant under $T \oplus F$, where T is the homomorphism of F in the gauge group.

Therefore, we are led to assume the existence of a CY with a stable, irreducible, and holomorphic $SU(5)$ bundle over it, admitting a freely acting discrete group F. Moreover, the topological numbers of this manifold after division with F are given by $m=n=3$, $\delta=1$, and $\varepsilon=6$. Let us comment here that the discrete symmetries used above in order to reduce the number of the Yukawa couplings should be respected by this CY manifold.

The present model clearly belongs to the class of models considered in [25]. For instance, suppose that F is a Z_3 which is embedded in a $T=Z_3$ identified with a discrete subgroup of the $U(1)$ appearing in the decomposition

$$
\begin{aligned}
&SU(5) \supset SU(3) \times SU(2) \times U(1), \\
&10 = (1,1)(6) + (\bar{3},1)(-4) + (3,2)(1), \\
&\bar{5} = (1,2)(-3) + (\bar{3},1)(2).
\end{aligned} \tag{8}
$$

Next recall that the gauge symmetries surviving after applying the Wilson flux breaking mechanism are those that commute with T. Then it is clear that the $SU(5)$ gauge symmetry of the model at hand breaks down to the standard model. One can go further and consult the Tables of [25] in order to attribute appropriate transformation properties to the various scalar multiplets, such as to make the model phenomenologically viable. As an example, consider that the scalar multiplets are invariant under the action of F, while they transform under the action of T according to $\exp(y\pi)$ where y is the hypercharge in (8). Then one can easily see that only the $(1,2)(-3)$ components coming from the $\bar{5}$ and the $(1,1)(6)$ coming from the 10 remain light. All the other components acquire superheavy masses of the order of the compactification scale. Therefore, in a natural way the model is provided with light Higgs doublets that can drive the spontaneous symmetry breakdown of $SU(2) \times U(1)$ down to $U(1)_{\text{em}}$ and, on the other hand, it is exorcised from the appearance of light "coloured scalars" that would lead to fast proton decay. Note that the above discrete symmetries do not affect the fermion supermultiplets [25].

Having described the basic strategy to make the model phenomenologically viable we postpone the full analysis of the various possibilities to a future publication [26]. For our purposes here we assume that the discrete symmetries involved permit only the existence of a pair of light Higgs doublets which is coupled only to the third family. Moreover, by adding soft breaking terms we can achieve supersymmetry breaking at the order of the electroweak scale. Then examining the evolution of the gauge couplings according to the renormalization group equations [27] we find

$$\sin^2\theta_W(M_Z)=0.233,\ M_X=2\cdot 10^{16},\ \alpha_{\rm em}^{-1}(M_Z)=127.9,$$
$$\alpha_s(M_Z)=0.120, \quad \text{and} \quad \alpha_X=0.0425, \tag{9}$$

in excellent agreement with the experimental values [4]

$$\sin^2\theta_W(M_Z)^{\rm exp}=0.2327\pm 0.0008,$$
$$\alpha_{\rm em}^{-1}(M_Z)^{\rm exp}=127.9\pm 0.2,$$
$$\alpha_s(M_Z)^{\rm exp}=0.118\pm 0.008. \tag{10}$$

Running now the renormalization group equations for the Yukawa couplings with the above values for α_X and M_X and initial values at M_X:

$$g_t^2=\tfrac{8}{5}(4\pi\alpha_X);\quad \bar g_b^2=\bar g_\tau^2=\tfrac{6}{5}(4\pi\alpha_X), \tag{11}$$

we find at M_W:

$$m(\text{top})=178.8\ \text{GeV},\quad m(\text{bottom})=3.1\ \text{GeV},$$
$$\text{and}\quad m(\text{tau})=1.8\ \text{GeV}. \tag{12}$$

As we can see, the model gives result for the tau and bottom masses in very good agreement with experiment, and predicts a high value for the mass of the top. Notice that these values are determined by the solution (7) to the finiteness conditions (6), and that although we have assumed that only the third family becomes massive, we do not expect the results to change considerably, since the third family terms dominate in the calculation.

3.B $N=1$, $SU(5)$ model with three fermion families and Higgs in the adjoint

This model has been considered before for two-loop [20, 21] as well as for all-loop finiteness [10]. The particle content consists of the following supermultiplets: three $(\bar 5+10)$, identified with the three supermultiplets describing the fermion families, four $(5+\bar 5)$, and one 24 considered as Higgs supermultiplets.

The first finiteness condition, (2), is, as before automatically met. In order to satisfy the second condition, (3), we have to examine the superpotential of the model. The most general $SU(5)$ invariant, $N=1$ cubic superpotential with the above particle content is:

$$\begin{aligned} W=&\tfrac{1}{2}g_{ija}\,10_i\,10_j\,H_a+\bar g_{ija}\,10_i\,\bar 5_j\,\bar H_a+\tfrac{1}{2}g'_{ijk}\,10_i\,\bar 5_j\,\bar 5_k\\ &+\tfrac{1}{2}q_{iab}\,10_i\,\bar H_a\,\bar H_b+f_{ab}\,\bar H_a\,24H_b+p(24)^3\\ &+h_{ia}\,\bar 5_i\,24H_a, \end{aligned} \tag{13}$$

where $i,j,k=1,\ldots,3$ and $a,b=1,\ldots,4$ and we have suppressed the $SU(5)$ indices. The 10_i's and $\bar 5_i$'s are the usual three families and 24 is the scalar superfield in the adjoint. The four $(5+\bar 5)$ Higgses are denoted by H_a, $\bar H_a$.

Then, (3) imposes the following relations among the Yukawa and gauge couplings:

$$\begin{aligned} &\bar H\colon 4\bar g_{ija}\,\bar g^{ijb}+\tfrac{24}{5}f_{ac}f^{bc}+4q_{iac}\,q^{ibc}=\tfrac{24}{5}g^2\,\delta^b_a,\\ &H\colon 3g_{ija}\,g^{ijb}+\tfrac{24}{5}f_{ca}f^{cb}+\tfrac{24}{5}h_{ia}\,h^{ib}=\tfrac{24}{5}g^2\,\delta^b_a,\\ &\bar 5\colon 4\bar g_{kia}\,\bar g^{kja}+\tfrac{24}{5}h_{ia}\,h^{ja}+4g'_{ikl}\,g'^{jkl}=\tfrac{24}{5}g^2\,\delta^j_i,\\ &10\colon 2\bar g_{ika}\,\bar g^{jka}+3g_{ika}\,g^{jka}+q_{iab}\,q^{jab}+g'_{kli}\,g'^{klj}=\tfrac{36}{5}g^2\,\delta^j_i,\\ &24\colon f_{ab}f^{ab}+\tfrac{21}{5}pp^*+h_{ia}\,h^{ia}=10g^2. \end{aligned} \tag{14}$$

In most of the previous studies of this model no attempt was made to find isolated and non-degenerate solutions. Their philosophy was rather in the opposite direction. They have used the freedom offered by the degenerate solutions in order to make specific ansatze that could lead to phenomenologically acceptable predictions. Following the lines prescribed in the previous model we impose additional symmetries on the model*. The new symmetries imposed on this model are again given in table 1 for 10_i, $\bar 5_i$ and H_a for $a=1,\ldots,4$. The terms in the superpotential which are invariant under the symmetries of the model are the terms with Yukawa couplings g_{iii}, $\bar g_{iii}$, f_{ii} and p.

We find the following solution of (14)

$$\begin{aligned} &g^2_{111}=g^2_{222}=g^2_{333}=\tfrac{8}{5}g^2,\quad \bar g^2_{111}=\bar g^2_{222}=\bar g^2_{333}=\tfrac{6}{5}g^2,\\ &f^2_{11}=f^2_{22}=f^2_{33}=0,\quad f^2_{44}=g^2;\quad p^2=\tfrac{15}{7}g^2. \end{aligned} \tag{15}$$

Therefore, we are in the same situation as in [21], i.e. each fermion family is coupled to a different Higgs. For simplicity, as in the previous models, we assume that only one pair of Higgs fields is light and acquires a v.e.v. which is coupled to the third family. This situation can easily be realised by adding appropriate mass terms. The solution of the doublet-triplet splitting problem in this model goes along the lines described in [21].

4 Finite models based on other gauge groups

There exist some more FUTs that have a chance to develop into realistic models. For instance, an inspection of the list of refs. [11, 12] suggests that the following models are worth to be examined:

1. An $SO(10)$ model with particle content consisting of eight 10, n 16 and $(8-n)$ $\overline{16}$ (with $5\le n\le 8$) supermultiplets. This model can accommodate an even number of fermion families and could result from a CY compactification as it was discussed in model A.
2. An E_6 model containing n 27 and $(12-n)$ $\overline{27}$ (with $7\le n\le 12$) supermultiplets which can accommodate an even number of fermion families.
3. An $SU(6)$ model with three 6, nine $\bar 6$ and one 35 supermultiplets. The model can describe three fermionic families, six Higgs in the fundamental, six Higgs in the antifundamental and one Higgs in the adjoint.

* See however [10] for an attempt to construct an all-loop finite model

5 Conclusions

We have discussed a number of one and two loop finite unified models. Emphasis was given in the construction of $SU(5)$, $N=1$ supersymmetric models which are finite in all orders before the spontaneous symmetry breaking.

In particular, in the case of $SU(5)$, $N=1$ supersymmetric models with three families the top quark mass is predicted to be 178.8 GeV. We have restricted our analysis to the case that only the third fermion family becomes massive after the electroweak symmetry breaking. The generalization to non zero masses for the remaining fermions and mixing angles is straightforward. However, due to the clear dominance of the third family, and in particular of the top quark mass, our prediction is not expected to change in a noticeable way.

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5.2 Reduction of couplings and heavy top quark in the minimal SUSY GUT

Title: Reduction of couplings and heavy top quark in the minimal SUSY GUT
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Comment (*Myriam Mondragón, George Zoupanos*)
To start with, it would have been natural to write this paper before the construction of $N = 1$ Finite Unified Models which were discussed in the previous subsection. This work is very interesting for a number of reasons. The $N = 1$ minimal supersymmeric $SU(5)$ was logically the minimal framework to discuss the reduction of coupling ideas in a realistic supersymmetric unification setup, the only known consistent framework to overcome the problem of quadratic divergencies of the SM and also the first unification attempt. Another interesting aspect of this study was to to examine to which extent the prediction ot the top quark mass of the Finite models was persisting in other GUTs as a more general feature of the reduction of couplings, which led to an exhaustive search for GYU in $N = 1$ supersymmetric GUTs. Finally, the $N = 1$ minimal supersymmetric $SU(5)$ GUT is a nice framework to realize physically and apply technically the idea of *partial reduction* initiated in subsections 3.3 and 3.5. More specifically, in the study of Finite models a complete reduction of couplings was achieved, which was not expected to be the case in the minimal supersymmetric $SU(5)$. On the other hand the method of partial reduction, already introduced in subsection 3.1 became more transparent, especially after the reduction equations had been replaced by the mathematically equivalent set of partial differential equations as described in subsections 3.3 and 3.5. Therefore, the minimal supersymmetric $SU(5)$ was a natural new framework for an innovative method to be applied. A rather interesting feature that emerged is that of all the possible solutions only two are asymptotically free, and both of them lie in the same RGI surface. Even more remarkable is that they lead to good phenomenology, compatible with the data available at the time.

In the future it is worth to have a fresh look to the reduction of couplings in the minimal $N = 1$ supersymmetric $SU(5)$, including the soft supersymmetry sector, in view of the results of the corresponding search in the MSSM to be discussed in subsection 5.10 and the updated experimental results on the top and bottom quark masses, as well as the discovery of the Higgs particle at LHC.

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NUCLEAR PHYSICS B

Reduction of couplings and heavy top quark in the minimal SUSY GUT

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Abstract

Out of 256 independent reduction solutions that can be found within the minimal supersymmetric SU(5) GUT, there are exactly two asymptotically free solutions which can restrict the top quark mass m_t and do not contradict the observed mass spectrum of the first two fermion generations. A numerical analysis shows that these two solutions lie on the same renormalization group invariant surface on which m_t and the bottom quark mass m_b assume relatively stable values for a given supersymmetry breaking scale m_{SUSY}. For $m_{\mathrm{SUSY}} = 200$ GeV with $\alpha_S(M_Z) = 0.12, \alpha_{\mathrm{em}}(M_Z) = (127.9)^{-1}$ and $m_\tau = 1.78$ GeV fixed, we find that on this surface m_t and m_b vary 2% and 3% around their central values 182 GeV and 5.3 GeV, respectively.

1. Introduction

The apparent success of the standard model in describing the elementary particles and their interactions seems to suggest us that most likely we are on the right track, and therefore we should be able to calculate some of its independent couplings. One of the most well-known ideas pointing towards relating a priori different couplings of the theory, such as gauge couplings as well as some of the Yukawa couplings, is

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the idea of unification [1-3]. In fact the minimal Georgi-Glashow SU(5) [2] model was very successful in qualitatively predicting the $\sin^2\theta_W$ and the mass ratio m_τ/m_b. Subsequently, the prediction on $\sin^2\theta_W$ was ruled out by more accurate measurements, but nevertheless we are left with the hope that the original qualitative agreement was not totally accidental. Indeed the most accurate measurements on the gauge coupling constants at LEP so far suggest that the minimal SU(5) GUT just has to be replaced by the minimal supersymmetric one [4].

The original unification philosophy relates the gauge and separately the Yukawa couplings, and therefore its logical extension is to attempt to relate the couplings of both sectors, *gauge–Yukawa unification*. As a consequence of such an extension it might become possible to understand why the top quark is so much heavier than the other fermions in the standard theory.

There exists a theoretical framework [5] within which one can study *all* possible relations amongst couplings in a renormalizable field theory, allowing to draw conclusions to *all* orders in perturbation. Specifically suppose that g and $\lambda_1,\ldots,\lambda_n$ are the gauge and Yukawa couplings of a GUT respectively. Then a natural extension of the unification idea would be that there exist relations among them of the type

$$\lambda_i = \lambda_i(g)\,, \quad i=1,\ldots,n\,. \tag{1}$$

Obviously, such relations are not compatible with renormalization group invariance in *general*; it is not true that if such relations hold at one renormalization point they also hold also at any other one. The reason is that the infinities associated with the full set of couplings cannot necessarily be removed in the reduced system. It was shown that the relations (1) can hold only if λ_i are solutions of the reduction equations [5,6]

$$\beta_g \frac{\mathrm{d}\lambda_i}{\mathrm{d}g} = \beta_{\lambda_i}\,, \quad i=1,\ldots,n\,. \tag{2}$$

Then by establishing and solving Eq. (2) in a given model one can find in an exhaustive way the relations amongst the couplings [5]. Applied to the standard theory [7,8], the coupling reduction, a mini gauge–Yukawa unification, indeed predicted a heavy top quark, though it now seems to be below the experimental lower bound [9].

The interest in the present work is twofold. First, applying the reduction philosophy under the mild assumption that the theory is asymptotically free, we would like to examine whether there exist relations as in Eq. (1) amongst the gauge and Yukawa couplings in the minimal supersymmetric SU(5) GUT that are consistent with the latest experimental data. If the reduction method would predict a top quark mass consistent with the experiment, we would be able to understand, at least technically, why it is so heavy. We then hope that there exist physical principles behind the successful gauge–Yukawa unification by means of reduction of couplings, which is achieved only in a technically natural way. The second one is to compare the present prediction with the one resulting from the SU(5) Finite Unified Theory (FUT), since it has been found [10] that the finite theory also predicts a large top quark mass [6]. Needless to say that

[6] We recall that in the latter case the unbroken SU(5) GUT is finite to all orders in the sense of vanishing β-functions [11]. Previous similar attempts [12] were claiming finiteness only at one- and two-loop levels.

both models have more predictive power than the models based on SO(10) and E_6 [13].

In Section 2 we recapitulate the relation between asymptotic freedom (AF) [14] and reduction of couplings [5,6]. While most of the material of this section is covered in one way of the other in the original papers [5,6] as well as in the two existing review articles [15,16], the notion of partial reduction is not, simply because it had not been worked out in a satisfactory manner at that time. In the course of time the method of partial reduction [7] has become more transparent, especially after the reduction equations (1) had been replaced by the mathematically equivalent set of partial differential equations [8,9], and thanks to many unpublished works of Zimmermann. Since various results on partial reduction of couplings have remained unpublished, we have decided, in order that the present paper is self-contained as much as possible, to go briefly through the idea of reduction of couplings and to classify the asymptotically free (supersymmetric) systems in terms of the terminology introduced in reduction of couplings.

The asymptotic behavior of ordinary GUTs have been investigated in the classic paper [17] for instance, while the investigation along the line of reduction of coupling constants [18] also exists. However, it turned out that due to the presence of the self-couplings of the scalar fields it is very difficult to find completely asymptotically free ordinary GUTs with realistic symmetry breaking pattern [17]. In contrast to these theories in supersymmetric Yang–Mills theories such couplings do not appear independently. Therefore, it is more appropriate to apply reduction of couplings—and in particular the method of partial reduction [7–9]—to supersymmetric GUTs, since this method allows freedom that can be used to reconcile the idea of reduction of couplings with experimental facts.

In Section 3 we consider the minimal supersymmetric GUT of Dimopoulos, Georgi and Sakai [4]. To investigate the asymptotic freedom property of the model we switch off the dimensional couplings and also the family mixing. We show that only two out of 256 reduction solutions can provide us with the possibility to get nontrivial information on the top quark mass m_t. We then apply the method of partial reduction and argue that even these two solutions lie on the same renormalization group invariant surface. Remarkably the top and bottom masses assume stable values on that surface. Another interesting result is that our low-energy predictions do not differ very much from those of the SU(5) FUT in which the reduction of couplings has also been applied [10].

2. Asymptotic freedom and reduction of couplings in $N = 1$ gauge theories

Let us consider a chiral, anomaly free, $N = 1$ globally supersymmetric gauge theory based on a group G with the gauge coupling constant g. The superpotential of the theory is given by

$$W = \frac{1}{2} m_{ij} \phi^i \phi^j + \frac{1}{6} C_{ijk} \phi^i \phi^j \phi^k , \tag{3}$$

where m_{ij} and C_{ijk} are gauge invariant tensors and the matter field ϕ^i transforms according to the irreducible representation R_i of the gauge group G. The renormalization constants associated with the superpotential (3), assuming that supersymmetry is preserved, are

$$\phi^{0i} = (Z^i_j)^{1/2} \phi^j , \quad m^0_{ij} = Z^{i'j'}_{ij} m_{i'j'} , \quad C^0_{ijk} = Z^{i'j'k'}_{ijk} C_{i'j'k'} . \tag{4}$$

The $N = 1$ non-renormalization theorem [19] ensures that there are no mass- and cubic-interaction-term infinities and therefore

$$\begin{aligned} Z^{ijk}_{i'j'k'} Z^{1/2\,i'}_{i''} Z^{1/2\,j'}_{j''} Z^{1/2\,k''}_{k'} &= \delta^i_{(i''} \delta^j_{j''} \delta^k_{k'')} , \\ Z^{ij}_{i'j'} Z^{1/2\,i'}_{i''} Z^{1/2\,j'}_{j''} &= \delta^i_{(i''} \delta^j_{j'')} . \end{aligned} \tag{5}$$

As a result, the only surviving possible infinities are the wave-function renormalization constants Z^i_j, i.e. one infinity for each field. The one-loop β-function of the gauge coupling g is given by [20]

$$\beta^{(1)}_g = \frac{\mathrm{d}g}{\mathrm{d}t} = \frac{g^3}{16\pi^2} \left[\sum_i l(R_i) - 3\, C_2(\mathrm{G}) \right] , \tag{6}$$

where $l(R_i)$ is the Dynkin index of R_i and $C_2(\mathrm{G})$ is the quadratic Casimir of the adjoint representation of the gauge group G. The β-functions of C_{ijk}, by virtue of the non-renormalization theorem [19], are related to the anomalous dimension matrix γ_{ij} of the matter fields ϕ^i as

$$\beta_{ijk} = \frac{\mathrm{d}C_{ijk}}{\mathrm{d}t} = C_{ijl}\, \gamma^l_k + C_{ikl}\, \gamma^l_j + C_{jkl}\, \gamma^l_i . \tag{7}$$

At one-loop level γ_{ij} is [20]

$$\gamma^{(1)}_{ij} = \frac{1}{32\pi^2} [\, C^{ikl} C_{jkl} - 2\, g^2 C_2(R_i) \delta_{ij} \,] , \tag{8}$$

where $C_2(R_i)$ is the quadratic Casimir of the representation R_i, and $C^{ijk} = C^*_{ijk}$.

Here we are interested in examining the reduction of the couplings of the asymptotically free softly broken supersymmetric gauge theories. Since dimensional coupling parameters such as masses and couplings of cubic scalar field terms do not influence the asymptotic freedom property of a theory, it is sufficient to take into account only the dimensionless supersymmetric couplings such as g and C_{ijk}. So we neglect the existence of dimensional parameters, and assume furthermore that C_{ijk} are real so that C^2_{ijk} always are positive numbers. For our purposes, it is convenient to work with the square of the couplings and to arrange C_{ijk} in such a way that they are covered by a single index i $(i = 1, \ldots, n)$:

$$\alpha = \frac{g^2}{4\pi} , \qquad \alpha_i = \frac{g_i^2}{4\pi} . \tag{9}$$

The evolution equations of α's in perturbation theory then take the form

$$\frac{d\alpha}{dt} = \beta = -\beta^{(1)}\alpha^2 + \dots ,$$
$$\frac{d\alpha_i}{dt} = \beta_i = -\beta_i^{(1)}\alpha_i\alpha + \sum_{j,k}\beta_{i,jk}^{(1)}\alpha_j\alpha_k + \dots , \tag{10}$$

where ... denotes the contributions from higher orders, and $\beta_{i,jk}^{(1)} = \beta_{i,kj}^{(1)}$.

Given the set of the evolution equations (10), we investigate its asymptotic freedom property, as follows. First we define [17,15]

$$\tilde{\alpha}_i \equiv \frac{\alpha_i}{\alpha}, \quad i = 1, \dots , n, \tag{11}$$

and derive from Eq. (10)

$$\alpha\frac{d\tilde{\alpha}_i}{d\alpha} = -\tilde{\alpha}_i + \frac{\beta_i}{\beta} = \left(-1 + \frac{\beta_i^{(1)}}{\beta^{(1)}}\right)\tilde{\alpha}_i$$
$$-\sum_{j,k}\frac{\beta_{i,jk}^{(1)}}{\beta^{(1)}}\tilde{\alpha}_j\tilde{\alpha}_k + \sum_{r=2}\left(\frac{\alpha}{\pi}\right)^{r-1}\tilde{\beta}_i^{(r)}(\tilde{\alpha}) , \tag{12}$$

where $\tilde{\beta}_i^{(r)}(\tilde{\alpha})$ $(r = 2, \dots)$ are power series of $\tilde{\alpha}$'s and can be computed from the rth loop β-functions. Then we assume that

$$\alpha \to 0 \quad \text{as} \quad t \to \infty , \tag{13}$$

which among other requires that $\beta^{(1)} > 0$, and we look for solutions of Eq. (12) that satisfy

$$\tilde{\alpha}_i \to \rho_i \quad (0 \leqslant \rho_i < \infty) \quad \text{as} \quad \alpha \to 0 . \tag{14}$$

If there exist such solutions $\tilde{\alpha}_i$, the assumption (13) is self-consistent and the system is asymptotically free to all orders in perturbation theory.

Because of the non-renormalization theorem [19], it is always possible to set any supersymmetric coupling constant equal to zero without contradicting renormalizability. However, in the following discussion, we assume that the couplings α_i are different from zero for phenomenological reasons. Note that this assumption does not necessarily imply that all the ρ's are different from zero. Instead it requires that, if $\rho_i = 0$ for some i, there must exist a nonvanishing solution α_i that asymptotically approaches zero as $\alpha \to 0$. Let us classify the various cases that might appear in the reduction of couplings of an asymptotically free theory.

(i) *Trivial case.*

This is the case with $\rho_i = 0$ $(i = 1, \dots , n)$, and the leading behavior of $\tilde{\alpha}_i$ is given by

$$\tilde{\alpha}_i = a_i\alpha^{\delta_{(i)}} + \dots \text{ with } a_i > 0 , \ \delta_{(i)} > 0 , \quad i = 1, \dots , n, \tag{15}$$

where ... indicates terms that decrease faster than α^{δ_i} as $\alpha \to 0$, and a_i are arbitrary positive constants. To find these solutions, we substitute the ansatz (15) into Eq. (12), and assume that the higher-order terms in α and $\tilde{\alpha}$'s can consistently be neglected. One easily finds that

$$\delta_{(i)} = -1 + \frac{\beta_i^{(1)}}{\beta^{(1)}} \tag{16}$$

so that $\beta_i^{(1)} > \beta^{(1)}$ has to be necessarily satisfied. In the case that the wave-function renormalization constants Z_i^j are diagonal, i.e. proportional to δ_i^j, the above condition is sufficient for the ansatz (15) to be the leading behavior in the asymptotic limit, in accord with the previous results [21].

(ii) *AF through nontrivial reduction.*

Asymptotic freedom can also be achieved by nontrivial reduction of coupling constants, and these solutions in general determine the upper bound for the trivial asymptotically-free solutions (15) (see for instance Refs. [8,22,23]). The nontrivial-reduction solutions [5] are power series solutions of Eq. (12)

$$\tilde{\alpha}_i = \rho_i + \sum_{r=2} \rho_i^{(r)} \alpha^{r-1}, \quad \rho_i > 0, \quad i = 1, \dots, n. \tag{17}$$

Substituting the ansatz (17) into Eq. (12), one can easily see that the expansion coefficients $\rho_i^{(r)}$ can be uniquely determined if [6,15]

$$\det M_{ij}(r) \neq 0 \quad \text{for all} \quad r = 1, \dots, \tag{18}$$

$$M_{ij}(r) \equiv \left(-1 + r + \frac{\beta_i^{(1)}}{\beta^{(1)}} \right) \delta_{ij} - 2 \sum_k \frac{\beta_{i,jk}^{(1)}}{\beta^{(1)}} \rho_k ,$$

where ρ_i are the nonzero solutions of [24,18,5,6]

$$\left(-1 + \frac{\beta_i^{(1)}}{\beta^{(1)}} \right) \rho_i - \sum_{j,k} \frac{\beta_{i,jk}^{(1)}}{\beta^{(1)}} \rho_j \rho_k = 0 . \tag{19}$$

If the condition (18) is not satisfied, that is, there is a vanishing eigenvalue for some r, the solution (17) generally has to be modified so as to contain fractional powers and logarithms of α [5,6,15,16]. But in very special cases [5,6,15,16] this does not happen so that (18) does not exhibit the necessary condition. Therefore, the uniqueness property of $\rho_i^{(r)}$ should be checked on a case by case basis if (18) is not satisfied. Obviously, if $\beta^{(1)} > 0$ and (17) is the solution of (12), the system is asymptotically free and contains only one independent coupling constant g.

As first noticed by Oehme [15], the nontrivial reduction solutions, ρ_i, exhibit the infrared stable fixed point of the evolution equations (12) in the *one-loop approximation*, the Pendleton–Ross fixed point [25]. In the framework of reduction of couplings, this point is used as the unstable ultraviolet fixed point and exists as such to all orders in perturbation theory, i.e. the existence of renormalization group trajectory that asymptotically approaches the point in the ultraviolet limit is mathematically ensured, while this point as an infrared fixed point may be an artifact of the lowest-order approximation and may have no sensible meaning in higher orders. In fact Zimmermann [22] has shown that in the infrared limit the ratios of couplings considered by Pendleton and Ross [25] in the standard model (that approach their fixed point in the one-loop approximation) diverge in the two-loop approximation.

The solutions "above" the nontrivial solutions are not asymptotically free in general. These asymptotically non-free solutions can of course be used (in lower orders in perturbations theory presumably) from some other reasons, e.g., to satisfy compositeness condition of Ref. [30]. However, they are irrelevant in constructing an asymptotically free GUT.

(iii) *AF through partial reduction.*

A partially reduced system [7-9] is a system in which only a part of coupling constants are reduced and exhibits a mixture of the systems discussed in (i) and (ii), as we will see shortly. Suppose we choose the solutions of Eq. (19) in the form

$$\rho_i = 0\,, \quad i = 1,\ldots,m\,,$$
$$\eta_i \equiv \rho_i > 0\,, \quad i = m+1,\ldots,n\,. \tag{20}$$

We then investigate the stability of the above set of solutions by calculating the eigenvalues of $M_{ij}(r=0)$:

$$M_{ij}(r=0)\ \xi_j^{(l)} = \delta_{(l)}\ \xi_i^{(l)}\,, \tag{21}$$

where $M_{ij}(r)$ is given in (18), and divide $\delta_{(l)}$ in the positive and negatives ones:

$$\delta_{(l_+)} > 0\,, \quad \delta(l_-) \leqslant 0\,. \tag{22}$$

To proceed, we assume that the leading behavior of $\tilde{\alpha}_i$ for $i = 1,\ldots,m$ in the asymptotic limit is given by

$$\tilde{\alpha}_i = \sum_{l_+} a_{l_+}\,[\alpha]^{\delta_{(l_+)}}\,\xi_i^{(l_+)} \neq 0 \quad \text{for } i = 1,\ldots,m\,, \tag{23}$$

where a_{l_+} are arbitrary constants. Note that the nonzero requirement on the right-hand side of Eq. (23) is nontrivial, and if $\tilde{\alpha}_i$ for some $i = 1,\ldots,m$ vanishes in the approximation above, we must re-arrange the superpotential, because we have been assuming that none of the superpotential couplings can be dropped. We then check whether the ansatz (23) really corresponds to the leading behavior by taking account into the higher-order terms in Eq. (12). As in the case of (i), if the wave-function renormalization constants are diagonal, the positivity of $\delta_{(l_+)}$ is sufficient for (23) to be a consistent approximation.

If all those conditions are satisfied, we may regard $\tilde{\alpha}_i$ $(i = 1,\ldots,m)$ as small perturbations to the undisturbed reduced system that is defined by α and $\tilde{\alpha}_i(\alpha)$ $(i = m+1,\ldots,n)$. The small (asymptotically-free) perturbations enter in such a way that the reduced couplings, i.e. $\tilde{\alpha}_i$ $(i = m+1,\ldots,n)$, become functions of α as well as of $\tilde{\alpha}_i$ $(i = 1,\ldots,m)$. It turned out [8,9] that, to investigate such partially reduced systems, it is most convenient to work with the partial differential equations which are mathematically equivalent to the reduction equations (12) (to avoid confusion, we let a, b run from 1 to m and i, j from $m+1$ to n):

$$\left\{ \tilde{\beta} \frac{\partial}{\partial \alpha} + \sum_{a=1}^{m} \tilde{\beta}_a \frac{\partial}{\partial \tilde{\alpha}_a} \right\} \tilde{\alpha}_i(\alpha, \tilde{\alpha}) = \tilde{\beta}_i(\alpha, \tilde{\alpha}) , \tag{24}$$

$$\tilde{\beta}_{i(a)} = \frac{\beta_{i(a)}}{\alpha^2} - \frac{\beta}{\alpha^2} \, \alpha_{\tilde{i(a)}} , \quad \tilde{\beta} \equiv \frac{\beta}{\alpha} .$$

We then look for solutions of the form

$$\tilde{\alpha}_i = \eta_i + \sum_{r=1} \left(\frac{\alpha}{\pi}\right)^{r-1} f_i^{(r)}(\tilde{\alpha}_a) , \tag{25}$$

where $f_i^{(r)}(\tilde{\alpha}_a)$ are supposed to be power series of $\tilde{\alpha}_a$ [7]. Inserting the ansatz (25) into Eq. (24), and assuming that $f_i^{(r)}$ are power series of $\tilde{\alpha}_a$, one finds that it is possible to obtain sufficient conditions for the uniqueness of $f_i^{(r)}$ in terms of the lowest-order coefficients. Since in most of the cases the wave-function renormalization constants are diagonal, we give here the sufficient conditions for that case:

$$\det N_{ij}(r, r_a) \neq 0 \quad \text{for all} \quad r-1 \, , \; r_a = 0, \dots ,$$

$$N_{ij}(r, r_a) \equiv \left\{ - \left(r + \sum_{a=1}^{m} r_a \, \delta_{(a)} \right) \beta^{(1)} + \beta_{i,ii}^{(1)} \eta_i \right\} \delta_{ij} - 2 \, \beta_{i,ij} \eta_i , \tag{26}$$

where we have used the fact

$$0 = -\beta^{(1)} + \beta_i^{(1)} - 2 \sum_{k \neq i} \beta_{i,ik}^{(1)} \eta_k - \beta_{i,ii}^{(1)} \eta_i ,$$

$$\delta_{(a)} = \frac{1}{\beta^{(1)}} \left[-\beta^{(1)} + \beta_a^{(1)} - 2 \sum_k \beta_{a,ak}^{(1)} \eta_k \right] . \tag{27}$$

The $\delta_{(a)}$ above is exactly the exponent for the leading behavior of $\tilde{\alpha}_a$ $(a = 1, \dots, m)$:

$$\tilde{\alpha}_a \sim [\alpha]^{\delta_{(a)}} \quad \text{as} \quad \alpha \to 0 . \tag{28}$$

Since $\delta_{(a)} > 0$ by assumption, we see from (28) that all $\tilde{\alpha}_i$ $(i = m+1, \dots, n)$ have to approach asymptotically zero as $\alpha \to 0$, implying that the partially reduced system (20) with the solutions (23) and (25) is asymptotically free.

We would like to emphasize once again that the reduction of couplings is not the infrared fixed point method of Pendleton and Ross [25]. In the case of partial reduction (which we have discussed above), the difference exists already at the one-loop level; the correction terms in the solutions (25), which can be systematically calculated and

[7] This particular type of solution can be motivated by requiring that in the limit of vanishing perturbations we obtain the undisturbed solutions [8,27].

approach zero in the ultraviolet limit, have nothing to do with infrared fixed points and do not vanish in the one-loop approximation.

3. Minimal AF supersymmetric SU(5) model

Let us consider the *minimal* $N = 1$ supersymmetric gauge model based on the group SU(5) [4]. Its particle content is then specified and has the following transformation properties under SU(5): three $(\overline{\mathbf{5}}+\mathbf{10})$-supermultiplets which accommodate three fermion families, one $(\mathbf{5}+\overline{\mathbf{5}})$ to describe the two Higgs supermultiplets appropriate for electroweak symmetry breaking and a $\mathbf{24}$-supermultiplet required to provide the spontaneous symmetry breaking of SU(5) down to SU(3) × SU(2) × U(1).

Since we are neglecting the dimensional parameters and furthermore ignoring the family mixing, the superpotential of the model is exactly given by

$$\begin{aligned} W = {} & \frac{1}{2}\{ g_u \mathbf{10}_1\,\mathbf{10}_1 + g_c \mathbf{10}_2\,\mathbf{10}_2 + g_t \mathbf{10}_3\,\mathbf{10}_3 \} H \\ & + \{ g_d \overline{\mathbf{5}}_1\,\mathbf{10}_1 + g_s \overline{\mathbf{5}}_2\,\mathbf{10}_2 + g_b \overline{\mathbf{5}}_3\,\mathbf{10}_3 \} \overline{H} + \frac{g_\lambda}{3}(\mathbf{24})^3 + g_f \overline{H}\,\mathbf{24}\,H , \end{aligned} \tag{29}$$

where $H, \overline{H}$ are the $\mathbf{5}, \overline{\mathbf{5}}$ Higgs supermultiplets and we have suppressed the SU(5) indices. The one-loop β-functions of the above couplings are given in (6) and (7) and for the present model are found to be the following:

$$\begin{aligned} \beta^{(1)} &= -\frac{3}{16\pi^2} g^3 , \\ \beta_U^{(1)} &= \frac{1}{16\pi^2}\left[-\frac{96}{5} g^2 + 9 g_U^2 + \frac{24}{5} g_f^2 + 4 g_D^2 \right] g_U , \quad U = u, c, t , \\ \beta_D^{(1)} &= \frac{1}{16\pi^2}\left[-\frac{84}{5} g^2 + 3 g_U^2 + \frac{24}{5} g_f^2 + 10 g_D^2 \right] g_D , \quad D = d, s, b , \\ \beta_\lambda^{(1)} &= \frac{1}{16\pi^2}\left[-30 g^2 + \frac{63}{5} g_\lambda^2 + 3 g_f^2 \right] g_\lambda , \\ \beta_f^{(1)} &= \frac{1}{16\pi^2}\left[-\frac{98}{5} g^2 + 3 \sum_{U=u,c,t} g_U^2 + 4 \sum_{D=d,s,b} g_D^2 + \frac{53}{5} g_f^2 + \frac{21}{5} g_\lambda^2 \right] g_f , \end{aligned} \tag{30}$$

in an obvious notation. According to the notation introduced in (11), let us define

$$\tilde{\alpha}_i \equiv \frac{\alpha_i}{\alpha} , \qquad \alpha_i = \frac{g_i^2}{4\pi} , \quad i = u, d, \ldots, \lambda, f . \tag{31}$$

In terms of these couplings, Eqs. (12) become

$$\begin{aligned} \alpha \frac{\mathrm{d}\tilde{\alpha}_U}{\mathrm{d}\alpha} &= \frac{27}{5}\tilde{\alpha}_U - 3\tilde{\alpha}_U^2 - \frac{4}{3}\tilde{\alpha}_U\tilde{\alpha}_D - \frac{8}{5}\tilde{\alpha}_U\tilde{\alpha}_f , \\ \alpha \frac{\mathrm{d}\tilde{\alpha}_D}{\mathrm{d}\alpha} &= \frac{23}{5}\tilde{\alpha}_D - \frac{10}{3}\tilde{\alpha}_D^2 - \tilde{\alpha}_D\tilde{\alpha}_U - \frac{8}{5}\tilde{\alpha}_D\tilde{\alpha}_f , \end{aligned}$$

$$\alpha\frac{\mathrm{d}\tilde{\alpha}_\lambda}{\mathrm{d}\alpha} = 9\tilde{\alpha}_\lambda - \frac{21}{5}\tilde{\alpha}_\lambda^2 - \tilde{\alpha}_\lambda\tilde{\alpha}_f ,$$
$$\alpha\frac{\mathrm{d}\tilde{\alpha}_f}{\mathrm{d}\alpha} = \frac{83}{15}\tilde{\alpha}_f - \frac{53}{15}\tilde{\alpha}_f^2 - \tilde{\alpha}_f\tilde{\alpha}_U - \frac{4}{3}\tilde{\alpha}_f\tilde{\alpha}_D - \frac{7}{5}\tilde{\alpha}_f\tilde{\alpha}_\lambda , \tag{32}$$

in the one-loop approximation. Given the above equations describing the evolution of the eight independent couplings (α_i, $i = u, d, \ldots, \lambda, f$), there exist $2^8 = 256$ nondegenerate solutions corresponding to vanishing ρ's as well as nonvanishing ones given by Eq. (19). As we emphasized in the introduction, we require the reduction solutions to yield some information on the top quark mass. The possibility to predict the top quark mass depends on an interplay between the vacuum expectation value of the two SU(2) Higgs doublets involved in the model and the known masses of the third generation (m_b, m_τ). For the case at hand we find that only the solutions with $\rho_t, \rho_b \neq 0$ are eligible, which in turn means that we are left with $2^6 = 64$ possibilities. We further require the solutions to be consistent with the observed fermion mass spectrum of the first two generations. This is possible only if the Yukawa couplings of the first two generations can be treated as small perturbations. This implies that we have to choose the solutions of the form

$$\rho_t, \rho_b \neq 0 \quad \text{and} \quad \rho_u = \rho_d = \rho_c = \rho_s = 0 . \tag{33}$$

There exist exactly four such solutions:

$$\begin{array}{lllll}
1: & \eta_t = \frac{132}{95}, & \eta_b = \frac{94}{95}, & \eta_\lambda = \frac{20645}{9576}, & \eta_f = -\frac{25}{456}, \\
2: & \eta_t = \frac{89}{65}, & \eta_b = \frac{63}{65}, & \eta_\lambda = \frac{15}{7}, & \rho_f = 0, \\
3: & \eta_t = \frac{2533}{2605}, & \eta_b = \frac{1491}{2605}, & \rho_\lambda = 0, & \eta_f = \frac{560}{521}, \\
4: & \eta_t = \frac{89}{65}, & \eta_b = \frac{63}{65}, & \rho_\lambda = 0, & \rho_f = 0,
\end{array} \tag{34}$$

where η's are nonvanishing ρ's. The solution 1 must be ruled out because $\eta_f = \rho_f < 0$ and therefore it is inconsistent with Eq. (9). The solution 2 also has to be ruled out because $\delta_f = -5/39 < 0$ (which is defined in Eq. (21)). Recall that a negative δ (see (28)) means that to obtain an asymptotically free system either we have to set the corresponding coupling, α_f in this case, identically equal to zero, or it should be nontrivially reduced. If $\alpha_f = 0$ we should search for a mechanism to provide the necessary doublet-triplet splitting in the $\mathbf{5}, \overline{\mathbf{5}}$ supermultiplets in order to make the model consistent with the experimental limits on proton decay. If α_f is nontrivially reduced it is the solution 1.

So we are left with two solutions and find

$$\begin{array}{ll}
3: & \delta_\lambda = \frac{4129}{521}, \quad \delta_u = \delta_c = \frac{9587}{2605}, \quad \delta_d = \delta_s = \frac{7503}{2605}, \\
4: & \delta_\lambda = 9, \quad \delta_f = \frac{112}{39}, \quad \delta_u = \delta_c = \frac{27}{5}, \quad \delta_d = \delta_s = \frac{23}{5},
\end{array} \tag{35}$$

which are both asymptotically free. Moreover, according to the criterion of the previous section, the solutions 3 and 4 give the possibility to obtain partial reductions, which we will do in the following.

That is, we look for solutions to Eq. (24) of the form

$$\tilde{\alpha}_{t,b,f} = \eta_{t,b,f} + \sum_{r,r_u,\ldots,r_\lambda} \left(\frac{\alpha}{\pi}\right)^r \left[\prod_{a=u,d,s,c,\lambda} (\tilde{\alpha}_a)^{r_a}\right] f_{t,b,f}^{(r,r_u,\ldots,r_\lambda)} \quad \text{for } 3,$$

$$\tilde{\alpha}_{t,b} = \eta_{t,b} + \sum_{r,r_u,\ldots,r_\lambda,r_f} \left(\frac{\alpha}{\pi}\right)^r \left[\prod_{a=u,d,s,c,\lambda,f} (\tilde{\alpha}_a)^{r_a}\right] f_{t,b}^{(r,r_u,\ldots,r_\lambda,r_f)} \quad \text{for } 4. \tag{36}$$

To see the uniqueness of the expansion coefficients f's we have to compute the matrix N (defined in Eq. (26)), and we find

$$\begin{aligned}
N_{tt} &= -\frac{3}{2\pi}\left[r + \frac{9587}{2605}(r_u+r_c) + \frac{7503}{2605}(r_d+r_s) + \frac{4129}{521}r_\lambda + \frac{7599}{2605}\right],\\
N_{bb} &= -\frac{3}{2\pi}\left[r + \frac{9587}{2605}(r_u+r_c) + \frac{7503}{2605}(r_d+r_s) + \frac{4129}{521}r_\lambda + \frac{994}{527}\right],\\
N_{ff} &= -\frac{3}{2\pi}\left[r + \frac{9587}{2605}(r_u+r_c) + \frac{7503}{2605}(r_d+r_s) + \frac{4129}{521}r_\lambda + \frac{5936}{1563}\right],\\
N_{tb} &= -\frac{3}{2\pi}\frac{10132}{7815}, \quad N_{tf} = -\frac{3}{2\pi}\frac{20264}{13025}, \quad N_{bt} = -\frac{3}{2\pi}\frac{1491}{2605},\\
N_{bf} &= -\frac{3}{2\pi}\frac{11928}{13175}, \quad N_{ft} = -\frac{3}{2\pi}\frac{560}{521}, \quad N_{fb} = -\frac{3}{2\pi}\frac{2240}{1563},
\end{aligned} \tag{37}$$

for the solution 3, while for the solution 4 we obtain

$$\begin{aligned}
N_{tt} &= -\frac{3}{2\pi}\left[r + \frac{27}{5}(r_u+r_c) + \frac{23}{5}(r_d+r_s) + 9r_\lambda + \frac{112}{39}r_f + \frac{801}{195}\right],\\
N_{bb} &= -\frac{3}{2\pi}\left[r + \frac{27}{5}(r_u+r_c) + \frac{23}{5}(r_d+r_s) + 9r_\lambda + \frac{112}{39}r_f + \frac{42}{13}\right],\\
N_{tb} &= -\frac{3}{2\pi}\frac{63}{65}, \quad N_{tb} = -\frac{3}{2\pi}\frac{356}{195}.
\end{aligned} \tag{38}$$

One can explicitly convince oneself that each of the corresponding determinants can never become zero for all $r, r_u, \ldots \geqslant 0$, from which we conclude that the solutions (36) are unique to all orders in perturbation theory.

We have also computed some lower-order terms within the one-loop approximation. For the solution 3 we find

$$\tilde{\alpha}_i = \eta_i + f_i^{(r_\lambda=1)}\,\tilde{\alpha}_\lambda + f_i^{(r_\lambda=2)}\,\tilde{\alpha}_\lambda^2 + \ldots \quad \text{for } i = t, b, f, \tag{39}$$

where

$$\eta_{t,b,f} = \frac{2533}{2605}, \frac{1491}{2605}, \frac{560}{521},$$

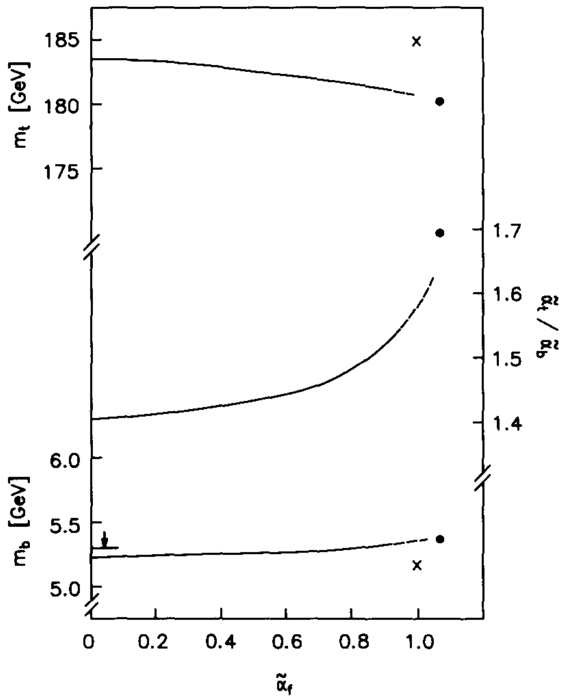

Fig. 1. The $\tilde{\alpha}_f$-dependence of $\tilde{\alpha}_t$ and $\tilde{\alpha}_b$ with $\tilde{\alpha}_\lambda = 0$ (Eqs. (41)). The values for the FUT [10] (Eqs.(44)) are indicated by $\times$. The $\bullet$ points corresponds to η's of the solution 3 (Eq. (40)).

$$f_{t,b,f}^{(r_\lambda=1)} \simeq 0.018\,,\ 0.012\,,\ -0.131\,, \qquad (40)$$
$$f_{t,b,f}^{(r_\lambda=2)} \simeq 0.005\,,\ 0.004\,,\ -0.021\,,$$

and for the solution 4

$$\tilde{\alpha}_i = \eta_i + f_i^{(r_f=1)}\,\tilde{\alpha}_f + f_i^{(r_\lambda=1)}\,\tilde{\alpha}_\lambda + f_i^{(r_f=1,r_\lambda=1)}\,\tilde{\alpha}_f\,\tilde{\alpha}_\lambda$$
$$+ f_i^{(r_f=2)}\,\tilde{\alpha}_f^2 + f_i^{(r_\lambda=2)}\,\tilde{\alpha}_\lambda^2 \ldots \quad \text{for } i = t, b\,, \qquad (41)$$

where

$$\begin{aligned} \eta_{t,b} &= \frac{89}{65}, \frac{63}{65}, & f_i^{(r_\lambda=1)} &= f_i^{(r_\lambda=2)} = 0\,, \\ f_{t,b}^{(r_f=1)} &\simeq -0.258,\ -0.213\,, & f_{t,b}^{(r_f=2)} &\simeq -0.055\,,\ -0.050\,, \\ f_{t,b}^{(r_f=1,r_\lambda=1)} &\simeq -0.021\,,\ -0.018\,. \end{aligned} \qquad (42)$$

In the solutions (39) and (41) we have suppressed the contributions from the Yukawa couplings of the first two generations because they are negligibly small.

Presumably, both solutions (39) and (42) are related; a numerical analysis on the solutions, as shown in Fig. 1, suggests that the solution 3 is a "boundary" of 4 as it often happens between a nontrivial and trivial reduction solutions [8]. If it is really so, then there is only one unique (partial) reduction solution in the minimal supersymmetric GUT that provides us with the possibility of predicting α_t. Note furthermore that not only α_t but also α_b is predicted in this reduction solution.

So far we have considered the unbroken minimal supersymmetric SU(5) model, and required the reduction solutions to yield relations among the couplings of the theory that may be phenomenologically viable. The next step is to relate the singled out solutions (39) and (41) with observable parameters. To this end, we apply the well-known renormalization group technique and regard these reduction solutions as the boundary conditions holding at the unification scale [9] in addition to the usual ones

$$\alpha_1 = \alpha_2 = \alpha_3; \quad \alpha_b = \alpha_\tau; \ldots . \tag{43}$$

Just below the unification scale we would like to obtain the standard SU(3) × SU(2) × $U(1)$ model while assuming that all the superpartners are decoupled below the Fermi scale. Then the standard model should be spontaneously broken down to SU(3) × $U(1)_{\mathrm{em}}$ due to v.e.v. of the two Higgs SU(2)-doublets contained in the $\mathbf{5}, \bar{\mathbf{5}}$-supermultiplets. One way to obtain the correct low energy theory is to add to the Lagrangian soft supersymmetry breaking terms and then to arrange the mass parameters in the superpotential along with the soft breaking terms so that the desired symmetry breaking pattern of the original SU(5) is really the preferred one, all the superpartners are unobservable at present energies, there is no contradiction with proton decay, and so forth (for instance, see Refs. [28]).

The largest theoretical uncertainty after all the above is done is the arbitrariness of the superpartner masses. To simplify our numerical analysis we would like to assume a unique threshold for all the superpartners. (We use the θ-function approximation to β-functions to take into account heavy particle decoupling.) Then we examine numerically the evolution of the gauge and Yukawa couplings including the two-loop effects, according to their renormalization group equations [20,29]. In Fig. 2 we plot the variation of m_t, m_b and $\tilde{\alpha}_t/\tilde{\alpha}_b$ versus $\tilde{\alpha}_f$ while using supersymmetry breaking scale $m_{\mathrm{SUSY}} = 200$ GeV, $\alpha_{\mathrm{S}}(M_Z) = 0.12$, $\alpha^{-1}(m_{\mathrm{GUT}}) = 24.2$ and $m_\tau = 1.78$ GeV as inputs. Since $\alpha(m_{\mathrm{GUT}})$ is fixed in this analysis, α_{em} is no longer a free parameter. We find that, for $\tilde{\alpha}_f \leqslant 0.9$, $\sin^2\theta_{\mathrm{W}}(M_Z)$ and $\alpha_{\mathrm{em}}(M_Z)$ are consistent with the experimental values [10]:

$$\sin^2\theta_{\mathrm{W}}(M_Z)_{\mathrm{exp}} = 0.2324 \pm 0.0008\,, \qquad \alpha_{\mathrm{em}}^{-1}(M_Z)_{\mathrm{exp}} = 127.9 \pm 0.2\,.$$

In Table 1 we present all the parameters of the (partially) reduced minimal susy SU(5) model for two distinct supersymmetry breaking scales; $m_{\mathrm{SUSY}} = 200$ GeV and 500 GeV (with $\tilde{\alpha}_f = 0.2$, $\tilde{\alpha}_\lambda = 0$). All the dimensionless parameters (except $\tan\beta$) are defined in the $\overline{\mathrm{MS}}$ scheme, and all the masses (except for m_{SUSY} and m_{GUT}) are pole masses.

[8] The solutions (41) are plotted as functions of $\tilde{\alpha}_f$ where we have set $\tilde{\alpha}_\lambda = 0$ because $\tilde{\alpha}_\lambda$-dependence is small as one can see from (41).

[9] For $\tilde{\alpha}_\lambda = 0, \tilde{\alpha}_f = 0.1$, for instance, these boundary conditions are: $\alpha \simeq 0.745\,\alpha_t \simeq 1.056\,\alpha_b$, which should be compared with the SO(10) type boundary conditions [13].

[10] If we take into account the corrections to $\sin^2\theta_{\mathrm{W}}(M_Z)$ that come from a large m_t, $-10^7\,[138^2 - (m_t/\mathrm{GeV})^2]$, the results that agree with the corrected one give slightly larger m_b, i.e. $(5.4 - 5.7)$ GeV.

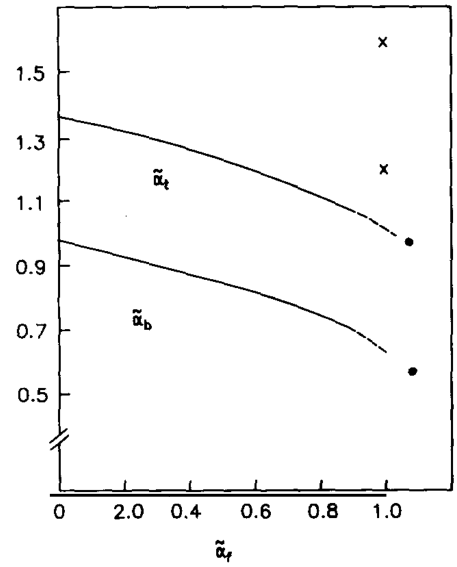

Fig. 2. The predictions of m_t, m_b and $\tilde{\alpha}_t/\tilde{\alpha}_b$ (m_{GUT}) as functions of $\tilde{\alpha}_f$. The inputs are: $m_{\mathrm{SUSY}} = 200$ GeV, $\alpha_S(M_Z) = 0.12$, $\alpha^{-1}(m_{\mathrm{GUT}}) = 24.2$ and $m_\tau = 1.78$ GeV. The points with $\times$ are FUT predictions [10], and those with $\bullet$ correspond to the edge of the solution 3 (Eq. (39)). The experimental upper bound on m_b is also indicated.

Table 1
The predictions for $m_{\mathrm{SUSY}} = 200$, 500 GeV, where we have used: $m_\tau = 1.78$ GeV, $\alpha_{\mathrm{em}}^{-1}(M_Z) = 127.9$ and $\alpha_S(M_Z) = 0.12$

m_{SUSY}	$\sin^2\theta_W(M_Z)$	$\alpha(m_{\mathrm{GUT}})$	$\tan\beta$	m_{GUT}	m_b	m_t
200 GeV	0.232	0.041	51.7	1.9×10^{16} GeV	5.2 GeV	183.1 GeV
500 GeV	0.231	0.041	51.7	1.9×10^{16} GeV	5.4 GeV	184.6 GeV

Note that all the quantities except m_{SUSY} in the Table 1 are predicted in the present model of gauge-Yukawa unification. Our predicted value of m_t is quite similar to that of the SO(10) model [13]. This similarity is certainly related to the quasi infrared-stable fixed point behavior of the Yukawa couplings [30]. But we would like to emphasize that our model of unification has more predictive power than the SO(10) model.

It is very interesting to compare the prediction above with that of the SU(5) FUT [10] which for this reason is included in Fig. 1. We recall that in the latter case the solution of the reduction equations resulted in the following relations among the various couplings

$$\tilde{\alpha}_U = \frac{8}{5}, \quad (U = u, c, t), \qquad \tilde{\alpha}_D = \frac{6}{5}, \quad (D = d, s, b),$$
$$\tilde{\alpha}_f = 1, \qquad \tilde{\alpha}_\lambda = \frac{15}{7}. \tag{44}$$

From Fig. 2 one concludes that it is very difficult to discriminate the predictions of partially reduced and finite SU(5) models [11].

4. Summary

The accurate measurements of the gauge coupling constants in the standard model may be suggesting that the electroweak and strong interactions can be unified within the framework of relativistic field theory. If it is really so, completely asymptotically free models have certainly more chance to become the consistent unified theory of those interactions because they presumably do not suffer from the theoretically serious problem of *triviality*. However, as realized in the early stage of developments on GUTs [17], asymptotic freedom and spontaneous symmetry breaking of a unifying gauge group down to $SU(3) \times U(1)_{em}$ through $SU(3) \times SU(2) \times U(1)$ cannot easily coexist. In this respect supersymmetric models are very different. However, supersymmetrizing an ordinary GUT (with $N = 1$) does not improve its predictability in general; the gauge and Yukawa sectors will be kept unrelated, and the family problem will remain still unsolved.

While we do not claim that the idea of reduction of couplings can solve these difficult problems, we recall that it provides a theoretical tool to reduce the number of the independent couplings in a given model without loosing its renormalizability and asymptotic freedom property [5,6]. In this paper we have worked out the reduction program for the minimal susy GUT and studied the interplay between its asymptotic freedom property and predictability. We have found that there exists only an extremely limited number of possibilities to reduce the model in a way that causes no conflict with the mass spectrum of the first two generations of fermions and can predict the top quark mass. Interestingly, the predicted top mass is not only consistent within the experimental bounds, but also can (hopefully) be tested soon. The prediction on m_t does not differ very much from that of the SU(5) FUT model [10], suggesting that the partially reduced model we have presented is not far from the finite one in a certain sense. It would be interesting to carry out similar programs in other models to observe differences or similarities, and this investigation might help in searching for the physical principles on which the reduction method is based on.

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[11] At first sight the relations (44) seem to cause problems with the light fermion mass spectrum. Note however that the finite model contains four $(\mathbf{5}+\bar{\mathbf{5}})$ Higgs supermultiplets so that it is technically possible to reproduce the known mass spectrum and generation mixing [10].

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5.3 Perturbative unification of soft supersymmetry-breaking

Title: Perturbative unification of soft supersymmetry-breaking terms
Authors: J. Kubo, M. Mondragon, G. Zoupanos
Journal: Phys. Letts. **B389** (1996) 523-532

Comment (*Myriam Mondragón, George Zoupanos*)
As we have seen in subsections 2.1 and 2.2 the reduction of couplings was originally formulated for massless theories. On the other hand the successful reduction and impressive predictions of the top and bottom quark masses of $N = 1$ $SU(5)$ GUTs (finite and minimal supersymmetric) require the introduction of a massive soft supersymmery breaking (SSB) sector to become realistic. The extension of the reduction of couplings to theories with massive parameters is not straightforward if one wants to keep the generality and the rigour on the same level as for the massless case. In this paper for simplicity a mass-independent renormalization scheme has been employed so that all the RG functions have only trivial dependencies on the dimensional parameters. Then the method suggested consists in searching for RGI relations among the SSB parameters, which are consistent with the perturbative renormalizability.

The method has been applied in the minimal GYU $N = 1$ supersymmetric $SU(5)$ model with the result that the SSB sector contains as the only arbitrary parameter the unified gaugino mass. Another characteristic feature of the findings of the analysis is that the set of the perturbatively unified SSB parameters differs significantly from the so-called universal SSB parameters, signaling already at that time the existence of a "sum rule" in GYU theories, as will be discussed later in subsections 5.5 and 5.6. The mass spectrum was then calculated using the experimental constraints known at the time and would have been ruled out now with the present LHC results. A new analysis, taking into account the recent B-physics results and including the radiative corrections coming from the supersymmetric spectrum for the bottom and tau masses, certainly would be very interesting and could lead to different spectrum to be compared with the recent findings at LHC on the Higgs mass and on the bounds of supersymmetric particles.

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Perturbative unification of soft supersymmetry-breaking terms

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Abstract

Perturbative unification of soft supersymmetry-breaking (SSB) parameters is proposed in gauge-Yukawa unified models. The method, which can be applied in any finite order in perturbation theory, consists in searching for renormalization group invariant relations among the SSB parameters, which are consistent with perturbative renormalizability. For the minimal gauge-Yukawa unified model based on $SU(5)$ we find that the low energy SSB sector contains a single arbitrary parameter, the unified gaugino mass. Within a certain approximation we find that the model predicts a superpartner spectrum which is consistent with the experimental data.

1. Introduction

The usual path chosen to reduce the independent parameters of a theory is the introduction of a symmetry. Grand Unified Theories (GUTs) are representative examples of such attempts. A natural gradual extension of the GUT idea, which preserves their successes and enhances the predictions, may be to attempt to relate the gauge and Yukawa couplings, or in other words, to achieve gauge-Yukawa Unification (GYU).

In recent papers, we have proposed an alternative way to achieve unification of couplings, which is based on the principles of reduction of couplings and finiteness [5]. These principles, which are formulated in perturbation theory, are not explicit symmetry principles, although they might imply symmetries. The former principle is based on the existence of renormalization group (RG) invariant relations among couplings, which do not necessarily result from a symmetry, but nevertheless preserve perturbative renormalizability. Similarly, the latter one is based on the fact that it is possible to find RG invariant relations among couplings that keep finiteness

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in perturbation theory. We have found that various supersymmetric GYU models predict mass values for the top and bottom quarks, M_t and M_b, which are consistent with the experimental data, and that under certain circumstances the different models can be distinguished from each other if M_t and M_b can be more accurately measured [2].

The most arbitrary part of a phenomenologically viable supersymmetric model is the breaking of supersymmetry. It is widely believed that the breaking of supersymmetry is soft whatever its origin is. If the model is coupled to supergravity, for instance, one can compute in principle the soft supersymmetry-breaking (SSB) terms. In fact, this is an attractive way to reduce the arbitrariness of the SSB terms, where the gravitino mass $m_{2/3}$ defines the scale of the supersymmetry-breaking [3].

In this letter, we would like to extend our unification idea to include the SSB sector. That is, we want to find RG invariant relations among the SSB parameters that are consistent with perturbative renormalizability [6]. To be definite, we will consider the minimal SUSY $SU(5)$ model with the GYU in the third generation [6]. We will find that, if one requires the breaking of the electroweak symmetry to occur in the desired manner, the SSB sector of the model can be completely fixed by the gaugino mass parameter M. It will turn out that the asymptotic freedom in the SSB sector of the gauge-Yukawa unified model can be achieved only through the reduction of the SSB parameters. We will then calculate within a certain approximation the SSB parameters of the minimal supersymmetric standard model (MSSM), which will turn out to be consistent with the experimental data. More details of our results will be published elsewhere.

2. Formalism

The reduction of couplings was originally formulated for massless theories on the basis of the Callan-Symanzik equation [7]. The extension to theories with massive parameters is not straightforward if one wants to keep the generality and the rigor on the same level as for the massless case; one has to fulfill a set of requirements coming from the renormalization group equations, the Callan-Symanzik equations, etc. along with the normalization conditions imposed on irreducible Green's functions [8]. There has been some progress in this direction [9]. Here, to simplify the situation, we would like to assume that a mass-independent renormalization scheme has been employed so that all the RG functions have only trivial dependencies of dimensional parameters.

To be general, we consider a renormalizable theory which contain a set of $(N+1)$ dimension-zero couplings, $\{\hat{g}_0, \hat{g}_1, \ldots, \hat{g}_N\}$, a set of L parameters with dimension one, $\{\hat{h}_1, \ldots, \hat{h}_L\}$, and a set of M parameters with dimension two, $\{\hat{m}_1^2, \ldots, \hat{m}_M^2\}$. The renormalized irreducible vertex function satisfies the RG equation

$$0 = \mathcal{D}\Gamma[\ \Phi's; \hat{g}_0, \hat{g}_1, \ldots, \hat{g}_N; \hat{h}_1, \ldots, \hat{h}_L; \hat{m}_1^2, \ldots, \hat{m}_M^2; \mu\ ], \tag{1}$$

$$\mathcal{D} = \mu\frac{\partial}{\partial\mu} + \sum_{i=0}^{N} \beta_i \frac{\partial}{\partial\hat{g}_i} + \sum_{a=1}^{L} \gamma_a^h \frac{\partial}{\partial\hat{h}_a} + \sum_{\alpha=1}^{M} \gamma_\alpha^{m^2} \frac{\partial}{\partial\hat{m}_\alpha^2} + \sum_J \Phi_I \gamma^{\phi I}{}_J \frac{\delta}{\delta\Phi_J}.$$

Since we assume a mass-independent renormalization scheme, the γ's have the form

$$\gamma_a^h = \sum_{b=1}^{L} \gamma_a^{h,b}(g_0, \ldots, g_N)\hat{h}_b,$$

$$\gamma_\alpha^{m^2} = \sum_{\beta=1}^{M} \gamma_\alpha^{m^2,\beta}(g_0, \ldots, g_N)\hat{m}_\beta^2 + \sum_{a,b=1}^{L} \gamma_\alpha^{m^2,ab}(g_0, \ldots, g_N)\hat{h}_a\hat{h}_b, \tag{2}$$

[6] A similar but different idea has been recently proposed in Refs. [4,5].

where $\gamma_a^{h,b}, \gamma_\alpha^{m^2,\beta}$ and $\gamma_a^{m^2,ab}$ are power series of the dimension-zero couplings g's in perturbation theory.

As in the massless case, we then look for conditions under which the reduction of parameters,

$$\hat{g}_i = \hat{g}_i(g) \quad (i = 1, \ldots, N), \tag{3}$$

$$\hat{h}_a = \sum_{b=1}^{P} f_a^b(g) h_b \quad (a = P+1, \ldots, L), \tag{4}$$

$$\hat{m}_\alpha^2 = \sum_{\beta=1}^{Q} e_\alpha^\beta(g) m_\beta^2 + \sum_{a,b=1}^{P} k_\alpha^{ab}(g) h_a h_b \quad (\alpha = Q+1, \ldots, M), \tag{5}$$

is consistent with the RG equation (1), where we assume that $g \equiv g_0$, $h_a \equiv \hat{h}_a \quad (1 \le a \le P)$ and $m_\alpha^2 \equiv \hat{m}_\alpha^2 \quad (1 \le \alpha \le Q)$ are independent parameters of the reduced theory. We find that the following set of equations has to be satisfied:

$$\beta_g \frac{\partial \hat{g}_i}{\partial g} = \beta_i \quad (i = 1, \ldots, N), \tag{6}$$

$$\beta_g \frac{\partial \hat{h}_a}{\partial g} + \sum_{b=1}^{P} \gamma_b^h \frac{\partial \hat{h}_a}{\partial h_b} = \gamma_a^h \quad (a = P+1, \ldots, L), \tag{7}$$

$$\beta_g \frac{\partial \hat{m}_\alpha^2}{\partial g} + \sum_{a=1}^{P} \gamma_a^h \frac{\partial \hat{m}_\alpha^2}{\partial h_a} + \sum_{\beta=1}^{Q} \gamma_\beta^{m^2} \frac{\partial \hat{m}_\alpha^2}{\partial m_\beta^2} = \gamma_\alpha^{m^2} \quad (\alpha = Q+1, \ldots, M). \tag{8}$$

Using Eq. (2) for γ's, one finds that Eqs. (6)–(8) reduce to

$$\beta_g \frac{df_a^b}{dg} + \sum_{c=1}^{P} f_a^c [\, \gamma_c^{h,b} + \sum_{d=P+1}^{L} \gamma_c^{h,d} f_d^b \,] - \gamma_a^{h,b} - \sum_{d=P+1}^{L} \gamma_a^{h,d} f_d^b = 0 \tag{9}$$
$$(a = P+1, \ldots, L; b = 1, \ldots, P),$$

$$\beta_g \frac{de_\alpha^\beta}{dg} + \sum_{\gamma=1}^{Q} e_\alpha^\gamma [\, \gamma_\gamma^{m^2,\beta} + \sum_{\delta=Q+1}^{M} \gamma_\gamma^{m^2,\delta} e_\delta^\beta \,] - \gamma_\alpha^{m^2,\beta} - \sum_{\delta=Q+1}^{M} \gamma_\alpha^{m^2,\delta} e_\delta^\beta = 0 \tag{10}$$
$$(\alpha = Q+1, \ldots, M; \beta = 1, \ldots, Q),$$

$$\begin{aligned} &\beta_g \frac{dk_\alpha^{ab}}{dg} + 2\sum_{c=1}^{P} (\, \gamma_c^{h,a} + \sum_{d=P+1}^{L} \gamma_c^{h,d} f_d^a \,) k_\alpha^{cb} + \sum_{\beta=1}^{Q} e_\alpha^\beta [\, \gamma_\beta^{m^2,ab} + \sum_{c,d=P+1}^{L} \gamma_\beta^{m^2,cd} f_c^a f_d^b \\ &+ 2 \sum_{c=P+1}^{L} \gamma_\beta^{m^2,cb} f_c^a + \sum_{\delta=Q+1}^{M} \gamma_\beta^{m^2,\delta} k_\delta^{ab} \,] - [\, \gamma_\alpha^{m^2,ab} + \sum_{c,d=P+1}^{L} \gamma_\alpha^{m^2,cd} f_c^a f_d^b \\ &+ 2 \sum_{c=P+1}^{L} \gamma_\alpha^{m^2,cb} f_c^a + \sum_{\delta=Q+1}^{M} \gamma_\alpha^{m^2,\delta} k_\delta^{ab} \,] = 0 \end{aligned} \tag{11}$$
$$(\alpha = Q+1, \ldots, M; a, b = 1, \ldots, P).$$

If these equations are satisfied, the irreducible vertex function of the reduced theory

$$\begin{aligned}&\Gamma_R[\ \Phi's; g; h_1,\ldots,h_P; m_1^2,\ldots,\hat{m}_Q^2;\mu\ ]\\ &\quad\equiv \Gamma[\ \Phi's; g,\hat{g}_1(g),\ldots,\hat{g}_N(g); h_1,\ldots,h_P,\hat{h}_{P+1}(g,h),\ldots,\hat{h}_L(g,h);\\ &\quad m_1^2,\ldots,\hat{m}_Q^2,\hat{m}_{Q+1}^2(g,h,m^2),\ldots,\hat{m}_M^2(g,h,m^2);\mu\ ]\end{aligned} \tag{12}$$

has the same renormalization group flow as the original one.

The requirement for the reduced theory to be perturbative renormalizable means that the functions $\hat{g}_i$, f_a^b, e_α^β and k_α^{ab}, defined in Eq. (3)–(5), should have a power series expansion in the primary coupling g:

$$\hat{g}_i = g\sum_{n=0}^{\infty}\rho_i^{(n)}g^n,\quad f_a^b = g\sum_{n=0}^{\infty}\eta_a^{b\ (n)}g^n,\quad e_\alpha^\beta=\sum_{n=0}^{\infty}\xi_\alpha^{\beta\ (n)}g^n,\quad k_\alpha^{ab}=\sum_{n=0}^{\infty}\chi_\alpha^{ab\ (n)}g^n, \tag{13}$$

To obtain the expansion coefficients, we insert the power series ansatz above into Eqs. (6), (9)–(11) and require that the equations are satisfied at each order in g. Note that the existence of a unique power series solution is a non-trivial matter: It depends on the theory as well as on the choice of the set of independent parameters. In a concrete model we will consider below, we will discuss this issue more in detail.

3. Application to the minimal SUSY $SU(5)$ GUT

3.1. The model and its RG functions

The three generations of quarks and leptons are accommodated by three chiral superfields in $\Psi^I(\mathbf{10})$ and $\Phi^I(\overline{\mathbf{5}})$, where I runs over the three generations. A $\Sigma(\mathbf{24})$ is used to break $SU(5)$ down to $SU(3)_C\times SU(2)_L\times U(1)_Y$, and $H(\mathbf{5})$ and $\overline{H}(\overline{\mathbf{5}})$ to describe the two Higgs superfields appropriate for electroweak symmetry breaking [10]. The superpotential of the model is [10] [7]

$$\begin{aligned}W &= \frac{g_t}{4}\epsilon^{\alpha\beta\gamma\delta\tau}\Psi^{(3)}_{\alpha\beta}\Psi^{(3)}_{\gamma\delta}H_\tau+\sqrt{2}g_b\Phi^{(3)\alpha}\Psi^{(3)}_{\alpha\beta}\overline{H}^\beta+\frac{g_\lambda}{3}\Sigma_\alpha^\beta\Sigma_\beta^\gamma\Sigma_\gamma^\alpha+g_f\overline{H}^\alpha\Sigma_\alpha^\beta H_\beta\\ &\quad+\frac{\mu_\Sigma}{2}\Sigma_\alpha^\gamma\Sigma_\gamma^\alpha+\mu_H\overline{H}^\alpha H_\alpha,\end{aligned} \tag{14}$$

where $\alpha,\beta,\ldots$ are the $SU(5)$ indices, and we have suppressed the Yukawa couplings of the first two generations. The Lagrangian containing the SSB terms is

$$\begin{aligned}-\mathcal{L}_{\rm soft} &= m_{H_u}^2\hat{H}^{*\alpha}\hat{H}_\alpha+m_{H_d}^2\hat{\overline{H}}^*_\alpha\hat{\overline{H}}^\alpha+m_\Sigma^2\hat{\Sigma}^{\dagger\ \alpha}_{\beta}\hat{\Sigma}_\alpha^\beta+\sum_{I=1,2,3}[\ m_{\Phi^I}^2\hat{\Phi}^{*\ (I)}_\alpha\hat{\Phi}^{(I)\alpha}+m_{\Psi^I}^2\hat{\Psi}^{\dagger\ (I)\alpha\beta}\hat{\Psi}^{(I)}_{\beta\alpha}\ ]\\ &\quad+\{\ \tfrac{1}{2}M\lambda\lambda+B_H\hat{\overline{H}}^\alpha\hat{H}_\alpha+B_\Sigma\hat{\Sigma}_\beta^\alpha\hat{\Sigma}_\alpha^\beta+h_f\hat{\overline{H}}^\alpha\hat{\Sigma}_\alpha^\beta\hat{H}_\beta+\frac{h_\lambda}{3}\hat{\Sigma}_\alpha^\beta\hat{\Sigma}_\beta^\gamma\hat{\Sigma}_\gamma^\alpha+\frac{h_t}{4}\epsilon^{\alpha\beta\gamma\delta\tau}\hat{\Psi}^{(3)}_{\alpha\beta}\hat{\Psi}^{(3)}_{\gamma\delta}\hat{H}_\tau\\ &\quad+\sqrt{2}h_b\hat{\Phi}^{(3)\alpha}\hat{\Psi}^{(3)}_{\alpha\beta}\hat{\overline{H}}^\beta+\text{h.c.}\},\end{aligned} \tag{15}$$

where a hat is used to denote the scalar component of each chiral superfield.

The RG functions of this model may be found in Refs. [6,11,5], and we employ the usual normalization of the RG functions, $dA/d\ln\mu=[\beta^{(1)}(A)\ \text{ or }\ \gamma^{(1)}(A)]/16\pi^2+\ldots$, where $\ldots$ are higher orders, and μ is the renormalization scale:

[7] We suppress the hat on the couplings from now on, which was used in the previous section to distinguish the independent parameters from the dependent ones.

$$\beta^{(1)}(g) = -3g^3, \quad \beta^{(1)}(g_t) = [-\tfrac{96}{5}g^2 + 9g_t^2 + \tfrac{24}{5}g_f^2 + 4g_b^2]\,g_t,$$
$$\beta^{(1)}(g_b) = [-\tfrac{84}{5}g^2 + 3g_t^2 + \tfrac{24}{5}g_f^2 + 10g_b^2]\,g_b,$$
$$\beta^{(1)}(g_\lambda) = [-30g^2 + \tfrac{63}{5}g_\lambda^2 + 3g_f^2]\,g_\lambda,$$
$$\beta^{(1)}(g_f) = [-\tfrac{98}{5}g^2 + 3g_t^2 + 4g_b^2 + \tfrac{53}{5}g_f^2 + \tfrac{21}{5}g_\lambda^2]\,g_f, \quad \gamma^{(1)}(M) = -6g^2M,$$
$$\gamma^{(1)}(\mu_\Sigma) = [-20g^2 + 2g_f^2 + \tfrac{42}{5}g_\lambda^2]\,\mu_\Sigma, \quad \gamma^{(1)}(\mu_H) = [-\tfrac{48}{5}g^2 + \tfrac{48}{5}g_f^2 + 4g_b^2 + 3g_t^2]\,\mu_H,$$
$$\gamma^{(1)}(B_H) = [-\tfrac{48}{5}g^2 + \tfrac{48}{5}g_f^2 + 4g_b^2 + 3g_t^2]\,B_H + [\tfrac{96}{5}g^2M + \tfrac{96}{5}h_fg_f + 8g_bh_b + 6g_th_t]\,\mu_H,$$
$$\gamma^{(1)}(B_\Sigma) = [-20g^2 + 2g_f^2 + \tfrac{42}{5}g_\lambda^2]\,B_\Sigma + [40g^2M + 4h_fg_f + \tfrac{84}{5}g_\lambda h_\lambda]\,\mu_\Sigma,$$
$$\gamma^{(1)}(h_t) = [-\tfrac{96}{5}g^2 + 9g_t^2 + \tfrac{24}{5}g_f^2 + 4g_b^2]\,h_t + [\tfrac{192}{5}Mg^2 + 18h_tg_t + 8h_bg_b + \tfrac{48}{5}h_fg_f]\,g_t,$$
$$\gamma^{(1)}(h_b) = [-\tfrac{84}{5}g^2 + 3g_t^2 + \tfrac{24}{5}g_f^2 + 10g_b^2]\,h_b + [\tfrac{168}{5}Mg^2 + 6h_tg_t + 20h_bg_b + \tfrac{48}{5}h_fg_f]\,g_b,$$
$$\gamma^{(1)}(h_\lambda) = [-30g^2 + \tfrac{63}{5}g_\lambda^2 + 3g_f^2]\,h_\lambda + [60Mg^2 + \tfrac{126}{5}h_\lambda g_\lambda + 6h_fg_f]\,g_\lambda,$$
$$\gamma^{(1)}(h_f) = [-\tfrac{98}{5}g^2 + 3g_t^2 + 4g_b^2 + \tfrac{53}{5}g_f^2 + \tfrac{21}{5}g_\lambda^2]\,h_f + [\tfrac{196}{5}Mg^2 + 6h_tg_t + 8h_bg_b + \tfrac{42}{5}h_\lambda g_\lambda + \tfrac{106}{5}h_fg_f]\,g_f,$$
$$\gamma^{(1)}(m_{H_d}^2) = -\tfrac{96}{5}g^2M^2 + \tfrac{48}{5}g_f^2(m_{H_u}^2 + m_{H_d}^2 + m_\Sigma^2) + 8g_b^2(m_{H_d}^2 + m_{\Psi^3}^2 + m_{\Phi^3}^2) + \tfrac{48}{5}h_f^2 + 8h_b^2,$$
$$\gamma^{(1)}(m_{H_u}^2) = -\tfrac{96}{5}g^2M^2 + \tfrac{48}{5}g_f^2(m_{H_u}^2 + m_{H_d}^2 + m_{\Sigma^3}^2) + 6g_t^2(m_{H_u}^2 + 2m_{\Psi^3}^2) + \tfrac{48}{5}h_f^2 + 6h_t^2,$$
$$\gamma^{(1)}(m_\Sigma^2) = -40g^2M^2 + 2g_f^2(m_{H_u}^2 + m_{H_d}^2 + m_\Sigma^2) + \tfrac{126}{5}g_\lambda^2m_\Sigma^2 + 2h_f^2 + \tfrac{42}{5}h_\lambda^2,$$
$$\gamma^{(1)}(m_{\Phi^3}^2) = -\tfrac{96}{5}g^2M^2 + 8g_b^2(m_{H_d}^2 + m_{\Psi^3}^2 + m_{\Phi^3}^2) + 8h_b^2,$$
$$\gamma^{(1)}(m_{\Psi^3}^2) = -\tfrac{144}{5}g^2M^2 + 6g_t^2(m_{H_u}^2 + 2m_{\Psi^3}^2) + 4g_b^2(m_{H_d}^2 + m_{\Psi^3}^2 + m_{\Phi^3}^2) + 6h_t^2 + 4h_b^2,$$
$$\gamma^{(1)}(m_{\Phi^{1,2}}^2) = -\tfrac{96}{5}g^2M^2, \quad \gamma^{(1)}(m_{\Psi^{1,2}}^2) = -\tfrac{144}{5}g^2M^2, \tag{16}$$

where g stands for the gauge coupling.

3.2. The reduction solution

We require that the reduced theory should contain the minimal number of the SSB parameters that are consistent with perturbative renormalizability. We will find that the set of the perturbatively unified SSB parameters significantly differ from the so-called universal SSB parameters.

Without loss of generality, one can assume that the gauge coupling g is the primary coupling. Note that the reduction solutions in the dimension-zero sector is independent of the dimensionfull sector (under the assumption of a mass independent renormalization scheme). It has been found [6] that there exist two asymptotically free (AF) solutions that make a gauge-Yukawa Unification possible in the present model:

$$a:\quad g_t = \sqrt{\tfrac{2533}{2605}}\,g + O(g^3), \quad g_b = \sqrt{\tfrac{1491}{2605}}\,g + O(g^3), \quad g_\lambda = 0, \quad g_f = \sqrt{\tfrac{560}{521}}\,g + O(g^3),$$
$$b:\quad g_t = \sqrt{\tfrac{89}{65}}\,g + O(g^3), \quad g_b = \sqrt{\tfrac{63}{65}}\,g + O(g^3), \quad g_\lambda = 0, \quad g_f = 0, \tag{17}$$

where the higher order terms denote uniquely computable power series in g. It has been also found that the two solutions in (17) describe the boundaries of an asymptotically free RG-invariant surface in the space of the couplings, on which g_λ and g_f can be different from zero. This observation has enabled us to obtain a partial reduction of couplings for which the g_λ and g_f can be treated as (non-vanishing) independent parameters without loosing AF. Later we have found [2] that the region on the AF surface consistent with the proton decay constraint has to be very close to the solution a. Therefore, we assume in the following discussion that we are exactly at the boundary defined by the solution a [8].

In the dimensionful sector, we seek the reduction of the parameters in the form (4) and (5). First, one can realize that the supersymmetric mass parameters, μ_Σ and μ_H, and the gaugino mass parameter M cannot be reduced; that is, there is no solution in the desired form. Therefore, they should be treated as independent parameters. We find the following lowest order reduction solution:

$$B_H = \tfrac{1029}{521}\,\mu_H M, \quad B_\Sigma = -\tfrac{3100}{521}\,\mu_\Sigma M, \tag{18}$$

$$\begin{aligned} &h_t = -g_t\, M, \quad h_b = -g_b\, M, \; h_f = -g_f\, M, \quad h_\lambda = 0, \\ &m^2_{H_u} = -\tfrac{569}{521} M^2, \quad m^2_{H_d} = -\tfrac{460}{521} M^2, \; m^2_\Sigma = \tfrac{1550}{521} M^2, \\ &m^2_{\Phi^3} = \tfrac{436}{521} M^2, \quad m^2_{\Phi^{1,2}} = \tfrac{8}{5} M^2, \quad m^2_{\Psi^3} = \tfrac{545}{521} M^2, \quad m^2_{\Psi^{1,2}} = \tfrac{12}{5} M^2. \end{aligned} \tag{19}$$

So, the gaugino mass parameter M plays a similar role as the gravitino mass $m_{2/3}$ in supergravity coupled GUTs and characterizes the scale of the supersymmetry-breaking.

In addition to the μ_Σ, μ_H and M, it is possible to include also B_H and B_Σ as independent parameters without changing the one-loop reduction solution (19).

3.3. Uniqueness of the reduction

We next address the question of whether the lowest-order solution given in (18) and (19) can be uniquely extended to a power series solution in higher orders. In Ref. [6], the uniqueness in the dimension-zero sector is proved, and so we assume here that the reduction in this sector has been performed.

Let us begin with the case of h_a $(a = t, b, f)$. We prove the uniqueness by induction; we assume that the reduction is unique to $O(g^{n-1})$ and show that the expansion coefficients in the next order can be uniquely calculated. We then insert the ansatz

$$h_a = -g_a M + \ldots + g g^n \eta_a^{(n)} M, \quad a = t, b, f, \tag{20}$$

along with the solution a in the dimension-zero sector (17), into the reduction Eq. (9) using Eq. (13). Then collecting terms of $O(g^{n+3})$, one obtains $\sum_{c=t,b,f} L_{ac}(n)\eta_c^{(n)} = \cdots$, where $\cdots$ in the r.h. side is known by assumption. One finds that the determinant,

$$\det L(n) = \tfrac{38423832921}{6786025} + \tfrac{21646499373}{6786025} n + \tfrac{1423971}{2605} n^2 + 27 n^3, \tag{21}$$

for integer $n > 0$ never vanishes, implying that the expansion coefficients $\eta_a^{(n)}$ can be uniquely calculated. Since the one-loop reduction (19) is unique, the η's exist uniquely to any finite order.

The uniqueness in the dimension-two sector proceeds similarly. Note that the uniqueness of the expansion coefficients for B_H, B_Σ, $m^2_{\Phi^{1,2}}$ and $m^2_{\Psi^{1,2}}$ can be easily shown, because their one-loop anomalous dimensions are such that there exists no mixing among the coefficients (see Eq. (16)). In the case of m^2_α $(\alpha = H_d, H_u, \Sigma, \Phi^3, \Psi^3)$, we have to do a similar investigation as for the h's. So we start with $m^2_\alpha = \xi_\alpha^{(0)} M^2 + \ldots + g^n \xi_\alpha^{(n)} M^2$, where [9] the lowest order coefficients $\xi_\alpha^{(0)}$ can be read off from (19), and we assume that the lower order terms denoted

[8] How to go away slightly from this boundary will be discussed elsewhere. Note that $g_\lambda = 0$ is inconsistent, but $g_\lambda <\sim 0.005$ has to be fulfilled to satisfy the proton decay constraint [2]. We expect that the inclusion of a small g_λ will not affect the prediction of the perturbative unification of the SSB parameters.

[9] As for the case of h_a's, we have assumed that the $\gamma(m^2)$'s are independent of the supersymmetric mass parameters μ_H and μ_Σ.

by ... are known. After some algebraic calculations, one finds that the $\xi_i^{(n)}$ also can be uniquely calculated to any finite order [10]

3.4. Asymptotic freedom (AF) and the stability of the reduction solution

If a reduction solution is unstable, the asymptotic freedom requirement and the requirement on a power series reduction solution are equivalent in general. In what follows, we show that the reduction solution (19) is an unstable asymptotically free solution and exhibits the Pendleton-Ross infrared fixed point [12]. That is, the AF requirement forces all the h_a's and m_α^2's to be reduced according to the reduction solution (19). On contrary, B_H and B_Σ behave asymptotically free, and their reduction solution (18) will turn out to be stable. To see these, we first derive the asymptotic behavior of the independent parameters, μ_Σ, μ_H and M:

$$\mu_\Sigma \sim g^{3100/1653}, \quad \mu_H \sim g^{-1029/521}, \quad M \sim g^2 \quad \text{as} \quad g \to 0, \tag{22}$$

where we have used Eq. (17) and $d/d\ln\mu = (-3g^3 + O(g^5))d/dg$. So, the μ_H does not vanish asymptotically. Note, however, that thanks to the AF in the gauge-Yukawa sector the asymptotic behavior given in (22) becomes exact in the ultraviolet limit. Moreover, in a mass independent renormalization scheme (which we are assuming throughout), the supersymmetric mass parameters μ_H and μ_Σ do not enter in the anomalous dimensions for h's and m^2's [13] so that the investigation below is not affected by the bad asymptotic behavior of μ_H. To proceed, we introduce $\tilde{h}_a \equiv h_a/M$ and $\tilde{m}_\alpha^2 \equiv m_\alpha^2/M^2$, and consider a solution near the reduction solution (19): $\tilde{h}_a(g) = -g_a + \Delta_a^h(g)$, $a = t, b, f$. Then we derive from Eq. (7) the linearized equations

$$\frac{d\Delta_a^h(g)}{dg} = \sum_{c=t,b,f} Y_{ac}\Delta_c^h(g)/g. \tag{23}$$

The asymptotic behavior of the system is dictated by the eigenvalues of the matrix Y, and one finds that the three basis vectors $\mathrm{v}_i^h(g)$ behave like

$$\mathrm{v}_i^h \sim g^{\lambda_i}, \quad \lambda_i = -11.64\ldots, -4.98\ldots, -3.61\ldots, \tag{24}$$

as $g \to 0$, where the λ_i's are the eigenvalues of Y, implying that the reduction solution for h_a's is ultraviolet unstable. One also sees that AF requires the h_a's to be reduced because $M \sim g^2$ as $g \to 0$.

The m^2-sector can be discussed similarly. Assuming that $\tilde{m}_\alpha^2(g) = \xi_{(0)}^5 + \Delta_\alpha^{m^2}(g)$, $\alpha = H_d, H_u, \Sigma, \Phi^{1,2,3}$, $\Psi^{1,2,3}$, and that the h_a's are reduced, we find that the eigenvalues of the matrix Z which enters in the linearized equations, $d\Delta_\alpha^{m^2}(g)/dg = \sum_{\beta=H_d,H_u,\Sigma,\Phi^3,\Psi^3} Z_{\alpha\beta}\Delta_\beta^{m^2}(g)/g$, are given by $(-14.64\ldots, -7.98\ldots, -6.61\ldots, -4, -4, -4, -4)$. Therefore, the reduction solution for m_α^2's is also ultraviolet unstable, and one, moreover, sees that the AF of m_α^2's is ensured only by the reduction (19) because $M^2 \sim g^4$ as $g \to 0$.

As for B_H and B_Σ, we find that as $g \to 0$,

$$B_H \simeq \tfrac{1029}{521}\mu_H M + c_H\, g^{1.97\ldots}, \quad B_\Sigma \simeq -\tfrac{3100}{521}\mu_\Sigma M + c_\Sigma\, g^{0.64\ldots} \tag{25}$$

near the reduction solution, where c's are integration constants. Therefore, the B's are asymptotically free ($\mu_H M \sim g^{0.024\ldots}$, $\mu_\Sigma M \sim g^{3.8\ldots}$), and so the reduction solution for the B's are asymptotically stable. This is good news, because, as we will see later, the reduction solution (19) including (18) is not consistent with the radiative breaking of the electroweak symmetry at low energy. To make the radiative breaking possible, we

[10] The approach of unifying the SSB parameters of Ref. [4] is based on a condition on the anomalous dimensions (the $P = Q/3$ condition). This condition is more restrictive than simply requiring the complete reduction of parameters, because the number of the anomalous dimensions usually exceeds that of parameters. It has turned out to be very difficult to satisfy the $P = Q/3$ condition in higher orders in non-finite theories [15].

have to treat B_H as an independent parameter. But, as we have just seen, this can be done without loosing AF of the model.

The solution (19) exhibits the one-loop infrared fixed point, which therefore could be used for the infrared-fixed-point approach [14]. This approach is based on the assumption that infrared fixed points found in first order in perturbation theory persist in higher orders and that the ratio of the compactification scale Λ_C (or the Planck scale M_P) to M_{GUT} is large enough for various parameters to come very close to their infrared values when running from Λ_C down to M_{GUT}. Therefore, this approach may yield similar results to ours, because the reduction solution in one-loop order (19) is the infrared fixed point. Here we would like to see how fast the desired infrared fixed point can be approached in our concrete model.

To this end, we assume that h_a, $a = t, b, f$ and m_α^2, $\alpha = H_d, H_u, \Sigma, \Phi^{1,2,3}, \Psi^{1,2,3}$ vanish at Λ_C, while we treat M as independent. The one-loop evolution of $m^2_{\Phi^{1,2}}$ and $m^2_{\Psi^{1,2}}$ can be discussed analytically:

$$\frac{m^2_{\Phi^{1,2}}}{M^2} = \frac{8}{5} + c_{\Phi^{1,2}} g^{-4}, \quad \frac{m^2_{\Psi^{1,2}}}{M^2} = \frac{12}{5} + c_{\Psi^{1,2}} g^{-4}, \tag{26}$$

where c's are integration constants. Imposing the above mentioned boundary condition at Λ_C, one finds at M_{GUT}

$$\frac{m^2_{\Phi^{1,2}}}{M^2} \simeq 0.25, 0.35, 0.52, \quad \frac{m^2_{\Psi^{1,2}}}{M^2} \simeq 0.37, 0.53, 0.79 \ \text{ for } \ \frac{\Lambda_C}{M_{GUT}} = 10^2, 10^3, 10^5, \tag{27}$$

respectively, where we have used $\alpha = g^2/4\pi = 0.04$ at M_{GUT}. Unfortunately, we see that the infrared fixed point, 1.6 and 2.4, is quite far from the approached points. We have checked numerically that this also holds for the other SSB parameters.

3.5. Prediction

Since the $SU(5)$ symmetry is spontaneously broken at M_{GUT}, the reduction relations (17)–(19) exhibit a boundary condition on the gauge and Yukawa couplings and also on the SSB parameters at this energy scale [11]. To make our unification idea and its consequence transparent, we shall make an oversimplifying assumption that below M_{GUT} their evolution is governed by the MSSM and that there exists a unique threshold M_{SUSY}, which we identify with M, for all superpartners of the MSSM, so that below M_{SUSY} the standard model (SM) is the correct effective theory. We recall that it is most convenient to fix $\tan\beta$ through the matching condition on the Yukawa couplings at M_{SUSY} in the gauge-Yukawa Unification scenario [6,2]. That is, the Higgs sector is partly fixed by the dimension-zero sector. This is the reason why the complete reduction in the dimensionfull sector, defined by (18) and (19), is inconsistent with the radiative breaking of the electroweak symmetry, as we will see below.

Since we are not stressing the accuracy of the approximation, we assume that the potential of the MSSM at $\mu = M$ takes the tree-level form. The minimization of the potential yields two conditions at M_{SUSY} [16],

$$0 = m^2_{H_d} - m^2_{H_u} + M_Z^2 \frac{1 - \tan^2\beta}{1 + \tan^2\beta} + B_H \frac{\tan^2\beta - 1}{\tan\beta}, \tag{28}$$

$$0 = 2\mu_H^2 + m^2_{H_d} + m^2_{H_u} + B_H \frac{\tan^2\beta + 1}{\tan\beta}, \tag{29}$$

where $\tan\beta = v_2/v_1$, $M_Z = (1/2)\sqrt{(3g_1^2/5 + g_2^2)(v_1^2 + v_2^2)}$, $v_{1,2} = (1/\sqrt{2})\langle \hat{H}_{d,u} \rangle$. Using the unification condition given by (18) and (19) under the assumption that M_Z and $\tan\beta$ at M_{SUSY} are given, these two

[11] Here we examine the evolution of these parameters according to their renormalization group equations in two-loop order for the gauge and Yukawa couplings and in one-loop order for the SSB parameters.

Table 1
Prediction of the SSB parameters

M_1 (TeV)	0.22	$m^2_{\tilde{L}_3}$ (TeV^2)	0.30
M_2 (TeV)	0.42	$m^2_{\tilde{\tau}}$ (TeV^2)	0.23
M_3 (TeV)	1.2	$m^2_{\tilde{Q}_3}$ (TeV^2)	1.1
h_t (TeV)	−0.89	$m^2_{\tilde{b}}$ (TeV^2)	0.95
h_b (TeV)	−0.88	$m^2_{\tilde{t}}$ (TeV^2)	0.93
h_τ (TeV)	−0.12	$m^2_{\tilde{L}_1} = m^2_{\tilde{L}_2}$ (TeV^2)	0.52
B_H (TeV^2)	−0.0027	$m^2_{\tilde{e}} = m^2_{\tilde{\mu}}$ (TeV^2)	0.64
μ_H (TeV)	± 0.94	$m^2_{\tilde{Q}_1} = m^2_{\tilde{Q}_2}$ (TeV^2)	1.9
$m^2_{H_d}$ (TeV^2)	−0.76	$m^2_{\tilde{d}} = m^2_{\tilde{s}}$ (TeV^2)	1.6
$m^2_{H_u}$ (TeV^2)	−0.90	$m^2_{\tilde{u}} = m^2_{\tilde{c}}$ (TeV^2)	1.8

conditions could fix the M and μ_H at M_{GUT}. Unfortunately, this is not the case. We have numerically checked that the unification condition given by (17)–(19) does not satisfy Eqs. (28) and (29). Therefore, we have to treat one of m_{H_u}, m_{H_d} and B_H as an independent parameter to make the radiative breaking at M_{SUSY} possible. From the discussion of Section 3.4 it is clear that the most natural choice is B_H, because this is the unique possibility to keep AF. In addition, the lowest order unification condition (19) remains the same; otherwise it would be modified.

We use

$$\alpha_1(M_Z) = 0.0169, \quad \alpha_2(M_Z) = 0.0337, \quad \alpha_\tau(M_Z) = 8.005 \times 10^{-6} \tag{30}$$

as input parameters and fix $M_{\mathrm{SUSY}} = M$ at 500 GeV. Then the prediction from the gauge-Yukawa Unification (17) is:

$$\begin{aligned} &M_t \simeq 1.8 \times 10^2 \text{ GeV}, \quad M_b \simeq 5.4 \text{ GeV}, \quad \alpha_3(M_Z) \simeq 0.12, \\ &M_{\mathrm{GUT}} \simeq 1.7 \times 10^{16} \text{ GeV}, \quad \alpha_{\mathrm{GUT}} \simeq 0.040, \quad \tan\beta(M_{\mathrm{SUSY}}) \simeq 48, \end{aligned} \tag{31}$$

where M_t and M_b are the physical top and bottom quark masses. These values suffer from corrections coming from different sources such as threshold effects, which are partly taken into account and estimated in Ref. [2]. In Table 1, we show the prediction of the SSB parameters.

For the SSB parameters above we have used the notation of Ref. [17]. Using these parameters, one can then compute the superpartner spectrum. We have checked that it is consistent with the experimental data. The LSP, for instance, is found to be a neutralino of ~ 220 GeV with a dominant component of the photino [12]. Details of our calculations and results will be presented elsewhere.

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5.4 Unification beyond GUTs: Gauge Yukawa unification (Lectures)

Title: Unification beyond GUTs: Gauge Yukawa unification
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Comment (*Myriam Mondragón, George Zoupanos*)
As has been already noted a natural extension of the GUT idea is to find a way to relate the gauge and Yukawa sectors of a theory, that is to achieve GYU. A symmetry which naturally relates the two sectors is supersymmetry, in particular $N = 2$ supersymmetry. However, as has been also noted earlier in a different context, $N = 2$ supersymmetric theories have serious phenomenological problems due to light mirror fermions. Also in superstring theories and in composite models there exist relations among the gauge and Yukawa couplings, but both kind of theories have phenomenological problems, which we are not going to address here.

There have been other attempts to relate the gauge and Yukawa sectors which we recall and update for completeness here, while the references are already in the lectures paper. One was proposed by Decker, Pestieau, and Veltman. By requiring the absence of quadratic divergencies in the SM, they found a relationship among the squared masses appearing in the Yukawa and in the gauge sectors of the theory. A very similar relation is obtained by applying naively in the SM the general formula derived from demanding spontaneous supersymmetry breaking via F-terms. In both cases a prediction for the top quark was possible only when it was permitted experimentally to assume the $M_H \ll M_{W,Z}$ with the result $M_t \simeq 69$ GeV. Otherwise there is only a quadratic relation among M_t and M_H. Using this relationship in the former case and a version of naturalness into account, i.e. that the quadratic corrections to the Higgs mass be at most equal to the physical mass, the Higgs mass is found to be ~ 260 GeV, for a top quark mass of around 176 GeV, in complete disagreement with the recent findings at LHC [14, 15].

A well known relation among gauge and Yukawa couplings is the Pendleton-Ross (P-R) infrared fixed point. The P-R proposal, involving the Yukawa coupling of the top quark g_t and the strong gauge coupling α_3, was that the ratio α_t/α_3, where $\alpha_t = g_t^2/4\pi$, has an infrared fixed point. This assumption predicted $M_t \sim 100$ GeV. In addition, it has been shown that the P-R conjecture is not justified at two-loops, since the ratio α_t/α_3 diverges in the infrared. Another interesting conjecture, made by Hill, is that α_t itself develops a quasi-infrared fixed point, leading to the prediction $M_t \sim 280$ GeV. The P-R and Hill conjectures have been done in the framework of the SM. The same conjectures within the Minimal Supersymmetric SM (MSSM) lead to the following relations:

$$M_t \approx 140 \text{ GeV} \sin\beta (\text{P-R}), \quad M_t \approx 200 \text{ GeV} \sin\beta (\text{Hill}),$$

where $\tan\beta = v_u/v_d$ is the ratio of the two vacuum expectation values (vev's) of the Higgs fields of the MSSM. From theoretical considerations one can expect

$$1 < \tan\beta < 50 \Leftrightarrow 1/\sqrt{2} < \sin\beta < 1.$$

This corresponds to

$$100 \text{ GeV} < M_t < 140 \text{ GeV} \;\; (\text{P-R}), \quad 140 \text{ GeV} < M_t < 200 \text{ GeV (Hill)}.$$

Thus, the MSSM P-R conjecture is ruled out, while within the MSSM, the Hill conjecture does not give a prediction for M_t, since the value of $\sin\beta$ is not fixed by other considerations. The Hill model can accommodate the correct value of $M_t \sim 173$ GeV for $\sin\beta \approx 0.865$ corresponding to $\tan\beta \approx 1.7$. Such small values, however, are strongly challenged if the newly discovered Higgs particle is identified with the lightest MSSM Higgs boson [19]. Only a very heavy scalar top spectrum with large mixing could accommodate such a small $\tan\beta$ value.

The consequence of GYU is that in the lowest order in perturbation theory the gauge and Yukawa couplings above M_{GUT} are related in the form

$$g_i = \kappa_i g_{GUT}, \quad i = 1, 2, 3, e, ..., \tau, b, t, \qquad (*)$$

where g_i (i = 1,...,t) stand for the gauge and Yukawa couplings, g_{GUT} is the unified coupling and we have neglected the Cabbibo-Kobayashi-Maskawa mixing of the quarks. So, eq. ($*$) corresponds to a set of boundary conditions on the renormalization group evolution for the effective theory below M_{GUT}, which we have assumed to be the MSSM. As we have seen in subsections 5.1 and 5.2 it is possible to construct supersymmetric GUTs with GYU in the third generation that can predict the bottom and top quark masses in accordance with the experimental data. This means that the top-bottom hierarchy could be explained in these models, in a similar way as the hierarchy of the gauge couplings of the SM can be explained if one assumes the existence of a unifying gauge symmetry at M_{GUT}. It is clear that the GYU scenario is the most predictive scheme as far as the mass of the top quark is concerned. It may be worth recalling the predictions for M_t of ordinary GUTs, in particular of supersymmetric $SU(5)$ and $SO(10)$. The MSSM with $SU(5)$ Yukawa boundary unification allows M_t to be anywhere in the interval between 100-200 GeV for varying $\tan\beta$, which is now a free parameter. Similarly, the MSSM with $SO(10)$ Yukawa boundary conditions, i.e. $t - b - \tau$ Yukawa Unification, gives M_t in the interval 160-200 GeV. In addition we have analyzed [20] the infrared quasi-fixed-point behaviour of the M_t prediction in some detail. In particular we have seen that the infrared value for large $\tan\beta$ depends on $\tan\beta$ and its lowest value is ~ 188 GeV. Comparing this with the experimental value $m_t = (173.2 \pm 0.9)$ GeV [13] we conclude that the present data on M_t cannot be explained from the infrared quasi-fixed-point behaviour alone (see Figure 4 of hep-ph/9703289). An estimate of the theoretical uncertainties involved in GYU has been done in ref [20]. Although a fresh look has to be done in the case of the minimal N =1 supersymmetric $SU(5)$, we can conclude that the studies on the GYU of the asymptotically non-free supersymmetric Pati-Salam [21] and asymptotically non-free $SO(10)$ [25] models have ruled them out on the basis of the top quark mass prediction.

It sould be emphasized once more that only one of the Finite Unified models (discussed in subsection 5.1 and which will be further discussed in sections 5.5, 5.8, 5.9) not only predicted correctly the top and bottom quark masses but in addition predicted the Higgs mass in striking agreement with the recent findings at LHC [14, 15].

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UNIFICATION BEYOND GUTs: GAUGE-YUKAWA UNIFICATION*

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Gauge-Yukawa Unification (GYU) is a renormalization group invariant functional relation among gauge and Yukawa couplings which holds beyond the unification point in Grand Unified Theories (GUTs). We present here various models where GYU is obtained by requiring the principles of finiteness and reduction of couplings. We examine the consequences of these requirements for the low energy parameters, especially for the top quark mass. The predictions are such that they clearly distinguish already GYU from ordinary GUTs. It is expected that it will be possible to discriminate among the various GYUs when more accurate measurements of the top quark mass are available.

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1. Introduction

The standard model (SM) is very accurate in describing the elementary particles and their interactions, but it has a large number of free parameters whose values are determined only experimentally.

To reduce the number of free parameters of a theory, and thus render it more predictive, one is usually led to introduce a symmetry. Grand Unified Theories (GUTs) are very good examples of such a procedure [1–3]. For instance, in the case of minimal $SU(5)$ it was possible to reduce the gauge couplings by one and give a prediction for one of them. GUTs can also relate the Yukawa couplings among themselves, again SU(5) provided an example of this by predicting the ratio M_τ/M_b [4] in SM. Unfortunately, requiring more gauge symmetry does not seem to help, since additional complications are introduced due to new degrees of freedom, in the ways and channels of breaking the symmetry, etc.

A natural extension of the GUT idea is to find a way to relate the gauge and Yukawa sectors of a theory, that is to achieve Gauge-Yukawa Unification (GYU). A symmetry which naturally relates the two sectors is supersymmetry, in particular $N = 2$ supersymmetry. It turns out, however, that $N = 2$ supersymmetric theories have serious phenomenological problems due to light mirror fermions. Also in superstring theories and in composite models there exist relations among the gauge and Yukawa couplings, but both kind of theories have phenomenological problems.

There have been other attempts to relate the gauge and Yukawa sectors. One was proposed by Decker, Pestieau, and Veltman [6]. By requiring the absence of quadratic divergences in the SM, they found a relationship between the squared masses appearing in the Yukawa and in the gauge sectors of the theory. A very similar relation is obtained by applying naively in the SM the general formula derived from demanding spontaneous supersymmetry breaking via F-terms [7]. In both cases a prediction for the top quark was possible only when it was permitted experimentally to neglect the M_H as compared to $M_{W,Z}$ with the result $M_t = 69$ GeV. Otherwise there is only a quadratic relation among M_t and M_H.

A well known relation among gauge and Yukawa couplings is the Pendleton-Ross (P-R) *infrared* fixed point [8]. The P-R proposal, involving the Yukawa coupling of the top quark g_t and the strong gauge coupling α_3, was that the ratio α_t/α_3, where $\alpha_t = g_t^2/4\pi$, has an infrared fixed point. This assumption predicted $M_t \sim 100$ GeV. In addition, it has been shown [9] that the P-R conjecture is not justified at two-loops, since then the ratio α_t/α_3 diverges in the infrared.

Another interesting conjecture, made by Hill [10], is that α_t itself develops a quasi-infrared fixed point, leading to the prediction $M_t \sim 280$ GeV.

The P-R and Hill conjectures have been done in the framework on the SM. The same conjectures within the minimal supersymmetric SM (MSSM) lead to the following relations:

$$M_t \simeq 140\ \mathrm{GeV}\ \sin\beta \quad (\mathrm{P-R})\,, \tag{1}$$

$$M_t \simeq 200\ \mathrm{GeV}\ \sin\beta \quad (\mathrm{Hill})\,, \tag{2}$$

where $\tan\beta = v_u/v_d$ is the ratio of the two VEV of the Higgs fields of the MSSM. We should stress that in this case there is no prediction for M_t, given that $\sin\beta$ is not fixed from other considerations.

In a series of papers [11–14, 63] we have proposed another way to relate the gauge and Yukawa sectors of a theory. It is based on the fact that within the framework of a renormalizable field theory, one can find renormalization group invariant (RGI) relations among parameters that can improve the calculability and the predictive power of a theory. We have considered models in which the GYU is achieved using the principles of reduction of couplings [17–21] and finiteness [11, 22–27, 33–36, 61]. These principles, which are formulated in perturbation theory, are not explicit symmetry principles, although they might imply symmetries. The former principle is based on the existence of RGI relations among couplings, which preserve perturbative renormalizability. Similarly, the latter one is based on the fact that it is possible to find RGI relations among couplings that keep finiteness in perturbation theory, even to all orders. Applying these principles one can relate the gauge and Yukawa couplings without introducing necessarily a symmetry, nevertheless improving the predictive power of a model.

It is worth noting that the above principles have been applied in supersymmetric GUTs for reasons that will be transparent in the following sections. We should also stress that our conjecture for GYU is by no means in conflict with the interesting proposals mentioned before (see also Ref. [60]), but it rather uses all of them, hopefully in a more successful perspective. For instance, the use of susy GUTs comprises the demand of the cancellation of quadratic divergences in the SM. Similarly, the very interesting conjectures about the infrared fixed points are generalized in our proposal, since searching for RGI relations among various couplings corresponds to searching for *fixed points* of the coupled differential equations obeyed by the various couplings of a theory.

2. Unification of couplings by the RGI method

Let us next briefly outline the idea of reduction of couplings. Any RGI relation among couplings (which does not depend on the renormalization scale μ explicitly) can be expressed, in the implicit form $\Phi(g_1, \cdots, g_A) =$

const., which has to satisfy the partial differential equation (PDE)

$$\mu \frac{d\Phi}{d\mu} = \vec{\nabla} \cdot \vec{\beta} = \sum_{a=1}^{A} \beta_a \frac{\partial \Phi}{\partial g_a} = 0 , \tag{3}$$

where β_a is the β-function of g_a. This PDE is equivalent to a set of ordinary differential equations, the so-called reduction equations (REs) [18],

$$\beta_g \frac{dg_a}{dg} = \beta_a , \ a = 1, \cdots, A , \tag{4}$$

where g and β_g are the primary coupling and its β-function, and the counting on a does not include g. Since maximally $(A-1)$ independent RGI "constraints" in the A-dimensional space of couplings can be imposed by the Φ_a's, one could in principle express all the couplings in terms of a single coupling g. The strongest requirement is to demand power series solutions to the REs,

$$g_a = \sum_{n=0} \rho_a^{(n)} g^{2n+1} , \tag{5}$$

which formally preserve perturbative renormalizability. Remarkably, the uniqueness of such power series solutions can be decided already at the one-loop level [18]. To illustrate this, let us assume that the β-functions have the form

$$\begin{aligned} \beta_a &= \frac{1}{16\pi^2}\Big[\sum_{b,c,d \neq g} \beta_a^{(1)\,bcd} g_b g_c g_d + \sum_{b \neq g} \beta_a^{(1)\,b} g_b g^2 \Big] + \cdots , \\ \beta_g &= \frac{1}{16\pi^2} \beta_g^{(1)} g^3 + \cdots , \end{aligned} \tag{6}$$

where $\cdots$ stands for higher order terms, and $\beta_a^{(1)\,bcd}$'s are symmetric in b, c, d. We then assume that the $\rho_a^{(n)}$'s with $n \leq r$ have been uniquely determined. To obtain $\rho_a^{(r+1)}$'s, we insert the power series (5) into the REs (4) and collect terms of $O(g^{2r+3})$ and find

$$\sum_{d \neq g} M(r)_a^d \, \rho_d^{(r+1)} = \text{lower order quantities} ,$$

where the r.h.s. is known by assumption, and

$$M(r)_a^d = 3 \sum_{b,c \neq g} \beta_a^{(1)\,bcd} \rho_b^{(1)} \rho_c^{(1)} + \beta_a^{(1)\,d} - (2r+1)\, \beta_g^{(1)} \delta_a^d , \tag{7}$$

$$0 = \sum_{b,c,d \neq g} \beta_a^{(1)\,bcd} \rho_b^{(1)} \rho_c^{(1)} \rho_d^{(1)} + \sum_{d \neq g} \beta_a^{(1)\,d} \rho_d^{(1)} - \beta_g^{(1)} \rho_a^{(1)} . \tag{8}$$

Therefore, the $\rho_a^{(n)}$'s for all $n > 1$ for a given set of $\rho_a^{(1)}$'s can be uniquely determined if $\det M(n)_a^d \neq 0$ for all $n \geq 0$.

As it will be clear later by examining specific examples, the various couplings in supersymmetric theories have easily the same asymptotic behaviour. Therefore searching for a power series solution of the form (5) to the REs (4) is justified. This is not the case in non-supersymmetric theories.

The possibility of coupling unification described in this section is without any doubt attractive because the "completely reduced" theory contains only one independent coupling, but it can be unrealistic. Therefore, one often would like to impose fewer RGI constraints, and this is the idea of partial reduction [19].

3. Partial reduction in $N = 1$ supersymmetric gauge theories

Let us consider a chiral, anomaly free, $N = 1$ globally supersymmetric gauge theory based on a group G with gauge coupling constant g. The superpotential of the theory is given by

$$W = \tfrac{1}{2}\, m_{ij}\, \phi_i\, \phi_j + \tfrac{1}{6}\, C_{ijk}\, \phi_i\, \phi_j\, \phi_k \ , \tag{9}$$

where m_{ij} and C_{ijk} are gauge invariant tensors and the matter field ϕ_i transforms according to the irreducible representation R_i of the gauge group G. The renormalization constants associated with the superpotential (9), assuming that supersymmetry is preserved, are

$$\phi_i^0 = (Z_i^j)^{(1/2)}\, \phi_j \ , \tag{10}$$

$$m_{ij}^0 = Z_{ij}^{i'j'}\, m_{i'j'} \ , \tag{11}$$

$$C_{ijk}^0 = Z_{ijk}^{i'j'k'}\, C_{i'j'k'} \ . \tag{12}$$

The $N = 1$ non-renormalization theorem [32] ensures that there are no mass and cubic-interaction-term infinities and therefore

$$\begin{aligned} Z_{ijk}^{i'j'k'}\, Z_{i'}^{1/2\, i''}\, Z_{j'}^{1/2\, j''}\, Z_{k'}^{1/2\, k''} &= \delta_{(i}^{i''}\, \delta_j^{j''} \delta_{k)}^{k''} \ , \\ Z_{ij}^{i'j'}\, Z_{i'}^{1/2\, i''}\, Z_{j'}^{1/2\, j''} &= \delta_{(i}^{i''}\, \delta_{j)}^{j''} \ . \end{aligned} \tag{13}$$

As a result the only surviving possible infinities are the wave-function renormalization constants Z_i^j, *i.e.*, one infinity for each field. The one-loop β-function of the gauge coupling g is given by [22]

$$\beta_g^{(1)} = \frac{dg}{dt} = \frac{g^3}{16\pi^2}\left[\sum_i l(R_i) - 3\, C_2(G)\right] , \tag{14}$$

where $l(R_i)$ is the Dynkin index of R_i and $C_2(G)$ is the quadratic Casimir of the adjoint representation of the gauge group G. The β-functions of C_{ijk}, by virtue of the non-renormalization theorem, are related to the anomalous dimension matrix γ_{ij} of the matter fields ϕ_i as:

$$\beta_{ijk} = \frac{dC_{ijk}}{dt} = C_{ijl}\,\gamma^l_k + C_{ikl}\,\gamma^l_j + C_{jkl}\,\gamma^l_i \,. \tag{15}$$

At one-loop level γ_{ij} is [22]

$$\gamma^{(1)}_{ij} = \frac{1}{32\pi^2}\left[C^{ikl}\,C_{jkl} - 2\,g^2\,C_2(R_i)\delta_{ij}\right], \tag{16}$$

where $C_2(R_i)$ is the quadratic Casimir of the representation R_i, and $C^{ijk} = C^*_{ijk}$. Since dimensional coupling parameters such as masses and couplings of cubic scalar field terms do not influence the asymptotic properties of a theory on which we are interested here, it is sufficient to take into account only the dimensionless supersymmetric couplings such as g and C_{ijk}. So we neglect the existence of dimensional parameters, and assume furthermore that C_{ijk} are real so that C^2_{ijk} always are positive numbers. For our purposes, it is convenient to work with the square of the couplings and to arrange C_{ijk} in such a way that they are covered by a single index i $(i = 1,\cdots,n)$:

$$\alpha = \frac{|g|^2}{4\pi}\,,\ \alpha_i = \frac{|g_i|^2}{4\pi}\,. \tag{17}$$

The evolution equations of α's in perturbation theory then take the form

$$\begin{aligned}\frac{d\alpha}{dt} &= \beta = -\beta^{(1)}\alpha^2 + \cdots\,,\\ \frac{d\alpha_i}{dt} &= \beta_i = -\beta^{(1)}_i\,\alpha_i\,\alpha + \sum_{j,k}\beta^{(1)}_{i,jk}\,\alpha_j\,\alpha_k + \cdots\,,\end{aligned} \tag{18}$$

where $\cdots$ denotes the contributions from higher orders, and $\beta^{(1)}_{i,jk} = \beta^{(1)}_{i,kj}$.

Given the set of the evolution equations (18), we investigate the asymptotic properties, as follows. First we define [17, 18]

$$\tilde{\alpha}_i \equiv \frac{\alpha_i}{\alpha}\,,\ i = 1,\cdots,n\,, \tag{19}$$

and derive from Eq. (18)

$$\begin{aligned}\alpha\frac{d\tilde{\alpha}_i}{d\alpha} &= -\tilde{\alpha}_i + \frac{\beta_i}{\beta} = \left(-1 + \frac{\beta^{(1)}_i}{\beta^{(1)}}\right)\tilde{\alpha}_i\\ &\quad -\sum_{j,k}\frac{\beta^{(1)}_{i,jk}}{\beta^{(1)}}\,\tilde{\alpha}_j\,\tilde{\alpha}_k + \sum_{r=2}\left(\frac{\alpha}{\pi}\right)^{r-1}\tilde{\beta}^{(r)}_i(\tilde{\alpha})\,,\end{aligned} \tag{20}$$

where $\tilde{\beta}_i^{(r)}(\tilde{\alpha})$ $(r = 2, \cdots)$ are power series of $\tilde{\alpha}$'s and can be computed from the r-th loop β-functions. Next we search for fixed points ρ_i of Eq. (19) at $\alpha = 0$. To this end, we have to solve

$$\left(-1 + \frac{\beta_i^{(1)}}{\beta^{(1)}}\right)\rho_i - \sum_{j,k} \frac{\beta_{i,jk}^{(1)}}{\beta^{(1)}} \rho_j \rho_k = 0 \,, \tag{21}$$

and assume that the fixed points have the form

$$\rho_i = 0 \text{ for } i = 1, \cdots, n' \,;\; \rho_i > 0 \text{ for } i = n'+1, \cdots, n \,. \tag{22}$$

We then regard $\tilde{\alpha}_i$ with $i \leq n'$ as small perturbations to the undisturbed system which is defined by setting $\tilde{\alpha}_i$ with $i \leq n'$ equal to zero. As we have seen, it is possible to verify at the one-loop level [18] the existence of the unique power series solution

$$\tilde{\alpha}_i = \rho_i + \sum_{r=2} \rho_i^{(r)} \alpha^{r-1} \,,\; i = n'+1, \cdots, n \tag{23}$$

of the reduction equations (20) to all orders in the undisturbed system. These are RGI relations among couplings and keep formally perturbative renormalizability of the undisturbed system. So in the undisturbed system there is only *one independent* coupling, the primary coupling α.

The small perturbations caused by nonvanishing $\tilde{\alpha}_i$ with $i \leq n'$ enter in such a way that the reduced couplings, *i.e.*, $\tilde{\alpha}_i$ with $i > n'$, become functions not only of α but also of $\tilde{\alpha}_i$ with $i \leq n'$. It turned out that, to investigate such partially reduced systems, it is most convenient to work with the partial differential equations

$$\left\{ \tilde{\beta} \frac{\partial}{\partial \alpha} + \sum_{a=1}^{n'} \tilde{\beta}_a \frac{\partial}{\partial \tilde{\alpha}_a} \right\} \tilde{\alpha}_i(\alpha, \tilde{\alpha}) = \tilde{\beta}_i(\alpha, \tilde{\alpha}) \,,$$
$$\tilde{\beta}_{i(a)} = \frac{\beta_{i(a)}}{\alpha^2} - \frac{\beta}{\alpha^2} \tilde{\alpha}_{i(a)} \,,\; \tilde{\beta} \equiv \frac{\beta}{\alpha} \,, \tag{24}$$

which are equivalent to the reduction equations (20), where we let a, b run from 1 to n' and i, j from $n'+1$ to n in order to avoid confusion. We then look for solutions of the form

$$\tilde{\alpha}_i = \rho_i + \sum_{r=2} \left(\frac{\alpha}{\pi}\right)^{r-1} f_i^{(r)}(\tilde{\alpha}_a) \,,\; i = n'+1, \cdots, n \,, \tag{25}$$

where $f_i^{(r)}(\tilde{\alpha}_a)$ are supposed to be power series of $\tilde{\alpha}_a$. This particular type of solution can be motivated by requiring that in the limit of vanishing perturbations we obtain the undisturbed solutions (23) [21, 28]. Again it is possible to obtain the sufficient conditions for the uniqueness of $f_i^{(r)}$ in terms of the lowest order coefficients.

4. The minimal asymptotically free SU(5) model

The minimal N=1 supersymmetric SU(5) model [29] is particularly interesting, being the the simplest GUT supported by the LEP data [5]. Here we will consider it as an attractive example of a partially reduced model. Its particle content is well defined and has the following transformation properties under SU(5): three $(\overline{\mathbf{5}}+\mathbf{10})$-supermultiplets which accommodate three fermion families, one $(\mathbf{5}+\overline{\mathbf{5}})$ to describe the two Higgs supermultiplets appropriate for electroweak symmetry breaking and a **24**-supermultiplet required to provide the spontaneous symmetry breaking of SU(5) down to SU(3)×SU(2)×U(1).

Since we are neglecting the dimensional parameters and the Yukawa couplings of the first two generations, the superpotential of the model is exactly given by

$$W = \tfrac{1}{2}\, g_t \mathbf{10}_3\, \mathbf{10}_3\, H + g_b\, \overline{\mathbf{5}}_3\, \mathbf{10}_3\, \overline{H} + g_\lambda\, (\mathbf{24})^3 + g_f\, \overline{H}\, \mathbf{24}\, H \ , \qquad (26)$$

where $H, \overline{H}$ are the $\mathbf{5}, \overline{\mathbf{5}}$-Higgs supermultiplets and we have suppressed the SU(5) indices. According to the notation introduced in Eq. (19), Eqs. (20) become

$$\begin{aligned}
\alpha\, \frac{d\tilde{\alpha}_t}{d\alpha} &= \frac{27}{5}\,\tilde{\alpha}_t - 3\,\tilde{\alpha}_t^2 - \frac{4}{3}\,\tilde{\alpha}_t\tilde{\alpha}_b - \frac{8}{5}\,\tilde{\alpha}_t\,\tilde{\alpha}_f \ ,\\
\alpha\, \frac{d\tilde{\alpha}_b}{d\alpha} &= \frac{23}{5}\,\tilde{\alpha}_b - \frac{10}{3}\,\tilde{\alpha}_b^2 - \tilde{\alpha}_b\tilde{\alpha}_t - \frac{8}{5}\,\tilde{\alpha}_b\,\tilde{\alpha}_f \ ,\\
\alpha\, \frac{d\tilde{\alpha}_\lambda}{d\alpha} &= 9\tilde{\alpha}_\lambda - \frac{21}{5}\,\tilde{\alpha}_\lambda^2 - \tilde{\alpha}_\lambda\,\tilde{\alpha}_f \ ,\\
\alpha\, \frac{d\tilde{\alpha}_f}{d\alpha} &= \frac{83}{15}\,\tilde{\alpha}_f - \frac{53}{15}\,\tilde{\alpha}_f^2 - \tilde{\alpha}_f\tilde{\alpha}_t - \frac{4}{3}\,\tilde{\alpha}_f\,\tilde{\alpha}_b - \frac{7}{5}\,\tilde{\alpha}_f\,\tilde{\alpha}_\lambda
\end{aligned} \qquad (27)$$

in the one-loop approximation. Given the above equations describing the evolution of the four independent couplings $(\alpha_i\ ,\ i = t, b, \lambda, f)$, there exist $2^4 = 16$ non-degenerate solutions corresponding to vanishing ρ's as well as non-vanishing ones given by Eq. (25). The possibility to predict the top quark mass depends on a nontrivial interplay between the vacuum expectation value of the two SU(2) Higgs doublets involved in the model and the known masses of the third generation $(m_b\ ,\ m_\tau)$. It is clear that only the solutions of the form

$$\rho_t\ ,\ \rho_b \neq 0 \qquad (28)$$

can predict the top and bottom quark masses.

There exist exactly four such solutions. The first solution is ruled out since it is inconsistent with Eq. (17), and the second one is ruled out since it does not satisfy the criteria to be asymptotically free. We are left with two asymptotically free solutions, which we label 3 and 4 (or AFUT3 and AFUT4, for asymptotically free unified theory). According to the criteria of section 3, these two solutions give the possibility to obtain partial reductions. To achieve this, we look for solutions [12] of the form Eq. (23) to both 3 and 4.

We present now the computation of some lower order terms within the one-loop approximation for the solutions. For solution 3:

$$\tilde{\alpha}_i = \eta_i + f_i^{(r_\lambda=1)}\,\tilde{\alpha}_\lambda + f_i^{(r_\lambda=2)}\,\tilde{\alpha}_\lambda^2 + \cdots \qquad \text{for } i = t, b, f\ , \tag{29}$$

where

$$\begin{aligned} \eta_{t,b,f} &= \frac{2533}{2605}\,,\ \frac{1491}{2605}\,,\ \frac{560}{521}\,,\\ f_{t,b,f}^{(r_\lambda=1)} &\simeq 0.018\,,\ 0.012\,,\ -0.131\,,\\ f_{t,b,f}^{(r_\lambda=2)} &\simeq 0.005\,,\ 0.004\,,\ -0.021\,. \end{aligned} \tag{30}$$

For the solution 4,

$$\begin{aligned} \tilde{\alpha}_i &= \eta_i + f_i^{(r_f=1)}\,\tilde{\alpha}_f + f_i^{(r_\lambda=1)}\,\tilde{\alpha}_\lambda + f_i^{(r_f=1,r_\lambda=1)}\,\tilde{\alpha}_f\,\tilde{\alpha}_\lambda\\ &\quad + f_i^{(r_f=2)}\,\tilde{\alpha}_f^2 + f_i^{(r_\lambda=2)}\,\tilde{\alpha}_\lambda^2 \cdots \ \text{ for } i = t, b\ , \end{aligned} \tag{31}$$

where

$$\begin{aligned} \eta_{t,b} &= \frac{89}{65}\,,\ \frac{63}{65}\,,\ f_i^{(r_\lambda=1)} = f_i^{(r_\lambda=2)} = 0\,,\\ f_{t,b}^{(r_f=1)} &\simeq -0.258\,,\ -0.213\,,\ f_{t,b}^{(r_f=1)} \simeq -0.258\,,\ -0.213\,,\\ f_{t,b}^{(r_f=2)} &\simeq -0.055\,, -0.050\,, f_{t,b}^{(r_f=1,r_\lambda=1)} \simeq -0.021\,, -0.018\,. \end{aligned} \tag{32}$$

In the solutions (29) and (31) we have suppressed the contributions from the Yukawa couplings of the first two generations because they are negligibly small.

Presumably, both solutions are related; a numerical analysis on the solutions [12] suggests that the solution 3 is a "boundary" of 4. If it is really so, then there is only one unique reduction solution in the minimal supersymmetric GUT that provides us with the possibility of predicting α_t. Note furthermore that not only α_t but also α_b is predicted in this reduction solution.

Just below the unification scale we would like to obtain the MSSM SU(3)×SU(2)×U(1) and one pair of Higgs doublets, and assume that all the superpartners are degenerate at the supersymmetry breaking scale, where the MSSM will be broken to the normal SM. Then the standard model should be spontaneously broken down to $SU(3)\times U(1)_{em}$ due to VEV of the two Higgs SU(2)-doublets contained in the $\mathbf{5}, \overline{\mathbf{5}}$-super-multiplets.

One way to obtain the correct low energy theory is to add to the Lagrangian soft supersymmetry breaking terms and to arrange the mass parameters in the superpotential along with the soft breaking terms so that the desired symmetry breaking pattern of the original SU(5) is really the preferred one, all the superpartners are unobservable at present energies, there is no contradiction with proton decay, and so forth. Then we study the evolution of the couplings at two loops respecting all the boundary conditions at $M_{\rm GUT}$.

5. Finiteness in $N = 1$ SUSY gauge theories

According to the discussion in Chapter 3, the non-renormalization theorem ensures there are no extra mass and cubic-interaction-term renormalizations, implying that the β-functions of C_{ijk} can be expressed as linear combinations of the anomalous dimensions γ_{ij} of ϕ^i. Therefore, all the one-loop β-functions of the theory vanish if $\beta_g^{(1)}$ and $\gamma_{ij}^{(1)}$, given in Eqs. (14) and (16) respectively, vanish, *i.e.*

$$\sum_i \ell(R_i) = 3C_2(G)\,, \tag{33}$$

$$C^{ikl}C_{jkl} = 2\delta^i_j g^2 C_2(R_i)\,. \tag{34}$$

A very interesting result is that the conditions (33,34) are necessary and sufficient for finiteness at the two-loop level [22].

In case supersymmetry is broken by soft terms, one-loop finiteness of the soft sector imposes further constraints on it [24]. In addition, the same set of conditions that are sufficient for one-loop finiteness of the soft breaking terms render the soft sector of they theory two-loop finite [25].

The one- and two-loop finiteness conditions (33), (34) restrict considerably the possible choices of the irreps. R_i for a given group G as well as the Yukawa couplings in the superpotential (9). Note in particular that the finiteness conditions cannot be applied to the supersymmetric standard model (SSM), since the presence of a U(1) gauge group is incompatible with the condition (33), due to $C_2[U(1)]= 0$. This naturally leads to the expectation that finiteness should be attained at the grand unified level only, the SSM being just the corresponding, low-energy, effective theory.

Another important consequence of one- and two-loop finiteness is that supersymmetry (most probably) can only be broken by soft breaking terms.

Indeed, due to the unacceptability of gauge singlets, F-type spontaneous symmetry breaking [30] terms are incompatible with finiteness, as well as D-type [31] spontaneous breaking which requires the existence of a U(1) gauge group.

A natural question to ask is what happens at higher loop orders. The answer is contained in a theorem [33] which states the necessary and sufficient conditions to achieve finiteness at all orders. Before we discuss the theorem let us make some introductory remarks. The finiteness conditions impose relations between gauge and Yukawa couplings. To require such relations which render the couplings mutually dependent at a given renormalization point is trivial. What is not trivial is to guarantee that relations leading to a reduction of the couplings hold at any renormalization point. As we have seen, the necessary, but also sufficient, condition for this to happen is to require that such relations are solutions to the REs

$$\beta_g \frac{d\lambda_{ijk}}{dg} = \beta_{ijk} \tag{35}$$

and hold at all orders. As we have seen, remarkably the existence of all-order solutions to (35) can be decided at the one-loop level.

Let us now turn to the all-order finiteness theorem [33], which states when a $N = 1$ supersymmetric gauge theory can become finite to all orders in the sense of vanishing β-functions, that is of physical scale invariance. It is based on (a) the structure of the supercurrent in $N = 1$ SYM [39, 40, 41], and on (b) the non-renormalization properties of $N = 1$ chiral anomalies [33, 34]. Details on the proof can be found in Refs. [33] and further discussion in Refs. [34–36]. Here, following mostly Ref. [36] we present a comprehensible sketch of the proof.

Consider a $N = 1$ supersymmetric gauge theory, with simple Lie group G. The content of this theory is given at the classical level by the matter supermultiplets S_i, which contain a scalar field ϕ_i and a Weyl spinor ψ_{ia}, and the gauge fields V_a, which contain a gauge vector field A^a_μ and a gaugino Weyl spinor λ^a_α.

Let us first recall certain facts about the theory:

(1) A massless $N = 1$ supersymmetric theory is invariant under a U(1) chiral transformation R under which the various fields transform as follows

$$A'_\mu = A_\mu,\ \ \lambda'_\alpha = \exp(-i\theta)\lambda_\alpha\ \ \phi' = \exp(-i\frac{2}{3}\theta)\phi,\ \ \psi'_\alpha = \exp(-i\frac{1}{3}\theta)\psi_\alpha,\ \cdots \tag{36}$$

The corresponding axial Noether current $J^\mu_R(x)$,

$$J^\mu_R(x) = \bar{\lambda}\gamma^\mu\gamma^5\lambda + \cdots, \tag{37}$$

is conserved classicaly, while in the quantum case is violated by the axial anomaly

$$\partial J_R^\mu = r(\epsilon^{\mu\nu\sigma\rho} F_{\mu\nu} F_{\sigma\rho} + \cdots). \tag{38}$$

From its known topological origin in ordinary gauge theories [37], one would expect the axial vector current J_R^μ to satisfy the Adler–Bardeen theorem [38] and receive corrections only at the one-loop level. Indeed it has been shown that the same non-renormalization theorem holds also in supersymmetric theories [34]. Therefore

$$r = \hbar \beta_g^{(1)}. \tag{39}$$

(2) The massless theory we consider is scale invariant at the classical level and, in general, there is a scale anomaly due to radiative corrections. The scale anomaly appears in the trace of the energy momentum tensor $T_{\mu\nu}$, which is traceless classically. It has the form

$$T_\mu^\mu \;=\; \beta_g F^{\mu\nu} F_{\mu\nu} + \cdots \tag{40}$$

(3) Massless, $N = 1$ supersymmetric gauge theories are classically invariant under the supersymmetric extension of the conformal group – the superconformal group. Examining the superconformal algebra, it can be seen that the subset of superconformal transformations consisting of translations, supersymmetry transformations, and axial R transformations is closed under supersymmetry, *i.e.* these transformations form a representation of supersymmetry. It follows that the conserved currents corresponding to these transformations make up a supermultiplet represented by an axial vector superfield called supercurrent [39] J,

$$J \equiv \{J_R'^\mu,\ Q_\alpha^\mu,\ T_\nu^\mu, ...\}, \tag{41}$$

where $J_R'^\mu$ is the current associated to R invariance, Q_α^μ is the one associated to supersymmetry invariance, and T_ν^μ the one associated to translational invariance (energy-momentum tensor).

The anomalies of the R current $J_R'^\mu$, the trace anomalies of the supersymmetry current, and the energy-momentum tensor, form also a second supermultiplet, called the supertrace anomaly

$$\begin{aligned} S &= \{\mathrm{Re}\, S,\ \mathrm{Im}\, S,\ S_\alpha\} \\ &= \{T_\mu^\mu,\ \partial_\mu J_R'^\mu,\ \sigma^\mu_{\alpha\dot\beta} \bar{Q}_\mu^{\dot\beta}\ +\ \cdots\}, \end{aligned} \tag{42}$$

where T_μ^μ in Eq. (40) and

$$\partial_\mu J_R'^\mu \;=\; \beta_g \epsilon^{\mu\nu\sigma\rho} F_{\mu\nu} F_{\sigma\rho} + \cdots \tag{43}$$

$$\sigma^\mu_{\alpha\dot\beta} \bar{Q}_\mu^{\dot\beta} \;=\; \beta_g \lambda^\beta \sigma^{\mu\nu}_{\alpha\beta} F_{\mu\nu} + \cdots \tag{44}$$

(4) It is very important to note that the Noether current defined in (37) is not the same as the current associated to R invariance that appears in the supermultiplet J in (41), but they coincide in the tree approximation. So starting from a unique classical Noether current $J^{\mu}_{R(\text{class})}$, the Noether current J^{μ}_{R} is defined as the quantum extension of $J^{\mu}_{R(\text{class})}$ which allows for the validity of the non-renormalization theorem. On the other hand J'^{μ}_{R}, is defined to belong to the supercurrent J, together with the energy-momentum tensor. The two requirements cannot be fulfilled by a single current operator at the same time.

Although the Noether current J^{μ}_{R} which obeys (38) and the current J'^{μ}_{R} belonging to the supercurrent multiplet J are not the same, there is a relation [33] between quantities associated with them

$$r = \beta_g(1 + x_g) + \beta_{ijk}x^{ijk} - \gamma_A r^A \,, \tag{45}$$

where r was given in Eq. (39). The r^A are the non-renormalized coefficients of the anomalies of the Noether currents associated to the chiral invariances of the superpotential, and — like r — are strictly one-loop quantities. The γ_A's are linear combinations of the anomalous dimensions of the matter fields, and x_g, and x^{ijk} are radiative correction quantities. The structure of equality (45) is independent of the renormalization scheme.

One-loop finiteness, *i.e.* vanishing of the β-functions at one-loop, implies that the Yukawa couplings λ_{ijk} must be functions of the gauge coupling g. To find a similar condition to all orders it is necessary and sufficient for the Yukawa couplings to be a formal power series in g, which is solution of the REs (35).

We can now state the theorem for all-order vanishing β-functions.

Theorem:

Consider an $N = 1$ supersymmetric Yang–Mills theory, with simple gauge group. If the following conditions are satisfied

1. There is no gauge anomaly.
2. The gauge β-function vanishes at one-loop

$$\beta_g^{(1)} = 0 = \sum_i l(R_i) - 3\,C_2(G). \tag{46}$$

3. There exist solutions of the form

$$\lambda_{ijk} = \rho_{ijk}g, \qquad \rho_{ijk} \in \mathbb{C} \tag{47}$$

to the conditions of vanishing one-loop matter fields anomalous dimensions

$$\gamma_j^{i\,(1)} = 0 = \frac{1}{32\pi^2}\,[\,C^{ikl}C_{jkl} - 2\,g^2\,C_2(R_i)\delta_{ij}]. \tag{48}$$

4. These solutions are isolated and non-degenerate when considered as solutions of vanishing one-loop Yukawa β-functions:

$$\beta_{ijk} = 0. \tag{49}$$

Then, each of the solutions (47) can be uniquely extended to a formal power series in g, and the associated super Yang-Mills models depend on the single coupling constant g with a β function which vanishes at all-orders.

It is important to note a few things: The requirement of isolated and non-degenerate solutions guarantees the existence of a formal power series solution to the reduction equations. The vanishing of the gauge β-function at one-loop, $\beta_g^{(1)}$, is equivalent to the vanishing of the R current anomaly (38). The vanishing of the anomalous dimensions at one-loop implies the vanishing of the Yukawa couplings β-functions at that order. It also implies the vanishing of the chiral anomaly coefficients r^A. This last property is a necessary condition for having β functions vanishing at all orders.

Proof:

Insert β_{ijk} as given by the REs into the relationship (45) between the axial anomalies coefficients and the β-functions. Since these chiral anomalies vanish, we get for β_g an homogeneous equation of the form

$$0 = \beta_g(1 + O(\hbar)). \tag{50}$$

The solution of this equation in the sense of a formal power series in $\hbar$ is $\beta_g = 0$, order by order. Therefore, due to the REs (35), $\beta_{ijk} = 0$ too.

Thus we see that finiteness and reduction of couplings are intimately related.

6. Finite SU(5) model

As a realistic example of the concepts presented in the previous section we consider a Finite Unified Model Based on SU(5). From the classification of theories with vanishing one-loop β function for the gauge coupling [23], one can see that using SU(5) as gauge group there exist only two candidate models which can accommodate three fermion generations. These models contain the chiral supermutiplets $\mathbf{5}$, $\overline{\mathbf{5}}$, $\mathbf{10}$, $\overline{\mathbf{5}}$, $\mathbf{24}$ with the multiplicities

$(6,9,4,1,0)$ and $(4,7,3,0,1)$, respectively. Only the second one contains a **24**-plet which can be used for spontaneous symmetry breaking (SSB) of SU(5) down to SU(3)×SU(2)×U(1). (For the first model one has to incorporate another way, such as the Wilson flux breaking to achieve the desired SSB of SU(5) [11]). Therefore, we would like to concentrate only on the second model.

To simplify the situation, we neglect the intergenerational mixing among the lepton and quark supermultiplets and consider the following SU(5) invariant cubic superpotential for the (second) model:

$$\begin{aligned} W &= \sum_{i=1}^{3}\sum_{\alpha=1}^{4}\left[\,\frac{1}{2}g_{i\alpha}^{u}\,\mathbf{10}_i\mathbf{10}_i H_\alpha + g_{i\alpha}^{d}\,\mathbf{10}_i\overline{\mathbf{5}}_i\,\overline{H}_\alpha\,\right] \\ &\quad + \sum_{\alpha=1}^{4} g_{\alpha}^{f}\,H_\alpha\,\mathbf{24}\,\overline{H}_\alpha + \frac{g^\lambda}{3}\,(\mathbf{24})^3\ , \ \text{with}\ \ g_{i\alpha}^{u,d}=0 \text{ for } i\neq\alpha\ , \qquad (51) \end{aligned}$$

where the $\mathbf{10}_i$'s and $\overline{\mathbf{5}}_i$'s are the usual three generations, and the four $(\mathbf{5}+\overline{\mathbf{5}})$ Higgses are denoted by H_α , $\overline{H}_\alpha$. The superpotential is not the most general one, but by virtue of the non-renormalization theorem, this does not contradict the philosophy of the coupling unification by the reduction method (a RG invariant fine tuning is a solution of the reduction equation). In the case at hand, however, one can find a discrete symmetry that can be imposed on the most general cubic superpotential to arrive at the non-intergenerational mixing [11]. This is given in Table I.

TABLE I

The charges of the $Z_7 \times Z_3$ symmetry

	$\mathbf{10}_1$	$\mathbf{10}_2$	$\mathbf{10}_3$	$\bar{5}_1$	$\bar{5}_2$	$\bar{5}_3$	H_1	H_2	H_3	H_4
Z_7	1	2	4	4	1	2	5	3	6	0
Z_3	1	2	0	0	0	0	1	2	0	0

Given the superpotential W, we can compute the β functions of the model. We denote the gauge coupling by g (with the vanishing one-loop β function), and our normalization of the β functions is as usual, *i.e.*,

$$dg_i/d\ln\mu \;=\; \beta_i^{(1)}/16\pi^2 + O(g^5),$$

where μ is the renormalization scale. We find:

$$
\begin{aligned}
\beta_g^{(1)} &= 0 \, , \\
\beta_{i\alpha}^{u(1)} &= \frac{1}{16\pi^2}\left[-\frac{96}{5}\,g^2 + 6\sum_{\beta=1}^{4}(g_{i\beta}^{u})^2 + 3\sum_{j=1}^{3}(g_{j\alpha}^{u})^2 + \frac{24}{5}\,(g_{\alpha}^{f})^2 \right. \\
&\left. +4\sum_{\beta=1}^{4}(g_{i\beta}^{d})^2\right] g_{i\alpha}^{u} \, , \\
\beta_{i\alpha}^{d(1)} &= \frac{1}{16\pi^2}\left[-\frac{84}{5}\,g^2 + 3\sum_{\beta=1}^{4}(g_{i\beta}^{u})^2 + \frac{24}{5}\,(g_{\alpha}^{f})^2 + 4\sum_{j=1}^{3}(g_{j\alpha}^{d})^2 \right. \\
&\left. +6\sum_{\beta=1}^{4}(g_{i\beta}^{d})^2\right] g_{i\alpha}^{d} \, , \qquad (52) \\
\beta^{\lambda(1)} &= \frac{1}{16\pi^2}\left[-30\,g^2 + \frac{63}{5}\,(g^{\lambda})^2 + 3\sum_{\alpha=1}^{4}(g_{\alpha}^{f})^2\right] g^{\lambda} \, , \\
\beta_{\alpha}^{f(1)} &= \frac{1}{16\pi^2}\left[-\frac{98}{5}\,g^2 + 3\sum_{i=1}^{3}(g_{i\alpha}^{u})^2 + 4\sum_{i=1}^{3}(g_{i\alpha}^{d})^2 + \frac{48}{5}\,(g_{\alpha}^{f})^2 \right. \\
&\left. +\sum_{\beta=1}^{4}(g_{\beta}^{f})^2 + \frac{21}{5}\,(g^{\lambda})^2\right] g_{\alpha}^{f} \, .
\end{aligned}
$$

We then regard the gauge coupling g as the primary coupling and solve the reduction equations (4) with the power series ansatz. One finds that the power series,

$$
\begin{aligned}
(g_{ii}^{u})^2 &= \frac{8}{5}g^2 + \dots \, , \; (g_{ii}^{d})^2 = \frac{6}{5}g^2 + \dots \, , \; (g^{\lambda})^2 = \frac{15}{7}g^2 + \dots \, , \\
(g_{4}^{f})^2 &= g^2 \, , \; (g_{\alpha}^{f})^2 = 0 + \dots \quad (\alpha = 1,2,3) \, , \qquad (53)
\end{aligned}
$$

exists uniquely, where ... indicates higher order terms and all the other couplings have to vanish. As we have done in the previous section, we can easily verify that the higher order terms can be uniquely computed.

Consequently, all the one-loop β functions of the theory vanish. Moreover, all the one-loop anomalous dimensions for the chiral supermultiplets,

$$
\begin{aligned}
\gamma_{\mathbf{10}i}^{(1)} &= \frac{1}{16\pi^2}\left[-\frac{36}{5}\,g^2 + 3\sum_{\beta=1}^{4}(g_{i\beta}^{u})^2 + 2\sum_{\beta=1}^{4}(g_{i\beta}^{d})^2\right] , \\
\gamma_{\overline{\mathbf{5}}i}^{(1)} &= \frac{1}{16\pi^2}\left[-\frac{24}{5}\,g^2 + 4\sum_{\beta=1}^{4}(g_{i\beta}^{d})^2\right] ,
\end{aligned}
$$

$$\gamma^{(1)}_{H_\alpha} = \frac{1}{16\pi^2}\left[-24\,g^2 + 3\,\sum_{i=1}^{3}(g^u_{i\alpha})^2 + \frac{24}{5}(g^f_\alpha)^2\right] , \tag{54}$$

$$\gamma^{(1)}_{\overline{H}_\alpha} = \frac{1}{16\pi^2}\left[-24\,g^2 + 4\,\sum_{i=1}^{3}(g^d_{i\alpha})^2 + \frac{24}{5}(g^f_\alpha)^2\right] ,$$

$$\gamma^{(1)}_{\mathbf{24}} = \frac{1}{16\pi^2}\left[-\frac{10}{5}\,g^2 + + \sum_{\alpha=1}^{4}(g^f_\alpha)^2 + \frac{21}{5}\,(g^\lambda)^2\right]$$

also vanish in the reduced system. As it has already been mentioned before, these conditions are necessary and sufficient for finiteness at the two-loop level [22].

In most of the previous studies of the present model [26, 27], however, the complete reduction of the Yukawa couplings, which is necessary for all-order-finiteness, was ignored. They have used the freedom offered by the degeneracy in the one- and two-loop approximations in order to make specific ansätze that could lead to phenomenologically acceptable predictions. In the above model, we found a diagonal solution for the Yukawa couplings, with each family coupled to a different Higgs. However, we may use the fact that mass terms do not influence the RG functions in a certain class of renormalization schemes, and introduce appropriate mass terms that permit us to perform a rotation in the Higgs sector such that only one pair of Higgs doublets, coupled to the third family, remains light and acquires a non-vanishing VEV [27]. Note that the effective coupling of the Higgs doublets to the first family after the rotation is very small avoiding in this way a potential problem with the proton lifetime [42]. Thus, effectively, we have at low energies the Minimal Supersymmetric Standard Model (MSSM) with only one pair of Higgs doublets satisfying the boundary conditions at $M_{\rm GUT}$

$$g_t^2 = \frac{8}{5}g^2 + O(g^4)\ ,\ g_b^2 = g_\tau^2 = \frac{6}{5}g^2 + O(g^4)\ , \tag{55}$$

where g_i $(i = t, b, \tau)$ are the top, bottom and tau Yukawa couplings of the MSSM, and the other Yukawa couplings should be regarded as free.

Adding soft breaking terms (which are supposed not to influence the β functions beyond $M_{\rm GUT}$), we can obtain supersymmetry breaking. The conditions on the soft breaking terms to preserve one-loop finiteness have been given already some time ago [24]. Recently, the same problem in two-loop orders has been addressed [25]. It is an open problem whether there exists a suitable set of conditions on the soft terms for all-loop finiteness.

7. Predictions of low energy parameters

In this section we will refine the predictions of the AFUT and FUT models, taking into account certain corrections and we will compare them with the experimental data.

As mentioned before, at low energies we want the MSSM, with one pair of Higgs doublets, and we will assume that at the supersymmetry breaking scale all the superpartners are degenerate.

Since the gauge symmetry is spontaneously broken below $M_{\rm GUT}$, the finiteness conditions in the case of the FUT model do not restrict the renormalization property at low energies, and all it remains is a boundary condition on the gauge and Yukawa couplings at $M_{\rm GUT}$, *i.e.*, Eq. (53). Clearly the same holds also in the AFUT models. So we examine the evolution of these couplings according to their renormalization group equations at two-loops with the corresponding boundary conditions at $M_{\rm GUT}$.

Below $M_{\rm GUT}$ their evolution is assumed to be governed by the MSSM. We further assume a unique threshold $M_{\rm SUSY}$ for all superpartners of the MSSM so that below $M_{\rm SUSY}$ the SM is the correct effective theory. We recall that $\tan\beta$ is usually determined in the Higgs sector, which however strongly depends on the supersymmetry breaking terms. Here we avoid this by using the tau mass M_τ as input, which means that we partly fix the Higgs sector indirectly. That is, assuming that

$$M_Z \ll M_t \ll M_{\rm SUSY} \ , \tag{56}$$

we require the matching condition at $M_{\rm SUSY}$ [43],

$$\begin{aligned}\alpha_t^{\rm SM} &= \alpha_t \sin^2\beta \ , \ \alpha_b^{\rm SM} = \alpha_b \cos^2\beta \ , \ \alpha_\tau^{\rm SM} = \alpha_\tau \cos^2\beta \ ,\\ \alpha_\lambda &= \tfrac{1}{4}(\tfrac{3}{5}\alpha_1 + \alpha_2)\cos^2 2\beta\end{aligned} \tag{57}$$

to be satisfied, where $\alpha_i^{\rm SM}$ $(i = t, b, \tau)$ are the SM Yukawa couplings and α_λ is the Higgs coupling. The MSSM threshold corrections to this matching condition [44, 45] will be discussed later. This is our definition of $\tan\beta$, and Eq. (57) fixes $\tan\beta$, because with a given set of the input parameters [46],

$$M_\tau = 1.777 \ {\rm GeV} \ , \ M_Z = 91.188 \ {\rm GeV} \ , \tag{58}$$

with [47]

$$\begin{aligned}\alpha_{\rm EM}^{-1}(M_Z) &= 127.9 + \frac{8}{9\pi}\log\frac{M_t}{M_Z} \ ,\\ \sin^2\theta_{\rm W}(M_Z) &= 0.2319 - 3.03\times10^{-5}T - 8.4\times10^{-8}T^2 \ ,\\ T &= M_t/[{\rm GeV}] - 165 \ ,\end{aligned} \tag{59}$$

the matching condition (57) and the GYU boundary condition at $M_{\rm GUT}$ can be satisfied only for a specific value of $\tan\beta$. Here M_τ, M_t, M_Z are pole masses, and the couplings are defined in the $\overline{\rm MS}$ scheme with six flavors. The translation from a Yukawa coupling into the corresponding mass follows according to

$$m_i = \frac{1}{\sqrt{2}} g_i(\mu)\, v(\mu)\ ,\ i = t, b, \tau \ \text{ with } \ v(M_Z) = 246.22 \text{ GeV}\ , \tag{60}$$

where $m_i(\mu)$'s are the running masses satisfying the respective evolution equation of two-loop order. The pole masses can be calculated from the running ones of course. For the top mass, we use [43, 44]

$$M_t = m_t(M_t)\left[1 + \frac{4}{3}\frac{\alpha_3(M_t)}{\pi} + 10.95\left(\frac{\alpha_3(M_t)}{\pi}\right)^2 + k_t\frac{\alpha_t(M_t)}{\pi}\right]\ , \tag{61}$$

where $k_t \simeq -0.3$ for the range of parameters we are concerned with in this paper [44]. Note that both sides of Eq. (61) contain M_t so that M_t is defined only implicitly. Therefore, its determination requires an iteration method. As for the tau and bottom masses, we assume that $m_\tau(\mu)$ and $m_b(\mu)$ for $\mu \leq M_Z$ satisfy the evolution equation governed by the $\mathrm{SU}(3)_{\rm C}\times\mathrm{U}(1)_{\rm EM}$ theory with five flavors and use

$$\begin{aligned} M_b &= m_b(M_b)\left[1 + \frac{4}{3}\frac{\alpha_{3(5\rm f)}(M_b)}{\pi} + 12.4\left(\frac{\alpha_{3(5\rm f)}(M_b)}{\pi}\right)^2\right]\ ,\\ M_\tau &= m_\tau(M_\tau)\left[1 + \frac{\alpha_{\rm EM(5f)}(M_\tau)}{\pi}\right]\ , \end{aligned} \tag{62}$$

where the experimental value of $m_b(M_b)$ is $(4.1 - 4.5)$ GeV [46]. The couplings with five flavors entered in Eq. (30) $\alpha_{3(5\rm f)}$ and $\alpha_{\rm EM(5f)}$ are related to α_3 and $\alpha_{\rm EM}$ by

$$\begin{aligned} \alpha^{-1}_{3(5\rm f)}(M_Z) &= \alpha_3^{-1}(M_Z) - \frac{1}{3\pi}\ln\frac{M_t}{M_Z}\ ,\\ \alpha^{-1}_{\rm EM(5f)}(M_Z) &= \alpha^{-1}_{\rm EM}(M_Z) - \frac{8}{9\pi}\ln\frac{M_t}{M_Z}\ . \end{aligned} \tag{63}$$

Using the input values given in eqs. (58) and (60), we find

$$\begin{aligned} m_\tau(M_\tau) &= 1.771 \text{ GeV}\ , m_\tau(M_Z) = 1.746 \text{ GeV}\ ,\\ \alpha^{-1}_{\rm EM(5f)}(M_\tau) &= 133.7\ , \end{aligned} \tag{64}$$

and from Eq. (60) we obtain

$$\alpha_\tau^{\rm SM}(M_Z) = \frac{g_\tau^2}{4\pi} = 8.005\times 10^{-6}\ , \tag{65}$$

which we use as an input parameter instead of M_τ.

The matching condition (57) suffers from the threshold corrections coming from the MSSM superpartners:

$$\alpha_i^{\rm SM} \to \alpha_i^{\rm SM}(1+\Delta_i^{\rm SUSY}) \ , \ i = 1,2,\ldots,\tau \, . \qquad (66)$$

It was shown that these threshold effects to the gauge couplings can be effectively parametrized by just one energy scale [48]. Accordingly, we can identify our $M_{\rm SUSY}$ with that defined in Ref. [48]. This ensures that there are no further one-loop threshold corrections to $\alpha_3(M_Z)$ when we calculate it as a function of $\alpha_{\rm EM}(M_Z)$ and $\sin^2\theta_W(M_Z)$.

The same scale $M_{\rm SUSY}$ does not describe threshold corrections to the Yukawa couplings, and they could cause large corrections to the fermion mass prediction [44, 45] [1]. For m_b, for instance, the correction can be as large as 50% for very large values of $\tan\beta$, especially in models with radiative gauge symmetry breaking and with supersymmetry softly broken by the universal breaking terms. As we will see, the SU(5)-FUT and AFUT models predict (with these corrections suppressed) values for the bottom quark mass that are rather close to the experimentally allowed region so that there is room only for small corrections. Consequently, if we want to break SU(2)×U(1) gauge symmetry radiatively, the models favor non-universal soft breaking terms [49].

To get an idea about the magnitude of the correction, we consider the case that all the superpartners have the same mass $M_{\rm SUSY}$ = 500 GeV with $M_{\rm SUSY} \gg \mu_H$ and $\tan\beta \geq 50$. Using Δ's given in Ref. [45], we find that the MSSM correction to the M_t prediction is ~ -1 % for this case. Comparing with the results of [45, 50], this may appear to be underestimated for other cases. Note, however, that there is a nontrivial interplay among the corrections between the M_t and M_b predictions for a given GYU boundary condition at $M_{\rm GUT}$ and the fixed pole tau mass, which has not been taken into account in refs. [45, 50]. In the following discussion, therefore, we regard the MSSM threshold correction to the M_t prediction as unknown and denote it by

$$\delta^{\rm MSSM} M_t \, . \qquad (67)$$

In the case of the AFUT models, the non-observation of proton decay favours a solution close to AFUT3.

In Table II we present the predictions for M_t for various $M_{\rm SUSY}$, in the case of the FUT model.

[1] It is possible to compute the MSSM correction to M_t directly, *i.e.*, without constructing an effective theory below $M_{\rm SUSY}$. In this approach, too, large corrections have been reported [50]. In the present paper, evidently, we are following the effective theory approach as *e.g.* Refs. [44, 45].

TABLE II

The predictions for different $M_{\rm SUSY}$ for FUT

$M_{\rm SUSY}$ [GeV]	$\alpha_3(M_Z)$	$\tan\beta$	$M_{\rm GUT}$ [GeV]	$m_b(M_b)$ [GeV]	M_t [GeV]
300	0.123	54.2	2.08×10^{16}	4.54	183.5
500	0.122	54.3	1.77×10^{16}	4.54	184.0
10^3	0.120	54.4	1.42×10^{16}	4.54	184.4

As we can see from the table, only negative MSSM corrections of at most ~ 10 % to $m_b(M_b)$ are allowed ($m_b^{\rm exp}(M_b) = (4.1 - 4.5)$ GeV), implying that FUT favors non-universal soft symmetry breaking terms as announced. The predicted M_t values are well below the infrared value [51], for instance 194 GeV for $M_{\rm SUSY} = 500$ GeV, so that the M_t prediction must be sensitive against the change of the boundary condition.

We recall that if one includes the threshold effects of superheavy particles [52], the GUT scale $M_{\rm GUT}$ at which α_1 and α_2 are supposed to meet is related to the mass of the superheavy $SU(3)_C$-triplet Higgs supermultiplets contained in H_α and $\overline{H}_\alpha$. These effects have therefore influence on the GYU boundary conditions.

TABLE III

The predictions for the AFUT model

$m_{\rm SUSY}$ [GeV]	$\alpha_3(M_Z)$	$\tan\beta$	$M_{\rm GUT}$ [GeV]	m_b [GeV]	m_t [GeV]
300	0.120	47.7	1.8×10^{16}	5.4	179.7
500	0.118	47.7	1.39×10^{16}	5.3	178.9

In Table III we present the predictions for the AFUT viable model (AFUT3). For these model the corrections mentioned above have been calculated [16] and are of the order of $\lesssim 2\%$. The threshold effects of the superheavy particles were estimated to be of the same order as in the gauge sector, which leads to an uncertainty of $\sim \pm 0.4$ GeV in M_t. The structure of the threshold effects in FUT is involved, but they are not arbitrary and probably determinable to a certain extent, because the mixing of the superheavy Higgses is strongly dictated by the fermion mass matrix of the MSSM. To bring these threshold effects under control is challenging. Here we assume that the magnitude of these effects is $\sim \pm 4$ GeV in M_t, which is estimated by comparing the minimal GYU model based on SU(5) [16].

Thus, for the FUT model the prediction for M_t [16] will be

$$M_t = (183 + \delta^{\rm MSSM} M_t \pm 5) \;\; \text{GeV} \;, \tag{68}$$

where the finite corrections coming from the conversion from the dimensional reduction scheme to the ordinary $\overline{\rm MS}$ in the gauge sector [62] are included, and those in the Yukawa sector are included as an uncertainty of $\sim \pm 1$ GeV. The MSSM threshold correction is denoted $\delta^{\rm MSSM} M_t$ which has been discussed in the previous section.

In the case of the AFUT model the prediction is [16]

$$M_t = (181 + \delta^{\rm MSSM} M_t \pm 3) \;\; \text{GeV} \;. \tag{69}$$

Comparing the M_t prediction above with the most recent experimental values [53],

$$\begin{aligned} M_{top} &= 176.8 \; \pm 4.4_{\rm stat} \; \pm 4.8_{\rm syst} \;\; \text{GeV} \quad \text{CDF} \\ M_{top} &= 169.0 \; \pm 8.0_{\rm stat} \; \pm 8.0_{\rm syst} \;\; \text{GeV} \quad \text{D0} \end{aligned} \tag{70}$$

we see it is consistent with the experimental data.

It is interesting to note that the consistency of the finiteness hypothesis is closely related to the fine structure of supersymmetry breaking and also to the Higgs sector, because these superpartner corrections to m_b can be kept small for appropriate supersymmetric spectrum characterized by very heavy squarks and/or small μ_H describing the mixing of the two Higgs doublets in the superpotential [2].

The predictions for M_t versus $M_{\rm SUSY}$ for the two sets of boundary conditions given above (AFUT3 and AFUT4) together with the corresponding predictions of the FUT model, are given in Figure 1. In a recent study [16], we have considered the proton decay constraint [55] to further reduce the parameter space of the model. It has been found that the model consistent with the non-observation of the proton decay should be very close to AFUT3, implying a better possibility to discriminate between the FUT and AFUT models, as one can see from Figure 1.

[2] The solution with small μ_H is favored by the experimental data and cosmological constraints [49]. The sign of this correction is determined by the relative sign of μ_H and the gluino mass parameter and is correlated with the chargino exchange contribution to the $b \to s\gamma$ decay [44]. The later has the same sign as the Standard Model and the charged Higgs contributions when the supersymmetric corrections to m_b are negative.

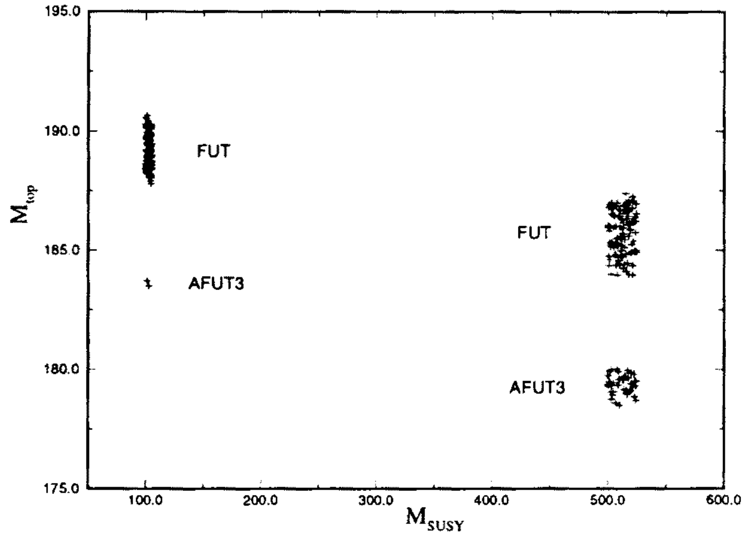

Fig. 1. M_t predictions of SU(5) FUT and AFUT3 models, for given M_{SUSY} around 100 and 500 GeV. For the FUT model $\tilde{\alpha}_t = 1.6$, $\tilde{\alpha}_b = 1.2$, and for AFUT3 $\tilde{\alpha}_t = 0.97$, $\tilde{\alpha}_b = 0.57$.

8. Asymptotically non-free supersymmetric Pati-Salam model

We present now a model where the reduction of couplings is applied, but that does not have a single gauge group, but a product of simple groups. In order for the RGI method for the gauge coupling unification to work, the gauge couplings should have the same asymptotic behavior. Note that this common behavior is absent in the standard model with three families. A way to achieve a common asymptotic behavior of all the different gauge couplings is to embed $\mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ to some non-abelian gauge group, as it was done in the previous sections. However, in this case still a major rôle in the GYU is due to the group theoretical aspects of the covering GUT. Here we would like to examine the power of RGI method by considering theories without covering GUTs. We found [13] that the minimal phenomenologically viable model is based on the gauge group of Pati and Salam [1]- $\mathcal{G}_{\mathrm{PS}} \equiv SU(4) \times SU(2)_R \times SU(2)_L$. We recall that $N = 1$ supersymmetric models based on this gauge group have been studied with renewed interest because they could in principle be derived from superstring [56].

In our supersymmetric, Gauge-Yukawa unified model based on $\mathcal{G}_{\mathrm{PS}}$ [13], three generations of quarks and leptons are accommodated by six chiral supermultiplets, three in $(\mathbf{4}, \mathbf{2}, \mathbf{1})$ and three $(\overline{\mathbf{4}}, \mathbf{1}, \mathbf{2})$, which we denote by

$\Psi^{(I)\mu\ i_R}$ and $\overline{\Psi}^{(I)i_L}_{\mu}$. ($I$ runs over the three generations, and μ, ν $(= 1,2,3,4)$ are the SU(4) indices while i_R , i_L $(= 1,2)$ stand for the $\mathrm{SU}(2)_{L,R}$ indices.) The Higgs supermultiplets in $(\mathbf{4},\mathbf{2},\mathbf{1})$, $(\overline{\mathbf{4}},\mathbf{2},\mathbf{1})$ and $(\mathbf{15},\mathbf{1},\mathbf{1})$ are denoted by $H^{\mu\ i_R}$, $\overline{H}_{\mu\ i_R}$ and Σ^{μ}_{ν}, respectively. They are responsible for the spontaneous symmetry breaking (SSB) of $\mathrm{SU}(4)\times\mathrm{SU}(2)_R$ down to $\mathrm{SU}(3)_C\times\mathrm{U}(1)_Y$. The SSB of $\mathrm{U}(1)_Y\times\mathrm{SU}(2)_L$ is then achieved by the nonzero VEV of $h_{i_R i_L}$ which is in $(\mathbf{1},\mathbf{2},\mathbf{2})$. In addition to these Higgs supermultiplets, we introduce $G^{\mu}_{\nu\ i_R i_L}$ $(\mathbf{15},\mathbf{2},\mathbf{2})$, ϕ $(\mathbf{1},\mathbf{1},\mathbf{1})$ and Σ'^{μ}_{ν} $(\mathbf{15},\mathbf{1},\mathbf{1})$. The $G^{\mu}_{\nu\ i_R i_L}$ is introduced to realize the $\mathrm{SU}(4)\times\mathrm{SU}(2)_R\times\ \mathrm{SU}(2)_L$ version of the Georgi-Jarlskog type ansatz [57] for the mass matrix of leptons and quarks while ϕ is supposed to mix with the right-handed neutrino supermultiplets at a high energy scale. With these things in mind, we write down the superpotential of the model W, which is the sum of the following superpotentials:

$$
\begin{aligned}
W_Y &= \sum_{I,J=1}^{3} g_{IJ}\,\overline{\Psi}^{(I)i_R}_{\mu}\,\Psi^{(J)\mu\ i_L}\ h_{i_R i_L}\ ,\\
W_{GJ} &= g_{GJ}\,\overline{\Psi}^{(2)i_R}_{\mu}\,G^{\mu}_{\nu\ i_R j_L}\,\Psi^{(2)\nu\ j_L}\ ,\\
W_{NM} &= \sum_{I=1,2,3} g_{I\phi}\ \epsilon_{i_R j_R}\,\overline{\Psi}^{(I)i_R}_{\mu}\ H^{\mu\ j_R}\,\phi\ ,\\
W_{SB} &= g_H\,\overline{H}_{\mu\ i_R}\,\Sigma^{\mu}_{\nu}\,H^{\nu\ i_R}+\frac{g_\Sigma}{3}\,\mathrm{Tr}\,[\ \Sigma^3\ ]+\frac{g_{\Sigma'}}{2}\,\mathrm{Tr}\,[\ (\Sigma')^2\,\Sigma\ ]\ ,\\
W_{TDS} &= \frac{g_G}{2}\,\epsilon^{i_R j_R}\epsilon^{i_L j_L}\,\mathrm{Tr}\,[\ G_{i_R i_L}\,\Sigma\,G_{j_R j_L}\ ]\ ,\\
W_M &= m_h\,h^2+m_G\,G^2+m_\phi\,\phi^2+m_H\,\overline{H}\,H+m_\Sigma\,\Sigma^2+m_{\Sigma'}\,(\Sigma')^2\ .
\end{aligned}
\tag{71}
$$

Although W has the parity, $\phi\to-\phi$ and $\Sigma'\to-\Sigma'$, it is not the most general potential, but, as we already mentioned, this does not contradict the philosophy of the coupling unification by the RGI method.

We denote the gauge couplings of $\mathrm{SU}(4)\times\mathrm{SU}(2)_R\times\ \mathrm{SU}(2)_L$ by α_4 , α_{2R} and α_{2L}, respectively. The gauge coupling for $\mathrm{U}(1)_Y$, α_1, normalized in the usual GUT inspired manner, is given by $1/\alpha_1\ =\ 2/5\alpha_4+3/5\alpha_{2R}$. In principle, the primary coupling can be any one of the couplings. But it is more convenient to choose a gauge coupling as the primary one because the one-loop β functions for a gauge coupling depends only on its own gauge coupling. For the present model, we use α_{2L} as the primary one. Since the gauge sector for the one-loop β functions is closed, the solutions of the fixed point equations (21) are independent on the Yukawa and Higgs couplings. One easily obtains $\rho_4^{(1)}=8/9$, $\rho_{2R}^{(1)}\ =\ 4/5$, so that the RGI relations (25) at the one-loop level become

$$\tilde{\alpha}_4 = \frac{\alpha_4}{\alpha_{2L}} = \frac{8}{9}\,,\ \tilde{\alpha}_1 = \frac{\alpha_1}{\alpha_{2L}} = \frac{5}{6}\,. \tag{72}$$

The solutions in the Yukawa-Higgs sector strongly depend on the result of the gauge sector. After slightly involved algebraic computations, one finds that most predictive solutions contain at least three vanishing $\rho_i^{(1)}$'s. Out of these solutions, there are two that exhibit the most predictive power and moreover they satisfy the neutrino mass relation $m_{\nu_\tau} > m_{\nu_\mu}\,,\ m_{\nu_e}$.

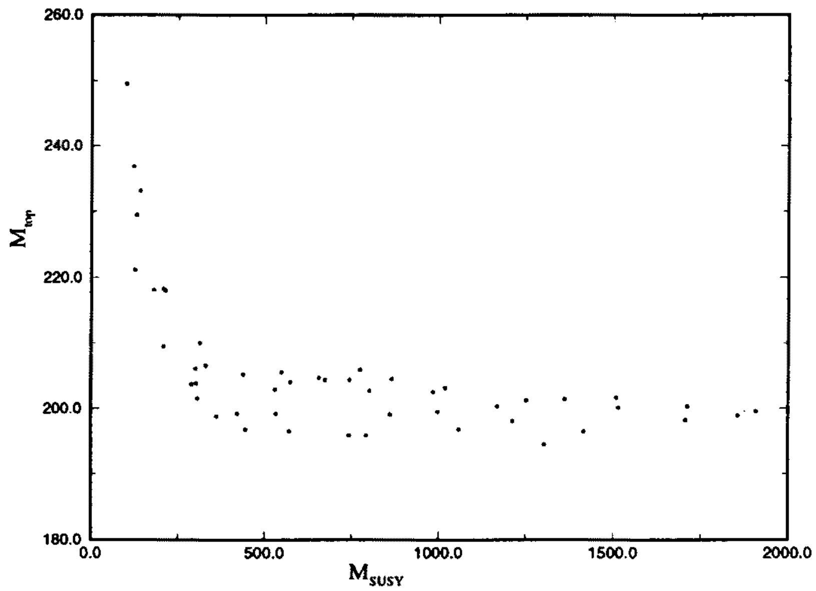

Fig. 2. The values for M_t predicted by the Pati-Salam model for different $M_{\rm SUSY}$ scales. Only the ones with $M_{\rm SUSY}$ beyond 400 GeV are realistic.

For the first solution we have $\rho_{1\phi}^{(1)} = \rho_{2\phi}^{(1)} = \rho_{\Sigma}^{(1)} = 0$, while for the second solution, $\rho_{1\phi}^{(1)} = \rho_{2\phi}^{(1)} = \rho_{G}^{(1)} = 0$, and one finds that for the cases above the power series solutions (25) take the form

$$\tilde{\alpha}_{GJ} \simeq \begin{cases} 1.67 - 0.05\tilde{\alpha}_{1\phi} + 0.004\tilde{\alpha}_{2\phi} - 0.90\tilde{\alpha}_{\Sigma} + \cdots \\ 2.20 - 0.08\tilde{\alpha}_{2\phi} - 0.05\tilde{\alpha}_{G} + \cdots \end{cases},$$

$$\tilde{\alpha}_{33} \simeq \begin{cases} 3.33 + 0.05\tilde{\alpha}_{1\phi} + 0.21\tilde{\alpha}_{2\phi} - 0.02\tilde{\alpha}_{\Sigma} + \cdots \\ 3.40 + 0.05\tilde{\alpha}_{1\phi} - 1.63\tilde{\alpha}_{2\phi} - 0.001\tilde{\alpha}_{G} + \cdots \end{cases},$$

$$\tilde{\alpha}_{3\phi} \simeq \begin{cases} 1.43 - 0.58\tilde{\alpha}_{1\phi} - 1.43\tilde{\alpha}_{2\phi} - 0.03\tilde{\alpha}_{\Sigma} + \cdots \\ 0.88 - 0.48\tilde{\alpha}_{1\phi} + 8.83\tilde{\alpha}_{2\phi} + 0.01\tilde{\alpha}_{G} + \cdots \end{cases},$$

$$\tilde{\alpha}_H \simeq \begin{cases} 1.08 - 0.03\tilde{\alpha}_{1\phi} + 0.10\tilde{\alpha}_{2\phi} - 0.07\tilde{\alpha}_\Sigma + \cdots \\ 2.51 - 0.04\tilde{\alpha}_{1\phi} - 1.68\tilde{\alpha}_{2\phi} - 0.12\tilde{\alpha}_G + \cdots \end{cases} ,$$

$$\tilde{\alpha}_\Sigma \simeq \begin{cases} --- \\ 0.40 + 0.01\tilde{\alpha}_{1\phi} - 0.45\tilde{\alpha}_{2\phi} - 0.10\tilde{\alpha}_G + \cdots \end{cases} ,$$

$$\tilde{\alpha}_{\Sigma'} \simeq \begin{cases} 4.91 - 0.001\tilde{\alpha}_{1\phi} - 0.03\tilde{\alpha}_{2\phi} - 0.46\tilde{\alpha}_\Sigma + \cdots \\ 8.30 + 0.01\tilde{\alpha}_{1\phi} + 1.72\tilde{\alpha}_{2\phi} - 0.36\tilde{\alpha}_G + \cdots \end{cases} ,$$

$$\tilde{\alpha}_G \simeq \begin{cases} 5.59 + 0.02\tilde{\alpha}_{1\phi} - 0.04\tilde{\alpha}_{2\phi} - 1.33\tilde{\alpha}_\Sigma + \cdots \\ --- \end{cases} . \tag{73}$$

We have assumed that the Yukawa couplings g_{IJ} except for g_{33} vanish. They can be included into RGI relations as small perturbations, but their numerical effects will be rather small.

The number N_H of the Higgses lighter than $M_{\rm SUSY}$ could vary from one to four while the number of those to be taken into account above $M_{\rm SUSY}$ is fixed at four. We have assumed here that $N_H = 1$. The dependence of the top mass on $M_{\rm SUSY}$ in this model is shown in Figure 2.

9. Asymptotically non-free SO(10) model

We will show in this section a model based on $SO(10)$ in which also the reduction of couplings can be applied [14].

We denote the hermitean SO(10)-gamma matrices by Γ_α , $\alpha = 1, \cdots, 10$. The charge conjugation matrix C satisfies $C = C^{-1}$, $C^{-1}\,\Gamma_\alpha^T\, C = -\Gamma_\alpha$, and the Γ_{11} is defined as $\Gamma_{11} \equiv (-i)^5\,\Pi_{\alpha=1}^{10}\Gamma_\alpha$ with $(\Gamma_{11})^2 = 1$. The chiral projection operators are given by $\mathcal{P}_\pm = \frac{1}{2}(\,1 \pm \Gamma_{11})$.

In SO(10) GUTs [3, 58], three generations of quarks and leptons are accommodated by three chiral supermultiplets in **16** which we denote by

$$\Psi^I(\mathbf{16}) \ \text{ with } \ \mathcal{P}_+\Psi^I = \Psi^I , \tag{74}$$

where I runs over the three generations and the spinor index is suppressed. To break SO(10) down to $\rm SU(3)_C{\times}SU(2)_L{\times}U(1)_Y$, we use the following set of chiral superfields:

$$S_{\{\alpha\beta\}}(\mathbf{54}) \ , \ A_{[\alpha\beta]}(\mathbf{45}) \ , \ \phi(\mathbf{16}) \ , \ \overline{\phi}(\overline{\mathbf{16}}) \ . \tag{75}$$

The two $\rm SU(2)_L$ doublets which are responsible for the spontaneous symmetry breaking (SSB) of $\rm SU(2)_L{\times}U(1)_Y$ down to $\rm U(1)_{EM}$ are contained in $H_\alpha(\mathbf{10})$. We further introduce a singlet φ which after the SSB of SO(10) will mix with the right-handed neutrinos so that they will become superheavy.

The superpotential of the model is given by

$$W = W_Y + W_{SB} + W_{HS} + W_{NM} + W_M \ , \tag{76}$$

where

$$
\begin{aligned}
W_Y &= \frac{1}{2}\sum_{I,J=1}^{3} g_{IJ}\,\Psi^I C\Gamma_\alpha \Psi^J\; H_\alpha\ ,\\
W_{SB} &= \frac{g_\phi}{2}\,\overline{\phi}\,\Gamma_{[\alpha\beta]}\,\phi\; A_{[\alpha\beta]} + \frac{g_S}{3!}\,\mathrm{Tr}\; S^3 + \frac{g_A}{2}\,\mathrm{Tr}\; A^2 S\ ,\\
W_{HS} &= \frac{g_{HS}}{2}\; H_\alpha\, S_{\{\alpha\beta\}}\, H_\beta\ ,\; W^I_{NM} \;=\; \sum_{I=1}^{3} g_{INM}\,\Psi^I\,\overline{\phi}\,\varphi\ ,\\
W_M &= \frac{m_H}{2}\,H^2 + m_\varphi\,\varphi^2 + m_\phi\,\overline{\phi}\phi + \frac{m_S}{2}\,S^2 + \frac{m_A}{2}\,A^2\ ,
\end{aligned}
\tag{77}
$$

and $\Gamma_{[\alpha\beta]} = i(\Gamma_\alpha\Gamma_\beta - \Gamma_\beta\Gamma_\alpha)/2$. As in the case of the SU(5) minimal model, the superpotential is not the most general one, but this does not contradict the philosophy of the coupling unification by the reduction method. W_{SB} is responsible for the SSB of SO(10) down to $\mathrm{SU(3)}_C\times\mathrm{SU(2)}_W\times\mathrm{U(1)}_Y$, and this can be achieved without breaking supersymmetry, while W_{HS} is responsible for the triplet-doublet splitting of H. The right-handed neutrinos obtain a superheavy mass through W_{NM} after the SSB, and the Yukawa couplings for the leptons and quarks are contained in W_Y. We assume that there exists a choice of soft supersymmetry breaking terms so that all the vacuum expectation values necessary for the desired SSB corresponds to the minimum of the potential.

Given the supermultiplet content and the superpotential W, we can compute the β functions of the model. The gauge coupling of SO(10) is denoted by g, and our normalization of the β functions is as usual, *i.e.*, $dg_i/d\ln\mu \;=\; \beta_i^{(1)}/16\pi^2 + O(g^5)$, where μ is the renormalization scale. We find:

$$
\begin{aligned}
\beta_g^{(1)} &= 7\,g^3\ ,\\
\beta_{g_T}^{(1)} &= g_T\,(\,14|g_T|^2 + \frac{27}{5}|g_{HS}|^2 + |g_{3NM}|^2 - \frac{63}{2}g^2\,)\ ,\\
\beta_{g_\phi}^{(1)} &= g_\phi(\,53|g_\phi|^2 + \frac{48}{5}|g_A|^2 + \frac{1}{2}|g_{1NM}|^2 + \frac{1}{2}|g_{2NM}|^2 + \frac{1}{2}|g_{3NM}|^2 - \frac{77}{2}g^2\,),\\
\beta_S^{(1)} &= g_S(\,\frac{84}{5}|g_S|^2 + 12|g_A|^2 + \frac{3}{2}|g_{HS}|^2 - 60g^2\,)\ ,\\
\beta_A^{(1)} &= g_A(\,16|g_\phi|^2 + \frac{28}{5}|g_S|^2 + \frac{116}{5}|g_A|^2 + \frac{1}{2}|g_{HS}|^2 - 52g^2\,)\ ,\\
\beta_{HS}^{(1)} &= g_{HS}(\,8|g_T|^2 + \frac{28}{5}|g_S|^2 + 4|g_A|^2 + \frac{113}{10}|g_{HS}|^2 - 38g^2\,)\ ,\\
\beta_{1NM}^{(1)} &= g_{1NM}(\,\frac{45}{2}|g_\phi|^2 + 9|g_{1NM}|^2 + \frac{17}{2}|g_{2NM}|^2 + \frac{17}{2}|g_{3NM}|^2 - \frac{45}{2}g^2\,)\ ,
\end{aligned}
$$

$$\beta^{(1)}_{2NM}=g_{2NM}(\frac{45}{2}|g_\phi|^2+\frac{17}{2}|g_{1NM}|^2+9|g_{2NM}|^2+\frac{17}{2}|g_{3NM}|^2-\frac{45}{2}g^2)\,,$$
$$\beta^{(1)}_{3NM}=g_{3NM}(5|g_T|^2+\frac{45}{2}|g_\phi|^2+\frac{17}{2}|g_{1NM}|^2+\frac{17}{2}|g_{2NM}|^2$$
$$+9|g_{3NM}|^2-\frac{45}{2}g^2)\,. \qquad (78)$$

We have assumed that the Yukawa couplings g_{IJ} except for $g_T \equiv g_{33}$ vanish. They can be included as small perturbations. Needless to say that the soft susy breaking terms do not alter the β functions above.

We find that there exist two independent solutions, A and B, that have the most predictive power, where we have chosen the SO(10) gauge coupling as the primary coupling:

$$\rho_T = \begin{cases} 163/60 & \simeq 2.717 \\ 0 \end{cases},\quad \rho_\phi = \begin{cases} 5351/9180 & \simeq 0.583 \\ 1589/2727 & \simeq 0.583 \end{cases},$$
$$\rho_S = \begin{cases} 152335/51408 & \simeq 2.963 \\ 850135/305424 & \simeq 2.783 \end{cases},\rho_A = \begin{cases} 31373/22032 & \simeq 1.424 \\ 186415/130896 & \simeq 1.424 \end{cases},$$
$$\rho_{HS} = \begin{cases} 7/81 & \simeq 0.086 \\ 170/81 & \simeq 2.099 \end{cases},\ \rho_{1NM}=\rho_{2NM}=\begin{cases} 191/204 & \simeq 0.936 \\ 191/303 & \simeq 0.630 \end{cases},$$
$$\rho_{3NM} = \begin{cases} 0 \\ 191/303 & \simeq 0.630 \end{cases} \quad \text{for} \begin{cases} A \\ B \end{cases}. \qquad (79)$$

Clearly, the solution B has less predictive power because $\rho_T = 0$. So, we consider below only the solution A, in which the coupling α_{3NM} should be regarded as a small perturbation because $\rho_{3NM} = 0$.

Given this solution it is possible to show, as in the case of SU(5), that the ρ's can be uniquely computed in any finite order in perturbation theory.

The corrections to the reduced couplings coming from the small perturbations up to and including terms of $O(\tilde{\alpha}^2_{3NM})$:

$$\tilde{\alpha}_T=(163/60-0.108\cdots\tilde{\alpha}_{3NM}+0.482\cdots\tilde{\alpha}^2_{3NM}+\cdots)+\cdots\,,$$
$$\tilde{\alpha}_\phi=(5351/9180+0.316\cdots\tilde{\alpha}_{3NM}+0.857\cdots\tilde{\alpha}^2_{3NM}+\cdots)+\cdots\,,$$
$$\tilde{\alpha}_S=(152335/51408+0.573\cdots\tilde{\alpha}_{3NM}+5.7504\cdots\tilde{\alpha}^2_{3NM}+\cdots)+\cdots\,,$$
$$\tilde{\alpha}_A=(31373/22032-0.591\cdots\tilde{\alpha}_{3NM}-4.832\cdots\tilde{\alpha}^2_{3NM}+\cdots)+\cdots\,,$$
$$\tilde{\alpha}_{HS}=(7/81-0.00017\cdots\tilde{\alpha}_{3NM}+0.056\cdots\tilde{\alpha}^2_{3NM}+\cdots)+\cdots\,,$$
$$\tilde{\alpha}_{1NM}=\tilde{\alpha}_{2NM}=(191/204-4.473\cdots\tilde{\alpha}_{3NM}+2.831\cdots\tilde{\alpha}^2_{3NM}+\cdots)+\cdots, \qquad (80)$$

where $\cdots$ indicates higher order terms which can be uniquely computed. In the partially reduced theory defined above, we have two independent couplings, α and α_{3NM} (along with the Yukawa couplings α_{IJ} , $I,J \neq T$).

At the one-loop level, Eq. (80) defines a line parametrized by $\tilde{\alpha}_{3NM}$ in the 7 dimensional space of couplings. A numerical analysis shows that this line blows up in the direction of $\tilde{\alpha}_S$ at a finite value of $\tilde{\alpha}_{3NM}$ [14]. So if we require $\tilde{\alpha}_S$ to remain within the perturbative regime (*i.e.*, $g_S \lesssim 2$, which means $\tilde{\alpha}_S \lesssim 8$ because $\alpha_{\rm GUT} \sim 0.04$), the $\tilde{\alpha}_{3NM}$ should be restricted to be below ~ 0.067. As a consequence, the value of $\tilde{\alpha}_T$ is also bounded

$$2.714 \lesssim \tilde{\alpha}_T \lesssim 2.736 \ . \tag{81}$$

This defines GYU boundary conditions holding at the unification scale $M_{\rm GUT}$ in addition to the group theoretic one, $\alpha_T = \alpha_t = \alpha_b = \alpha_\tau$. The value of $\tilde{\alpha}_T$ is practically fixed so that we may assume that $\tilde{\alpha}_T = 163/60 \simeq 2.72$, which is the unperturbed value.

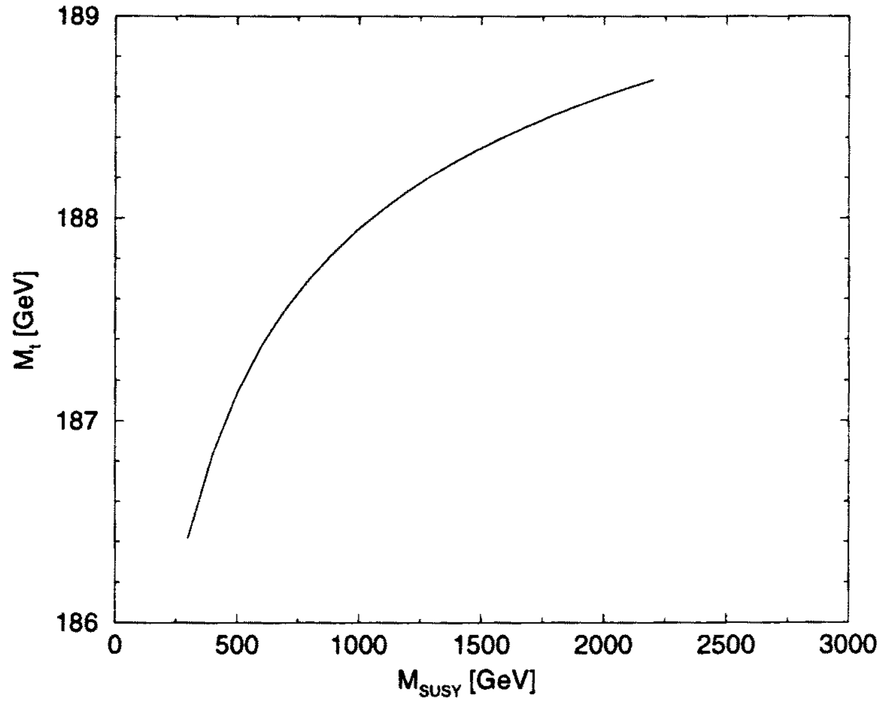

Fig. 3. M_t prediction versus $M_{\rm SUSY}$ for $\tilde{\alpha}_T = 2.717$.

Figure 3 shows the prediction for M_t in this model for different values of the supersymmetry breaking scale $M_{\rm SUSY}$. It is worth noticing that the value for M_t predicted is below its infrared value ($M_{\rm top-IR} \sim 189$ GeV) [14], but it is slightly above the recent experimental values (70).

10. Conclusions

As a natural extension of the unification of gauge couplings provided by all GUTs and the unification of Yukawa couplings, we have introduced the idea of Gauge–Yukawa Unification. GYU is a functional relationship among the gauge and Yukawa couplings provided by some principle. In our studies GYU has been achieved by applying the principles of reduction of couplings and finiteness. The consequence of GYU is that in the lowest order in perturbation theory the gauge and Yukawa couplings above $M_{\rm GUT}$ are related in the form

$$g_i = \kappa_i \, g_{\rm GUT} \, , \quad i = 1, 2, 3, e, \cdots, \tau, b, t \, , \tag{82}$$

where g_i $(i = 1, \cdots, t)$ stand for the gauge and Yukawa couplings, $g_{\rm GUT}$ is the unified coupling, and we have neglected the Cabibbo–Kobayashi–Maskawa mixing of the quarks. So, Eq. (82) exhibits a set of boundary conditions on the renormalization group evolution for the effective theory below $M_{\rm GUT}$, which we have assumed to be the MSSM. We have shown [15, 16] that it is possible to construct some supersymmetric GUTs with GYU in the third generation that can predict the bottom and top quark masses in accordance with the recent experimental data [53]. This means that the top-bottom hierarchy could be explained in these models, in a similar way as the hierarchy of the gauge couplings of the SM can be explained if one assumes the existence of a unifying gauge symmetry at $M_{\rm GUT}$.

It is clear that the GYU scenario is the most predictive scheme as far as the mass of the top quark is concerned. It may be worth recalling the predictions for M_t of ordinary GUTs, in particular of supersymmetric SU(5) and SO(10). The MSSM with SU(5) Yukawa boundary unification allows M_t to be anywhere in the interval between 100-200 GeV for varying $\tan\beta$, which is now a free parameter. Similarly, the MSSM with SO(10) Yukawa boundary conditions, *i.e.* $t - b - \tau$ Yukawa Unification gives M_t in the interval 160-200 GeV. We have analyzed [16] the infrared quasi-fixed-point behaviour of the M_t prediction in some detail. In particular we have seen that the *infrared value* for large $\tan\beta$ depends on $\tan\beta$ and its lowest value is $\sim$ 188 GeV. Comparing this with the experimental value (70) we may conclude that the present data on M_t cannot be explained from the infrared quasi-fixed-point behaviour alone (see Figure 4).

Clearly, to exclude or verify different GYU models, the experimental as well as theoretical uncertainties have to be further reduced. One of the largest theoretical uncertainties in FUT results from the not-yet-calculated threshold effects of the superheavy particles. Since the structure of the superheavy particles is basically fixed, it will be possible to bring these threshold effects under control, which will reduce the uncertainty of the

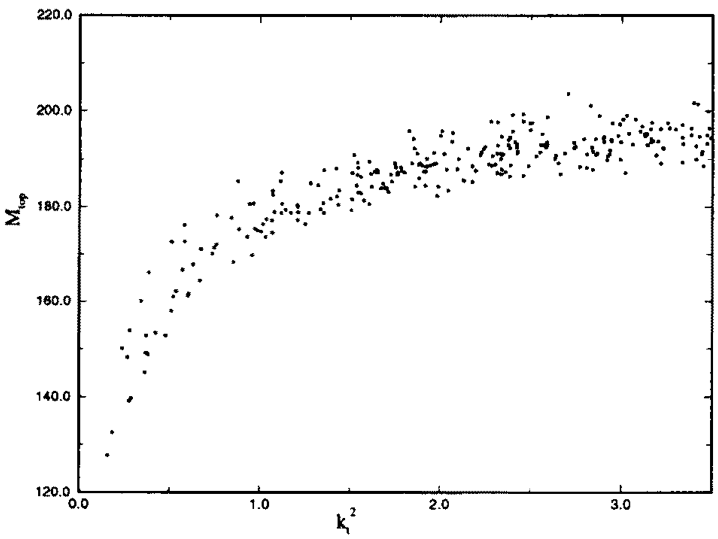

Fig. 4. The dependence of the top mass M_t with k_t^2, at fixed $M_{\rm SUSY} = 500$ GeV. As we can see, after $k_t^2 \sim 2.0$ the top mass goes to its infrared fixed point value.

M_t prediction. We have been regarding $\delta^{\rm MSSM} M_t$ as unknown because we do not have sufficient information on the superpartner spectra. Recently, however, we have demonstrated [63] how to extend the principle of reduction of couplings in a way as to include the dimensionfull parameteres. As a result, it is in principle possible to predict the superpartner spectra as well as the rest of the massive parameters of a theory.

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5.5 Constraints on finite soft supersymmetry-breaking terms

Title: Constraints on finite soft supersymmetry-breaking terms.
Authors: T. Kobayashi, J. Kubo, M. Mondragon, G. Zoupanos
Journal: Nucl. Phys. **B511** (1998) 45-68

Comment (*Myriam Mondragón, George Zoupanos*)
This is one of the most important and complete papers written on the subject of Finite Unified Theories and their predictions. An important point is that a new $N = 1$ Finite $SU(5)$ model was suggested, which (a) is more economical in the number of free parameters as compared to the original discussed in subsection 5.1 (it contains two instead of three parameters in its SSB), and (b) the new Finite model gives more accurate predictions for the top and bottom quark masses as seen today. At the time both Finite $SU(5)$ models were consistent with experimental data, but in a more recent analysis that will be presented in subsection 5.8 only a version of the second one survives in the comparison with the updated top and bottom quark mass measurements.

Another important issue discussed in the present paper concerns the "sum rule" for the soft scalar masses at two loops. To be more specific a number of problems appeared in finite unified theories using the attractive "universal" set of soft scalar masses. For instance, (i) the universality predicted that the lightest supersymmetric particle was a charged particle, namely the superpartner of the τ lepton $\tilde{\tau}$, (ii) the MSSM with universal soft scalar masses was inconsistent with the standard radiative electroweak symmetry breaking, and (iii) which is the worst of all, the universal soft scalar masses lead to charge and/or colour breaking minima deeper than the standard vacuum. Naturally there have been attempts to relax this constraint. First an interesting observation was made that in a general $N = 1$ gauge-Yukawa unified (GYU) theories there exists a RGI "sum rule" for the soft scalar masses at one-loop, which obviously holds for the finite theories too. In the present paper it was found that in finite theories the "sum rule" remains RGI at two-loops with the surprising result that it does not change from its one-loop form, i.e. it does not receive two-loop corrections. In addition, some interesting remarks were done concerning the relation of the sum rule to certain string models.

Eventually in the present paper it was presented a complete analysis of the two Finite Unifite $SU(5)$ theories and their phenomenological consequences. The MSSM with the finiteness boundary conditions at the unification scale was examined by studying the evolution of the dimensionless parameters at two loops and the dimensionful at one loop. As a result it was determined the parameter space that was safe of the various phenomenological problems mentioned above and was predicted the supersymmetric spectrum and the Higgs masses. This analysis was the basis for the more detailed and updated one that will be discussed in the subsection 5.8.

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Nuclear Physics B 511 (1998) 45–68

NUCLEAR PHYSICS B

Constraints on finite soft supersymmetry-breaking terms

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Abstract

Requiring the soft supersymmetry-breaking (SSB) parameters in finite gauge-Yukawa unified models to be finite up to and including two-loop order, we derive a two-loop sum rule for the soft scalar masses. It is shown that this sum rule coincides with that of a certain class of string models in which the massive string states are organized into $N = 4$ supermultiplets. We investigate the SSB sector of two finite $SU(5)$ models. Using the sum rule which allows non-universality of the SSB terms and requiring that the lightest superparticle is neutral, we constrain the parameter space of the SSB sector in each model. © 1998 Elsevier Science B.V.

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1. Introduction

The standard model (SM) has a large number of free parameters whose values are determined only experimentally. To reduce the number of these free parameters, and thus render it more predictive, one is usually led to enlarge the symmetry of the SM. For instance, unification of the SM forces based on the $SU(5)$ GUT [1] predicted one of the gauge couplings [1] as well as the mass of the bottom quark [2]. Now it seems that LEP data is suggesting that the symmetry of the unified theory should be further enlarged and become $N = 1$ globally supersymmetric [3].

Relations among gauge and Yukawa couplings, which are missing in ordinary GUTs, could be a consequence of a further unification provided by a more fundamental theory at the Planck scale. Moreover, it might be possible that some of these relations are renormalization group invariant (RGI) below the Planck scale so that they are exactly preserved down to the GUT scale $M_{\rm GUT}$. In fact, one of our motivations in this paper is to point out such indication in the soft supersymmetry-breaking (SSB) sector in supersymmetric unified theories.

In our recent studies [4–6], we have been searching for RGI relations among gauge and Yukawa couplings in various supersymmetric GUTs. Thus, the idea of gauge–Yukawa unification (GYU) [4–6] relies not only on a symmetry principle, but also on the principle of reduction of couplings [7,8] (see also Ref. [9]). This principle is based on the existence of RGI relations among couplings, which do not necessarily result from a symmetry, but nevertheless preserve perturbative renormalizability or even finiteness. Here we would like to focus on finite unified theories [10–21,4,6].

Supersymmetry seems to be essential for a successful GYU, but, as it is for any realistic supersymmetric model, the breaking of supersymmetry has to be understood. We recall that the SSB parameters have dimensions greater than or equal to one and it is possible to treat dimensional couplings along the line of GYU [22,23], which shows that the SSB sector of a GYU model is controlled by the unified gaugino mass M. As for one- and two-loop finite SSB terms, only the universal solution for the SSB terms [10,19] is known so far. So another motivation of this paper is to re-investigate the conditions for the two-loop finite SSB terms and to express them in terms of simple sum rules for these parameters. We will indeed find that the universal solution can be relaxed for the SSB terms to be finite up to and including the two-loop corrections, and we will derive the two-loop corrected sum rule for the soft scalar masses. We will comment on the possibility of all-order-finite SSB terms.

The authors of Refs. [25,26,23] have pointed out that the universal soft scalar masses also appear for dilaton-dominated supersymmetry breaking in 4D superstring models [27–29]. Ibánez [25] (see also Ref. [26]) gives a possible superstring interpretation to it. We shall examine whether or not the two-loop corrected sum rule can also be obtained in some string model. We will indeed find that there is a class of 4D orbifold models in which exactly the same sum rule is satisfied. It may be worth-mentioning that not only in finite GYU models, but also in non-finite GYU models the same soft scalar-mass sum rule is satisfied at the one-loop level [30]. In Ref. [30] a possible answer to why this happens is speculated.

Motivated by the fact that the universal choice for the SSB terms can be relaxed, we will investigate the SSB sector of two finite $SU(5)$ models. The SSB parameters of these models are constrained by the sum rule and also by the requirement that the electroweak gauge symmetry is radiatively broken [31]. We will find that there is a parameter range for each model in which the lightest superparticle (LSP) is a neutralino, which will be compared with the case of the universal SSB parameters. The lightest Higgs turns out to be ~ 120 GeV.

2. Two-loop finiteness and Soft scalar-mass sum rule

2.1. Two-loop finite SSB terms

Various groups [24,19] have independently computed the coefficients of the two-loop RG functions for SSB parameters. [4] Here we would like to use them to re-investigate their two-loop finiteness and derive the two-loop soft scalar-mass sum rule.

The superpotential is

$$W = \tfrac{1}{6}Y^{ijk}\Phi_i\Phi_j\Phi_k + \tfrac{1}{2}\mu^{ij}\Phi_i\Phi_j, \tag{1}$$

along with the Lagrangian for SSB terms,

$$-\mathcal{L}_{\rm SB} = \tfrac{1}{6}h^{ijk}\phi_i\phi_j\phi_k + \tfrac{1}{2}b^{ij}\phi_i\phi_j + \tfrac{1}{2}(m^2)^j_i\phi^{*i}\phi_j + \tfrac{1}{2}M\lambda\lambda + \text{H.c.} \tag{2}$$

Since we would like to consider only finite theories here, we assume that the gauge group is a simple group and the one-loop β function of the gauge coupling g (A.1) vanishes, i.e.

$$b \equiv T(R) - 3C(G) = 0. \tag{3}$$

We also assume that the reduction equation

$$\beta_Y^{ijk} = \beta_g \frac{dY^{ijk}}{dg} \tag{4}$$

admits power series solutions of the form

$$Y^{ijk} = g\sum_{n=0} \rho^{ijk}_{(n)} g^{2n}, \tag{5}$$

where β_g and β_Y^{ijk} are β functions of g and Y^{ijk}, respectively. According to the finiteness theorem of Ref. [17], the theory is then finite [5] to all orders in perturbation theory, if the one-loop anomalous dimensions $\gamma_i^{(1)j}$ given in (A.2) vanish, i.e. if

[4] The RG functions [11,12,24,23,19] are given in Appendix A for completeness.

[5] Finiteness here means only for dimensionless couplings, i.e. g and Y^{ijk}.

$$\frac{1}{2}\sum_{p,q}\rho_{ipq(0)}\rho^{jpq}_{(0)} - 2\delta^j_i C(i) = 0 \tag{6}$$

is satisfied, where we have inserted Y^{ijk} in (4) into $\gamma_i^{(1)j}$. We recall that if the conditions (3) and (6) are satisfied, the two-loop expansion coefficients in (5), $\rho^{ijk}_{(1)}$, vanish [19]. (From (A.6) ad (A.7) we see that the two-loop coefficients $\beta_g^{(2)}$ and $\gamma_j^{(2)i}$ vanish if $\beta_g^{(1)}$ and $\gamma_j^{(1)i}$ vanish.) Further, the one- and two-loop finiteness for h^{ijk} can be achieved by [11,19]

$$h^{ijk} = -MY^{ijk} + \ldots = -M\rho^{ijk}_{(0)}g + O(g^5), \tag{7}$$

which can be seen from (A.9) if one uses Eq. (6). Note further that the $O(g^3)$ term is absent in (7). As for b^{ij} there is no constraint; b^{ij} is finite if Eqs. (6) and (7) are satisfied, which can be seen from the one- and two-loop coefficients of the β function for b^{ij}(A.5) and (A.10).

Now, to obtain the two-loop sum rule for soft scalar masses, we assume that the lowest order coefficients $\rho^{ijk}_{(0)}$ and also $(m^2)^i_j$ satisfy the diagonality relations

$$\rho_{ipq(0)}\rho^{jpq}_{(0)} \propto \delta^j_i \text{ for all } p \text{ and } q \text{ and } (m^2)^i_j = m^2_j\delta^i_j, \tag{8}$$

respectively. Then one finds that

$$\begin{aligned}[\beta^{(1)}_{m^2}]^j_i &= \rho_{ipq(0)}\rho^{jpq}_{(0)}(m^2_i/2 + m^2_j/2 + m^2_p + m^2_q)g^2 \\ &+(\rho_{ipq(0)}\rho^{jpq}_{(0)} - 8\delta^j_i C(i))MM^\dagger g^2 + O(g^6),\end{aligned} \tag{9}$$

where we have used $\rho^{jpq}_{(1)} = 0$ (which implies that the $O(g^4)$ term in (9) is absent). Using the condition (6), the diagonality relations (8) and also the soft scalar-mass sum rule (which we are going to prove)

$$\frac{m^2_i + m^2_j + m^2_k}{MM^\dagger} = 1 + \frac{g^2}{16\pi^2}\Delta^{(1)} + O(g^4) \quad \text{for } i,j,k \text{ with } \rho^{ijk}_{(0)} \neq 0, \tag{10}$$

we find that Eq. (9) can be written as

$$[\beta^{(1)}_{m^2}]^j_i = 4\delta^j_i MM^\dagger C(i)\Delta^{(1)}\frac{g^4}{16\pi^2} + O(g^6). \tag{11}$$

We will find shortly that the two-loop correction term $\Delta^{(1)}$ is given by

$$\Delta^{(1)} = -2\sum_l[(m^2_l/MM^\dagger) - \tfrac{1}{3}]T(R_l). \tag{12}$$

Therefore, the $\Delta^{(1)}$ vanishes for the universal choice

$$m^2_i = \kappa_i MM^\dagger \text{with} \kappa_i = \tfrac{1}{3} \text{ for all } i, \tag{13}$$

in accord with the previous findings of Ref. [19]. The result agrees also with that of Ref. [10] on $N = 4$ theory; $N = 4$ theory contains three $N = 1$ chiral superfields in the adjoint representation, which means $T(R_i) = C(G)$ $(i = 1,2,3)$. If $\kappa_1 + \kappa_2 + \kappa_1 = 1$ is satisfied, we obtain

$$\Delta^{(1)}(N=4) = -2\sum_{l=1}^{3}[\kappa_l - \tfrac{1}{3}]C(G) = 0. \tag{14}$$

To see that $\Delta^{(1)}$ is really given by Eq. (12) for two-loop finiteness of m_i^2, we recall that the two-loop β function for m_i^2 (A.11) can be nicely organized as [23]

$$\begin{aligned}[\beta_{m^2}^{(2)}]_i^j = (&A_{(\gamma)in}^{jp}\gamma_p^{(1)n} + A_{(m^2)in}^{jp}[\beta_{m^2}^{(1)}]_p^n + A_{(g)i}^{j}\beta_g^{(1)} \\ &+A_{(h)in}^{jp}[h^{nrq}Y_{lrq} + 4M\delta_p^n g^2 C(n)] + 4g^4 C(i) S' M M^\dagger \delta_i^j) + \text{H.c.},\end{aligned} \tag{15}$$

where

$$\begin{aligned}S' &= \sum_l (m_l^2/MM^\dagger)T(R_l) - C(G) \\ &= \sum_l [(m_l^2/MM^\dagger) - \tfrac{1}{3}]T(R_l) \quad \text{for } \sum_l T(R_l) = 3C(G),\end{aligned} \tag{16}$$

and the coefficients A are given in (A.11). Using the one-loop finiteness conditions (which are ensured by Eqs. (3), (6), (7) and (10)), we finally obtain

$$[\beta_{m^2}^{(2)}]_i^j = +8g^4 C(i) M M^\dagger S' \delta_i^j. \tag{17}$$

It is now easy to see that this term can be canceled by the $O(g^4)$ contribution to $[\beta_{m^2}^{(1)}]_i^j$ (which is given in (11)) if $\Delta^{(1)}$ is exactly given by Eq. (12). Note that we have not shown that the sum rule (10) is the unique solution for $[\beta_{m^2}^{(2)}]_i^j$. That is, we have only shown that the sum rule (10) is a solution to

$$\rho_{ipq(0)}\rho_{(0)}^{jpq}\left[\frac{m_i^2 + m_p^2 + m_q^2}{|M|^2} - 1\right] = -8S'\delta_i^j C(i), \tag{18}$$

but not in the opposite way. The question of whether the sum rule is the unique solution to (18) depends on the concrete model of course. We will address the question when discussing concrete finite models and find that the sum rule (10) is the unique solution for these models.

Since S' will be of $O(C(G))$, the two-loop correction term in the sum rule (10) may be estimated as

$$\frac{g^2}{16\pi^2}\Delta^{(1)} \sim \frac{\alpha_{\rm GUT}}{\pi}C(G). \tag{19}$$

If, however, the soft scalar masses are close to the universal one (13), the correction is small. In the concrete example of the $SU(5)$ finite models which we will consider below, it will turn out that the soft scalar masses should differ from the universal one if we require that the LSP is a neutralino. But the two-loop correction term $\Delta^{(1)}$ happens to vanish exactly, no matter how large the deviation from the universal choice of the soft scalar masses is.

2.2. Coincidence

It has been known [23,25,26] that the universal soft scalar masses which preserve their two-loop finiteness also appear for dilaton-dominated supersymmetry breaking in 4D superstring models [27-29]. Ibánez [25] (see also Ref. [26]) gives a possible superstring interpretation and argues that for dilaton dominance to work, the soft SSB terms have to be independent of the particular choice of compactification and consistent with any possible compactification, in particular with a toroidal compactification preserving $N = 4$ supersymmetry. Given that the universality of the soft scalar masses can be relaxed (as we have shown above), we would like to examine whether or not the two-loop corrected sum rule (10) can also be obtained in some string model. To this end, we consider a specific class of orbifold models with three untwisted moduli T_1, T_2, T_3 (which exist for instance in (0,2) symmetric abelian orbifold construction always). We then assume that some non-perturbative superpotential which breaks supersymmetry exists and that the dilation S and the moduli T_a play a dominant role for supersymmetry breaking. The Kähler potential K and the gauge kinetic function f in this case assume the generic form

$$K = -\ln(S+S^*) - \sum_{a=1}^{3} \ln(T_a+T_a^*) + \sum_i \Pi_{a=1}^3 (T_a+T_a^*)^{n_i^a} |\Phi_i|^2, \quad f = kS, \tag{20}$$

where the n_i^a stand for modular weights and are fractional numbers, and k is the Kac-Moody level [32-34]. The SSB parameters [6] in this class of models are given by [29,36-38,25,26]

$$M = \sqrt{3} m_{3/2} \sin\theta, \qquad m_i^2 = m_{3/2}^2 \left(1 + 3\cos^2\theta \sum_{a=1}^{3} n_i^a \Theta_a^2 \right), \tag{21}$$

$$h^{ijk} = -\sqrt{3} Y^{ijk} m_{3/2} \left[\sin\theta + \cos\theta \sum_{a=1}^{3} \Theta_a (u^a + n_i^a + n_j^a + n_k^a) \right], \tag{22}$$

where θ and Θ_a (which parametrize the unknown mechanism of supersymmetry breaking [29]) are defined as $F^S/Y = \sqrt{3} m_{3/2} \sin\theta$ and $F^{T_a}/(T_a+T_a^*) = \sqrt{3} m_{3/2} \cos\theta \Theta_a$ with $\sum_{a=1}^3 \Theta_a^2 = 1$. In Eq. (22) we have assumed Y^{ijk} to be independent of S and T_a. It is straightforward to see that the tree-level form of the sum rule (10) [29,37,25,26,39] [7] is satisfied, if

$$\mathbf{n}_i + \mathbf{n}_j + \mathbf{n}_k = -\mathbf{u} \equiv -(1,1,1). \tag{23}$$

Note that the condition (23) ensures that $K + \ln|W|^2$ is invariant under the duality transformation

[6] Since the SSB parameters b^{ij} are not constrained by two-loop finiteness, we do not consider them here.
[7] We call the soft scalar-mass sum rule (10) without the two-loop correction term the tree-level sum rule.

$$T_a \longrightarrow \frac{a_a T_a - i b_a}{i c_a T_a + d_a}, \tag{24}$$

where a_a, b_a, c_a and d_a are integers satisfying $a_a d_a - b_a c_a = 1$. The Kähler potential K (20) belongs to the general class of the Kähler potentials that lead to the tree-level sum rule [30]. When gauge symmetries break, we generally have D-term contributions to the soft scalar masses. Such D-term contributions, however, do not appear in the sum rule, because each D-term contribution is proportional to the charge of the matter field Φ_i [40].

We now would like to extend our discussion so as to include the two-loop correction in the sum rule (10). In superstrings, the correction to the tree-level relations among the SSB terms can be computed by using the fact that the target-space modular anomaly [41,42,27] is canceled by the Green–Schwarz mechanism [43] and from the threshold correction coming from the massive sates [44,45]. The Green–Schwarz mechanism induces a non-trivial transformation of S under the duality transformation, which implies that the Kähler potential for the dilaton S has to be modified to the duality-invariant Kähler potential [41,27],

$$-\ln Y, \qquad Y \equiv S + S^* - \sum_{a=1}^{3} \frac{\delta^a_{\rm GS}}{8\pi^2} \ln(T_a + T_a^*), \tag{25}$$

where $\delta^a_{\rm GS}$ is the Green–Schwarz coefficient [41,27]. This correction alters the tree-level formulae for h^{ijk} and m_i^2, while the threshold correction coming from the massive sates modifies the tree-level gauge kinetic function $f = S$ and hence changes the tree-level formula for the gaugino mass M. The requirement of the vanishing cosmological constant leads to the redefinition of the Goldstino parameters [36–38] as

$$\frac{1}{Y}\left(F^S - \sum_a \frac{\delta^a_{\rm GS}/8\pi^2}{T_a + T_a^*}\right) F^{T_a} = \sqrt{3} m_{3/2} \sin\theta, \tag{26}$$

$$\frac{F^{T_a}}{T_a + T_a^*} = \sqrt{3} m_{3/2} \cos\theta \tilde{\Theta}_a, \tag{27}$$

where

$$\tilde{\Theta}_a = (1 - \delta^a_{\rm GS}/Y 8\pi^2)^{-1/2} \Theta_a, \tag{28}$$

and Θ_a is defined in (22). Note that the quantum modification (27) does not change the tree-level relation for h^{ijk} (22) at all, which coincides with the two-loop result (7). This motivates us to assume that the relation for M also remains unchanged, which is true only if the contribution to the gauge kinetic function f coming from the massive states [45] are absent. It is known [45] (see also Ref. [27]) that such situation appears for the class of orbifold models in which the massive states are organized into $N = 4$ supermultiplets,[8] and one can easily convince oneself that if the condition (23) is

[8] The absence of the threshold effects coming from $N = 4$ massive supermultiplets has been first observed in an $N = 4$ Yang–Mills theory with spontaneously broken gauge symmetry [46].

satisfied, the tree-level sum rule for m_i^2 is modified to

$$\frac{m_i^2+m_j^2+m_k^2}{|M|^2}-1=\frac{\cos^2\theta}{\sin^2\theta}\left(1-\sum_{a=1}^{3}\tilde{\Theta}_a^2\right). \tag{29}$$

In this case the duality anomaly should be canceled only by the Green–Schwarz mechanism, implying that [41,27]

$$\delta_{\rm GS}^a=-C(G)+\sum_l T(R_l)(1+2n_l^a). \tag{30}$$

After a straightforward calculation one then finds two identities

$$\left(1-\sum_{a=1}^{3}\tilde{\Theta}_a^2\right)\left[1-\frac{g^2}{24\pi^2}\sum_l T(R_l)\right]$$
$$=-\frac{g^2}{16\pi^2}\left[2\sum_l T(R_l)\left(\frac{1}{3}+\sum_{a=1}^{3}\tilde{\Theta}_a^2 n_l^a\right)+\frac{b}{3}\sum_{a=1}^{3}\tilde{\Theta}_a^2\right], \tag{31}$$

$$\sum_l T(R_l)\left(\frac{m_l^2}{|M|^2}-\frac{1}{3}\right)=\frac{\cos^2\theta}{\sin^2\theta}\sum_l T(R_l)\left(\frac{1}{3}+\sum_{a=1}^{3}\tilde{\Theta}_a^2 n_l^a\right), \tag{32}$$

where we have used $Y=2/g^2$. Using these identities, one can convince oneself that the two-loop corrected sum rule (10) coincides with the sum rule (29) of the orbifold model up to and including $O(g^2)$ terms. For finite theories ($b=0$) it is possible to express the sum rule (29) in terms of field theory quantities only:

$$\frac{m_i^2+m_j^2+m_k^2}{|M|^2}-1=\frac{\sum_l T(R_l)(m_l^2/|M|^2-\frac{1}{3})}{C(G)-8\pi^2/g^2}. \tag{33}$$

It is remarkable that in this combination of the SSB terms the quantities such as the Goldstino angle parameterizing unknown supersymmetry breaking disappear. Since the sum rule (33) can be seen as an exact result, we conjecture that the sum rule (33) and the tree-level form of the relation $h^{ijk}=-MY^{ijk}(g)$ are also exact results in field theory that result from the finiteness of the SSB parameters.

2.3. Comment

We next would like to comment on the possibility of having all-order finite SSB terms. To begin with we recall that the RG functions are renormalization-scheme dependent starting at two-loop order. This is true, even if we assume that a mass-independent renormalization scheme is employed, except for the gauge coupling β function. Therefore, it could be possible to find a renormalization scheme in which all the higher order coefficients of the β functions (except for the gauge coupling β function) vanish. Since we know the two-loop RG functions explicitly, we would like to see whether we can find a renormalization scheme in which all the RG functions beyond the two-loop vanish. To simplify the problem, we assume that all the supersymmetric, massive parameters are set equal to zero and that Y^{ijk} and h^{jik} have been reduced in favor of g and M. Suppose that we have found reparametrizations of g, M and m^2 such that the β functions, except for β_g and β_{m^2}, beyond the two-loop order vanish. We then ask ourselves whether or not it is possible to find a reparametrization of m_i^2's of the form

$$m_i^2 \to m_i^2 + \frac{g^4}{16\pi^2} K_i \quad \text{with } K_i = r_{ij} m_j^2 + p_i |M|^2, \tag{34}$$

where r_{ij} and p_i are numbers, such that the three-loop β functions for m_i^2's vanish.[9] Inserting (34) into the one-loop β function (A.4), we see that the three-loop terms in the β function should be canceled by the term

$$\rho_{ipq(0)} \rho_{(0)}^{jpq} (K_i + K_j + K_k), \tag{35}$$

where we have used Eq. (4). Recall that because of the diagonality condition (8) the terms given above are proportional to δ_i^j and so the total number of these terms, N, is exactly the number of chiral superfields present in the theory. It is clear that if these N terms are linearly independent, the three-loop contributions in the β functions for m_i^2's can be canceled by them.

This algebraic question is very much related to the question of whether or not the sum rule is the unique solution to the two-loop finiteness, because it depends on the explicit form of $\rho_{(0)}^{ijk}$. One can convince oneself that if the sum rule is the unique solution to the two-loop finiteness and the sum rule does not fix $m_i^2/|M|^2$ completely, the N terms given in (35) are not linearly independent. In this case, it is not clear from the beginning that three-loop terms in the β function can be canceled by (35); one has to compute explicitly the three-loop contributions to see it. In the concrete models we will consider later, these N terms (35) are not linearly independent. The string-inspired result (33) should have a non-trivial meaning in this case; it suggests that the three-loop contributions can be canceled by a reparametrization of m_i^2, because the reparametrization defined by

$$m_i^2 \to m_i'^2 = m_i^2 - \frac{1}{3} \frac{\sum_l T(R_l)(m_l^2 - |M|^2/3)}{C(G) - 8\pi^2/g^2} \tag{36}$$

can bring the "exact" result (33) into the tree-level form. If, on the other hand, the sum rule is the unique solution to the two-loop finiteness and the sum rule fixes $m_i^2/|M|^2$ completely, the N terms (35) are linearly independent. We can then cancel all the three-loop contributions, which then can be continued to arbitrary order.

[9] It is possible to find a reparametrization of m_i^2 and then to make $\beta_{m^2}^{(2)}$ zero.

3. Finite theories based on $SU(5)$

3.1. General comments

From the classification of theories with vanishing one-loop gauge β function [13] one can easily see that there exist only two candidate possibilities to construct $SU(5)$ GUTs with three generations. These possibilities require that the theory should contain as matter fields the chiral supermultiplets $\mathbf{5}, \overline{\mathbf{5}}, \mathbf{10}, \overline{\mathbf{5}}, \mathbf{24}$ with the multiplicities $(6, 9, 4, 1, 0)$ and $(4, 7, 3, 0, 1)$, respectively. Only the second one contains a **24**-plet which can be used to provide the spontaneous symmetry breaking (SB) of $SU(5)$ down to $SU(3) \times SU(2) \times U(1)$. For the first model one has to incorporate another way, such as the Wilson flux breaking mechanism, to achieve the desired SB of $SU(5)$ [4]. Therefore, for a self-consistent field theory discussion we would like to concentrate only on the second possibility.

It is clear, at least for the dimensionless couplings, that the matter content of a theory is only a necessary condition for all-order finiteness. Therefore, there exist, in principle, various finite models for a given matter content. However, during the early studies [14,15], the theorem [17] that guarantees all-order finiteness and requires the existence of power series solution to any finite order in perturbation theory was not known. The theorem introduces new constraints, in particular requires that the solution to the one-loop finiteness conditions should be non-degenerate and isolated. In most studies the freedom resulted as a consequence of the degeneracy in the one- and two-loop solutions has been used to make specific ansätze that could lead to phenomenologically acceptable predictions. Note that the existence of such freedom is incompatible with the power series solutions [7,17].

Taking into account the new constraints an all-order finite $SU(5)$ model has been constructed [4], which among others successfully predicted the bottom and the top quark masses [4,6]. The latter is due to the gauge-and-Yukawa-of-the-third-generation unification [4–6] which has been achieved. In general the predictive power of a finite $SU(5)$ model depends on the structure of the superpotential and on the way the four pairs of Higgs quintets and anti-quintets mix to produce the two Higgs doublets of the minimal supersymmetric standard model (MSSM). Given that the finiteness conditions do not restrict the mass terms, there is a lot of freedom offered by this sector of the theory in mixing the four pairs of Higgs fields. As a result, it was possible in the early studies (a) to provide the adequate doublet–triplet splitting in the pair of **5** and $\overline{\mathbf{5}}$ which couple to ordinary fermions so as to suppress the proton decay induced by the coloured triplets and (b) to introduce angles in the gauge–Yukawa relations suppressing in this way the strength of the Yukawa couplings. Concerning requirement (b) one has to recall that at that time it was very unpleasant to have a top mass prediction at $O(150–200)$ GeV; the popular top quark mass was at $O(40)$ GeV. The above was most clearly stated in Ref. [15] and has been revived [21] taking into account the recent data. However, it is clear that using the large freedom offered by the Higgs mass parameter space in requiring condition (b) one strongly diminishes the beauty of a finite theory. Consequently, this freedom was abandoned in the recent studies of the all-loop finite $SU(5)$ model [4] and only condition (a) was kept as a necessary requirement.

3.2. Models

A predictive gauge–Yukawa unified $SU(5)$ model which is finite to all orders, in addition to the requirements mentioned already, should also have the following properties.

(i) One-loop anomalous dimensions are diagonal, i.e. $\gamma_i^{(1)j} \propto \delta_i^j$, according to assumption (8).

(ii) Three fermion generations, $\overline{\mathbf{5}}_i$ $(i = 1,2,3)$, obviously should not couple to $\mathbf{24}$. This can be achieved for instance by imposing $B - L$ conservation.

(iii) The two Higgs doublets of the MSSM should mostly be made out of a pair of Higgs quintets and anti-quintets, which couple to the third generation.

In the following we discuss two versions of the all-order finite model.

A: The model of Ref. [4].

B: A slight variation of model A, which can also be obtained from the class of the models suggested by Kazakov et al. [20] with a modification to suppress non-diagonal anomalous dimensions.

The quark mixing can be accommodated in these models, but for simplicity we neglect the intergenerational mixing and postpone the interesting problem of predicting the mixings to a future publication.

The superpotential that describes the two models takes the form [4,20]

$$
\begin{aligned}
W = & \sum_{i=1}^{3} [\tfrac{1}{2} g_i^u \mathbf{10}_i \mathbf{10}_i H_i + g_i^d\, \mathbf{10}_i \overline{\mathbf{5}}_i \overline{H}_i] + g_{23}^u \mathbf{10}_2 \mathbf{10}_3 H_4 \\
& + g_{23}^d \mathbf{10}_2 \overline{\mathbf{5}}_3 \overline{H}_4 + g_{32}^d \mathbf{10}_3 \overline{\mathbf{5}}_2 \overline{H}_4 + \sum_{a=1}^{4} g_a^f H_a \mathbf{24} \overline{H}_a + \frac{g^\lambda}{3} (\mathbf{24})^3 , \qquad (37)
\end{aligned}
$$

where H_a and $\overline{H}_a$ $(a = 1,\dots,4)$ stand for the Higgs quintets and anti-quintets. Given the superpotential W, we can now compute the γ functions of the model, from which we then compute the β functions. We find

$$
\begin{aligned}
\gamma_{\mathbf{10}_1}^{(1)} &= \frac{1}{16\pi^2}\left[-\frac{36}{5}\, g^2 + 3\,(g_1^u)^2 + 2\,(g_1^d)^2\right], \\
\gamma_{\mathbf{10}_2}^{(1)} &= \frac{1}{16\pi^2}\left[-\frac{36}{5}\, g^2 + 3\,(g_2^u)^2 + 2\,(g_2^d)^2 + 3\,(g_{23}^u)^2 + 2\,(g_{23}^d)^2\right], \\
\gamma_{\mathbf{10}_3}^{(1)} &= \frac{1}{16\pi^2}\left[-\frac{36}{5}\, g^2 + 3\,(g_3^u)^2 + 2\,(g_3^d)^2 + 3\,(g_{23}^u)^2 + 2\,(g_{32}^d)^2\right], \\
\gamma_{\overline{\mathbf{5}}_1}^{(1)} &= \frac{1}{16\pi^2}\left[-\frac{24}{5}\, g^2 + 4\,(g_1^d)^2\right], \\
\gamma_{\overline{\mathbf{5}}_2}^{(1)} &= \frac{1}{16\pi^2}\left[-\frac{24}{5}\, g^2 + 4\,(g_2^d)^2 + 4\,(g_{32}^d)^2\right],
\end{aligned}
$$

$$\gamma^{(1)}_{\overline{5}_3} = \frac{1}{16\pi^2}\left[-\frac{24}{5}g^2 + 4(g^d_3)^2 + 4(g^d_{23})^2\right],$$

$$\gamma^{(1)}_{H_i} = \frac{1}{16\pi^2}\left[-\frac{24}{5}g^2 + 3(g^u_i)^2 + \frac{24}{5}(g^f_i)^2\right], \quad i = 1,2,3,$$

$$\gamma^{(1)}_{\overline{H}_i} = \frac{1}{16\pi^2}\left[-\frac{24}{5}g^2 + 4(g^d_i)^2 + \frac{24}{5}(g^f_i)^2\right], \quad i = 1,2,3,$$

$$\gamma^{(1)}_{H_4} = \frac{1}{16\pi^2}\left[-\frac{24}{5}g^2 + 6(g^u_{23})^2 + \frac{24}{5}(g^f_4)^2\right],$$

$$\gamma^{(1)}_{\overline{H}_4} = \frac{1}{16\pi^2}\left[-\frac{24}{5}g^2 + 4(g^d_{23})^2 + 4(g^d_{32})^2 + \frac{24}{5}(g^f_4)^2\right],$$

$$\gamma^{(1)}_{24} = \frac{1}{16\pi^2}\left[-10g^2 + \sum_{a=1}^{4}(g^f_a)^2 + \frac{21}{5}(g^\lambda)^2\right]. \tag{38}$$

The non-degenerate and isolated solutions to $\gamma^{(1)}_i = 0$ for the models $\{A, B\}$ are

$$(g^u_1)^2 = \{\tfrac{8}{5}, \tfrac{8}{5}\}g^2, \quad (g^d_1)^2 = \{\tfrac{6}{5}, \tfrac{6}{5}\}g^2, \quad (g^u_2)^2 = (g^u_3)^2 = \{\tfrac{8}{5}, \tfrac{4}{5}\}g^2,$$
$$(g^d_2)^2 = (g^d_3)^2 = \{\tfrac{6}{5}, \tfrac{3}{5}\}g^2, \quad (g^u_{23})^2 = \{0, \tfrac{4}{5}\}g^2, \quad (g^d_{23})^2 = (g^d_{32})^2 = \{0, \tfrac{3}{5}\}g^2,$$
$$(g^\lambda)^2 = \tfrac{15}{7}g^2, \quad (g^f_2)^2 = (g^f_3)^2 = \{0, \tfrac{1}{2}\}g^2, \quad (g^f_1)^2 = 0, \quad (g^f_4)^2 = \{1, 0\}g^2. \tag{39}$$

We have explicitly checked that these solutions (39) are also the solutions of the reduction equation (4) and that they can be uniquely extended to the corresponding power series solutions (4).[10] Consequently, these models are finite to all orders.

After the reduction of couplings (39) the symmetry of W (37) is enhanced: For model A one finds that the superpotential has the $\mathbb{Z}_7 \times \mathbb{Z}_3 \times \mathbb{Z}_2$ discrete symmetry

$$\begin{aligned}
&\overline{5}_1 : (4,0,1), \quad \overline{5}_2 : (1,0,1), \quad \overline{5}_3 : (2,0,1),\\
&\mathbf{10}_1 : (1,1,1), \quad \mathbf{10}_2 : (2,2,1), \quad \mathbf{10}_3 : (4,0,1),\\
&H_1 : (5,1,0), \quad H_2 : (3,2,0), \quad H_3 : (6,0,0),\\
&\overline{H}_1 : (-5,-1,0), \quad \overline{H}_2 : (-3,-2,0), \quad \overline{H}_3 : (-6,0,0),\\
&H_4 : (0,0,0), \quad \overline{H}_4 : (0,0,0), \quad \mathbf{24} : (0,0,0),
\end{aligned} \tag{40}$$

while for model B one finds $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4$ defined as

$$\begin{aligned}
&\overline{5}_1 : (1,0,0), \quad \overline{5}_2 : (0,1,0), \quad \overline{5}_3 : (0,0,1),\\
&\mathbf{10}_1 : (1,0,0), \quad \mathbf{10}_2 : (0,1,0), \quad \mathbf{10}_3 : (0,0,1),\\
&H_1 : (2,0,0), \quad H_2 : (0,2,0), \quad H_3 : (0,0,2),\\
&\overline{H}_1 : (-2,0,0), \quad \overline{H}_2 : (0,-2,0), \quad \overline{H}_3 : (0,0,-2),\\
&H_4 : (0,3,3), \quad \overline{H}_4 : (0,-3,-3), \quad \mathbf{24} : (0,0,0),
\end{aligned} \tag{41}$$

[10] The coefficients in (39) are slightly different from those in models considered in Refs. [20].

where the numbers in parenthesis stand for the charges under the discrete symmetries.

The main difference between models A and B is that three pairs of Higgs quintets and anti-quintets couple to the **24** for B so that it is not necessary [20] to mix them with H_4 and $\overline{H}_4$ in order to achieve the triplet-doublet splitting after SB of $SU(5)$. This enhances the predicitivity, because then the mixing of the three pairs of Higgses are strongly constrained to fit the phenomenology of the first two generations [20].

Before we go to present our analysis on low-energy predictions of the models, we would like to discuss the structure of the sum rule for the soft scalar masses for each case. According to (8), we recall that they are supposed to be diagonal. From the one-loop finiteness for the soft scalar masses, we obtain (there are $\{10, 13\}$ equations for 15 unknown $\kappa^{(0)}$'s):

$$\kappa^{(0)}_{H_i} = 1 - 2\kappa^{(0)}_{\mathbf{10}_i}, \qquad \kappa^{(0)}_{\overline{H}_i} = 1 - \kappa^{(0)}_{\mathbf{10}_i} - \kappa^{(0)}_{\overline{\mathbf{5}}_i} \quad (i = 1,2,3),$$
$$\kappa^{(0)}_{H_4} = \tfrac{2}{3} - \kappa^{(0)}_{\overline{H}_4}, \qquad \kappa^{(0)}_{\mathbf{24}} = \tfrac{1}{3} \quad \text{for A,} \tag{42}$$

and

$$\kappa^{(0)}_{H_1} = 1 - 2\kappa^{(0)}_{\mathbf{10}_1}, \qquad \kappa^{(0)}_{H_2} = \kappa^{(0)}_{H_3} = \kappa^{(0)}_{H_4} = 1 - 2\kappa^{(0)}_{\mathbf{10}_3},$$
$$\kappa^{(0)}_{\overline{H}_1} = 1 - \kappa^{(0)}_{\mathbf{10}_1} - \kappa^{(0)}_{\overline{\mathbf{5}}_1}, \qquad \kappa^{(0)}_{\overline{H}_2} = \kappa^{(0)}_{\overline{H}_3} = \kappa^{(0)}_{\overline{H}_4} = -\tfrac{1}{3} + 2\kappa^{(0)}_{\mathbf{10}_3},$$
$$\kappa^{(0)}_{\overline{\mathbf{5}}_2} = \kappa^{(0)}_{\overline{\mathbf{5}}_3} = \tfrac{4}{3} - 3\kappa^{(0)}_{\mathbf{10}_3}, \qquad \kappa^{(0)}_{\mathbf{10}_2} = \kappa^{(0)}_{\mathbf{10}_3}, \kappa^{(0)}_{\mathbf{24}} = \tfrac{1}{3} \quad \text{for B,} \tag{43}$$

where we have defined

$$\frac{m_i^2}{|M|^2} = \kappa_i^{(0)} + \frac{g^2}{16\pi^2}\kappa_i^{(1)} + \ldots, \qquad i = \mathbf{10}_1, \mathbf{10}_2, \ldots, \mathbf{24}. \tag{44}$$

We then use the solution (39) to calculate the actual value for S' by using Eq. (16), which expresses the two-loop correction to the sum rule. Surprisingly, it turns out for both models that

$$S' = 0. \tag{45}$$

That is, the one-loop sum rule in the present models is not corrected in two-loop order.

Next we would like to address the question of whether the sum rule (10) is the unique solution to the two-loop finiteness. To this end, we recall that the two-loop finiteness for the soft scalar masses follows if Eq. (18), i.e.

$$\rho_{ipq(0)}\rho^{jpq}_{(0)}(\kappa_i^{(1)} + \kappa_p^{(1)} + \kappa_q^{(1)}) = -8C(i)\sum_l [\kappa_p^{(0)} - \tfrac{1}{3}]T(R_l) = -8C(i)S', \tag{46}$$

is satisfied. There are 15 equations for 15 unknown $\kappa^{(1)}$'s. We find that the solution is not unique; it can be parameterized by $\{7, 4\}$ parameters for a given S' which is zero for the present models. For instance,

$$\kappa^{(1)}_{H_i} = -2S' - 2\kappa^{(1)}_{\mathbf{10}_i}, \qquad \kappa^{(1)}_{\overline{H}_i} = -2S' - \kappa^{(1)}_{\overline{\mathbf{5}}_i} - \kappa^{(1)}_{\mathbf{10}_i} \quad (i = 1,2,3),$$
$$\kappa^{(1)}_{H_4} = -\frac{4S'}{3} - \kappa^{(1)}_{\overline{H}_4}, \qquad \kappa^{(1)}_{24} = -\frac{2S'}{3} \quad \text{for A,} \tag{47}$$

and

$$
\begin{aligned}
&\kappa^{(1)}_{H_1} = -2S' - 2\kappa^{(1)}_{\mathbf{10}_1}, \qquad \kappa^{(1)}_{24} = -\frac{2S'}{3}, \qquad \kappa^{(1)}_{\mathbf{10}_2} = \kappa^{(1)}_{\mathbf{10}_3},\\
&\kappa^{(1)}_{H_2} = \kappa^{(1)}_{H_3} = \kappa^{(1)}_{H_4} = -2S' - 2\kappa^{(1)}_{\mathbf{10}_3}, \qquad \kappa^{(1)}_{\overline{H}_2} = \kappa^{(1)}_{\overline{H}_3} = \kappa^{(1)}_{\overline{H}_4} = \frac{2S'}{3} + 2\kappa^{(1)}_{\mathbf{10}_3},\\
&\kappa^{(1)}_{\overline{H}_1} = -2S' - \kappa^{(1)}_{\overline{\mathbf{5}}_1} - \kappa^{(1)}_{\mathbf{10}_1}, \qquad \kappa^{(1)}_{\overline{\mathbf{5}}_2} = \kappa^{(1)}_{\overline{\mathbf{5}}_3} = -\frac{8S'}{3} - 3\kappa^{(1)}_{\mathbf{10}_3} \quad \text{for B.}
\end{aligned}
\tag{48}
$$

As one can easily see that the $\kappa^{(1)}$'s satisfy

$$
\kappa^{(1)}_i + \kappa^{(1)}_j + \kappa^{(1)}_k = -2S' = 0, \tag{49}
$$

which shows that the sum rule (10) in the present models is the unique solution to two-loop finiteness.

4. Predictions of low energy parameters

Since the gauge symmetry is spontaneously broken below $M_{\rm GUT}$, the finiteness conditions do not restrict the renormalization property at low energies, and all it remains are boundary conditions on the gauge and Yukawa couplings (39) and the $h = -MY$ relation (7) and the soft scalar-mass sum rule (10) at $M_{\rm GUT}$. So we examine the evolution of these parameters according to their renormalization group equations at two loops for dimensionless parameters and at one loop for dimensional ones with these boundary conditions. Below $M_{\rm GUT}$ their evolution is assumed to be governed by the MSSM. We further assume a unique supersymmetry breaking scale M_s so that below M_s the SM is the correct effective theory.

We recall that $\tan\beta$ is usually determined in the Higgs sector. However, it has turned out that in the case of GYU models it is convenient to define $\tan\beta$ by using the matching condition at M_s [47],

$$
\begin{aligned}
&\alpha^{\rm SM}_t = \alpha_t \sin^2\beta, \qquad \alpha^{\rm SM}_b = \alpha_b \cos^2\beta, \qquad \alpha^{\rm SM}_\tau = \alpha_\tau \cos^2\beta,\\
&\quad \alpha_\lambda = \tfrac{1}{4}\left(\tfrac{3}{5}\alpha_1 + \alpha_2\right)\cos^2 2\beta,
\end{aligned}
\tag{50}
$$

where $\alpha^{\rm SM}_i$ $(i = t, b, \tau)$ are the SM Yukawa couplings and α_λ is the Higgs coupling $(\alpha_I = g_I^2/4\pi^2)$. With a given set of input parameters [48],

$$
M_\tau = 1.777\ \text{GeV}, \qquad M_Z = 91.188\ \text{GeV}, \tag{51}
$$

with [49]

$$
\begin{aligned}
\alpha^{-1}_{\rm EM}(M_Z) &= 127.9 + \frac{8}{9\pi}\log\frac{M_t}{M_Z},\\
\sin^2\theta_{\rm W}(M_Z) &= 0.2319 - 3.03\times 10^{-5}T - 8.4\times 10^{-8}T^2,\\
T &= M_t/[\text{GeV}] - 165,
\end{aligned}
\tag{52}
$$

Table 1
The predictions for different M_s for model A

M_s [GeV]	$\alpha_{3(5f)}(M_Z)$	$\tan\beta$	$M_{\rm GUT}$ [GeV]	M_b [GeV]	M_t [GeV]
300	0.123	54.1	2.2×10^{16}	5.3	183
500	0.122	54.2	1.9×10^{16}	5.3	183
10^3	0.120	54.3	1.5×10^{16}	5.2	184

Table 2
The predictions for different M_s for model B

M_s [GeV]	$\alpha_{3(5f)}(M_Z)$	$\tan\beta$	$M_{\rm GUT}$ [GeV]	M_b [GeV]	M_t [GeV]
800	0.120	48.2	1.5×10^{16}	5.4	174
10^3	0.119	48.2	1.4×10^{16}	5.4	174
1.2×10^3	0.118	48.2	1.3×10^{16}	5.4	174

the matching condition (50) and the GYU boundary condition at $M_{\rm GUT}$ can be satisfied only for a specific value of $\tan\beta$. Here M_τ, M_t, M_Z are pole masses, and the couplings above are defined in the $\overline{\rm MS}$ scheme with six flavors. Under the assumptions specified above, it is possible without knowing the details of the scalar sector of the MSSM to predict various parameters such as the top quark mass [4–6]. We present them for model A in Table 1 and for model B in Table 2. Comparing, for instance, the M_t predictions above with the most recent experimental value [50],

$$M_t = (175.6 \pm 5.5)\ \text{GeV}, \tag{53}$$

and recalling that the theoretical values for M_t given in the tables may suffer from a correction of less than $\sim 4\%$ [6], we see that they are consistent with the experimental data. (For more details, see Ref. [6], where various corrections on the predictions of GYU models such as the MSSM threshold corrections are estimated. [11])

Now we come to the SSB sector. As mentioned, we impose at $M_{\rm GUT}$ the $h = -MY$ relation (7) and the soft scalar-mass sum rule (10), i.e. (42) and (47) for model A, and (43) and (48) for model B, and calculate their low-energy values. To make our unification idea and its consequence transparent, we shall make an oversimplifying assumption that the unique supersymmetry breaking scale M_s can be set equal to the unified gaugino mass M at $M_{\rm GUT}$. That is, we calculate the SSB parameters at $M_s = M$ from which we then compute the spectrum of the superpartners by using the tree-level formulae. [12] Since $\tan\beta$ by the dimension-zero sector because of GYU, one should examine each time whether GYU and the sum rule are consistent with the radiative breaking of the electroweak symmetry [31]. This consistency can be achieved, though not always, by using the freedom to fix the b term and the supersymmetric mass term μ which remain unconstrained by finiteness.

[11] The GUT threshold corrections in the $SU(5)$ finite model are given in Ref. [21].
[12] For the lightest Higgs mass we include radiative corrections.

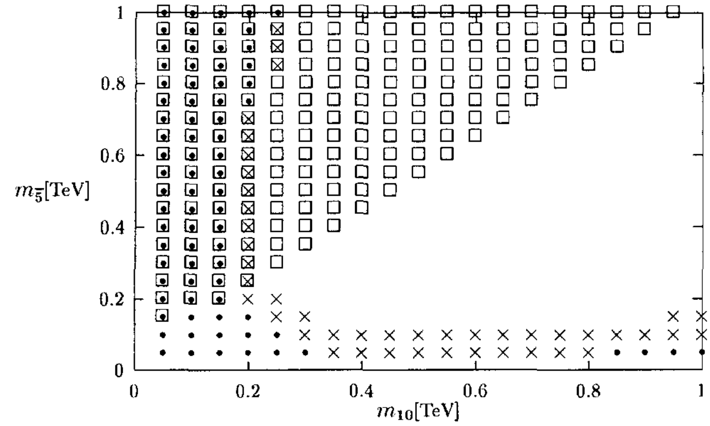

Fig. 1. The region without squares, dots and crosses yields a neutralino as the LSP for model A with $M = 0.3$ TeV.

As we can see from (42) and (43), the structure of the sum rules for the two models is different. Recall that the MSSM Higgs doublets, H_u and H_d, mostly stem from the third Higgses H_3 and $\overline{H}_3$.[13] Therefore, the scalar masses m_i^2 with $i = H_1, H_2, \overline{H}_1, \overline{H}_2$ do not enter into the low-energy sector, implying that $m_{\mathbf{10}_1}^2, m_{\bar{\mathbf{5}}_1}^2, m_{\mathbf{10}_2}^2$ and $m_{\bar{\mathbf{5}}_2}^2$ for model A, and $m_{\mathbf{10}_1}^2$ and $m_{\bar{\mathbf{5}}_1}^2$ for the B, respectively, are free parameters. So in following discussions we would like to focus on the third-generation scalar masses. The relevant sum rules at the GUT scale are thus given by

$$m_{H_u}^2 + 2m_{\mathbf{10}}^2 = m_{H_d}^2 + m_{\bar{\mathbf{5}}}^2 + m_{\mathbf{10}}^2 = M^2 \quad \text{for A,} \tag{54}$$

$$m_{H_u}^2 + 2m_{\mathbf{10}}^2 = M^2, \qquad m_{H_d}^2 - 2m_{\mathbf{10}}^2 = -\frac{M^2}{3}, \qquad m_{\bar{\mathbf{5}}}^2 + 3m_{\mathbf{10}}^2 = \frac{4M^2}{3} \quad \text{for B,} \tag{55}$$

where we use as free parameters $m_{\bar{\mathbf{5}}} \equiv m_{\bar{\mathbf{5}}_3}$ and $m_{\mathbf{10}} \equiv m_{\mathbf{10}_3}$ for model A, and $m_{\mathbf{10}}$ for B, in addition to M.

First we present the result for model A. We look for the parameter space in which the lighter s-tau mass squared $m_{\tilde{\tau}}^2$ is larger than the lightest neutralino mass squared m_{χ}^2 (which is the LSP). In Figs. 1, 2 and 3 we show this region in the $m_{\bar{\mathbf{5}}}$–$m_{\mathbf{10}}$ plane for $M = M_s = 0.3, 0.5$ and 1 TeV, respectively. The region with open squares does not lead to a successful radiative electroweak symmetry breaking, and the region with dots and crosses defines the region with $m_{\tilde{\tau}}^2 < 0$ and $m_{\tilde{\tau}}^2 < m_{\chi}^2$, respectively.

In Fig. 4 we show $m_{\tilde{\tau}}^2$ and m_{χ}^2 for the universal choice $m_{\mathbf{10}}^2 = m_{\bar{\mathbf{5}}}^2 = M^2/3$ at M_{GUT}. We find that there is no region of $M_s = M$ below O(few) TeV in which $m_{\tilde{\tau}}^2 > m_{\chi}^2$ is satisfied. In Table 3 we present the s-spectrum and the lightest Higgs mass m_h of model A with $M = 0.5$ TeV, $m_{\bar{\mathbf{5}}} = 0.3$ TeV and $m_{\mathbf{10}} = 0.5$ TeV. (Radiative corrections are included in m_h.)

[13] For model A, this is an assumption as we have discussed, while for B this is a consequence of the unitarity of the mixing matrix of the three Higgses [20].

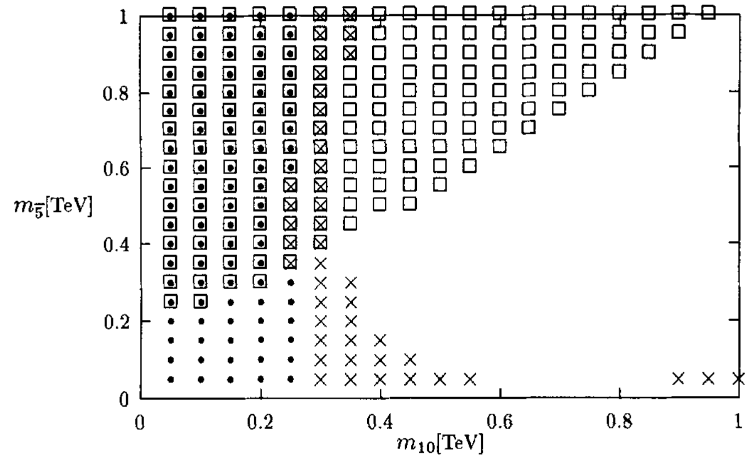

Fig. 2. The same as Fig. 1 with M = 0.5 TeV.

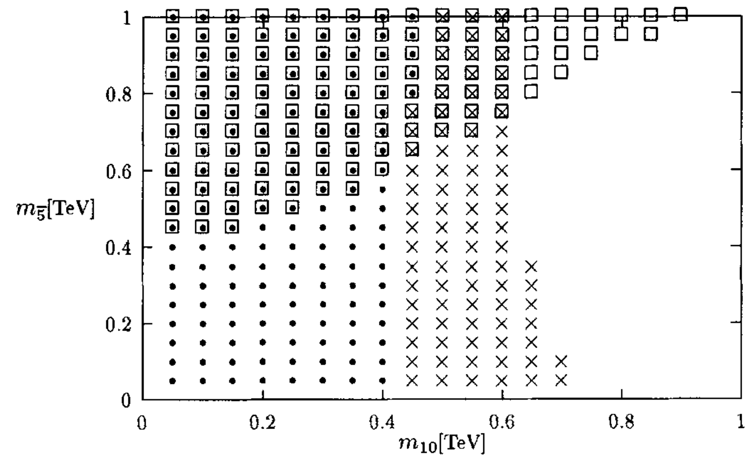

Fig. 3. The same as Fig. 1 with M = 1 TeV.

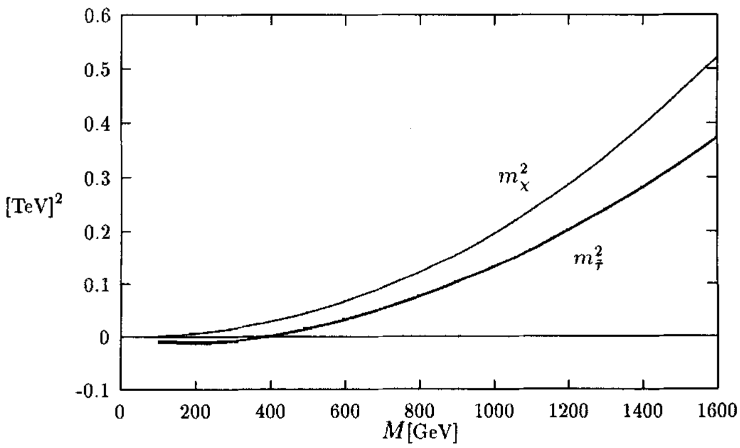

Fig. 4. $m_{\tilde{\tau}}^2$ and m_χ^2 for the universal choice of the soft scalar masses.

Table 3
A representative example of the predictions for the s-spectrum for model A

$m_\chi = m_{\chi_1}$ (TeV)	0.22	$m_{\tilde{b}_2}$ (TeV)	1.06
m_{χ_2} (TeV)	0.41	$m_{\tilde{\tau}} = m_{\tilde{\tau}_1}$ (TeV)	0.33
m_{χ_3} (TeV)	0.93	$m_{\tilde{\tau}_2}$ (TeV)	0.54
m_{χ_4} (TeV)	0.94	$m_{\tilde{\nu}_1}$ (TeV)	0.41
$m_{\chi_1^\pm}$ (TeV)	0.41	m_A (TeV)	0.44
$m_{\chi_2^\pm}$ (TeV)	0.94	$m_{H^\pm}$ (TeV)	0.45
$m_{\tilde{t}_1}$ (TeV)	0.92	m_H (TeV)	0.44
$m_{\tilde{t}_2}$ (TeV)	1.08	m_h (TeV)	0.12
$m_{\tilde{b}_1}$ (TeV)	0.86		

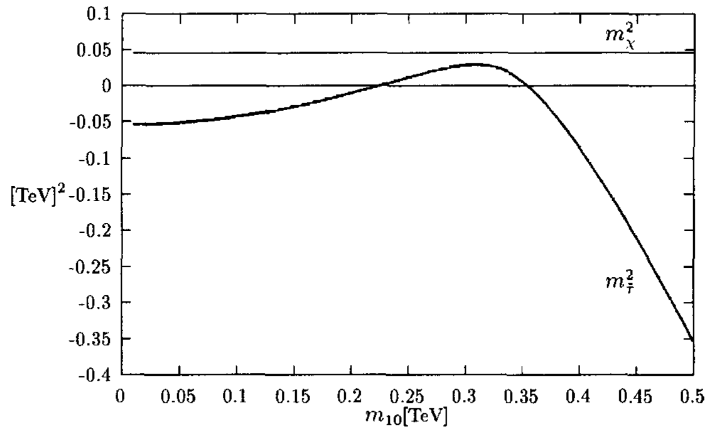

Fig. 5. $m_{\tilde{\tau}}^2$ and m_χ^2 against m_{10}^2 for $M = 0.5$ TeV.

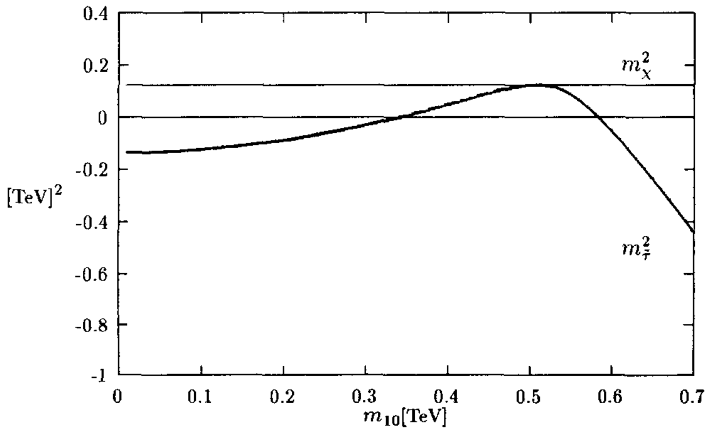

Fig. 6. The same as Fig. 4 for $M = 0.8$ TeV.

The model B has only two free SSB parameters m_{10} and $M = (M_s)$. For a fixed M, the neutralino masses are independent of m_{10}, while $m_{\tilde{\tau}}$ depends on it. Shown are $m_{\tilde{\tau}}^2$ and m_χ^2 as function of m_{10} in Figs. 5, 6 and 7 for $M = 0.5, 0.8$ and 1 TeV.

In Fig. 8 we plot the maximal value of $m_{\tilde{\tau}}^2$, denoted by Max$(m_{\tilde{\tau}}^2)$, and m_χ^2 for different values of M, which should be compared with Fig. 9 in which we plot the case of the

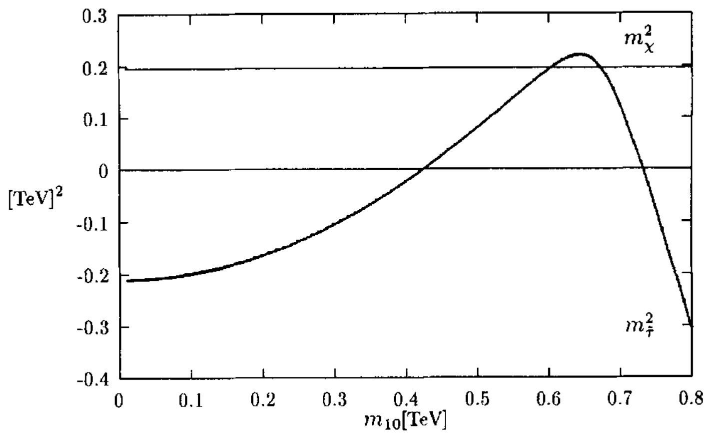

Fig. 7. The same as Fig. 4 for M = 1 TeV.

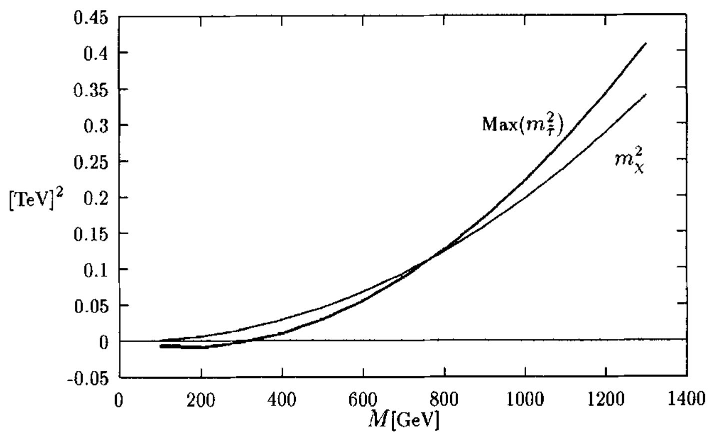

Fig. 8. Max$(m_{\tilde\tau}^2)$ and m_χ^2 as function of M.

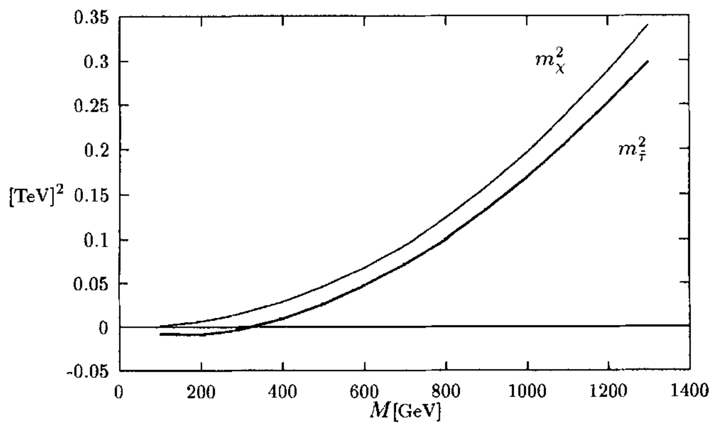

Fig. 9. $m_{\tilde\tau}^2$ and m_χ^2 as function of M for the universal choice.

Table 4
A representative example of the predictions of the s-spectrum for model B

$m_\chi = m_{\chi_1}$ (TeV)	0.44	$m_{\tilde{b}_2}$ (TeV)	1.79
m_{χ_2} (TeV)	0.84	$m_{\tilde{\tau}} = m_{\tilde{\tau}_1}$ (TeV)	0.47
m_{χ_3} (TeV)	1.38	$m_{\tilde{\tau}_2}$ (TeV)	0.69
m_{χ_4} (TeV)	1.39	$m_{\tilde{\nu}_1}$ (TeV)	0.62
$m_{\chi_1^\pm}$ (TeV)	0.84	m_A (TeV)	0.74
$m_{\chi_2^\pm}$ (TeV)	1.39	$m_{H^\pm}$ (TeV)	0.75
$m_{\tilde{t}_1}$ (TeV)	1.59	m_H (TeV)	0.74
$m_{\tilde{t}_2}$ (TeV)	1.82	m_h (TeV)	0.12
$m_{\tilde{b}_1}$ (TeV)	1.56		

universal choice of the scalar masses.

As Fig. 8 shows, M has to be relatively large to satisfy the constraint $m_{\tilde{\tau}}^2 > m_\chi^2$ for model B. We find, also for this model, that there is no region of M below O(few) TeV for the universal choice in which $m_{\tilde{\tau}}^2 > m_\chi^2$ is satisfied. In Table 4 we give a representative prediction for the s-spectrum for model B, where we have used M = 1 TeV and $m_{10} = 0.65$ TeV.

5. Conclusion

In this paper we have re-investigated the two-loop finiteness conditions for the SSB parameters in softly broken $N = 1$ supersymmetric Yang–Mills theories with a simple gauge group and found that the previously known result [11,19] on the $h = -MY$ relation (7) is necessary while the universal solution for the soft scalar masses can be continuously deformed to the sum rule (10).

Since it is known [25,26,23] that the universal soft scalar masses appear for dilaton-dominated supersymmetry breaking in 4D superstring models, we have examined whether or not the two-loop corrected soft scalar-mass sum rule can also be obtained in some string model. We have indeed found that the same sum rule is satisfied in a certain class of string models in which the massive string states are organized into $N = 4$ supermultiplets so that they do not contribute to the quantum modification of the gauge kinetic function. Since not only in finite GYU models, but also in non-finite GYU models the same soft scalar-mass sum rule is satisfied at least at the one-loop level [30], we believe that there exists something non-trivial behind these coincidences.

Motivated by these facts, we have investigated the SSB sector of two finite $SU(5)$ models A and B. We have found out that the two-loop corrections to the sum rule is absent in these models. Since we do not know why this happens, it is an accident to us. Finally we have investigated the low-energy sector of these models. Using the sum rule and requiring that the LSP is neutral, we have constrained the parameter space of the low-energy SSB sector in each model and calculated the spectrum of the superparticles. We have found that model A allows relatively light superparticles while in model B they are heavier than ~ 0.5 TeV. The mass of the lightest Higgs is ~ 120 GeV.

Taking into account all these results, we would like to conclude that the finite models we have considered are not only academically attractive, but also making interesting predictions which are consistent with the present experimental knowledge.

Acknowledgements

We thank D. Lüst and D. Suematsu for discussions.

Appendix A

The RG functions which we have used in the text are defined as

$$\frac{d}{dt}g = \beta_g = \sum_{n=1} \frac{1}{(16\pi^2)^n}\,\beta_g^{(n)}, \qquad \frac{d}{dt}M = \beta_M = \sum_{n=1} \frac{1}{(16\pi^2)^n}\,\beta_M^{(n)},$$

$$\frac{d}{dt}Y^{ijk} = \beta_Y^{ijk} = Y^{ijp}\sum_{n=1} \frac{1}{(16\pi^2)^n}\,\gamma_p^{(n)\,k} + (k \leftrightarrow i) + (k \leftrightarrow j),$$

$$\frac{d}{dt}h^{ijk} = \beta_h^{ijk} = \sum_{n=1} \frac{1}{(16\pi^2)^n}\,[\beta_h^{(n)}]^{ijk},$$

$$\frac{d}{dt}(m^2)^j_i = [\beta_{m^2}]^j_i = \sum_{n=1} \frac{1}{(16\pi^2)^n}\,[\beta_{m^2}^{(n)}]^j_i,$$

$$\frac{d}{dt}b^{ij} = \beta_b^{ij} = \sum_{n=1} \frac{1}{(16\pi^2)^n}\,\beta_b^{(n)ij},$$

where we assume that the gauge group is a simple group. The coefficients of the one- and two-loop RG functions [11,12,24,23,19] are

$$\beta_g^{(1)} = g^3[T(R) - 3C(G)], \qquad \beta_M^{(1)} = 2M\beta_g^{(1)}/g, \tag{A.1}$$

$$\gamma_i^{(1)j} = \tfrac{1}{2}Y_{ipq}Y^{jpq} - 2\delta_i^j\, g^2\, C(i), \qquad \chi_j^{(1)i} = h^{imn}Y_{jmn} + 4Mg^2C(i)\delta^i_j, \tag{A.2}$$

$$[\beta_h^{(1)}]^{ijk} = \tfrac{1}{2}h^{ijl}Y_{lmn}Y^{mnk} + Y^{ijl}Y_{lmn}h^{mnk} - 2(h^{ijk} - 2MY^{ijk})\, g^2\, C(k) \\ +(k \leftrightarrow i) + (k \leftrightarrow j), \tag{A.3}$$

$$[\beta_{m^2}^{(1)}]^j_i = \tfrac{1}{2}Y_{ipq}Y^{pqn}(m^2)^j_n + \tfrac{1}{2}Y^{jpq}Y_{pqn}(m^2)^n_i + 2Y_{ipq}Y^{jpr}(m^2)^q_r \\ +h_{ipq}h^{jpq} - 8\delta_i^j MM^\dagger\, g^2\, C(i), \tag{A.4}$$

$$\beta_b^{(1)ij} = b^{il}\gamma_l^{(1)j} + \mu^{il}\chi_l^{(1)i} + (i \leftrightarrow j), \tag{A.5}$$

$$\beta_g^{(2)} = 2g^2C(G)\beta_g^{(1)} - 2g^3d^{-1}(G)\sum_i C(i)\gamma_i^{(1)i}, \tag{A.6}$$

$$\gamma_j^{(2)i} = 2gC(i)\delta^i_j\beta_g^{(1)} - [Y_{jmn}Y^{mpi} + 2g^2C(j)\delta^p_j\delta^i_n]\gamma_p^{(1)n}, \tag{A.7}$$

$$\beta_M^{(2)} = 8gC(G)\beta_g^{(1)}M + g^2d^{-1}(G)\sum_i C(i)[-4\gamma_i^{(1)i}M + 2\chi_i^{(1)i}], \tag{A.8}$$

$$\begin{aligned}
\beta_h^{(2)ijk} = & -[h^{ijl}Y_{lmn}Y^{mpk} + 2Y^{ijl}Y_{lmn}h^{mpk} - 4g^2MY^{ijp}C(n)\delta_n^k]\gamma_p^{(1)n} \\
& -2g^2[h^{ijl}\gamma_l^{(1)k} + Y^{ijl}\chi_l^{(1)k}]C(k) + g(2h^{ijk} - 8MY^{ijk})C(k)\beta_g^{(1)} \\
& -Y^{ijl}Y_{lmn}Y^{pmk}\chi_p^{(1)n} + (k \leftrightarrow i) + (k \leftrightarrow j), \qquad (A.9)\\
\beta_b^{(2)ij} = & [-b^{il}Y_{lmn}Y^{mpj} - 2\mu^{il}Y_{lmn}h^{pmj} - Y^{ijl}Y_{lmn}b^{mp} + 4g^2MC(i)\mu^{ip}\delta_j^n]\gamma_p^{(1)n} \\
& -[\mu^{il}Y_{lmn}Y^{mpj} + \tfrac{1}{2}Y^{ijl}Y_{lmn}\mu^{mp}]\chi_p^{(1)n} - 2g^2C(i)[b^{il}\gamma_l^{(1)j} + \mu^{il}\chi_l^{(1)j}] \\
& +2gC(i)\beta_g^{(1)}[b^{ij} - 2\mu^{ij}] + (i \leftrightarrow j), \qquad (A.10)\\
[\beta_{m^2}^{(2)}]_i^j = & -[(m^2)_i^l Y_{lmn}Y^{mpj} + \tfrac{1}{2}Y_{ilm}Y^{jpm}(m^2)_n^l + \tfrac{1}{2}Y_{inm}Y^{jlm}(m^2)_l^p + Y_{iln}Y^{jrp}(m^2)_r^l \\
& +h_{iln}h^{jlp} + 4g^2|M|^2C(j)\delta_n^j\delta_i^p + 2g^2\sum_A (R_A)_i^j(R_Am^2)_n^p]\gamma_p^{(1)n} \\
& +[2g^2M^\dagger C(i)\delta_n^j\delta_i^p - h_{iln}Y^{jlp}]\chi_p^{(1)n} \\
& -\tfrac{1}{2}[Y_{iln}Y^{jlp} + 2g^2C(i)\delta_n^j\delta_i^p][\beta_{m^2}^{(1)}]_p^n \\
& +4|M|^2C(i)\delta_i^j[3g\beta_g^{(1)} + g^4S'] + \text{H.c.}, \qquad (A.11)
\end{aligned}$$

where S' is defined in Eq. (16). Further references may be found for instance in Ref. [47].

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5.6 Further all loop results in softly broken supersymmetric gauge theories

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Comment (*Myriam Mondragón, George Zoupanos*)
In this paper substantial progress has been achieved concerning the soft supersymmetry breaking sector of $N = 1$ supersymmetric gauge theories inluding the finite ones. In particular, the RGI sume rule discussed in subsection 5.5 up to two-loops was extended to all orders in perturbation theory. More specifically, recalling and extending our comments on 5.5 we observe that a RGI sum rule for the soft scalar masses exists in lower orders: it results from the independent analysis of the SSB sector of a $N = 1$ supersymmetric GYU; in one-loop for the non-finite case [26] and in two-loops for the finite case (subsection 5.5). The sum rule appears to have significant phenomenological consequences and in particular manages to overcome the unpleasant predictions of the previously known "universal" finiteness condition for the soft scalar masses.

The general feeling was that hardly one could find RGI relations in the SSB sector of $N = 1$ supersymmetric theories includind the finite ones beyond the two-loop order. However despite the negative expectations a very interesting progress has been achieved concerning the renormalization properties of the SSB parameters. The developement was based on the powerful supergraph method for studying supersymmetric theories which has been applied to the softly broken ones by using the "spurion" external space-time independent superfields. According to this method a softly broken supersymmetric gauge theory is considered as a supersymmetric one in which the various parameters such as couplings and masses have been promoted to external superfields that acquire "vacuum expectation values". Then based on this method certain relations among the soft term renormalization and that of an unbroken supersymmetric theory were derived. In particular the β-functions of the parameters of the softly broken theory are expressed in terms of partial differential operators involving the dimensionless parameters of the unbroken theory. A crucial aspect in the whole strategy for solving the set of coupled differential equations so as to be able to express all parameters in a RGI way, was to transform the partial differential operators involved to total derivative operators. It is definitely possible to do this on the RGI surface defined by the solution of the reduction equations. Using the above tools, in the present work we proved that the sum rule for the soft scalar massses is RGI to all-orders for both the general as well as for the finite case. Finally, the exact β-function for the soft scalar masses in the Novikov-Shifman-Vainstein-Zakharov (NSVZ) scheme for the softly broken supersymmetric QCD was obtained for the first time. The above method and results are of significant importance in the application of the reduction method in the MSSM and lead to important results and significant predictions, which will be discussed later in subsection 5.10.

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Further all-loop results in softly-broken supersymmetric gauge theories

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Abstract

It is proven that the recently found, renormalization-group invariant sum rule for the soft scalar masses in softly-broken $N=1$ supersymmetric gauge-Yukawa unified theories can be extended to all orders in perturbation theory. In the case of finite unified theories, the sum rule ensures the all-loop finiteness in the soft supersymmetry breaking sector. As a byproduct the exact β function for the soft scalar masses in the Novikov-Shifman-Vainstein-Zakharov (NSVZ) scheme for softly-broken supersymmetric QCD is obtained. It is also found that the singularity appearing in the sum rule in the NSVZ scheme exactly coincides with that which has been previously found in a certain class of superstring models in which the massive string states are organized into $N=4$ supermultiplets.

1. Introduction

The plethora of free parameters of the, very successful otherwise, Standard Model (SM), can be interpreted as signaling the existence of a more fundamental Physics picture in higher scales, whose remnants appear as free parameters in the SM. In fact after several decades of experience in searching for such a fundamental theory, which in principle could explain everything that is observed today in terms of very few parameters, it seems more realistic to expect that only parts of the fundamental theory are uncovered at various higher scales; maybe the full fundamental theory can only be found close to the Planck scale. The usual theoretical strategy to search for new Physics beyond the SM is to construct more symmetric theories, e.g. Grand Unified Theories (GUTs) at higher scales and subsequently test their predictions against the measured low energy parameters. A representa-

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tive candidate for carrying some of the information of the fundamental theory at intermediate scales is the $N=1$ globally supersymmetric $SU(5)$ GUT, given its predictive power for certain low energy free parameters of the SM.

In our recent studies [1–10] [4], we have developed another complementary strategy in searching for a more fundamental theory possibly at Planck scale and its consequences that could be missing in ordinary GUTs. Our method consists of hunting for renormalization group invariant (RGI) relations among couplings holding below the Planck scale and which therefore are exactly preserved down to the GUT scale. This programme applied in the dimensionless couplings of supersymmetric GUTs such as gauge and Yukawa couplings had already certain success by predicting correctly, among others, the top quark mass in the finite [1,4] and in the minimal [2,4] $N=1$ supersymmetric $SU(5)$-GUTs.

An impressive aspect of the RGI relations is that one can guarantee their validity to all-orders in perturbation theory by studying the uniqueness of the resulting relations at one-loop, as was proven in the early days of the programme of *reduction of couplings* [8].

Although supersymmetry seems to be an essential feature for a successful realization of the above programme, its breaking has to be understood too in this framework, which has the ambition to supply the SM with predictions for several of its free parameters. Therefore, the search for RGI relations was naturally extended to the soft supersymmetry breaking (SSB) sector of these theories [12,5], which involve parameters with dimension one and two. In the case of nonfinite theories, the method to prove the existence of reduction of couplings to all-loop [8–10] can be easily extended for the RGI relations among dimensional parameters [5] if use of a mass-independent renormalization scheme (RS) is assumed [5]. In contrast to this, for the case of finite theories the elegant way of Ref. [14] to show finiteness (which is based on a consideration of renormalization of certain anomalies) cannot be simply applied; reduction of couplings is merely one of the conditions for finiteness. The proof of the all-order finiteness is certainly less involved to be performed in a particular RS in which various properties of the RG functions are known and can be assumed [15]. Using the recent results [16–19] on the renormalization properties of the SSB sector in the supersymmetric version of the minimal subtraction scheme, Kazakov [20] has pursued that line of the thought and shown the finiteness in the SSB sector [6]. Soon later Jack, Jones and Pickering [23] have generalized Kazakov's idea [20] so as to find RGI relations among the SSB parameters in the nonfinite case.

Note that in the formulation of references above the SSB parameters are expressed in terms of the unified gauge coupling g and the unified gaugino mass parameter M only, which may appear as a too strong constraint on the SSB sector for a given phenomenological model. Therefore, there has been attempts [6,7] to relax this constraint without loosing RGI. An interesting observation resulting from the independent analysis of the SSB sector of a $N=1$ supersymmetric gauge-Yukawa unified theory is the existence of a RGI sum rule for the soft scalar- masses in lower orders; in one-loop for the nonfinite case [6] and in two-loop for the finite case [7]. The sum rule appears to have significant phenomenological consequences and in particular manages to overcome the unpleasant predictions of the previously known ''universal'' finiteness condition for the soft scalar masses [21,22]. The universal soft scalar masses apart from their simplicity they were appealing for a number of reasons (a) they are part of the constraints that preserve finiteness up to two-loop [21,22], (b) they appear to be RGI under a certain constraint, known as the $P=1/3Q$ condition [12], in more general supersymmetric gauge theories, and (3) they appear in the dilaton dominated supersymmetry breaking superstring scenarios [24]. In the latter case, since the dilaton couples in a universal manner to all particles the universality of soft scalar masses appears as a quite model independent feature. Unfortunately, further studies have exhibited a number of problems attributed to the universality of soft scalar masses. For instance (1) in finite unified theories the

[4] For an extended discussion and a complete list of references, see Ref. [11].
[5] The proof is also possible without any assumption on a particular RS [13].
[6] Finiteness in this sector in lower orders are shown in Refs. [21,22]

universality leads to a charged particle, the superpartner of τ, the s-τ, to be the lightest supersymmetric particle [25,7], (2) the MSSM with universal soft scalar masses is inconsistent with radiative electroweak symmetry breaking [26] and (3) worst of all the dilaton dominated limit leads to charge and/or colour breaking minima deeper than the standard vacuum [27]. Therefore, the sum rule is a welcome possibility. Furthermore, it was shown that the same sum rule is satisfied in a certain class of 4D orbiford models, at least at the tree-level for the nonfinite [6] and in two-loop order for the finite case [7] if the massive string states are organized into $N=4$ supermultiplets so that they do not contribute to the quantum modification of the gauge kinetic function [28].

The purpose of the present paper is to prove the existence of the RGI soft scalar-mass sum rule to all-orders for the nonfinite as well as for the finite case, based on the recent developments on the renormalization properties of the SSB sector of the $N=1$ supersymmetric gauge theories. As an interesting byproduct we obtain the exact β function for the soft scalar masses in the Novikov-Shifman-Vainstein-Zakharov (NSVZ) scheme [29] for softly-broken $N=1$ supersymmetric QCD.

2. Recent results on the renormalization of the SSB parameters

Most of the recent interesting progress [17–20,23] on the renormalization properties of the SSB parameters is based conceptually and technically on the work of Ref. [16]. In Ref. [16] the powerful supergraph method [30] for studying supersymmetric theories has been applied to the softly-broken ones by using the ''spurion'' external space-time independent superfields [31]. In the latter method a softly-broken supersymmetric gauge theory is considered as a supersymmetric one in which the various parameters such as couplings and masses have been promoted to external superfields that acquire ''vacuum expectation values''. Based on this method the relations among the soft term renormalization and that of an unbroken supersymmetric gauge theory have been derived.

To be more specific, following the notation of Ref. [23], in an $N=1$ supersymmetric gauge theory with superpotential

$$W(\Phi)=\tfrac{1}{6}Y^{ijk}\Phi_i\Phi_j\Phi_k+\tfrac{1}{2}\mu^{ij}\Phi_i\Phi_j \tag{1}$$

the SSB part L_{SSB} can be written as [16]

$$L(\Phi,W)=-\left(\int d^2\theta\eta\left(\tfrac{1}{6}h^{ijk}\Phi_i\Phi_j\Phi_k+\tfrac{1}{2}b^{ij}\Phi_i\Phi_j+\tfrac{1}{2}MW_A^{\alpha}W_{A\alpha}\right)+\text{h.c.}\right)$$
$$-\int d^4\theta\tilde{\eta}\eta\overline{\Phi^j}(m^2)^i_j(e^{2gV})^k_i\Phi_k\,, \tag{2}$$

where $\eta=\theta^2$, $\tilde{\eta}=\tilde{\theta}^2$ are the external spurion superfields and θ, $\tilde{\theta}$ are the usual grassmannian parameters, and M is the gaugino mass. The β functions of the M,h and m^2 parameters are found to be

$$\beta_M=2\mathscr{O}\left(\frac{\beta_g}{g}\right), \tag{3}$$

$$\beta_h^{ijk}=\gamma_l{}^ih^{ljk}+\gamma_l{}^jh^{ilk}+\gamma_l{}^kh^{ijl}-2\gamma_1{}^i{}_lY^{ljk}-2\gamma_1{}^j{}_lY^{ilk}-2\gamma_1{}^k{}_lY^{ijl}\,, \tag{4}$$

$$(\beta_{m^2})^i{}_j=\left[\Delta+X\frac{\partial}{\partial g}\right]\gamma^i{}_j\,, \tag{5}$$

$$\mathscr{O}=\left(Mg^2\frac{\partial}{\partial g^2}-h^{lmn}\frac{\partial}{\partial Y^{lmn}}\right), \tag{6}$$

$$\Delta = 2\mathcal{O}\mathcal{O}^* + 2|M|^2 g^2 \frac{\partial}{\partial g^2} + \tilde{Y}_{lmn}\frac{\partial}{\partial Y_{lmn}} + \tilde{Y}^{lmn}\frac{\partial}{\partial Y^{lmn}}\,, \tag{7}$$

where $(\gamma_1)^i{}_j = \mathcal{O}\gamma^i{}_j$, $Y_{lmn} = (Y^{lmn})^*$, and

$$\tilde{Y}^{ijk} = (m^2)^i{}_l Y^{ljk} + (m^2)^j{}_l Y^{ilk} + (m^2)^k{}_l Y^{ijl}\,. \tag{8}$$

Note that the X term in (5) is explicitly known only in the lowest order [22,32]:

$$X^{(2)} = -\frac{Sg^3}{8\pi^2}\,, \quad S\delta_{AB} = (m^2)^k{}_l (R_A R_B)^l{}_k - |M|^2 C(G)\,\delta_{AB}\,. \tag{9}$$

We do not consider the b parameters in the following discussions, because they do not enter into the β functions of the other quantities at all. Moreover they are finite if the other quantities are finite.

In order to express the h and m^2 parameters in terms of g and M in a RG invariant way, we have to solve the set of coupled reduction equations [8–10]. The key point in the strategy of Refs. [20,23] to solve the reduction equations is the assumption that the differential operators $\mathcal{O}$ and Δ given in Eqs. (6) and (7) become total derivative operators on the RG invariant surface which is defined by the solution of the reduction solutions. Although we consider this assumption as a subtle one and the extent of its validity requiring further clarification, we accept it throughout our analysis.

Observe that the β functions of the SSB parameters are obtained by applying the differential operators, $\mathcal{O}$ and Δ, on the RG functions, β_g and $\gamma^j{}_i$, of the unbroken theory, and note next that in a finite theory Y^{ijk} is a power series of g and that β_g as well as $\gamma^j{}_i$ have to identically vanish. But in general we expect that

$$\frac{\partial\gamma^j{}_i(g,Y,Y^*)}{\partial Y}\Big|_{Y=Y(g),Y^*=Y^*(g)} \neq 0 \quad \text{or} \quad \frac{\partial\gamma^j{}_i(g,Y,Y^*)}{\partial g}\Big|_{Y=Y(g),Y^*=Y^*(g)} \neq 0\,, \tag{10}$$

even if $\gamma^j{}_i(g,Y(g),Y^*(g))$ vanishes. However, one easily sees that

$$\begin{aligned}\frac{d\gamma^j{}_i}{dg}(g,Y=Y(g),Y^*=Y^*(g)) &= \frac{\partial\gamma^j{}_i(g,Y,Y^*)}{\partial g}\Big|_{Y=Y(g),Y^*=Y^*(g)} \\ &+ \frac{\partial\gamma^j{}_i(g,Y,Y^*)}{\partial Y}\Big|_{Y=Y(g),Y^*=Y^*(g)}\frac{dY(g)}{dg} + \frac{\partial\gamma^j{}_i(g,Y,Y^*)}{\partial Y^*}\Big|_{Y=Y(g),Y^*=Y^*(g)}\frac{dY^*(g)}{dg} = 0\,,\end{aligned} \tag{11}$$

if $\gamma^j{}_i(g,Y=Y(g),Y^*=Y^*(g)) = 0$. Kazakov [20] examining the finite case was searching for a RG invariant surface on which the differential operators $\mathcal{O}$ and Δ can be written as total derivative terms.

In Ref. [23] the general case has been considered and has been further assumed that

$$\gamma^j{}_i = \gamma_i\delta^j{}_i\,, \tag{12}$$

$$(m^2)^j{}_i = m_i^2\delta^j{}_i\,, \tag{13}$$

$$Y^{ijk}\frac{\partial}{\partial Y^{ijk}} = Y^{*ijk}\frac{\partial}{\partial Y^{*ijk}} \text{ on the space of the RG functions}\,, \tag{14}$$

and has been shown that if

$$h^{ijk} = -M(Y^{ijk})' \equiv -M\frac{dY^{ijk}(g)}{d\ln g}\,, \tag{15}$$

$$m_i^2 = |M|^2\Big\{-\big(1+\tilde{X}(g)\big)\big(g/\beta_g\big)\big(\gamma_i(g)\big) + \tfrac{1}{2}\big[\big(g/\beta_g\big)\gamma_i(g)\big]'\Big\} \tag{16}$$

are satisfied, then the differential operators $\mathscr{O}$ and Δ can be written as

$$\mathscr{O} = \frac{M}{2} \frac{d}{d\ln g} , \qquad (17)$$

$$\Delta = |M|^2 \left[\frac{1}{2} \frac{d^2}{d(\ln g)^2} + \left(1 + \tilde{X}(g)/g\right) \frac{d}{d\ln g} \right], \qquad (18)$$

where

$$g\tilde{X}(g) = \frac{1}{|M|^2} X\left(g, Y(g), Y^*(g), h(M,g), h^*(M,g), m^2\left(|M|^2, g\right)\right). \qquad (19)$$

Eqs. (17) and (18) can be derived from

$$\frac{d\ln Y^{ijk}}{d\ln g} = (\ln Y^{ijk})' = \left(g/\beta_g\right)\left[\gamma_i(g) + \gamma_j(g) + \gamma_k(g)\right], \qquad (20)$$

which follows assuming the reduction equation

$$\beta_g \frac{dY^{ijk}(g)}{dg} = \beta^{ijk} = Y^{ijk}(g)\left[\gamma_i(g) + \gamma_j(g) + \gamma_k(g)\right]. \qquad (21)$$

Note that so far Eq. (15) is a solution of the reduction equation (i.e. RG invariant), but Eq. (16) is not. At the final step, Jack et al. in Ref. [23] require that Eq. (16), too, is RG invariant, which fixes $\tilde{X}(g)$ uniquely up to a term related to an integration constant. This integration constant term is then fixed by comparing it with the lowest order result in Eq. (9). They found

$$\tilde{X}(g) = \tfrac{1}{2}\left(\ln(\beta_g/g)\right)' - 1. \qquad (22)$$

Note that there is no perturbative computation of X beyond two-loop. Therefore Eq. (22) may be understood as a prediction of perturbative computation of X. If one inserts $\tilde{X}$ above into Eq. (16), one obtains

$$m_i^2 = \tfrac{1}{2}|M|^2\left(g/\beta_g\right)\left(\gamma_i(g)\right)', \qquad (23)$$

which together with (15) is the final result of Ref. [23].

3. New results

Next let us consider the sum rules for soft scalar masses [6,7]. In turn, we assume neither (16) nor (23). But we assume that Y^{ijk} and h^{ijk} are already reduced, where h^{ijk} is given in Eq. (15), as well as that (12)–(14) hold. Suppose that the sum rule takes the form

$$m_i^2 + m_j^2 + m_k^2 = |M|^2 F^M_{ijk}(g) + \sum_l m_l^2 F^l_{ijk}(g). \qquad (24)$$

We require, as in Ref. [20,23], that Δ acting on γ_i can be written as a total derivative operator, and we find that

$$F^M_{ijk}(g) = \left(1 + \tilde{X}^M(g)\right)(\ln Y^{ijk})' + \tfrac{1}{2}(\ln Y^{ijk})'', \quad F^l_{ijk}(g) = \tilde{X}^l(g)(\ln Y^{ijk})' \qquad (25)$$

have to be satisfied, where

$$|M|^2 g\tilde{X}^M(g) + \sum_l m_l^2 g\tilde{X}^l(g) = X\left(g, Y(g), Y^*(g), h(M,g), h^*(M,g), m^2\right). \qquad (26)$$

Then we have

$$\beta_{m_i^2} = \Delta\gamma_i = \left\{ |M|^2 \left[\frac{1}{2}\frac{d^2}{d(\ln g)^2} + \left(1+\tilde{X}^M(g)\right)\frac{d}{d\ln g}\right] + \sum_l m_l^2 \tilde{X}^l(g)\frac{d}{d\ln g}\right\}\gamma_i(g)\,, \tag{27}$$

which vanishes if $\gamma_i(g)=0$. Therefore Eq. (24) with (25) is the desired sum rule for the finite theories. Since in two-loop order $(\ln Y^{ijk})'=1$, $(\ln Y^{ijk})''=0$ and X is given by Eq. (9), we reproduce our previous result [7]

$$m_i^2 + m_j^2 + m_k^2 = |M|^2 + \tilde{X}^{(2)}\,, \tag{28}$$

where $\tilde{X}^{(2)}$ [22,32] is given in (9). The general case is more involved. Following Ref. [23] we require that the sum rule (24) with F^M and F^l given in (25) is RG invariant in the general case, too. That is, the reduction equation of the form [5]

$$\mathscr{D}\left[m_i^2 + m_j^2 + m_k^2 - |M|^2 F^M_{ijk}(g) - \sum_l m_l^2 F^l_{ijk}\right] = 0 \tag{29}$$

has to be satisfied, where

$$\mathscr{D} \equiv \beta_g \frac{\partial}{\partial g} + \beta_M \frac{\partial}{\partial M} + \beta_M^* \frac{\partial}{\partial M^*} + \sum_l \beta_{m_l^2}\frac{\partial}{\partial m_l^2}\,. \tag{30}$$

The equation above implies that

$$\begin{aligned}
&\beta_{m_i^2} + \beta_{m_j^2} + \beta_{m_k^2} \\
&\quad = \left\{ |M|^2 \left[\frac{1}{2}\frac{d^2}{d(\ln g)^2} + \left(1+\tilde{X}^M(g)\right)\frac{d}{d\ln g}\right] + \sum_l m_l^2 \tilde{X}^l(g)\frac{d}{d\ln g}\right\}\left[\gamma_i(g)+\gamma_j(g)+\gamma_k(g)\right] \\
&\quad = |M|^2 \{ 2(\beta_g/g)' \left[(1+\tilde{X}^M)(\ln Y^{ijk})' + \tfrac{1}{2}(\ln Y^{ijk})''\right] \\
&\qquad + (\beta_g/g)\left[(\tilde{X}^M)'(\ln Y^{ijk})' + (1+\tilde{X}^M)(\ln Y^{ijk})'' + \tfrac{1}{2}(\ln Y^{ijk})'''\right] \\
&\qquad + \sum_l \tilde{X}^l (\ln Y^{ijk})'\left[\tfrac{1}{2}(\gamma_l)'' + (1+\tilde{X}^M)(\gamma_l)'\right] \\
&\qquad + \sum_l m_l^2 \left\{ (\beta_g/g)\left[(\tilde{X}^l)'(\ln Y^{ijk})' + \tilde{X}^l(\ln Y^{ijk})''\right] + \tilde{X}^l(\ln Y^{ijk})'\sum_m (\gamma_m)'\tilde{X}^m\right\},
\end{aligned} \tag{31}$$

where use has been made of Eqs. (3), (5), (20), (27) and

$$\mathscr{O} = \tfrac{1}{2} M \frac{d}{d\ln g}\,. \tag{32}$$

The Eq. (31) is satisfied if

$$\left[(\beta_g/g)\tilde{X}^M\right]' + \sum_l \tilde{X}^l \left[\tfrac{1}{2}(\gamma_l)'' + (1+\tilde{X}^M)(\gamma_l)'\right] = \tfrac{1}{2}(\beta_g/g)'' - (\beta_g/g)'\,, \tag{33}$$

$$\tilde{X}^i(\beta_g/g)' - (\tilde{X}^i)'(\beta_g/g) = \tilde{X}^i \sum_l \tilde{X}^l(\gamma_l)' \tag{34}$$

are satisfied. It seems a highly non trivial task to solve these nonlinear ordinary differential equations. On the other hand, there is another constraint coming from the result of [23], given in Eq. (22), for which it is assumed that m_i^2 are also reduced in favor of g and M: It reads

$$\sum_l \tilde{X}^l(\gamma_l)' = -2(1+\tilde{X}^M)(\beta_g/g) + (\beta_g/g)'\,. \tag{35}$$

It can be however shown that Eq. (35) follows from Eq. (33) and (34) so that Eq. (35) is not an independent condition that has to be satisfied by $\tilde{X}^M$ and $\tilde{X}^l$. For a given β_g, it may be in principle possible to solve Eqs. (33), (34) to find $\tilde{X}^M(g)$ and $\tilde{X}^l(g)$. We find that this set of non-linear differential equations can be solved for the β function of Novikov et al. [29] which is given by

$$\beta_g^{\rm NSVZ} = \frac{g^3}{16\pi^2}\left[\frac{\sum_l T(R_l)(1-\gamma_l/2) - 3C(G)}{1-g^2C(G)/8\pi^2}\right], \tag{36}$$

because $\beta_g^{\rm NSVZ}$ has a certain singularity at

$$g^2 = \frac{8\pi^2}{C(G)}\,. \tag{37}$$

We assume that $\tilde{X}^M$ and $\tilde{X}^l$ have a singularity of the form

$$\tilde{X}^M \sim \left(C(G) - 8\pi^2/g^2\right)^{-a},\ \tilde{X}^l \sim \left(C(G) - 8\pi^2/g^2\right)^{-a_l}, \tag{38}$$

and that $\gamma_l(g)$ has no singularity at $g^2 = 8\pi^2/C(G)$. To find a and a_l we derive from Eqs. (34) and (35)

$$(\ln \tilde{X}^l)' = \tilde{X}^M + 1\,, \tag{39}$$

which requires that $a=1$. From Eq. (35) we find that

$$1 \le a_l \le 2\,. \tag{40}$$

Further we find from Eqs. (33) and (35) that the leading singularity should be canceled without the $\tilde{X}^l$ terms in these equations, which fixes a_l also to be one. It is then straightforward to find the desired solution:

$$\tilde{X}^M_{\rm NSVZ} = -\frac{C(G)}{C(G) - 8\pi^2/g^2}\,, \tag{41}$$

$$\tilde{X}^l_{\rm NSVZ} = \frac{T(R_l)}{C(G) - 8\pi^2/g^2}\,, \tag{42}$$

where we have used

$$\sum_l \gamma_l^{\rm NSVZ} T(R_l) = \left(\beta_g^{\rm NSVZ}/g\right)\left(C(G) - \frac{8\pi^2}{g^2}\right) + \tfrac{1}{2}\left[\sum_l T(R_l) - 3C(G)\right]. \tag{43}$$

Therefore, the sum rule (24) in the NSVZ scheme takes form

$$m_i^2 + m_j^2 + m_k^2 = |M|^2\left\{\frac{1}{1-g^2C(G)/(8\pi^2)}\frac{d\ln Y^{ijk}}{d\ln g} + \frac{1}{2}\frac{d^2\ln Y^{ijk}}{d(\ln g)^2}\right\} + \sum_l \frac{m_l^2 T(R_l)}{C(G)-8\pi^2/g^2}\frac{d\ln Y^{ijk}}{d\ln g}\,. \tag{44}$$

This result should be compared with the superstring inspired result for the finite case [7] (i.e. $3C(G) = T(R) = \sum_l T(R_l)$)

$$m_i^2 + m_j^2 + m_k^2 = |M|^2\frac{1}{1-g^2C(G)/(8\pi^2)} + \sum_l \frac{m_l^2 T(R_l)}{C(G)-8\pi^2/g^2}\,, \tag{45}$$

which is valid in a certain class of orbifold models in which the massive string states are organized into $N=4$ supermultiplets, so that they do not contribute to the quantum modification of the kinetic function [28]. So if

$(\ln Y^{ijk})' = 1$, the RG invariant expressions (15) and (45) exactly coincide with the corresponding ones in the superstring models in this particular case.

As a byproduct we obtain the exact β function for m^2 in the NSVZ scheme:

$$\beta_{m_i^2}^{\rm NSVZ} = \left[|M|^2 \left\{ \frac{1}{1-g^2 C(G)/(8\pi^2)} \frac{d}{d\ln g} + \frac{1}{2} \frac{d^2}{d(\ln g)^2} \right\} + \sum_l \frac{m_l^2 T(R_l)}{C(G) - 8\pi^2/g^2} \frac{d}{d\ln g} \right] \gamma_i^{\rm NSVZ} , \tag{46}$$

where we have used Eq. (27), (41) and (42). Note that $\beta_{m_i^2}^{\rm NSVZ}$ assumes the form given in the r.h.s. of Eq. (46) only on the RG invariant surface defined by $Y = Y(g)$ and eq. (15) in the space of parameters. In theories without Yukawa couplings such as supersymmetric QCD, the β function above is valid in the unconstrained space of parameters, and the NSVZ β function above cannot be derived from the result of [23].

4. Conclusions

In the present paper we have shown to all orders in perturbation theory the existence of the RGI sum rule (24) for the soft scalar masses in the SSB sector of $N = 1$ supersymmetric gauge theories exhibiting gauge-Yukawa unification. The all-loop sum rule (24) with (25) substitutes the universal soft scalar masses (which leads to phenomenological problems), while the previously known relation among h's, Y's M and g still hold to all-loop [20,23]. Particularly interesting is the fact that the finite unified theories, which could be made all-loop finite in the supersymmetric sector [14,15,1] can be made completely *finite*, i.e. including the SSB sector, in terms of the soft scalar-mass sum rule (24), generalizing the recent result of Kazakov [20] and relaxing his finiteness conditions.

This very appealing theoretical result complements nicely the successful earlier prediction of the top quark mass [1,2,4] and the recent prediction of the Higgs masses and the s-spectrum [7].

In the NSVZ scheme, the sum rule can be written in a more explicit form (see (44)), exhibiting a definite singularity at $g^2 = 8\pi^2/C(G)$. The same singular behavior in the exact sum rule (45) in a certain class of superstring models has been observed [7]. This result seems to be suggesting a hint for a possible connection among the two kinds of theories.

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5.7 Finite $SU(N)^k$ unification

Title: Finite $SU(N)^k$ unification
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Comment (*Myriam Mondragón, George Zoupanos*)
This is a very interesting investigation since it provides the first example of a Finite Unified Theory based on gauge groups which are not simple. The best model, which is based on the gauge group $SU(3)^3$, is a very attractive gauge theory since being the maximal subgroup of E_6 it has been discussed in several investigations of GUTs, especially in the $N = 1$ supersymmetric ones based on exceptional groups. Moreover, it is a natural GUT obtained from the $N = 1$, 10-dimensional E_8 gauge group of the heterotic string theory [27, 28] and, surpisingly, is the theory obtained in realistic four-dimensional models in which the extra dimensions are non-commutative (fuzzy) manifolds [18].

In the present paper we examined the possibility of constructing realistic Finite Unified Theories based on product gauge groups. In particular, we considered $N = 1$ supersymmetric theories, with gauge groups of the type $SU(N)^1 \times SU(N)^2 \times ... \times SU(N)^k$, with n_f copies (number of families) of the supersymmetric multiplets $(N, \bar{N}, ..., 1) + (1, N, \bar{N}, ..., 1) + +... + (\bar{N}, 1, 1, ..., N)$. The first and very interesting result is that a simple examination of the one-loop β-function coefficient in the renormalization group equation of each $SU(N)$ leads to the result that finiteness at one-loop requires the existence of three families of quarks and leptons for any N and k, which also implies that if one fixes the number of families at three the theory is automatically finite. Then, from phenomenological considerations an $SU(3)^3$ model is singled out. In turn an all-loop and a two-loop finite model based on this gauge group were examined and the predictions concerning the third generation quark masses, the Higgs masses, and the supersymmetric spectrum were found. Although at the time this work was done the prediction of the top quark mass was in agreement with the corresponding experimental measurements, the latest experimental results [13] are challenging this prediction. The same holds now for the prediction of the Higgs mass, which was found to be $\sim 130 - 132$ GeV. There exist however ways to overcome these problems. For instance, so far in the analysis the masses of the new particles of all families appearing in the model were taken to be at the M_{GUT} scale. Taking into account new thresholds for these exotic particles below M_{GUT} one can hope to find a phenomenologically viable parameter space. The details of the predictions of the $SU(3)^3$ are currently under a careful re-analysis in view of the new value of the top-quark mass, the measured Higgs mass the possible new thresholds for the exotic particles, as well as different intermediate gauge symmetry breakings.

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Finite $\mathrm{SU}(N)^k$ unification

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Abstract: We consider $N = 1$ supersymmetric gauge theories based on the group $\mathrm{SU}(N)_1 \times \mathrm{SU}(N)_2 \times \cdots \times \mathrm{SU}(N)_k$ with matter content $(N, N^*, 1, \ldots, 1) + (1, N, N^*, \ldots, 1) + \cdots + (N^*, 1, 1, \ldots, N)$ as candidates for the unification symmetry of all particles. In particular we examine to which extent such theories can become finite and we find that a necessary condition is that there should be exactly three families. We discuss further some phenomenological issues related to the cases $(N, k) = (3, 3)$, $(3, 4)$, and $(4, 3)$, in an attempt to choose those theories that can become also realistic. Thus we are naturally led to consider the $\mathrm{SU}(3)^3$ model which we first promote to an all-loop finite theory and then we study its additional predictions concerning the top quark mass, Higgs mass and supersymmetric spectrum.

Keywords: Supersymmetry Phenomenology, Beyond Standard Model, Supersymmetric Standard Model, GUT.

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Contents

1. Introduction

In the last years there has been a large and sustained effort from the theoretical particle physics community to produce a unified description of all interactions. Two main frameworks have emerged during this endeavor, namely the superstring and the non-commutative geometry. The two approaches, although at a different stage of development, have common unification targets and share similar hopes for exhibiting improved renormalization properties in the ultraviolet (UV) regime as compared to ordinary field theories. Moreover, the two frameworks came closer by the observation that a natural realization of non-commutativity of space appears in the string theory context of D-branes in the presence of a constant background antisymmetric field [1, 2]. However, despite the importance of having frameworks to discuss quantum gravity in a self-consistent way and possibly to construct there finite theories, it is very interesting to search for the minimal realistic framework where finiteness can appear; the history of physics taught us that new ideas might work but usually this happens in the minimal setting. Furthermore, it is interesting to note that non-commutative gauge theories instead of being finite exhibit a curious mixing between the short and long distance modes in their loop expansion, called UV/IR mixing. For a theory to be finite in this framework it has to be finite beforehand in the continuum [3]. The aim for finiteness fulfilling an old theoretical dream remained central in various theoretical efforts over decades even in unrealistic frameworks from the particle physics point of view, such as the supersymmetric N=4 gauge theories starting in '80s [4], up to the AdS/CFT correspondence [5, 6, 7] observed in $AdS_5 \times S_5$ compactification of type IIB superstrings.

In a different context, the main goal expected from a unified description of interactions by the particle physics community is to understand the present day large number of free parameters of the Standard Model (SM) in terms of a few fundamental ones. In other words: to achieve *reduction of couplings* at a more fundamental level.

In our recent studies [8]–[10] we have developed a complementary strategy in searching for a more fundamental theory possibly at the Planck scale, whose basic ingredients are GUTs and supersymmetry, but its consequences certainly go beyond the known ones. Our method consists of hunting for renormalization group invariant (RGI) relations holding below the Planck scale, possibly set or required in a more fundamental theory, which in turn are preserved down to the GUT scale. This programme, called Gauge-Yukawa unification scheme, applied in the dimensionless couplings of supersymmetric GUTs, such as gauge and Yukawa couplings, had already noticable successes by predicting correctly, among others, the top quark mass in the finite SU(5) GUTs. An impressive aspect of the RGI relations is that one can guarantee their validity to all-orders in perturbation theory by studying the uniqueness of the resulting relations at one-loop, as was proven in the early days of the programme of *reduction of couplings* [11]. Even more remarkable is the fact that it is possible to find RGI relations among couplings that guarantee finiteness to all-orders in perturbation theory [12]–[14], including the soft supersymmetry breaking sector.

Here we examine the construction of realistic FUTs based on product gauge groups. In particular we point out that finiteness actually determines the number of families n_f in a class of supersymmetric SU$(N)^k$ gauge theories, namely $n_f = 3$ regardless of N and k. The case $N = 4$ and $k = 3$ was first pointed in ref. [15], and that of arbitrary N and $k = 3$ was discussed in ref. [16], both from the string point of view. Concerning the soft supersymmetry breaking sector of these latter models, although in principle it could be understood too in the same framework under certain assumptions [15, 17, 18], the explicit construction is still missing.

Our search for realistic FUTs based on product groups leads us to choose a supersymmetric SU$(3)^3$ model, which we subsequently promote to an all-loop finite theory, whose predictions we examine further.

The rest of the paper is organised as follows: in section 2 we review the method of reduction of couplings and recall how it is applied in $N = 1$ supersymmetric gauge theories in order to obtain all-loop finite gauge theories. In section 3 we describe the extension of finiteness in the case of soft supersymmetry breaking terms. Section 4 is devoted to a search for realistic FUTs based on product groups, out of which an SU$(3)^3$ supersymmetric gauge theory with three families is singled out. This theory then is further discussed in detail in section 5. Section 6 contains the predictions of the SU$(3)^3$ FUT concerning the top quark mass, the Higgs boson masses and the supersymmetric spectrum.

2. Reduction of couplings and finiteness in $N = 1$ supersymmetric gauge theories

Let us first recall the basic issues concerning *reduction of couplings*, in the case of dimensionless couplings and *finiteness* of $N = 1$ supersymmtric theories.

A RGI relation among couplings g_i,

$$\mathcal{F}(g_1, \ldots, g_N) = 0\,, \tag{2.1}$$

has to satisfy the partial differential equation

$$\mu \, \frac{d\mathcal{F}}{d\mu} = \sum_{i=1}^{N} \beta_i \, \frac{\partial \mathcal{F}}{\partial g_i} = 0 \,, \tag{2.2}$$

where β_i is the β-function of g_i. There exist $(N-1)$ independent $\mathcal{F}$'s, and finding the complete set of these solutions is equivalent to solve the so-called reduction equations (REs) [11],

$$\beta_g \left(\frac{dg_i}{dg} \right) = \beta_i \ , \ i = 1, \ldots, N \,, \tag{2.3}$$

where g and β_g are the primary coupling and its β-function. Using all the $(N-1)\,\mathcal{F}$'s to impose RGI relations, one can in principle express all the couplings in terms of a single coupling g. The complete reduction, which formally preserves perturbative renormalizability, can be achieved by demanding a power series solution, whose uniqueness can be investigated at the one-loop level.

In order to discuss finiteness, it seems unavoidable that we should consider supersymmetric gauge theories. Let us then consider a chiral, anomaly free, $N = 1$ globally supersymmetric gauge theory based on a group G with gauge coupling constant g. The superpotential of the theory is given by

$$W = \frac{1}{2} \, m^{ij} \, \Phi_i \, \Phi_j + \frac{1}{6} \, C^{ijk} \, \Phi_i \, \Phi_j \, \Phi_k \,, \tag{2.4}$$

where m^{ij} and C^{ijk} are gauge invariant tensors and the matter field Φ_i transforms according to the irreducible representation R_i of the gauge group G. All the one-loop β-functions of the theory vanish if $\beta_g^{(1)}$ and all the anomalous dimensions of the superfields $\gamma_i^{j(1)}$ vanish, i.e.

$$\sum_i \ell(R_i) = 3C_2(G) \,, \qquad \frac{1}{2} C_{ipq} C^{jpq} = 2\delta_i^j g^2 C_2(R_i) \,, \tag{2.5}$$

where $l(R_i)$ is the Dynkin index of R_i, and $C_2(G)$, $C_2(R_i)$ are respectively the quadratic Casimir invariant of the adjoint representation of G, and of the R_i representation. A natural question to ask is what happens at higher loop orders. A very interesting result is that the conditions (2.5) are necessary and sufficient for finiteness at the two-loop level [19, 20].

The one- and two-loop finiteness conditions (2.5) restrict considerably the possible choices of the irreps. R_i for a given group G as well as the Yukawa couplings in the superpotential (2.4) [21]–[23]. Note in particular that the finiteness conditions cannot be applied to the supersymmetric standard model (SSM), since the presence of a U(1) gauge group is incompatible with the first of the conditions (2.5), due to $C_2[\mathrm{U}(1)] = 0$. This leads to the expectation that finiteness should be attained at the grand unified level only, the SSM being just the corresponding, low-energy, effective theory.

The finiteness conditions impose relations between gauge and Yukawa couplings. Therefore, we have to guarantee that such relations leading to a reduction of the couplings hold at any renormalization point. The necessary, but also sufficient, condition for this to happen is to require that such relations are solutions to the reduction equations (REs) to all orders. Specifically there exists a very interesting theorem [12] which guarantees the vanishing of the β-functions to all orders in perturbation theory, if we demand reduction of couplings, and that all the one-loop anomalous dimensions of the matter field in the completely and uniquely reduced theory vanish identically.

3. Soft supersymmetry breaking in $N = 1$ FUTS

The above described method of reducing the dimensionless couplings has been extended [24] to the soft supersymmetry breaking (SSB) dimensionful parameters of $N = 1$ supersymmetric theories. In addition it was found [25] that RGI SSB scalar masses in general Gauge-Yukawa unified models satisfy a universal sum rule at one-loop, which was subsequently extended first up to two-loops [9] and then to all-loops [26].

To be more specific, consider the superpotential given by (2.4) along with the lagrangian for SSB terms

$$-\mathcal{L}_{\rm SB} = \frac{1}{6}\, h^{ijk}\, \phi_i\phi_j\phi_k + \frac{1}{2}\, b^{ij}\, \phi_i\phi_j + \frac{1}{2}\, (m^2)^j_i\, \phi^{*\,i}\phi_j + \frac{1}{2}\, M\, \lambda\lambda + \text{h.c.}\,, \tag{3.1}$$

where the ϕ_i are the scalar parts of the chiral superfields Φ_i , λ are the gauginos and M their unified mass. Since we would like to consider only finite theories here, we assume that the one-loop β-function of the gauge coupling g vanishes. We also assume that the reduction equations admit power series solutions of the form $C^{ijk} = g \sum_{n=0} \rho^{ijk}_{(n)} g^{2n}$. According to the finiteness theorem of ref. [12], the theory is then finite to all orders in perturbation theory, if, among others, the one-loop anomalous dimensions $\gamma_i^{j(1)}$ vanish. The one- and two-loop finiteness for h^{ijk} can be achieved [19, 27] by imposing the condition

$$h^{ijk} = -MC^{ijk} + \cdots = -M\rho^{ijk}_{(0)}\, g + O(g^5)\,. \tag{3.2}$$

In addition, it was found [9] that one and two-loop finiteness requires that the following two-loop sum rule for the soft scalar masses has to be satisfied

$$\frac{(\ m_i^2 + m_j^2 + m_k^2\ )}{MM^{\dagger}} = 1 + \frac{g^2}{16\pi^2}\, \Delta^{(2)} + O(g^4)\,, \tag{3.3}$$

where $\Delta^{(2)}$ is the two-loop correction,

$$\Delta^{(2)} = -2\sum_l \left[\left(\frac{m_l^2}{MM^{\dagger}}\right) - \left(\frac{1}{3}\right)\right]\ T(R_l)\,, \tag{3.4}$$

which vanishes for the universal choice [27]. Furthermore, it was found [28] that the relation

$$h^{ijk} = -M(C^{ijk})' \equiv -M\frac{dC^{ijk}(g)}{d\ln g}\,, \tag{3.5}$$

among couplings is all-loop RGI. Moreover, the progress made using the spurion technique leads to all-loop relations among SSB β-functions [10, 28] and [30]–[32], which allowed to find the all-loop RGI sum rule [26] in the Novikov-Shifman-Vainstein-Zakharov scheme [33].

4. Search for realistic FUTs based on product gauge groups

Let us now examine the possibility of constructing realistic FUTs based on product gauge groups. Consider the gauge group $\mathrm{SU}(N)_1 \times \mathrm{SU}(N)_2 \times \cdots \times \mathrm{SU}(N)_k$ with n_f copies of the supersymmetric multiplet $(N, N^*, 1, \ldots, 1) + (1, N, N^*, \ldots, 1) + \cdots + (N^*, 1, 1, \ldots, N)$. The one-loop β-function coefficient in the renormalization-group equation of each $\mathrm{SU}(N)$ gauge coupling is simply given by

$$b = \left(-\frac{11}{3} + \frac{2}{3}\right) N + n_f \left(\frac{2}{3} + \frac{1}{3}\right) \left(\frac{1}{2}\right) 2N = -3N + n_f N\,. \tag{4.1}$$

This means that $n_f = 3$ is a solution of the equation $b = 0$, independently of the values of N and k. Since $b = 0$ is a necessary condition for a finite field theory, the existence of three families of quarks and leptons is natural in such models. (This is true of course only if the matter content is exactly as given above. Other $\mathrm{SU}(N)^k$ models exist with very different, and rather *ad hoc*, supermultiplet structure. They are not included in our discussion.)

Next let us examine if this class of models can meet the obvious requirements in every unified theory, namely (i) that it leads to the SM or the MSSM at low energies, and (ii) that it predicts correctly $sin^2\theta_W$.

Let $N = 3$ and $k = 3$, then we have the well-known example of $\mathrm{SU}(3)_C \times \mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ [34, 35], with quarks transforming as

$$q = \begin{pmatrix} d & u & h \\ d & u & h \\ d & u & h \end{pmatrix} \sim (3, 3^*, 1)\,, \qquad q^c = \begin{pmatrix} d^c & d^c & d^c \\ u^c & u^c & u^c \\ h^c & h^c & h^c \end{pmatrix} \sim (3^*, 1, 3)\,, \tag{4.2}$$

and leptons transforming as

$$\lambda = \begin{pmatrix} N & E^c & \nu \\ E & N^c & e \\ \nu^c & e^c & S \end{pmatrix} \sim (1, 3, 3^*)\,. \tag{4.3}$$

If we switch the first and third rows of q^c together with the first and third columns of λ, we obtain the alternative left-right model first proposed in ref. [36] in the context of superstring-inspired E_6. The breaking down of $\mathrm{SU}(3)^3$ to $\mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{U}(1)_{Y_L+Y_R}$ is achieved with the (3,3) entry of λ, and the further breaking of $\mathrm{SU}(2)_R \times \mathrm{U}(1)_{Y_L+Y_R}$ to $\mathrm{U}(1)_Y$ with the (3,1) entry.

Let $N = 3$ and $k = 4$, then one example is the extension to include the chiral color of ref. [37]. Here $\mathrm{SU}(3)_C$ is split up into $\mathrm{SU}(3)_{CL}$ and $\mathrm{SU}(3)_{CR}$. This implies the existence of a neutral supermultiplet η transforming as (N^*, N) under these two groups. Let $\langle\eta_{11}\rangle = \langle\eta_{22}\rangle = \langle\eta_{33}\rangle$, then $\mathrm{SU}(3)_{CL} \times \mathrm{SU}(3)_{CR}$ breaks back down to $\mathrm{SU}(3)_C$ as desired. However at this scale,

$$\alpha_s^{-1} = \alpha_{sL}^{-1} + \alpha_{sR}^{-1} \tag{4.4}$$

and since α_{sL} and α_{sR} are to be unified with α_L and α_R, the predicted value of α_s would be too small. Thus this is not a candidate model of unification, unless the particle content is also extended [38], in which case finiteness would be lost.

Another possibility to consider is the quartification model of ref. [39]. Here unification is possible but only in the nonsupersymmetric case. In fact, $\sin^2\theta_W = 1/3$ instead of the canonical 3/8, and the unification scale of this model is only 4×10^{11} GeV.

Let us now turn to the interesting $N = 4$ and $k = 3$ case [15]. The obvious choice is $SU(4)_C \times SU(4)_L \times SU(4)_R$, where $SU(4)_C$ is the Pati-Salam color gauge group [40]. In that case, the quarks and leptons should transform as

$$f = \begin{pmatrix} d & u & y & x \\ d & u & y & x \\ d & u & y & x \\ e & \nu & a & v \end{pmatrix} \sim (4, 4^*, 1)\,, \qquad f^c = \begin{pmatrix} d^c & d^c & d^c & e^c \\ u^c & u^c & u^c & \nu^c \\ y^c & y^c & y^c & a^c \\ x^c & x^c & x^c & v^c \end{pmatrix} \sim (4^*, 1, 4)\,. \tag{4.5}$$

We see immediately that there have to be new heavy particles, i.e. the x and y quarks and the v and a leptons. In addition, we need to consider the $h \sim (1, 4, 4^*)$ supermultiplet.

The unification of quarks and leptons within $SU(4)_C$ implies that their electric charge Q should be given by

$$Q = \frac{1}{2}(B - L) + I_{3L} + I_{3R}\,. \tag{4.6}$$

However, the electric charges of the new heavy particles are not yet fixed. This is because SU(4) contains two disjoint SU(2) subgroups, one of which may be the usual $SU(2)_L$ or $SU(2)_R$, but the other is new. Therefore, another valid formula for Q is given by

$$Q = \frac{1}{2}(B - L) + I_{3L} + I_{3R} + I'_{3L} + I'_{3R}\,. \tag{4.7}$$

The quarks and leptons do not transform under $SU(2)'_L$ or $SU(2)'_R$, so their electric charges are not affected.

Using eq. (4.6), the charges of f, f^c, and h are respectively

$$Q_f = \begin{pmatrix} -1/3 & 2/3 & 1/6 & 1/6 \\ -1/3 & 2/3 & 1/6 & 1/6 \\ -1/3 & 2/3 & 1/6 & 1/6 \\ -1 & 0 & -1/2 & -1/2 \end{pmatrix}, \tag{4.8}$$

$$Q_{f^c} = \begin{pmatrix} 1/3 & 1/3 & 1/3 & 1 \\ -2/3 & -2/3 & -2/3 & 0 \\ -1/6 & -1/6 & -1/6 & 1/2 \\ -1/6 & -1/6 & -1/6 & 1/2 \end{pmatrix}, \tag{4.9}$$

$$Q_h = \begin{pmatrix} 0 & 1 & 1/2 & 1/2 \\ -1 & 0 & -1/2 & -1/2 \\ -1/2 & 1/2 & 0 & 0 \\ -1/2 & 1/2 & 0 & 0 \end{pmatrix}. \tag{4.10}$$

Using eq. (4.7), they are instead

$$Q_f = \begin{pmatrix} -1/3 & 2/3 & -1/3 & 2/3 \\ -1/3 & 2/3 & -1/3 & 2/3 \\ -1/3 & 2/3 & -1/3 & 2/3 \\ -1 & 0 & -1 & 0 \end{pmatrix}, \tag{4.11}$$

$$
Q_{f^c} = \begin{pmatrix} 1/3 & 1/3 & 1/3 & -1 \\ -2/3 & -2/3 & -2/3 & 0 \\ 1/3 & 1/3 & 1/3 & -1 \\ -2/3 & -2/3 & -2/3 & 0 \end{pmatrix}, \tag{4.12}
$$

$$
Q_h = \begin{pmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix}. \tag{4.13}
$$

The two different charge assignments result in two different values of

$$
\sin^2\theta_W = \frac{\sum I_{3L}^2}{\sum Q^2} \tag{4.14}
$$

at the unification scale. Whereas it is equal to 3/8 as usual in the former, it becomes 3/14 in the latter, which is not realistic. Therefore we will discuss further only the case with the charge assignments of eqs. (4.8–4.10).

Since we do not admit any other matter supermultiplets, the symmetry breaking of $\mathrm{SU}(4)_C \times \mathrm{SU}(4)_L \times \mathrm{SU}(4)_R$ must be achieved with the vacuum expectation values of the neutral scalar components of f, f^c, and h. The best we can do is to let all the (3,3), (3,4), (4,3), and (4,4) entries of h acquire vacuum expectation values, but then the $\mathrm{SU}(4)^3$ symmetry is only broken down to $\mathrm{SU}(4)_C \times \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{U}(1)_{L+R}$. The extra unwanted U(1) is necessarily present because in the decomposition of $\mathrm{SU}(4)_L$ and $\mathrm{SU}(4)_R$ to their $\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}(1)$ subgroups, the diagonal subgroup $\mathrm{U}(1)_{L+R}$ cannot be broken by the representation $(1, 4, 4^*)$. This problem persists even after the breaking of $\mathrm{SU}(4)_C \times \mathrm{SU}(2)_R$ by the (2,4) entry of f^c to $\mathrm{SU}(3)_C \times \mathrm{U}(1)_Y$.

Since the unbroken U(1) couples to all particles, including the known quarks and leptons, this model cannot be viable phenomenologically. We are thus forced to conclude that $\mathrm{SU}(4)_C \times \mathrm{SU}(4)_L \times \mathrm{SU}(4)_R$ with only the matter content of f, f^c, and h is not a suitable candidate for a finite theory of all particles.

There is another important constraint for a realistic $\mathrm{SU}(N)^k$ theory of quarks and leptons, i.e. the proper masses must be obtained. Excluding naturally nonrenormalizable terms in the superpotential, then only bilinear and trilinear terms are allowed. For the matter content assumed here, it would be zero unless $N = 3$ or $k = 3$. (We exclude $N = 2$ or $k = 2$ for obvious reasons.) If $N = 3$, then we have an invariant from the product of three $(3, 3^*)$ supermultiplets. If $k = 3$, then the invariant $(N, N^*, 1)(1, N, N^*)(N^*, 1, N)$ may be formed. Therefore, this discussion leads us naturally to the case $\mathrm{SU}(3)^3$.

5. An all-loop $\mathrm{SU}(3)^3$ FUT

Here we will discuss in some detail the supersymmetric $\mathrm{SU}(3)^3$ FUT with three families. In general a supersymmetric E_6 model in four dimensions is easily obtained in compactifications of a ten-dimensional E_8, appearing in the heterotic string, over Calabi-Yau spaces [41]. Even more interesting is the possibility to obtain softly broken supersymmetric E_6 type models via coset space dimensional reduction [42, 43] in compactifications using non-symmetric coset spaces [44]. Subsequently the $SU(3)^3$ can emerge using the Wilson fluxes [41, 45] in a straightforward way. What is less obvious to obtain is the spontaneous symmetry breaking of $SU(3)^3$ down to the MSSM, however it has been done already some time ago [46]. It requires introducing eight superfield of the type (λ, q, q^c) and five corresponding mirror superfields $(\bar{\lambda}, \bar{q}, \bar{q}^c)$. The details of this construction are given in ref. [46]. Therefore what remains as an open question is how to obtain the complete and detailed chain of breakings of the ten-dimensional E_8 down to the four-dimensional MSSM, but this is deeply related to the most fundamental problem of string theory, and will not be addressed further here. For our purposes, following [46], we consider a supersymmetric $SU(3)^3$ model with three families holding between the Planck M_P and the unification M_{GUT} scales, which breaks spontaneously down to the MSSM at M_{GUT}.

In order for all the gauge couplings to be equal at M_{GUT}, as is suggested by the LEP results [47], the cyclic symmetry Z_3 must be imposed, i.e.

$$q \to \lambda \to q^c \to q\,, \tag{5.1}$$

where q and q^c are given in eq. (4.2) and λ in eq. (4.3). Then, according to the discussion in section 3, the first of the finiteness conditions (2.5) for one-loop finiteness, namely the vanishing of the gauge β-function is satisfied.

Next let us consider the second condition, i.e. the vanishing of the anomalous dimensions of all superfields. To do that first we have to write down the superpotential. If there is just one family, then there are only two trilinear invariants, which can be constructed respecting the symmetries of the theory, and therefore can be used in the superpotential as follows

$$f\ Tr(\lambda q^c q) + \frac{1}{6} f'\ \epsilon_{ijk}\epsilon_{abc}(\lambda_{ia}\lambda_{jb}\lambda_{kc} + q^c_{ia}q^c_{jb}q^c_{kc} + q_{ia}q_{jb}q_{kc})\,. \tag{5.2}$$

In this case, the condition for vanishing anomalous dimension of each superfield is given by [12, 13, 8, 9, 10]

$$\frac{1}{2}(3|f|^2 + 2|f'|^2) = 2\left(\frac{4}{3}g^2\right)\,. \tag{5.3}$$

Quark and leptons obtain masses when the scalar parts of the superfields $(\tilde{N}, \tilde{N}^c)$ obtain vacuum expectation values (vevs),

$$m_d = f\langle\tilde{N}\rangle,\ \ m_u = f\langle\tilde{N}^c\rangle\,, \qquad m_e = f'\langle\tilde{N}\rangle,\ \ m_\nu = f'\langle\tilde{N}^c\rangle\,. \tag{5.4}$$

With three families, the most general superpotential contains 11 f couplings, and 10 f' couplings, subject to 9 conditions, due to the vanishing of the anomalous dimensions of each superfield. The conditions are the following

$$\sum_{j,k} f_{ijk}(f_{ljk})^* + \frac{2}{3}\sum_{j,k} f'_{ijk}(f'_{ljk})^* = \frac{16}{9}g^2\delta_{il}\,, \tag{5.5}$$

where

$$f_{ijk} = f_{jki} = f_{kij}\,, \tag{5.6}$$

$$f'_{ijk} = f'_{jki} = f'_{kij} = f'_{ikj} = f'_{kji} = f'_{jik}\,. \tag{5.7}$$

Quarks and leptons receive masses when the scalar part of the superfields $\tilde{N}_{1,2,3}$ and $\tilde{N}^c_{1,2,3}$ obtain vevs as follows

$$(\mathcal{M}_d)_{ij} = \sum_k f_{kij} \langle \tilde{N}_k \rangle \,, \qquad (\mathcal{M}_u)_{ij} = \sum_k f_{kij} \langle \tilde{N}^c_k \rangle \,, \tag{5.8}$$

$$(\mathcal{M}_e)_{ij} = \sum_k f'_{kij} \langle \tilde{N}_k \rangle \,, \qquad (\mathcal{M}_\nu)_{ij} = \sum_k f'_{kij} \langle \tilde{N}^c_k \rangle \,. \tag{5.9}$$

Since we want to have, among other conditions, gauge coupling unification, we will assume that the particle content of our finite $\mathrm{SU}(3)^3$ model below M_U is that of the MSSM with three fermion families, but only two Higgs doublets. Therefore we have to choose the linear combinations $\tilde{N}^c = \sum_i a_i \tilde{N}^c_i$ and $\tilde{N} = \sum_i b_i \tilde{N}_i$ to play the role of the two Higgs doublets, which will be responsible for the electroweak symmetry breaking. This can be done by choosing appropriately the masses in the superpotential [23], since they are not constrained by the finiteness conditions. Moreover, we choose that the two Higgs doublets are predominately coupled to the third generation. Then these two Higgs doublets couple to the three families differently, thus providing the freedom to understand their different masses and mixings.

Assuming for our purposes here that all f' couplings vanish[1] an isolated solution eq. (5.5) is

$$f^2 = f^2_{111} = f^2_{222} = f^2_{333} = \frac{16}{9} g^2 \,. \tag{5.10}$$

Hence we start at M_{GUT} with different Yukawa couplings for all the quarks

$$f_t = f a_3 \,, \qquad f_c = f a_2 \,, \qquad f_u = f a_1 \,, \tag{5.11}$$

$$f_b = f b_3 \,, \qquad f_s = f b_2 \,, \qquad f_d = f b_1 \,, \tag{5.12}$$

which is similar to the MSSM except that f is fixed by finiteness at M_{GUT}, and $a_3 \simeq 1$, $b_3 \simeq 1$, by construction, and therefore we have that $f_t \simeq f_b \simeq f$ at M_{GUT}. As for the lepton masses, because all f' couplings have been fixed to be zero at this order, in principle they are expected to appear radiatively induced by the scalar lepton masses appearing in the SSB sector of the theory. Unfortunately though, due to the finiteness conditions (3.2) they cannot appear radiatively and remain as a problem for further study. On the other hand it should be stressed that we can certainly let f' be non-vanishing in eq. (5.5) and thus introduce lepton masses in the model. Then the real price to be paid is basically aesthetic since the model in turn becomes finite only up to two-loops since the corresponding solution of eq. (5.5) is not an isolated one any more. However, given that the analysis we do in the next section takes into account RGEs up to two-loops, there is no practical cost in introducing non-zero f'. We include this possibility in our analysis in section 6.

Although we present the results of a more complete analysis in the next section, we find instructive to describe here the situation concerning the top quark mass prediction at

[1] In supersymmetric theories this can always be done due to the non-renormalization theorem [48], which guarantees that these terms will not appear radiatively. In general this is not the case in the presence of supersymmetry breaking terms, however finiteness imposes tight conditions in this respect too.

one-loop level ignoring the SSB sector. In this approximate analysis, we run the MSSM renormalization group equations at one-loop, using our boundary condition $f^2 = (16/9)g^2$ at the M_{GUT} scale as follows

$$8\pi^2 \left(\frac{dg_3^2}{dt}\right) = -3g_3^4\,, \tag{5.13}$$

$$8\pi^2 \left(\frac{dg_2^2}{dt}\right) = g_2^4\,, \tag{5.14}$$

$$8\pi^2 \left(\frac{dg_1^2}{dt}\right) = \frac{33}{5}g_1^4\,, \tag{5.15}$$

$$8\pi^2 \left(\frac{df_t^2}{dt}\right) = f_t^2\left(6f_t^2 + f_b^2 - \frac{16}{3}g_3^2 - 3g_2^2 - \frac{13}{15}g_1^2\right), \tag{5.16}$$

$$8\pi^2 \left(\frac{df_b^2}{dt}\right) = f_b^2\left(6f_b^2 + f_t^2 - \frac{16}{3}g_3^2 - 3g_2^2 - \frac{7}{15}g_1^2\right). \tag{5.17}$$

The g_i^2s are easily solved as functions of $t = \ln(M_{GUT}/M)$:

$$\alpha_3(M)^{-1} = \alpha_3(M_{\rm GUT})^{-1} - \left(\frac{3}{2\pi}\right)\ln\left(\frac{M_{\rm GUT}}{M}\right), \tag{5.18}$$

$$\alpha_2(M)^{-1} = \alpha_2(M_{\rm GUT})^{-1} + \left(\frac{1}{2\pi}\right)\ln\left(\frac{M_{\rm GUT}}{M}\right), \tag{5.19}$$

$$\alpha_1(M)^{-1} = \alpha_1(M_{\rm GUT})^{-1} + \left(\frac{33}{10\pi}\right)\ln\left(\frac{M_{\rm GUT}}{M}\right), \tag{5.20}$$

where $\alpha_i = g_i^2/4\pi$. Using the MSSM boundary conditions from the unification of the gauge couplings at one-loop and the constraints of the present model we have

$$\alpha_i(M_{\rm GUT}) = 0.0413\,, \tag{5.21}$$

$$\alpha_t(M_{\rm GUT}) = \alpha_b(M_{\rm GUT}) = \left(\frac{16}{9}\right)\alpha_i(M_{\rm GUT})\,. \tag{5.22}$$

Then we integrate the two differential equations (5.16) and (5.17), from $t = \ln(M_{\rm GUT}/M_{\rm EW})$ to $t = 0$, to determine f_t and f_b at the electroweak scale M_{EW}. Then $m_t = f_t v_u$ and $m_b = f_b v_d$, with v_u and V_d satisfying the condition $v_u^2 + v_d^2 = v^2$, $v = 174.3$ GeV. Thus given m_b, we can obtain m_t.

6. Predictions and conclusions

The gauge symmetry $\mathrm{SU}(3)^3$ is spontaneously broken down to the MSSM at $M_{\rm GUT}$, and the finiteness conditions do not restrict the renormalization properties at low energies. Therefore, below M_{GUT} all couplings and masses of the theory run according to the RGEs of the MSSM. The remnants of the all-loop FUT $\mathrm{SU}(3)^3$ are the boundary conditions on the gauge and Yukawa couplings (5.10), the $h = -MC$ relation, and the soft scalar-mass sum rule (3.3) at $M_{\rm GUT}$, which, when applied to the present model, takes the form

$$m_{H_u}^2 + m_{\tilde{t}^c}^2 + m_{\tilde{q}}^2 = M^2 \tag{6.1}$$

$$m_{H_d}^2 + m_{\tilde{b}^c}^2 + m_{\tilde{q}}^2 = M^2\,. \tag{6.2}$$

Thus we examine the evolution of these parameters according to their RGEs up to two-loops for dimensionless parameters and at one-loop for dimensionful ones imposing the corresponding boundary conditions. We further assume a unique supersymmetry breaking scale M_s (defined as the average of the mass of the stops) and therefore below that scale the effective theory is just the SM.

We consider two versions of the model:

I) The all-loop finite one in which f' vanishes and eq. (5.10) holds.

II) A two-loop finite version, in which we keep f' non-vanishing in eq. (5.5), and we use it to introduce the lepton masses.

The predictions for the top quark mass m_t are $\sim 183\,\text{GeV}$ for $\mu < 0$ in model I, whereas for model II it is $176 - -179\,\text{GeV}$ for $\mu < 0$, and $170 - -173\,\text{GeV}$ for $\mu > 0$. Recall that the bottom quark mass m_b is an input in FUT I and m_τ in FUT II.

Comparing these predictions with the most recent experimental value $m_t^{exp} = (178.0 \pm 4.3)\,\text{GeV}$ [49], and recalling that the theoretical values for m_t may suffer from a correction of $\sim 4\%$ [10], we see that they are consistent with the experimental data.

In the SSB sector, besides the constraints imposed by finiteness we further require

1. successful radiative electroweak symmetry breaking, and
2. $m^2_{\tilde{\tau},\tilde{b},\tilde{t}} > 0$.

As an additional constraint, we take into account the $BR(b \to s\gamma)$ [50]. We do not take into account, though, constraints coming from the muon anomalous magnetic moment (g-2) in this work, which would exclude a small region of the parameter space.

Our numerical analysis shows the following results for the two models: In the case of FUT I it is possible to find regions of parameter space which comply with all the above requirements both for the case where we have universal boundary conditions ($m_i^2 = m_j^2 = m_k^2 = M^2/3$), and for the case where we apply the sum rule eq. (3.3). In the case of universal boundary conditions and $\mu < 0$, $m_t \sim 183$ GeV, the Higgs mass is $\sim 131 - 132$ GeV, $\tan\beta \sim 50 - 51$, and the spectrum is rather heavy, the allowed region of parameter space starting with an LSP which is a neutralino $m_{\chi^0} \sim 825$ GeV for a value of $M \sim 1800\,\text{GeV}$. In the case the sum rule is applied we have one more free parameter, which is $m_{\tilde{q}^c} = m_{\tilde{q}}$ at the GUT scale. In this case we obtain a $\tan\beta \sim 47 - 54$, and the Higgs mass is $\sim 130 - 132$ GeV. The main difference between the universal boundary conditions and the sum rule comes in the sparticle spectrum, which can now start with an LSP at $m_{\chi^0} \sim 450\,\text{GeV}$, for a boundary condition of $M \sim 1800\,\text{GeV}$. In the case that $\mu > 0$ we do not find solutions which satisfy all the above requirements.

In the second version of the model FUT II, we have the following boundary conditions for the Yukawa couplings

$$f^2 = r\left(\frac{16}{9}\right)g^2\,, \tag{6.3}$$

$$f'^2 = (1-r)\left(\frac{8}{3}\right)g^2\,. \tag{6.4}$$

In this case, we do not have an all-loop finite model, since the solution is a parametric one, but it is the price we pay to give masses to the leptons. As for the boundary conditions of the soft scalars, we only have the universal case. This is because, applying the sum rule (3.3) to the superpotential with $f' \neq 0$ implies that $m_q^2 = m_{q^c}^2 = m_{H^{u,d}}^2 = M^2/3$, which is again the universal boundary condition. For the numerical analysis we fix the m_τ mass to obtain m_t and m_b. Taking $\mu < 0$, and for the experimentally allowed value of $m_b(m_b) = 4.1 - 4.4\,\mathrm{GeV}$ [51], the value of m_t goes from $\sim 176 - 179\,\mathrm{GeV}$. In this case $\tan\beta \sim 48 - 53$, and $m_H \sim 122 - 129\,\mathrm{GeV}$, with a charged LSP $m_{\tilde\tau} \sim 400 - 1000\,\mathrm{GeV}$, depending directly on the value of M, which varies from $\sim 1200 - 2200\,\mathrm{GeV}$ in this case.

Now for $\mu > 0$, the value of m_t compatible with the experimentally allowed value of m_b, goes from $\sim 170 - 173\,\mathrm{GeV}$, clearly the preferred value being the latter. For this range of values of m_t we obtain $\tan\beta \sim 58 - 62$, and $m_H \sim 120 - 125\,\mathrm{GeV}$, also with a charged LSP $m_{\tilde\tau} \sim 300 - 600\,\mathrm{GeV}$, again depending directly on the value of M, which varies from $\sim 1300 - 2000\,\mathrm{GeV}$.

We could go further and consider another version of the $\mathrm{SU}(3)^3$ model. For instance, if we impose global SU(3) as a family symmetry [16, 52], then there is only one Yukawa coupling in the superpotential, which leads to the following unique relation among Yukawa and gauge couplings

$$f^2 = \frac{8}{9} g^2 \,. \tag{6.5}$$

However both $\mathcal{M}_u$ and $\mathcal{M}_d$ in eq. (5.8) must now be antisymmetric in family space, resulting in one zero and two equal mass eigenvalues for each, which is not a realistic case. Note moreover, that the terms proportional to f' in the superpotential eq. (5.2) are not allowed to appear in the cases of refs. [15, 16] unless $N = 3$, and therefore they share the problem of the FUT I model, where we have chosen $f' = 0$.

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5.8 Confronting finite unified theories with low energy phenomenology

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Authors: S. Heinemeyer, M. Mondragon, G. Zoupanos
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Comment (*Sven Heinemeyer*)
After many years of theoretical preparation, finite unified theories were ready to be confronted with phenomenology and experimental results: the present paper is devoted to this aim. From the classification of theories with vanishing one-loop gauge β function, one can see that there exist only two candidate possibilities to construct $SU(5)$ GUTs with three generations. These possibilities require that the theory should contain as matter fields the chiral supermultiplets $\mathbf{5}$, $\mathbf{5}$, $\mathbf{10}$, $\bar{\mathbf{5}}$, $\mathbf{24}$ with the multiplicities (6, 9, 4, 1, 0) and (4, 7, 3, 0, 1), respectively. Only the second one contains a 24-plet which can be used to provide the spontaneous symmetry breaking of $SU(5)$ down to $SU(3) \times SU(2) \times U(1)$. The particle content of the models under consideration consists of the following supermultiplets: three ($\bar{\mathbf{5}} + \mathbf{10}$), needed for each of the three generations of quarks and leptons, four ($\bar{\mathbf{5}} + \mathbf{5}$) and one $\mathbf{24}$ considered as Higgs supermultiplets. When the gauge group of the finite GUT is broken the theory is no longer finite, and one then assumes that one is left with the MSSM.
Two versions of the model were possible originally, labeled **A** and **B**. The main difference between model **A** and model **B** is that two pairs of Higgs quintets and anti-quintets couple to the $\mathbf{24}$ in **B**, so that it is not necessary to mix them with H_4 and $\bar{H}_4$ in order to achieve the triplet-doublet splitting after the symmetry breaking of $SU(5)$.
Confronting those two models with the quark mass predictions for m_t showed that only model **B** can accomodate a top quark mass of about 173 GeV, while model **A** predicted consistently $m_t \sim 183$ GeV. Investigating the two signs of the μ parameter revealed that only $\mu < 0$ predicts a bottom quark mass value in the correct range, whereas the positive sign of μ results in m_b values more than 1 GeV too high. In this way the $SU(5)$ model **FUTB** was singled out as the only phenomenological viable option. Confronting the model predictions with the measured value of BR($b \to s\gamma$) and the (then valid) upper limit on BR($B_s \to \mu^+\mu^-$) further restricted the allowed parameter space.
The "surviving" parameter space was then used to predict the Higgs and the SUSY spectrum to be expected in the LHC searches. The light MSSM Higgs boson mass was predicted in a very narrow range of

$$M_h^{\text{predicted}} = 121 \ldots 126 \text{ GeV} ,$$

to which a ± 3 GeV theory uncertainty has to be added. The mass scale of the heavy Higgs bosons was predicted to be between ~ 500 GeV and the multi-10-TeV range. The lightest observable SUSY particle, either the light scalar tau or the second lightest neutralino, was predicted in the range between 500 GeV and ~ 4000 GeV, where the lighter regions was prefered by the prediction of cold dark matter. Finally, the colored particles were predicted in the range between ~ 2 TeV and ~ 15 TeV, where only the lighter part of the spectrum would allow a discovery at the LHC. These predictions now eagerly awaited the start of the LHC and the experimental data on Higgs and SUSY searches.

Confronting finite unified theories with low-energy phenomenology

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Confronting finite unified theories with low-energy phenomenology

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Abstract: Finite Unified Theories (FUTs) are $N = 1$ supersymmetric Grand Unified Theories that can be made all-loop finite. The requirement of all-loop finiteness leads to a severe reduction of the free parameters of the theory and, in turn, to a large number of predictions. FUTs are investigated in the context of low-energy phenomenology observables. We present a detailed scanning of the all-loop finite SU(5) FUTs, where we include the theoretical uncertainties at the unification scale and we apply several phenomenological constraints. Taking into account the restrictions from the top and bottom quark masses, we can discriminate between different models. Including further low-energy constraints such as B physics observables, the bound on the lightest Higgs boson mass and the cold dark matter density, we determine the predictions of the allowed parameter space for the Higgs boson sector and the supersymmetric particle spectrum of the selected model.

Keywords: Supersymmetric Standard Model, GUT, Supersymmetry Phenomenology.

Contents

1. Introduction

A large and sustained effort has been done in the recent years aiming to achieve a unified description of all interactions. Out of this endeavor two main directions have emerged as the most promising to attack the problem, namely, the superstring theories and non-commutative geometry. The two approaches, although at a different stage of development, have common unification targets and share similar hopes for exhibiting improved renormalization properties in the ultraviolet(UV) as compared to ordinary field theories. Moreover the two frameworks came closer by the observation that a natural realization of non-commutativity of space appears in the string theory context of D-branes in the presence of a constant background antisymmetric field [1]. However, despite the importance of having frameworks to discuss quantum gravity in a self-consistent way and possibly to construct there finite theories, it is very interesting to search for the minimal realistic framework in which finiteness can take place. In addition the main goal expected from a unified description of interactions by the particle physics community is to understand the present day large number of free parameters of the Standard Model (SM) in terms of a few fundamental ones. In other words, to achieve *reduction of couplings* at a more fundamental level. A complementary, and certainly not contradicting, program has been developed [2 − 4] in searching for a more fundamental theory possibly at the Planck scale called Finite Unified Theories (FUTs), whose basic ingredients are field theoretical Grand Unified Theories (GUTs) and supersymmetry (SUSY), but its consequences certainly go beyond the known ones.

Finite Unified Theories are $N = 1$ supersymmetric GUTs which can be made finite even to all-loop orders, including the soft supersymmetry breaking sector. The method to construct GUTs with reduced independent parameters [5, 6] consists of searching for renormalization group invariant (RGI) relations holding below the Planck scale, which in turn are preserved down to the GUT scale. Of particular interest is the possibility to find RGI relations among couplings that guarantee finiteness to all-orders in perturbation theory [7, 8]. In order to achieve the latter it is enough to study the uniqueness of the solutions to the one-loop finiteness conditions [7 − 9]. The constructed *finite unified* $N = 1$ supersymmetric GUTs, using the above tools, predicted correctly from the dimensionless sector (Gauge-Yukawa unification), among others, the top quark mass [2]. The search for RGI relations and finiteness has been extended to the soft supersymmetry breaking sector (SSB) of these theories [10 − 19], which involves parameters of dimension one and two. Eventually, the full theories can be made all-loop finite and their predictive power is extended to the Higgs sector and the SUSY spectrum. This, in turn, allows to make predictions for low-energy precision and astrophysical observables. The purpose of the present article is to do an exhaustive search of these latter predictions of the SU(5) finite models, taking into account the restrictions resulting from the low-energy observables. Then we present the predictions of the models under study for the parameter space that is still allowed after taking the phenomenological restrictions into account. Here we focus on the Higgs boson sector and the SUSY spectrum.

In our search we consider the restrictions imposed on the parameter space of the models due to the following observables: the 3rd generation quark masses, rare b decays, BR($b \to s\gamma$) and BR($B_s \to \mu^+\mu^-$), as well as the mass of the lightest $\mathcal{CP}$-even Higgs boson, M_h. Present data on these observables already provide interesting information about the allowed SUSY mass scales. The non-discovery of the Higgs boson at LEP [20, 21] excludes a part of the otherwise allowed parameter space. However the non-discovery of supersymmetric particles at LEP does not impose any restrictions on the parameter space of the models, given that their SUSY spectra turn out to be very heavy anyway. An important further constraint is provided by the density of dark matter in the Universe, which is tightly constrained by WMAP and other astrophysical and cosmological data [22], assuming that the dark matter consists largely of neutralinos [23]. We also briefly discuss the implication from the anomalous magnetic moment of the muon, $(g-2)_\mu$. Other recent analyses of GUT based models confronted with low-energy observables and dark matter constraints can be found in refs. [24, 25].

In this context we first review the sensitivity of each observable to indirect effects of supersymmetry, taking into account the present experimental and theoretical uncertainties. Later on we investigate the part of parameter space in the FUT models under consideration that is still allowed taking into account all low-energy observables.

In section 2 of the paper we review the conditions of finiteness in $N = 1$ SUSY gauge theories. The consequences of finiteness for the soft SUSY-breaking terms are discussed in section 3. The two SU(5) FUT models that emerge are briefly presented in section 4. In section 5 we discuss different precision observables, including the cold dark matter constraint. section 6 contains the analysis of the parts of parameter space that survive all constraints and the final predictions of the models. We conclude with section 7.

2. Reduction of couplings and finiteness in $N = 1$ SUSY gauge theories

Here we review the main points and ideas concerning the *reduction of couplings* and *finiteness* in $N = 1$ supersymmetric theories. A RGI relation among couplings g_i, $\Phi(g_1, \cdots, g_N)=0$, has to satisfy the partial differential equation $\mu \; d\Phi/d\mu = \sum_{i=1}^{N} \beta_i \, \partial\Phi/\partial g_i = 0$, where β_i is the β-function of g_i. There exist $(N-1)$ independent Φ's, and finding the complete set of these solutions is equivalent to solve the so-called reduction equations (REs) [5], $\beta_g \, (dg_i/dg) = \beta_i \, , \; i = 1, \ldots, N$, where g and β_g are the primary coupling (in favor of which all other couplings are reduced) and its β-function. Using all the $(N-1)\,\Phi$'s to impose RGI relations, one can in principle express all the couplings in terms of a single coupling g. The complete reduction, which formally preserves perturbative renormalizability, can be achieved by demanding a power series solution, whose uniqueness can be investigated at the one-loop level.

Finiteness can be understood by considering a chiral, anomaly free, $N = 1$ globally supersymmetric gauge theory based on a group G with gauge coupling constant g. The superpotential of the theory is given by

$$W = \frac{1}{2} \, m^{ij} \, \Phi_i \, \Phi_j + \frac{1}{6} \, C^{ijk} \, \Phi_i \, \Phi_j \, \Phi_k \, , \tag{2.1}$$

where m^{ij} (the mass terms) and C^{ijk} (the Yukawa couplings) are gauge invariant tensors and the matter field Φ_i transforms according to the irreducible representation R_i of the gauge group G.

The one-loop β-function of the gauge coupling g is given by

$$\beta_g^{(1)} = \frac{dg}{dt} = \frac{g^3}{16\pi^2} \, [\sum_i \, \ell(R_i) - 3 \, C_2(G) \,] \, , \tag{2.2}$$

where $\ell(R_i)$ is the Dynkin index of R_i and $C_2(G)$ is the quadratic Casimir of the adjoint representation of the gauge group G. The β-functions of C^{ijk}, by virtue of the non-renormalization theorem, are related to the anomalous dimension matrix γ_i^j of the matter fields Φ_i as

$$\beta_C^{ijk} = \frac{d}{dt} \, C^{ijk} = C^{ijp} \sum_{n=1} \frac{1}{(16\pi^2)^n} \, \gamma_p^{k(n)} + (k \leftrightarrow i) + (k \leftrightarrow j) \; . \tag{2.3}$$

At one-loop level γ_i^j is given by

$$\gamma_i^{j(1)} = \frac{1}{2} C_{ipq} C^{jpq} - 2\, g^2\, C_2(R_i) \delta_i^j \, , \tag{2.4}$$

where $C_2(R_i)$ is the quadratic Casimir of the representation R_i, and $C^{ijk} = C^*_{ijk}$.

All the one-loop β-functions of the theory vanish if the β-function of the gauge coupling $\beta_g^{(1)}$, and the anomalous dimensions $\gamma_i^{j(1)}$, vanish, i.e.

$$\sum_i \ell(R_i) = 3C_2(G) \, , \; \frac{1}{2} C_{ipq} C^{jpq} = 2\delta_i^j g^2 C_2(R_i). \tag{2.5}$$

A very interesting result is that the conditions (2.5) are necessary and sufficient for finiteness at the two-loop level [9, 13].

The one- and two-loop finiteness conditions (2.5) restrict considerably the possible choices of the irreducible representations R_i for a given group G as well as the Yukawa couplings in the superpotential (2.1). Note in particular that the finiteness conditions cannot be applied to the supersymmetric standard model (SSM). The presence of a U(1) gauge group, whose $C_2[\mathrm{U}(1)] = 0$, makes impossible to satisfy the condition (2.5). This leads to the expectation that finiteness should be attained at the grand unified level only, the SSM being just the corresponding low-energy, effective theory.

The finiteness conditions impose relations between gauge and Yukawa couplings. Therefore, we have to guarantee that such relations leading to a reduction of the couplings hold at any renormalization point. The necessary, but also sufficient, condition for this to happen is to require that such relations are solutions to the reduction equations (REs) to all orders. The all-loop order finiteness theorem of [7] is based on: (a) the structure of the supercurrent in $N = 1$ SYM and on (b) the non-renormalization properties of $N = 1$ chiral anomalies. Alternatively, similar results can be obtained [8, 26] using an analysis of the all-loop NSVZ gauge beta-function [27].

3. Soft supersymmetry breaking and finiteness

The above described method of reducing the dimensionless couplings has been extended [10] to the soft supersymmetry breaking (SSB) dimensionful parameters of $N = 1$ supersymmetric theories. In addition it was found [11] that RGI SSB scalar masses in general Gauge-Yukawa unified models satisfy a universal sum rule at one-loop, which was subsequently extended first up to two-loops [3] and then to all-loops [12].

To be more specific, consider the superpotential given by (2.1) along with the Lagrangian for SSB terms

$$-\mathcal{L}_{\mathrm{SB}} = \frac{1}{6}\, h^{ijk}\, \phi_i \phi_j \phi_k + \frac{1}{2}\, b^{ij}\, \phi_i \phi_j + \frac{1}{2}\, (m^2)^j_i\, \phi^{*\,i} \phi_j + \frac{1}{2}\, M\, \lambda\lambda + \mathrm{h.c.} \, , \tag{3.1}$$

where the ϕ_i are the scalar parts of the chiral superfields Φ_i , λ are the gauginos and M their unified mass. Since we would like to consider only finite theories here, we assume that the one-loop β-function of the gauge coupling g vanishes. We also assume that the reduction equations admit power series solutions of the form $C^{ijk} = g\,\sum_{n=0} \rho^{ijk}_{(n)} g^{2n}$. According to the finiteness theorem of ref. [7], the theory is then finite to all orders in perturbation theory, if, among others, the one-loop anomalous dimensions $\gamma_i^{j(1)}$ vanish. The one- and two-loop finiteness for h^{ijk} can be achieved [9, 13] by imposing the condition

$$h^{ijk} = -MC^{ijk} + \cdots = -M\rho^{ijk}_{(0)}\,g + O(g^5)\,. \tag{3.2}$$

In addition, it was found [3] that one and two-loop finiteness requires that the following two-loop sum rule for the soft scalar masses has to be satisfied

$$\frac{(\ m_i^2 + m_j^2 + m_k^2\ )}{MM^\dagger} = 1 + \frac{g^2}{16\pi^2}\,\Delta^{(2)} + O(g^4)\,, \tag{3.3}$$

where $\Delta^{(2)}$ is the two-loop correction,

$$\Delta^{(2)} = -2\sum_i \left[\left(\frac{m_i^2}{MM^\dagger}\right) - \left(\frac{1}{3}\right)\right]\ \ell(R_i)\,, \tag{3.4}$$

which vanishes for the universal choice [13], as well as in the models we consider in the next section. Furthermore, it was found [14] that the relation

$$h^{ijk} = -Mg(C^{ijk})' \equiv -Mg\frac{dC^{ijk}(g)}{d\ln g}\,, \tag{3.5}$$

among couplings is all-loop RGI. Moreover, the progress made using the spurion technique leads to all-loop relations among SSB β-functions [4, 14] and [16 − 19], which allowed to find the all-loop RGI sum rule [12] in the Novikov-Shifman-Vainstein-Zakharov scheme [27].

4. Finite unified theories

Finite Unified Theories (FUTs) have always attracted interest for their intriguing mathematical properties and their predictive power. One very important result is that the one-loop finiteness conditions (2.5) are sufficient to guarantee two-loop finiteness [28]. A classification of possible one-loop finite models was done independently by several authors [29]. The first one and two-loop finite SU(5) model was presented in [30], and shortly afterwards the conditions for finiteness in the soft SUSY-breaking sector at one-loop [9] were given. In [31] a one and two-loop finite SU(5) model was presented where the rotation of the Higgs sector was proposed as a way of making it realistic. The first all-loop finite theory was studied in [2], without taking into account the soft breaking terms. Finite soft breaking terms and the proof that one-loop finiteness in the soft terms also implies two-loop finiteness was done in [13]. The inclusion of soft breaking terms in a realistic model was done in [33] and their finiteness to all-loops studied in [34], although the universality of the soft breaking terms lead to a charged LSP. This fact was also noticed in [35], where the inclusion of an extra parameter in the Higgs sector was introduced to alleviate it. The derivation of the sum-rule in the soft supersymmetry breaking sector and the proof that it can be made all-loop finite were done in [12, 36, 30, 31], allowing thus for the construction of all-loop finite realistic models.

From the classification of theories with vanishing one-loop gauge β function [29], one can easily see that there exist only two candidate possibilities to construct SU(5) GUTs with three generations. These possibilities require that the theory should contain as matter fields the chiral supermultiplets $\mathbf{5}$, $\overline{\mathbf{5}}$, $\mathbf{10}$, $\overline{\mathbf{5}}$, $\mathbf{24}$ with the multiplicities $(6, 9, 4, 1, 0)$ and $(4, 7, 3, 0, 1)$, respectively. Only the second one contains a **24**-plet which can be used to provide the spontaneous symmetry breaking (SB) of SU(5) down to $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$. For the first model one has to incorporate another way, such as the Wilson flux breaking mechanism to achieve the desired SB of SU(5) [2]. Therefore, for a self-consistent field theory discussion we would like to concentrate only on the second possibility.

The particle content of the models we will study consists of the following supermultiplets: three $(\overline{\mathbf{5}} + \mathbf{10})$, needed for each of the three generations of quarks and leptons, four $(\overline{\mathbf{5}} + \mathbf{5})$ and one $\mathbf{24}$ considered as Higgs supermultiplets. When the gauge group of the finite GUT is broken the theory is no longer finite, and we will assume that we are left with the MSSM.

Therefore, a predictive Gauge-Yukawa unified SU(5) model which is finite to all orders, in addition to the requirements mentioned already, should also have the following properties:

1. One-loop anomalous dimensions are diagonal, i.e., $\gamma_i^{(1)\,j} \propto \delta_i^j$.
2. The three fermion generations, in the irreducible representations $\overline{\mathbf{5}}_i, \mathbf{10}_i$ $(i = 1, 2, 3)$, should not couple to the adjoint $\mathbf{24}$.
3. The two Higgs doublets of the MSSM should mostly be made out of a pair of Higgs quintet and anti-quintet, which couple to the third generation.

In the following we discuss two versions of the all-order finite model. The model of ref. [2], which will be labeled **A**, and a slight variation of this model (labeled **B**), which can also be obtained from the class of the models suggested in ref. [37] with a modification to suppress non-diagonal anomalous dimensions [3].

The superpotential which describes the two models before the reduction of couplings takes places is of the form [2, 36, 30, 31]

$$W = \sum_{i=1}^{3} \left[\frac{1}{2} g_i^u \, \mathbf{10}_i \mathbf{10}_i H_i + g_i^d \, \mathbf{10}_i \overline{\mathbf{5}}_i \, \overline{H}_i \right] + g_{23}^u \, \mathbf{10}_2 \mathbf{10}_3 H_4 \qquad (4.1)$$
$$+ g_{23}^d \, \mathbf{10}_2 \overline{\mathbf{5}}_3 \, \overline{H}_4 + g_{32}^d \, \mathbf{10}_3 \overline{\mathbf{5}}_2 \, \overline{H}_4 + \sum_{a=1}^{4} g_a^f \, H_a \, \mathbf{24} \, \overline{H}_a + \frac{g^\lambda}{3} \, (\mathbf{24})^3 \, ,$$

where H_a and $\overline{H}_a$ $(a = 1, \ldots, 4)$ stand for the Higgs quintets and anti-quintets.

We will investigate two realizations of the model, labelled **A** and **B**. The main difference between model **A** and model **B** is that two pairs of Higgs quintets and anti-quintets couple to the **24** in **B**, so that it is not necessary to mix them with H_4 and $\overline{H}_4$ in order to achieve the triplet-doublet splitting after the symmetry breaking of SU(5) [3]. Thus, although the particle content is the same, the solutions to eq. (2.5) and the sum rules are different, which will reflect in the phenomenology, as we will see.

	$\overline{\mathbf{5}}_1$	$\overline{\mathbf{5}}_2$	$\overline{\mathbf{5}}_3$	$\mathbf{10}_1$	$\mathbf{10}_2$	$\mathbf{10}_3$	H_1	H_2	H_3	H_4	$\overline{H}_1$	$\overline{H}_2$	$\overline{H}_3$	$\overline{H}_4$	$\mathbf{24}$
Z_7	4	1	2	1	2	4	5	3	6	-5	-3	-6	0	0	0
Z_3	0	0	0	1	2	0	1	2	0	-1	-2	0	0	0	0
Z_2	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0

Table 1: Charges of the $Z_7 \times Z_3 \times Z_2$ symmetry for Model **FUTA**.

4.1 FUTA

After the reduction of couplings the symmetry of the superpotential W (4.1) is enhanced. For model **A** one finds that the superpotential has the $Z_7 \times Z_3 \times Z_2$ discrete symmetry with the charge assignment as shown in table 1, and with the following superpotential

$$W = \sum_{i=1}^{3} \left[\frac{1}{2} g_i^u \, \mathbf{10}_i \mathbf{10}_i H_i + g_i^d \, \mathbf{10}_i \overline{\mathbf{5}}_i \, \overline{H}_i \right] + g_4^f \, H_4 \, \mathbf{24} \, \overline{H}_4 + \frac{g^\lambda}{3} (\mathbf{24})^3 \,, \qquad (4.2)$$

The non-degenerate and isolated solutions to $\gamma_i^{(1)} = 0$ for model **FUTA**, which are the boundary conditions for the Yukawa couplings at the GUT scale, are:

$$\begin{aligned}
&(g_1^u)^2 = \frac{8}{5}\, g^2 \,, && (g_1^d)^2 = \frac{6}{5}\, g^2 \,, && (g_2^u)^2 = (g_3^u)^2 = \frac{8}{5}\, g^2 \,, && \quad (4.3)\\
&(g_2^d)^2 = (g_3^d)^2 = \frac{6}{5}\, g^2 \,, && (g_{23}^u)^2 = 0 \,, && (g_{23}^d)^2 = (g_{32}^d)^2 = 0 \,, && \\
&(g^\lambda)^2 = \frac{15}{7} g^2 \,, && (g_2^f)^2 = (g_3^f)^2 = 0 \,, && (g_1^f)^2 = 0 \,, && (g_4^f)^2 = g^2 \,.
\end{aligned}$$

In the dimensionful sector, the sum rule gives us the following boundary conditions at the GUT scale for this model [36, 30, 31]:

$$m^2_{H_u} + 2 m^2_{\mathbf{10}} = m^2_{H_d} + m^2_{\overline{\mathbf{5}}} + m^2_{\mathbf{10}} = M^2 \,, \qquad (4.4)$$

and thus we are left with only three free parameters, namely $m_{\overline{\mathbf{5}}} \equiv m_{\overline{\mathbf{5}}_3}$, $m_{\mathbf{10}} \equiv m_{\mathbf{10}_3}$ and M.

4.2 FUTB

Also in the case of **FUTB** the symmetry is enhanced after the reduction of couplings. The superpotential has now a $Z_4 \times Z_4 \times Z_4$ symmetry with charges as shown in table 2 and with the following superpotential

$$\begin{aligned}
W = & \sum_{i=1}^{3} \left[\frac{1}{2} g_i^u \, \mathbf{10}_i \mathbf{10}_i H_i + g_i^d \, \mathbf{10}_i \overline{\mathbf{5}}_i \, \overline{H}_i \right] + g_{23}^u \, \mathbf{10}_2 \mathbf{10}_3 H_4 \qquad (4.5) \\
& + g_{23}^d \, \mathbf{10}_2 \overline{\mathbf{5}}_3 \, \overline{H}_4 + g_{32}^d \, \mathbf{10}_3 \overline{\mathbf{5}}_2 \, \overline{H}_4 + g_2^f \, H_2 \, \mathbf{24} \, \overline{H}_2 + g_3^f \, H_3 \, \mathbf{24} \, \overline{H}_3 + \frac{g^\lambda}{3} (\mathbf{24})^3 \,,
\end{aligned}$$

	$\overline{\mathbf{5}}_1$	$\overline{\mathbf{5}}_2$	$\overline{\mathbf{5}}_3$	$\mathbf{10}_1$	$\mathbf{10}_2$	$\mathbf{10}_3$	H_1	H_2	H_3	H_4	$\overline{H}_1$	$\overline{H}_2$	$\overline{H}_3$	$\overline{H}_4$	$\mathbf{24}$
Z_4	1	0	0	1	0	0	2	0	0	0	-2	0	0	0	0
Z_4	0	1	0	0	1	0	0	2	0	3	0	-2	0	-3	0
Z_4	0	0	1	0	0	1	0	0	2	3	0	0	-2	-3	0

Table 2: Charges of the $Z_4 \times Z_4 \times Z_4$ symmetry for Model **FUTB**.

For this model the non-degenerate and isolated solutions to $\gamma_i^{(1)} = 0$ give us:

$$\begin{aligned}
&(g_1^u)^2 = \frac{8}{5}\, g^2\,, && (g_1^d)^2 = \frac{6}{5}\, g^2\,, && (g_2^u)^2 = (g_3^u)^2 = \frac{4}{5}\, g^2\,, && \\
&(g_2^d)^2 = (g_3^d)^2 = \frac{3}{5}\, g^2\,, && (g_{23}^u)^2 = \frac{4}{5}\, g^2\,, && (g_{23}^d)^2 = (g_{32}^d)^2 = \frac{3}{5}\, g^2\,, && \\
&(g^\lambda)^2 = \frac{15}{7} g^2\,, && (g_2^f)^2 = (g_3^f)^2 = \frac{1}{2}\, g^2\,, && (g_1^f)^2 = 0\,, && (g_4^f)^2 = 0\,,
\end{aligned} \tag{4.6}$$

and from the sum rule we obtain:

$$\begin{aligned}
m_{H_u}^2 + 2m_{\mathbf{10}}^2 &= M^2\,, \qquad m_{H_d}^2 - 2m_{\mathbf{10}}^2 = -\frac{M^2}{3}\,, \\
m_{\overline{\mathbf{5}}}^2 + 3m_{\mathbf{10}}^2 &= \frac{4M^2}{3}\,,
\end{aligned} \tag{4.7}$$

i.e., in this case we have only two free parameters $m_{\mathbf{10}} \equiv m_{\mathbf{10}_3}$ and M for the dimensionful sector.

As already mentioned, after the SU(5) gauge symmetry breaking we assume we have the MSSM, i.e. only two Higgs doublets. This can be achieved by introducing appropriate mass terms that allow to perform a rotation of the Higgs sector [31, 2, 32, 30], in such a way that only one pair of Higgs doublets, coupled mostly to the third family, remains light and acquire vacuum expectation values. To avoid fast proton decay the usual fine tuning to achieve doublet-triplet splitting is performed. Notice that, although similar, the mechanism is not identical to minimal SU(5), since we have an extended Higgs sector.

Thus, after the gauge symmetry of the GUT theory is broken we are left with the MSSM, with the boundary conditions for the third family given by the finiteness conditions, while the other two families are basically decoupled.

We will now examine the phenomenology of such all-loop Finite Unified theories with SU(5) gauge group and, for the reasons expressed above, we will concentrate only on the third generation of quarks and leptons. An extension to three families, and the generation of quark mixing angles and masses in Finite Unified Theories has been addressed in [38], where several examples are given. These extensions are not considered here. Realistic Finite Unified Theories based on product gauge groups, where the finiteness implies three generations of matter, have also been studied [39].

5. Restrictions from the low-energy observables

Since the gauge symmetry is spontaneously broken below M_{GUT}, the finiteness conditions do not restrict the renormalization properties at low energies, and all it remains are boundary conditions on the gauge and Yukawa couplings (4.3) or (4.6), the $h = -MC$ relation (3.2), and the soft scalar-mass sum rule (3.3) at $M_{\rm GUT}$, as applied in the two models. Thus we examine the evolution of these parameters according to their RGEs up to two-loops for dimensionless parameters and at one-loop for dimensionful ones with the relevant boundary conditions. Below $M_{\rm GUT}$ their evolution is assumed to be governed by the MSSM. We further assume a unique supersymmetry breaking scale $M_{\rm SUSY}$ (which we define as the geometrical average of the stop masses) and therefore below that scale the effective theory is just the SM. This allows to evaluate observables at or below the electroweak scale.

In the following, we briefly describe the low-energy observables used in our analysis. We discuss the current precision of the experimental results and the theoretical predictions. We also give relevant details of the higher-order perturbative corrections that we include. We do not discuss theoretical uncertainties from the RG running between the high-scale parameters and the weak scale. At present, these uncertainties are expected to be less important than the experimental and theoretical uncertainties of the precision observables.

As precision observables we first discuss the 3rd generation quark masses that are leading to the strongest constraints on the models under investigation. Next we apply B physics and Higgs-boson mass constraints. Parameter points surviving these constraints are then tested with the cold dark matter (CDM) abundance in the early universe. We also briefly discuss the anomalous magnetic moment of the muon.

5.1 The quark masses

Since the masses of the (third generation) quarks are no free parameters in our model but predicted in terms of the GUT scale parameters and the τ mass, m_t and m_b are (as it turns out the most restrictive) precision observables for our analysis. For the top-quark mass we use the current experimental value for the pole mass [40]

$$m_t^{\rm exp} = 170.9 \pm 1.8 \text{ GeV} . \tag{5.1}$$

For the bottom-quark mass we use the value at the bottom-quark mass scale or at M_Z [41]

$$\overline{m}_b(m_b) = 4.25 \pm 0.1 \text{ GeV} \quad \text{or} \quad \overline{m}_b(M_Z) = 2.82 \pm 0.07 \text{ GeV} . \tag{5.2}$$

It should be noted that a numerically important correction appears in the relation between the bottom-quark mass and the bottom Yukawa coupling (that also enters the corresponding RGE running). The leading $\tan\beta$-enhanced corrections arise from one-loop contributions with gluino-sbottom and chargino-stop loops. We include the leading effects via the quantity Δ_b [42] (see also refs. [43 − 45]). Numerically the correction to the relation between the bottom-quark mass and the bottom Yukawa coupling is usually by far the dominant part of the contributions from the sbottom sector (see also refs. [46, 47]). In the limit of large soft SUSY-breaking parameters and $\tan\beta \gg 1$, Δ_b is given by [42]

$$\Delta_b = \frac{2\alpha_s}{3\,\pi}\, m_{\tilde g}\, \mu\, \tan\beta \,\times\, I(m_{\tilde b_1}, m_{\tilde b_2}, m_{\tilde g}) + \frac{\alpha_t}{4\,\pi}\, A_t\, \mu\, \tan\beta \,\times\, I(m_{\tilde t_1}, m_{\tilde t_2}, |\mu|)\,, \tag{5.3}$$

where the gluino mass is denoted by $m_{\tilde{g}}$ and $\alpha_f \equiv h_f^2/(4\pi)$, h_f being a fermion Yukawa coupling. The function I is defined as

$$I(a,b,c) = \frac{1}{(a^2-b^2)(b^2-c^2)(a^2-c^2)} \left(a^2 b^2 \log\frac{a^2}{b^2} + b^2 c^2 \log\frac{b^2}{c^2} + c^2 a^2 \log\frac{c^2}{a^2} \right) \sim \frac{1}{\max(a^2,b^2,c^2)} \ . \tag{5.4}$$

A corresponding correction of $\mathcal{O}(\alpha_\tau)$ has been included for the relation between the τ lepton mass and the τ Yukawa coupling. However, this correction is much smaller than the one given in eq. (5.3).

The Δ_b corrections are included by the replacement

$$\overline{m}_b \ \rightarrow \ \frac{\overline{m}_b}{1+\Delta_b} \,, \tag{5.5}$$

resulting in a resummation of the leading terms in $\mathcal{O}(\alpha_s \tan\beta)$ and $\mathcal{O}(\alpha_t \tan\beta)$ to all-orders. Expanding eq. (5.5) to first or second order gives an estimate of the effect of the resummation of the Δ_b terms and has been used as an estimate of unknown higher-order corrections (see below).

5.2 The decay $b \to s\gamma$

Since this decay occurs at the loop level in the SM, the MSSM contribution might *a priori* be of similar magnitude. A recent theoretical estimate of the SM contribution to the branching ratio at the NNLO QCD level is [48]

$$\mathrm{BR}(b \to s\gamma) = (3.15 \pm 0.23) \times 10^{-4} \ . \tag{5.6}$$

It should be noted that the error estimate for $\mathrm{BR}(b \to s\gamma)$ is still under discussion [49], and that other SM contributions to $b \to s\gamma$ have been calculated [50]. These corrections are small compared with the theoretical uncertainty quoted in eq. (5.6).

For comparison, the present experimental value estimated by the Heavy Flavour Averaging Group (HFAG) is [51, 52]

$$\mathrm{BR}(b \to s\gamma) = (3.55 \pm 0.24^{+0.09}_{-0.10} \pm 0.03) \times 10^{-4}, \tag{5.7}$$

where the first error is the combined statistical and uncorrelated systematic uncertainty, the latter two errors are correlated systematic theoretical uncertainties and corrections respectively.

Our numerical results have been derived with the $\mathrm{BR}(b \to s\gamma)$ evaluation provided in refs. [53 – 55], incorporating also the latest SM corrections provided in ref. [48]. The calculation has been checked against other codes [56 – 58]. Concerning the total error in a conservative approach we add linearly the errors of eqs. (5.6) and (5.7) as well an intrinsic SUSY error of 0.15×10^{-4} [25].

5.3 The decay $B_s \to \mu^+\mu^-$

The SM prediction for this branching ratio is $(3.4 \pm 0.5) \times 10^{-9}$ [59], and the present experimental upper limit from the Fermilab Tevatron collider is 5.8×10^{-8} at the 95% C.L. [60], still providing the possibility for the MSSM to dominate the SM contribution. The current Tevatron sensitivity, being based on an integrated luminosity of about 2 fb^{-1}, is expected to improve somewhat in the future. In ref. [60] an estimate of the future Tevatron sensitivity of 2×10^{-8} at the 90% C.L. has been given, and a sensitivity even down to the SM value can be expected at the LHC. Assuming the SM value, i.e. $\text{BR}(B_s \to \mu^+\mu^-) \approx 3.4 \times 10^{-9}$, it has been estimated [61] that LHCb can observe 33 signal events over 10 background events within 3 years of low-luminosity running. Therefore this process offers good prospects for probing the MSSM.

For the theoretical prediction we use the code implemented in ref. [56] (see also ref. [62]), which includes the full one-loop evaluation and the leading two-loop QCD corrections. We are not aware of a detailed estimate of the theoretical uncertainties from unknown higher-order corrections.

5.4 The lightest MSSM Higgs boson mass

The mass of the lightest $\mathcal{CP}$-even MSSM Higgs boson can be predicted in terms of the other SUSY parameters. At the tree level, the two $\mathcal{CP}$-even Higgs boson masses are obtained as a function of M_Z, the $\mathcal{CP}$-odd Higgs boson mass M_A, and $\tan\beta$. We employ the Feynman-diagrammatic method for the theoretical prediction of M_h, using the code `FeynHiggs` [63 – 66], which includes all relevant higher-order corrections. The status of the incorporated results can be summarized as follows. For the one-loop part, the complete result within the MSSM is known [67, 68]. Concerning the two-loop effects, their computation is quite advanced, see ref. [65] and references therein. They include the strong corrections at $\mathcal{O}(\alpha_t\alpha_s)$ and Yukawa corrections at $\mathcal{O}(\alpha_t^2)$ to the dominant one-loop $\mathcal{O}(\alpha_t)$ term, and the strong corrections from the bottom/sbottom sector at $\mathcal{O}(\alpha_b\alpha_s)$. For the $b/\tilde{b}$ sector corrections also an all-order resummation of the $\tan\beta$ -enhanced terms, $\mathcal{O}(\alpha_b(\alpha_s\tan\beta)^n)$, is known. The current intrinsic error of M_h due to unknown higher-order corrections have been estimated to be [65, 69 – 71]

$$\Delta M_h^{\text{intr,current}} = 3 \text{ GeV}. \tag{5.8}$$

The lightest MSSM Higgs boson is the models under consideration is always SM-like (see also refs. [72, 73]). Consequently, the current LEP bound of $M_h^{\text{exp}} > 114.4$ GeV at the 95% C.L. can be taken over [20, 21].

5.5 Cold dark matter density

Finally we discuss the impact of the cold dark matter (CDM) density. It is well known that the lightest neutralino, being the lightest supersymmetric particle (LSP), is an excellent candidate for CDM [23]. Consequently we demand that the lightest neutralino is indeed the LSP. Parameters leading to a different LSP are discarded.

The current bound, favored by a joint analysis of WMAP and other astrophysical and cosmological data [22], is at the $2\,\sigma$ level given by the range

$$0.094 < \Omega_{\rm CDM} h^2 < 0.129 \ . \tag{5.9}$$

Assuming that the cold dark matter is composed predominantly of LSPs, the determination of $\Omega_{\rm CDM} h^2$ imposes very strong constraints on the MSSM parameter space. As will become clear below, no model points fulfill the strict bound of eq. (5.9). On the other hand, many model parameters would yield a very large value of $\Omega_{\rm CDM}$. It should be kept in mind that somewhat larger values might be allowed due to possible uncertainties in the determination of the SUSY spectrum (as they might arise at large $\tan\beta$, see below).

However, on a more general basis and not speculating about unknown higher-order uncertainties, a mechanism is needed in our model to reduce the CDM abundance in the early universe. This issue could, for instance, be related to another problem, that of neutrino masses. This type of masses cannot be generated naturally within the class of finite unified theories that we are considering in this paper, although a non-zero value for neutrino masses has clearly been established [41]. However, the class of FUTs discussed here can, in principle, be easily extended by introducing bilinear R-parity violating terms that preserve finiteness and introduce the desired neutrino masses [102]. R-parity violation [103] would have a small impact on the collider phenomenology presented here (apart from fact the SUSY search strategies could not rely on a 'missing energy' signature), but remove the CDM bound of eq. (5.9) completely. The details of such a possibility in the present framework attempting to provide the models with realistic neutrino masses will be discussed elsewhere. Other mechanisms, not involving R-parity violation (and keeping the 'missing energy' signature), that could be invoked if the amount of CDM appears to be too large, concern the cosmology of the early universe. For instance, "thermal inflation" [74] or "late time entropy injection" [75] could bring the CDM density into agreement with the WMAP measurements. This kind of modifications of the physics scenario neither concerns the theory basis nor the collider phenomenology, but could have a strong impact on the CDM derived bounds.

Therefore, in order to get an impression of the *possible* impact of the CDM abundance on the collider phenomenology in our models under investigation, we will analyze the case that the LSP does contribute to the CDM density, and apply a more loose bound of

$$\Omega_{\rm CDM} h^2 < 0.3 \ . \tag{5.10}$$

(Lower values than the ones permitted by eq. (5.9) are naturally allowed if another particle than the lightest neutralino constitutes CDM.) For our evaluation we have used the code `MicroMegas` [56].

5.6 The anomalous magnetic moment of the muon

We finally comment on the status and the impact of the anomalous magnetic moment of the muon, $a_\mu \equiv \frac{1}{2}(g-2)_\mu$. The SM prediction for a_μ (see refs. [76–79] for reviews) depends on the evaluation of QED contributions, the hadronic vacuum polarization and light-by-light

(LBL) contributions. The evaluations of the hadronic vacuum polarization contributions using e^+e^- and τ decay data give somewhat different results. The latest estimate based on e^+e^- data [80] is given by:

$$a_\mu^{\rm theo} = (11\,659\,180.5 \pm 4.4_{\rm had} \pm 3.5_{\rm LBL} \pm 0.2_{\rm QED+EW}) \times 10^{-10}, \tag{5.11}$$

where the source of each error is labeled. We note that the new e^+e^- data sets that have recently been published in refs. [81 − 83] have been partially included in the updated estimate of $(g-2)_\mu$.

The SM prediction is to be compared with the final result of the Brookhaven $(g-2)_\mu$ experiment E821 [84], namely:

$$a_\mu^{\rm exp} = (11\,659\,208.0 \pm 6.3) \times 10^{-10}, \tag{5.12}$$

leading to an estimated discrepancy [80, 85]

$$a_\mu^{\rm exp} - a_\mu^{\rm theo} = (27.5 \pm 8.4) \times 10^{-10}, \tag{5.13}$$

equivalent to a 3.3-σ effect (see also refs. [78, 86, 87]). In order to illustrate the possible size of corrections, a simplified formula can be used, in which relevant supersymmetric mass scales are set to a common value, $M_{\rm SUSY} = m_{\tilde\chi^\pm} = m_{\tilde\chi^0} = m_{\tilde\mu} = m_{\tilde\nu_\mu}$. The result in this approximation is given by

$$a_\mu^{\rm SUSY,1L} = 13 \times 10^{-10} \left(\frac{100 \text{ GeV}}{M_{\rm SUSY}}\right)^2 \tan\beta \, {\rm sign}(\mu). \tag{5.14}$$

It becomes obvious that $\mu < 0$ is already challenged by the present data on a_μ. However, a heavy SUSY spectrum with $\mu < 0$ results in a $a_\mu^{\rm SUSY}$ prediction very close to the SM result. Since the SM is not regarded as excluded by $(g-2)_\mu$, we also still allow both signs of μ in our analysis.

Concerning the MSSM contribution, the complete one-loop result was evaluated a decade ago [88]. In addition to the full one-loop contributions, the leading QED two-loop corrections have also been evaluated [89]. Further corrections at the two-loop level have been obtained in refs. [90, 91], leading to corrections to the one-loop result that are $\lesssim 10\%$. These corrections are taken into account in our analysis according to the approximate formulae given in refs. [90, 91].

6. Final predictions

In this section we present the predictions of the models **FUTA** and **FUTB** with ($\mu > 0$ and $\mu < 0$), whose theoretically restricted parameter space due to finiteness has been further reduced by requiring correct electroweak symmetry breaking and the absence of charge or color breaking minima. We furthermore demand that the bounds discussed in the previous section are also fulfilled, see the following subsections. We have performed a scan over the GUT scale parameters, where we take as further input the τ mass, $m_\tau =$ 1.777 GeV. This allows us to extract the value of v_u, and then, using the relation $M_Z^2 =$

$\frac{1}{2}\sqrt{(3g_1^2/5+g_2^2)(v_u^2+v_d^2)}$, $v_{u,d}=1/\sqrt{2}\langle H_{u,d}\rangle$, we can extract the value of v_d. In this way it is possible to predict the masses of the top and bottom quarks, and the value of $\tan\beta$. As already mentioned, we take into account the large radiative corrections to the bottom mass, see eq. (5.5), as well as the ones to the tau mass. We have furthermore estimated the corrections to the top mass in our case and found them to be negligible, so they are not included in our analysis. As a general result for both models and both signs of μ we have a heavy SUSY mass spectrum, and $\tan\beta$ always has a large value of $\tan\beta\sim 44-56$.

6.1 Results vs. quark masses

The first low-energy constraint applied are the top- and bottom-quark masses as given in section 5.1. In figure 1 we present the predictions of the models concerning the bottom quark mass. The steps in the values for **FUTA** are due to the fact that fixed values of M were taken, while the other parameters m_5 and m_{10} were varied. However, this selected sampling of the parameter space is sufficient for us to draw our conclusions, see below.

We present the predictions for $\overline{m}_b(M_Z)$, to avoid unnecessary errors coming from the running from M_Z to the m_b pole mass, which are not related to the predictions of the present models. As already mentioned in section 5.1, we estimated the effect of the unknown higher order corrections. For such large values of $\tan\beta$, see above, in the case of **FUTB** for the bottom mass they are $\sim 8\%$, whereas for **FUTA** they can go to $\sim 30\%$ (these uncertainties are slightly larger for $\mu>0$ than for $\mu<0$). Although these theoretical uncertainties are not shown in the graphs, they have been taken into the account in the analysis of $\overline{m}_b$, by selecting only the values that comply with the value of the bottom mass within this theoretical error.

From the bounds on the $\overline{m}_b(M_Z)$ mass, we can see from figure 1 that the region $\mu>0$ is excluded both for **FUTA** and **FUTB** while for $\mu<0$ both models lie partially within the experimental limits.

In figure 2 we present the predictions of the models **FUTA** and **FUTB** concerning the top quark pole mass. We recall that the theoretical predictions of m_t have an uncertainty of $\sim 4\%$ [92]. The current experimental value is given in eq. (5.1). This clearly favors **FUTB** while **FUTA** corresponds to m_t values that are somewhat outside the experimental range, even taking theoretical uncertainties into account. Thus m_t and $\overline{m}_b(M_Z)$ together single out **FUTB** with $\mu<0$ as the most favorable solution. From section 5.6 it is obvious that $\mu<0$ is already challenged by the present data on a_μ. However, a heavy SUSY spectrum as we have here (see above and section 6.3) with $\mu<0$ results in a $a_\mu^{\rm SUSY}$ prediction very close to the SM result. Since the SM is not regarded as excluded by $(g-2)_\mu$, we continue with our analysis of **FUTB** with $\mu<0$.

6.2 Results for precision observables and CDM

For the remaining model, **FUTB** with $\mu<0$, we compare the predictions for BR($b\to s\gamma$), BR($B_s\to\mu^+\mu^-$) and M_h with their respective experimental constraints, see sections 5.2 – 5.4. First, in figure 3 we show the predictions for BR($b\to s\gamma$) vs. BR($B_s\to\mu^+\mu^-$) for all the points of **FUTB** with $\mu<0$. The gray (red) points in the lower left corner fulfill the

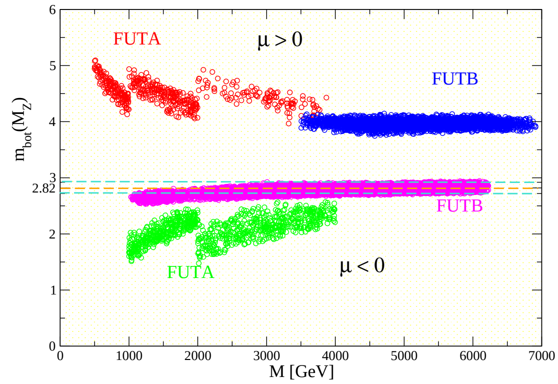

Figure 1: $\overline{m}_b(M_Z$ < 0 and $\mu > 0$.

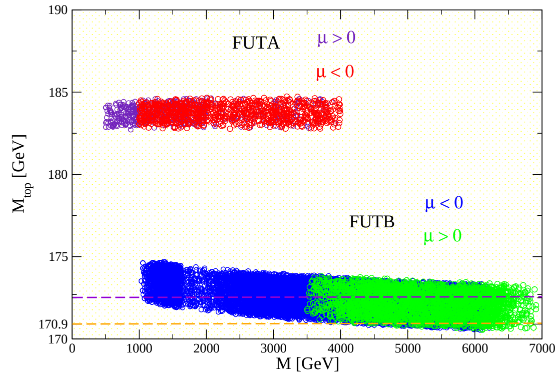

Figure 2: m_t as function of M for models **FUTA** and **FUTB**, for $\mu < 0$ and $\mu > 0$.

B physics constraints as given in sections 5.2, 5.3. Shown also in black are the parameter points that fulfill the loose CDM constraint of eq. (5.10), which can be found in the whole B physics allowed area.

In the second step we test the compatibility with the Higgs boson mass constraints and the CDM bounds. In figure 4 we show M_h (as evaluated with `FeynHiggs` [63 – 66]) as a function of M for **FUTB** with $\mu < 0$. Only the points that also fulfill the B physics bounds are included. The prediction for the Higgs boson mass is constrained to the interval $M_h = 118 \ldots 129$ GeV (including the intrinsic uncertainties of eq. (5.8)), thus fulfilling automatically the LEP bounds [20, 21]. Furthermore indicated in figure 4 by the darker (red) points is the parameter space that in addition fulfills the CDM constraint as given in eq. (5.10). The loose bound permits values of M from ~ 1000 GeV to about ~ 3000 GeV. The strong CDM bound, eq. (5.9), on the other hand, is not fulfilled by any data point, where the points with lowest $\Omega_{\rm CDM}h^2 \sim 0.2$ can be found for $M \gtrsim 1500$ GeV. As mentioned in section 5.5, the CDM bounds should be viewed as "additional" constraints (when

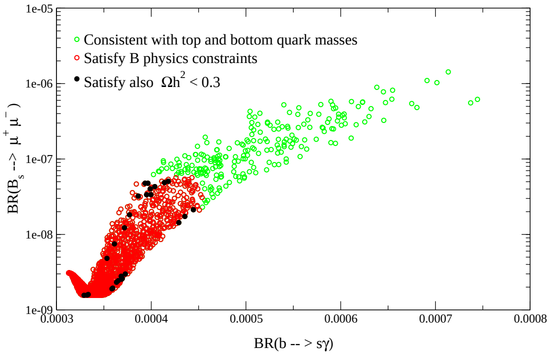

Figure 3: BR($b \to s\gamma$) vs. BR($B_s \to \mu^+\mu^-$). In green (light gray) are the points consistent with the top and bottom quark masses, in red (gray) are the subset of these that fulfill the B physics constraints, and in black the ones that also satisfy the CDM loose constraint.

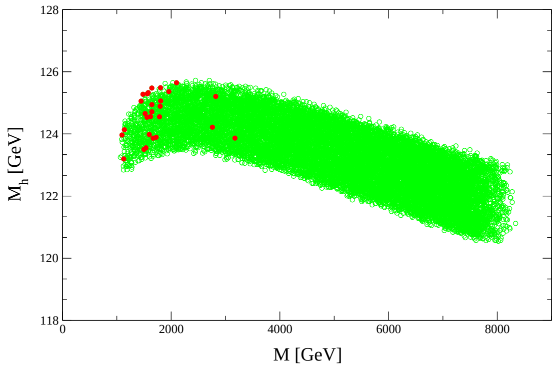

Figure 4: M_h is shown as a function of M. The light (green) points fulfill the B physics constraints. The darker (red) dots in addition satisfy the loose CDM constraint of eq. (5.10).

investigating the collider phenomenology). But even taking eq. (5.10) at face value, due to possible larger uncertainties in the calculation of the SUSY spectrum as outlined above, the CDM constraint (while strongly reducing the allowed parameter space) does not exclude the model. Within the current calculation data points which are in strict agreement with eq. (5.9) violate the B physics constraints.

6.3 The heavy Higgs and SUSY spectrum

The gray (red) points shown in figure 3 are the prediction of the finite theories once confronted with low-energy experimental data. In order to assess the discovery potential of the LHC [93, 94] and/or the ILC [95–98] we show the corresponding predictions for the most relevant SUSY mass parameters. In figure 5 we plot the mass of the lightest observable SUSY particle (LOSP) as function of M, that comply with the B physics constraints, as explained above. The darker (red) points fulfill in addition the loose CDM

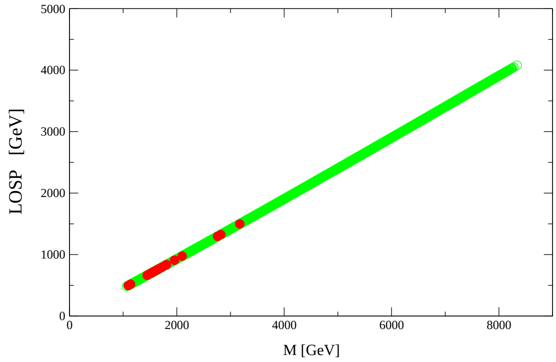

Figure 5: The mass of the LOSP is presented as a function of M. Shown are only points that fulfill the B physics constraints. The dark (red) dots in addition also satisfy the loose CDM constraint of eq. (5.10).

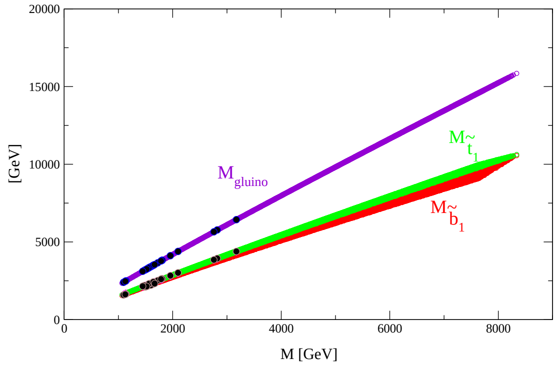

Figure 6: The mass of various colored particles are presented as a function of M. Shown are only points that fulfill the B physics constraints, the black ones satisfy also the loose CDM constraint.

constraint eq. (5.10). The LOSP is either the light scalar τ or the second lightest neutralino (which is close in mass with the lightest chargino). One can see that the masses are outside the reach of the LHC and also the ILC. Neglecting the CDM constraint, even higher particle masses are allowed.

More relevant for the LHC are the colored particles. Therefore, in figure 6 we show the masses of various colored particles: $m_{\tilde{t}_1}$, $m_{\tilde{b}_1}$ and $m_{\tilde{g}}$. The masses show a nearly linear dependence on M. Assuming a discovery reach of $\sim$ 2.5 TeV yields a coverage up to $M \lesssim 2$ TeV. This corresponds to the largest part of the CDM favored parameter space. All these particles are outside the reach of the ILC. Disregarding the CDM bounds, see section 5.5, on the other hand, results in large parts of the parameter space in which no SUSY particle can be observed neither at the LHC nor at the ILC.

We now turn to the predictions for the Higgs boson sector of **FUTB** with $\mu < 0$. In figure 7 we present the prediction for M_h vs. M_A, with the same color code as in figure 5.

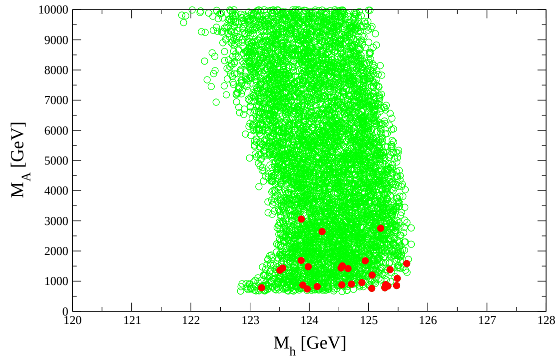

Figure 7: M_A vs M_h, with the same color code as in figure 5.

We have truncated the plot at about $M_A = 10$ TeV. The parameter space allowed by B physics extends up to ~ 30 TeV. The values that comply with the CDM constraints are in a relatively light region of M_A with $M_A \lesssim 4000$ GeV. However, taking figures 4 and 7 into account, the LHC and the ILC will observe only a light Higgs boson, whereas the heavy Higgs bosons remain outside the LHC or ILC reach.

There might be the possibility to distinguish the light MSSM Higgs boson from the SM Higgs boson by its decay characteristics. It has been shown that the ratio

$$\frac{\mathrm{BR}(h \to b\bar{b})}{\mathrm{BR}(h \to WW^*)} \tag{6.1}$$

is the most powerful discriminator between the SM and the MSSM using ILC measurements [99, 100]. We assume an experimental resolution of this ratio of $\sim 1.5\%$ at the ILC [101]. In figure 8 we show the ratio as a function of M with the same color code as in figure 5. It can be seen that up to $M \lesssim 2$ TeV a deviation from the SM ratio of more than $3\,\sigma$ can be observed. This covers most of the CDM favored parameter space. Neglecting the CDM constraint, i.e. going to higher values of M, results in a light Higgs boson that is indistinguishable from a SM Higgs boson.

Finally, in table 3 we present a representative example of the values obtained for the SUSY and Higgs boson masses for Model **FUTB** with $\mu < 0$. The masses are typically large, as already mentioned, with the LOSP starting from $\gtrsim 1000$ GeV.

It should be kept in mind that although we present the results that are consistent with the (loose) CDM constraints, the present model considers only the third generation of (s)quarks and (s)leptons. A more complete analysis will be given elsewhere when flavor mixing will be taken into account, see e.g. ref. [38]. A similar remark concerns the neutrino masses and mixings. It is well known that they can be introduced via bilinear R-parity violating terms [103] which preserve finiteness. In this case the dark matter candidate will not be the lightest neutralino, but could be another one, e.g. the axion.

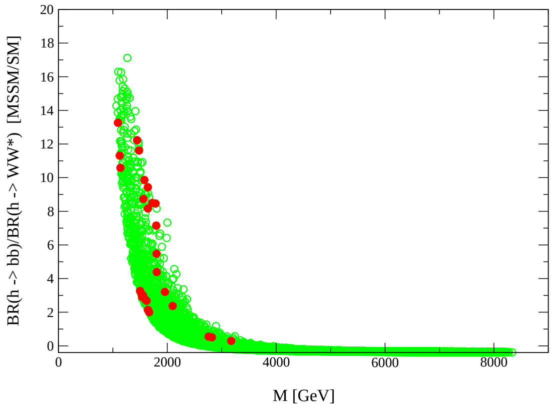

Figure 8: BR($h \to b\bar{b}$)/BR($h \to WW^*$) [MSSM/SM] (expressed in terms of σ with a resolution of 1.5% (see text)) is shown as a function of M. The color code is the same as in figure 5.

m_t	172	$\overline{m}_b(M_Z)$	2.7
$\tan\beta =$	46	α_s	0.116
$m_{\tilde{\chi}_1^0}$	796	$m_{\tilde{\tau}_2}$	1268
$m_{\tilde{\chi}_2^0}$	1462	$m_{\tilde{\nu}_3}$	1575
$m_{\tilde{\chi}_3^0}$	2048	μ	-2046
$m_{\tilde{\chi}_4^0}$	2052	B	4722
$m_{\tilde{\chi}_1^\pm}$	1462	M_A	870
$m_{\tilde{\chi}_2^\pm}$	2052	$M_{H^\pm}$	875
$m_{\tilde{t}_1}$	2478	M_H	869
$m_{\tilde{t}_2}$	2804	M_h	124
$m_{\tilde{b}_1}$	2513	M_1	796
$m_{\tilde{b}_2}$	2783	M_2	1467
$m_{\tilde{\tau}_1}$	798	M_3	3655

Table 3: A representative spectrum of **FUTB** with $\mu < 0$. All masses are in GeV.

7. Conclusions

In the present paper we have examined the predictions of two $N = 1$ supersymmetric and moreover all-loop finite SU(5) unified models, leading after the spontaneous symmetry breaking at the Grand Unification scale to the *finiteness-constrained MSSM*.

The finiteness conditions in the supersymmetric part of the unbroken theory lead to relations among the dimensionless couplings, i.e. *gauge-Yukawa unification*. In addition the finiteness conditions in the SUSY-breaking sector of the theories lead to a tremendous reduction of the number of the independent soft SUSY-breaking parameters leaving one model (**A**) with three and another (**B**) with two free parameters. Therefore the *finiteness-*

constrained MSSM consists of the well known MSSM with boundary conditions at the Grand Unification scale for its various dimensionless and dimensionful parameters inherited from the all-loop finiteness unbroken theories. Obviously these lead to an extremely restricted and, consequently, very predictive parameter space of the MSSM.

In the present paper the finiteness constrained parameter space of MSSM is confronted with the existing low-energy phenomenology such as the top and bottom quark masses, B physics observables, the bound on the lightest Higgs boson mass and constraints from the cold dark matter abundance in the universe. In the first step the result of our parameter scan of the finiteness restricted parameter space of MSSM, after applying the quark mass constraints and including theoretical uncertainties at the unification scale, singles out the *finiteness-constrained MSSM* coming from the model (**B**) with $\mu < 0$ (yielding $(g-2)_\mu$ values similar to the SM). This model was further restricted by applying the B physics constraints. The remaining parameter space then automatically fulfills the LEP bounds on the lightest MSSM Higgs boson with $M_h = 118 \ldots 129$ GeV (including already the intrinsic uncertainties). In the final step the CDM measurements have been imposed. Considering the CDM constraints it should be kept in mind that modifications in the model are possible (non-standard cosmology or R-parity violating terms that preserve finiteness) that would have only a small impact on the collider phenomenology. Therefore the CDM relic abundance should be considered as an "additional" constraint, indicating its *possible* impact. In general, a relatively heavy SUSY and Higgs spectrum at the few TeV level has been obtained, where the lower range of masses yield better agreement with the CDM constraint. The mass of the lightest observable SUSY particle (the lightest slepton or the second lightest neutralino) is larger than 500 GeV, which remains unobservable at the LHC and the ILC. The charged SUSY particles start at around 1.5 TeV and grow nearly linearly with M. Large parts of the CDM favored region results in masses of stops and sbottoms below ~ 2.5 TeV and thus might be detectable at the LHC. The measurement of branching ratios of the lightest Higgs boson to bottom quarks and W bosons at the ILC shows a deviation to the SM results of more than $3\,\sigma$ for values of $M \lesssim 2.5$ TeV, again covering most of the CDM favored region.

In conclusion, **FUTB** with $\mu < 0$, fulfilling the existing constraints from quark masses, B physics observables, Higgs boson searches and CDM measurements, results at a heavy SUSY spectrum and large $\tan\beta$. Nonetheless, colored particles are likely to be observed in the range of ~ 2 TeV at the LHC. The ILC could measure a deviation in the branching ratios of the lightest Higgs boson. However, neglecting the CDM constraint allows larger values of M. This results in a heavier SUSY spectrum, outside the reach of the LHC and the ILC. In this case also the lightest Higgs boson is SM-like.

Acknowledgments

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5.9 Finite theories after the discovery of a Higgs-like boson at the LHC

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Comment (*Sven Heinemeyer*)
Before the start-up of the LHC the idea of finite unified theories, using the $SU(5)$ gauge group, resulted in only one viable model (s. subsect. 5.8). Investigating the model properties yielded a clear prediction for the Higgs and the SUSY spectrum. The light MSSM Higgs boson mass was predicted in a very narrow range of

$$M_h^{\text{predicted}} = 121 \ldots 126 \text{ GeV} \;, \qquad (*)$$

to which a ± 3 GeV theory uncertainty has to be added. The mass scale of the heavy Higgs bosons was predicted to be between ~ 500 GeV and the multi-10-TeV range. The lightes observable SUSY particle, either the light scalar tau or the second lightest neutralino, was predicted in the range between 500 GeV and ~ 4000 GeV, where the lighter regions was prefered by the prediction of cold dark matter. Finally, the colored particles were predicted in the range between ~ 2 TeV and ~ 15 TeV, where only the lighter part of the spectrum would allow a discovery at the LHC. These predictions now eagerly awaited the start of the LHC and the experimental data on Higgs and SUSY searches.
The spectacular discovery of a Higgs-like particle with a mass around $M_H \simeq 126$ GeV, which has been announced by ATLAS [14] and CMS [15], marks a milestone of an effort that has been ongoing for almost half a century and opens up a new era of particle physics. Both ATLAS and CMS reported a clear excess in the two photon channel, as well as in the $ZZ^{(*)}$ channel. The discovery is further corroborated, though not with high significance, by the $WW^{(*)}$ channel and by the final Tevatron results [24]. The combined sensitivity in each of the LHC experiments reaches more than $5\,\sigma$. Remarkably, the measured value agrees quite precisely with the value predicted by the SU(5) finite unified theory as given in eq. $(*)$. Consequently, as a crucial new ingredient one has to take into account the recent discovery of a Higgs boson with a mass measurement of

$$M_h \sim 126.0 \pm 1 \pm 2 \text{ GeV} \;,$$

where ± 1 comes from the experimental error and ± 2 corresponds to the theoretical error, and see how this affects the SUSY spectrum. Constraining the allowed values of the Higgs mass this way puts a limit on the allowed values of the other mass parameters of the model. Furthermore, no direct observation of SUSY particles has been detected, and the lower limits on the SUSY spectrum have to be taken into account in a realistic evaluation of the model predictions.
Without any M_h restrictions the coloured SUSY particles have masses above ~ 1.8 TeV in agreement with the non-observation of those particles at the LHC. Including the Higgs mass constraints in general favors the lower part of the SUSY particle mass spectra, but also cuts away the very low values. Neglecting the theory uncertainties of M_h permits SUSY masses which would remain unobservable at the LHC, the ILC or CLIC. On the other hand, large parts of the allowed spectrum of the lighter scalar tau or the lighter neutralinos might be accessible at CLIC with $\sqrt{s} = 3$ TeV. Including the theory uncertainties, even higher masses are permitted, further weakening the discovery potential of the LHC and future e^+e^- colliders.

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Finite theories after the discovery of a Higgs-like boson at the LHC

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ABSTRACT

Finite Unified Theories (FUTs) are $N = 1$ supersymmetric Grand Unified Theories (GUTs) which can be made finite to all-loop orders, based on the principle of reduction of couplings, and therefore are provided with a large predictive power. Confronting the predictions of $SU(5)$ FUTs with the top and bottom quark masses and other low-energy experimental constraints a light Higgs boson mass in the range $M_h \sim 121$–126 GeV was predicted, in striking agreement with the recent discovery of a Higgs-like state around ~ 125.7 GeV at ATLAS and CMS. Furthermore the favoured model, a finiteness constrained version of the MSSM, naturally predicts a relatively heavy spectrum with coloured supersymmetric particles above ~ 1.5 TeV, consistent with the non-observation of those particles at the LHC. Restricting further the best FUT's parameter space according to the discovery of a Higgs-like state and B-physics observables we find predictions for the rest of the Higgs masses and the s-spectrum.

1. Introduction

The success of the Standard Model (SM) of Elementary Particle Physics has recently been confirmed by the observation of a state compatible with an (SM-like) Higgs boson at the LHC [1]. Still, the number of free parameters of the SM points towards the possibility that it is the low energy limit of a more fundamental theory. One of the most studied extensions of the SM is the Minimal Supersymmetric Standard Model (MSSM) [2], where one particular realization is the constrained MSSM (CMSSM) [3] with only five free parameters. Recent LHC results discard some regions of the CMSSM and point towards a heavy spectrum in case this particular version of SUSY is realized in nature [4].

Searching for renormalization group invariant (RGI) relations [5–16] holding below the Planck scale down to the GUT scale provides a different strategy to search for a more fundamental theory, whose basic ingredients are GUTs and supersymmetry (SUSY), and with far reaching consequences [6–9]. An outstanding feature of the use of RGI relations is that one can guarantee their validity to all-orders in perturbation theory by studying the uniqueness of the resulting relations at one-loop [10]. Even more remarkable is the fact that it is possible to find RGI relations among couplings that guarantee finiteness to all-orders in perturbation theory [11].

The Gauge–Yukawa unification scheme, based in RGI relations applied in the dimensionless couplings of supersymmetric GUTs, such as gauge and Yukawa couplings, had noticeable successes by predicting correctly the top quark mass in the finite [6] and in the minimal $N = 1$ supersymmetric $SU(5)$ GUTs [7]. Finite Unified Theories are $N = 1$ supersymmetric GUTs which can be made finite to all-loop orders, including the soft-SUSY breaking sector (for reviews and detailed references see [9,12–15]), which involves parameters of dimension one and two. Taking into account the restrictions resulting from the low-energy observables, it was possible to extend the predictive power of the RGI method to the Higgs sector and the SUSY spectrum. The Higgs boson mass thus eventually predicted [16]

$$M_h \simeq 121\text{–}126\ \text{GeV} \tag{1}$$

is in agreement with the recent discovery Higgs-like state at the LHC [1]. As further features a heavy SUSY spectrum and large values of $\tan\beta$ (the ratio of the two vacuum expectation values) were found [16].

In this Letter, first we review two $SU(5)$-based finite SUSY models and their predictions, taking into account the restrictions resulting from the low-energy observables [16]. Only one model survives all the phenomenological constraints. Then we extend our previous analysis by imposing more recent constraints resulting from the bounds on $\mathrm{BR}(B_s \to \mu^+\mu^-)$. Moreover, as the

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crucial new ingredient we interpret the newly discovered particle at ~ 126 GeV as the lightest MSSM Higgs boson and we analyse which constraints imposes the measured value of the Higgs boson mass on the predictions of the SUSY spectrum.

2. Finiteness

Finiteness can be understood by considering a chiral, anomaly free, $N = 1$ globally supersymmetric gauge theory based on a group G with gauge coupling constant g. The superpotential of the theory is given by

$$W = \frac{1}{2} m^{ij} \Phi_i \Phi_j + \frac{1}{6} C^{ijk} \Phi_i \Phi_j \Phi_k, \tag{2}$$

where m^{ij} (the mass terms) and C^{ijk} (the Yukawa couplings) are gauge invariant tensors and the matter field Φ_i transforms according to the irreducible representation R_i of the gauge group G. All the one-loop β-functions of the theory vanish if the β-function of the gauge coupling $\beta_g^{(1)}$, and the anomalous dimensions of the Yukawa couplings $\gamma_i^{j(1)}$, vanish, i.e.

$$\sum_i \ell(R_i) = 3C_2(G), \qquad \frac{1}{2} C_{ipq} C^{jpq} = 2\delta_i^j g^2 C_2(R_i), \tag{3}$$

where $\ell(R_i)$ is the Dynkin index of R_i, and $C_2(G)$ is the quadratic Casimir invariant of the adjoint representation of G. These conditions are also enough to guarantee two-loop finiteness [17]. A striking fact is the existence of a theorem [11] that guarantees the vanishing of the β-functions to all-orders in perturbation theory. This requires that, in addition to the one-loop finiteness conditions (3), the Yukawa couplings are reduced in favour of the gauge coupling to all-orders (see [15] for details). Alternatively, similar results can be obtained [18] using an analysis of the all-loop NSVZ gauge beta-function [19].

Next consider the superpotential given by (2) along with the Lagrangian for soft supersymmetry breaking (SSB) terms

$$\begin{aligned} -\mathcal{L}_{\rm SB} &= \frac{1}{6} h^{ijk} \phi_i \phi_j \phi_k + \frac{1}{2} b^{ij} \phi_i \phi_j \\ &\quad + \frac{1}{2} (m^2)_i^j \phi^{*i} \phi_j + \frac{1}{2} M \lambda \lambda + \text{h.c.}, \end{aligned} \tag{4}$$

where the ϕ_i are the scalar parts of the chiral superfields Φ_i, λ are the gauginos and M their unified mass, h^{ijk} and b^{ij} are the trilinear and bilinear dimensionful couplings respectively, and $(m^2)_i^j$ the soft scalars masses. Since we would like to consider only finite theories here, we assume that the gauge group is a simple group and the one-loop β-function of the gauge coupling g vanishes. We also assume that the reduction equations admit power series solutions of the form

$$C^{ijk} = g \sum_n \rho_{(n)}^{ijk} g^{2n}. \tag{5}$$

According to the finiteness theorem of Refs. [11,20], the theory is then finite to all-orders in perturbation theory, if, among others, the one-loop anomalous dimensions $\gamma_i^{j(1)}$ vanish. The one- and two-loop finiteness for h^{ijk} can be achieved through the relation [21]

$$h^{ijk} = -M C^{ijk} + \cdots = -M \rho_{(0)}^{ijk} g + O(g^5), \tag{6}$$

where $\cdots$ stand for higher order terms.

In addition it was found that the RGI SSB scalar masses in Gauge–Yukawa unified models satisfy a universal sum rule at one-loop [22]. This result was generalized to two-loops for finite theories [14], and then to all-loops for general Gauge–Yukawa and finite unified theories [23]. From these latter results, the following soft scalar-mass sum rule is found [14]

$$\frac{(m_i^2 + m_j^2 + m_k^2)}{M M^\dagger} = 1 + \frac{g^2}{16\pi^2} \Delta^{(2)} + O(g^4) \tag{7}$$

for i, j, k with $\rho_{(0)}^{ijk} \neq 0$, where $m_{i,j,k}^2$ are the scalar masses and $\Delta^{(2)}$ is the two-loop correction

$$\Delta^{(2)} = -2 \sum_l \left[\left(m_l^2 / M M^\dagger \right) - (1/3) \right] \ell(R_l), \tag{8}$$

which vanishes for the universal choice, i.e. when all the soft scalar masses are the same at the unification point. This correction also vanishes in the models considered here.

3. $SU(5)$ finite unified theories

Finite Unified Models have been studied for already two decades. A realistic two-loop finite $SU(5)$ model was presented in [24], and shortly afterwards the conditions for finiteness in the soft susy breaking sector at one-loop [17] were given. Since finite models usually have an extended Higgs sector, in order to make them viable a rotation of the Higgs sector was proposed [25]. The first all-loop finite theory was studied in [6], without taking into account the soft breaking terms. Naturally, the concept of finiteness was extended to the soft breaking sector, where also one-loop finiteness implies two-loop finiteness [21], and then finiteness to all-loops in the soft sector of realistic models was studied [26,27], although the universality of the soft breaking terms lead to a charged lightest SUSY particle (LSP). This fact was also noticed in [28], where the inclusion of an extra parameter in the Higgs sector was introduced to alleviate it. With the derivation of the sum rule in the soft supersymmetry breaking sector and the proof that it can be made all-loop finite the construction of all-loop phenomenologically viable finite models was made possible [14,23].

Here we will examine two all-loop Finite Unified Theories with $SU(5)$ gauge group, where the reduction of couplings has been applied to the third generation of quarks and leptons. An extension to three families, and the generation of quark mixing angles and masses in Finite Unified Theories has been addressed in [29], where several examples are given. These extensions are not considered here. Realistic Finite Unified Theories based on product gauge groups, where the finiteness implies three generations of matter, have also been studied [30].

The particle content of the models we will study consists of the following supermultiplets: three $(\overline{\mathbf{5}} + \mathbf{10})$, needed for each of the three generations of quarks and leptons, four $(\overline{\mathbf{5}} + \mathbf{5})$ and one $\mathbf{24}$ considered as Higgs supermultiplets. When the gauge group of the finite GUT is broken the theory is no longer finite, and we will assume that we are left with the MSSM.

Thus, a predictive Gauge–Yukawa unified $SU(5)$ model which is finite to all-orders, in addition to the requirements mentioned already, should also have the following properties:

1. One-loop anomalous dimensions are diagonal, i.e., $\gamma_i^{(1)j} \propto \delta_i^j$.
2. Three fermion generations, in the irreducible representations $\overline{\mathbf{5}}_i, \mathbf{10}_i$ $(i = 1, 2, 3)$, which obviously should not couple to the adjoint $\mathbf{24}$.
3. The two Higgs doublets of the MSSM should mostly be made out of a pair of Higgs quintet and anti-quintet, which couple to the third generation.

The two versions of the all-order finite model we will discuss here are the following: The model of [6], which will be labelled **A**, and a slight variation of this model (labelled **B**), which can also be obtained from the class of the models suggested in [26] with a modification to suppress non-diagonal anomalous dimensions.

The superpotential which describes the two models, which we will label **A** and **B**, takes the form [6,14]

$$\begin{aligned} W = {} & \sum_{i=1}^{3}\left[\frac{1}{2}g_i^u \mathbf{10}_i\mathbf{10}_i H_i + g_i^d \mathbf{10}_i\bar{\mathbf{5}}_i\overline{H}_i\right] \\ & + g_{23}^u \mathbf{10}_2\mathbf{10}_3 H_4 + g_{23}^d \mathbf{10}_2\bar{\mathbf{5}}_3\overline{H}_4 + g_{32}^d \mathbf{10}_3\bar{\mathbf{5}}_2\overline{H}_4 \\ & + \sum_{a=1}^{4} g_a^f H_a \mathbf{24}\overline{H}_a + \frac{g^\lambda}{3}(\mathbf{24})^3, \end{aligned} \tag{9}$$

where H_a and $\overline{H}_a$ $(a = 1,\dots,4)$ stand for the Higgs quintets and anti-quintets.

The main difference between model **A** and model **B** is that two pairs of Higgs quintets and anti-quintets couple to the **24** in **B**, so that it is not necessary to mix them with H_4 and $\overline{H}_4$ in order to achieve the triplet-doublet splitting after the symmetry breaking of $SU(5)$ [14]. Thus, although the particle content is the same, the solutions to the finiteness equations and the sum rules are different, which has repercussions in the phenomenology.

FUTA

After the reduction of couplings the symmetry of the superpotential W (9) is enhanced (for details see [31]). The superpotential for this model is

$$\begin{aligned} W = {} & \sum_{i=1}^{3}\left[\frac{1}{2}g_i^u \mathbf{10}_i\mathbf{10}_i H_i + g_i^d \mathbf{10}_i\bar{\mathbf{5}}_i\overline{H}_i\right] \\ & + g_4^f H_4 \mathbf{24}\overline{H}_4 + \frac{g^\lambda}{3}(\mathbf{24})^3. \end{aligned} \tag{10}$$

The non-degenerate and isolated solutions to $\gamma_i^{(1)} = 0$ for model **FUTA**, which are the boundary conditions for the Yukawa couplings at the GUT scale, are

$$\begin{aligned} & (g_1^u)^2 = \frac{8}{5}g^2, \quad (g_1^d)^2 = \frac{6}{5}g^2, \quad (g_2^u)^2 = (g_3^u)^2 = \frac{8}{5}g^2, \\ & (g_2^d)^2 = (g_3^d)^2 = \frac{6}{5}g^2, \quad (g_{23}^u)^2 = 0, \quad (g_{23}^d)^2 = (g_{32}^d)^2 = 0, \\ & (g^\lambda)^2 = \frac{15}{7}g^2, \quad (g_2^f)^2 = (g_3^f)^2 = 0, \\ & (g_1^f)^2 = 0, \quad (g_4^f)^2 = g^2. \end{aligned} \tag{11}$$

In the dimensionful sector, the sum rule gives us the following boundary conditions at the GUT scale for this model [14]:

$$m_{H_u}^2 + 2m_{\mathbf{10}}^2 = m_{H_d}^2 + m_{\bar{\mathbf{5}}}^2 + m_{\mathbf{10}}^2 = M^2, \tag{12}$$

and thus we are left with only three free parameters, namely $m_{\bar{\mathbf{5}}} \equiv m_{\bar{\mathbf{5}}_3}$, $m_{\mathbf{10}} \equiv m_{\mathbf{10}_3}$ and M.

FUTB

Also in the case of **FUTB** the symmetry is enhanced after the reduction of couplings, with the following superpotential [31]

$$\begin{aligned} W = {} & \sum_{i=1}^{3}\left[\frac{1}{2}g_i^u \mathbf{10}_i\mathbf{10}_i H_i + g_i^d \mathbf{10}_i\bar{\mathbf{5}}_i\overline{H}_i\right] + g_{23}^u \mathbf{10}_2\mathbf{10}_3 H_4 \\ & + g_{23}^d \mathbf{10}_2\bar{\mathbf{5}}_3\overline{H}_4 + g_{32}^d \mathbf{10}_3\bar{\mathbf{5}}_2\overline{H}_4 + g_2^f H_2\mathbf{24}\overline{H}_2 \\ & + g_3^f H_3\mathbf{24}\overline{H}_3 + \frac{g^\lambda}{3}(\mathbf{24})^3. \end{aligned} \tag{13}$$

For this model the non-degenerate and isolated solutions to $\gamma_i^{(1)} = 0$ give us

$$\begin{aligned} & (g_1^u)^2 = \frac{8}{5}g^2, \quad (g_1^d)^2 = \frac{6}{5}g^2, \quad (g_2^u)^2 = (g_3^u)^2 = \frac{4}{5}g^2, \\ & (g_2^d)^2 = (g_3^d)^2 = \frac{3}{5}g^2, \quad (g_{23}^u)^2 = \frac{4}{5}g^2, \\ & (g_{23}^d)^2 = (g_{32}^d)^2 = \frac{3}{5}g^2, \quad (g^\lambda)^2 = \frac{15}{7}g^2, \\ & (g_2^f)^2 = (g_3^f)^2 = \frac{1}{2}g^2, \quad (g_1^f)^2 = 0, \quad (g_4^f)^2 = 0, \end{aligned} \tag{14}$$

and from the sum rule we obtain

$$\begin{aligned} & m_{H_u}^2 + 2m_{\mathbf{10}}^2 = M^2, \\ & m_{H_d}^2 - 2m_{\mathbf{10}}^2 = -\frac{M^2}{3}, \\ & m_{\bar{\mathbf{5}}}^2 + 3m_{\mathbf{10}}^2 = \frac{4M^2}{3}, \end{aligned} \tag{15}$$

i.e., in this case we have only two free parameters $m_{\mathbf{10}} \equiv m_{\mathbf{10}_3}$ and M for the dimensionful sector.

As already mentioned, after the $SU(5)$ gauge symmetry breaking we assume we have the MSSM, i.e. only two Higgs doublets. This can be achieved by introducing appropriate mass terms that allow to perform a rotation of the Higgs sector [6,24,25,32], in such a way that only one pair of Higgs doublets, coupled mostly to the third family, remains light and acquire vacuum expectation values. To avoid fast proton decay the usual fine tuning to achieve doublet-triplet splitting is performed. Notice that, although similar, the mechanism is not identical to minimal $SU(5)$, since we have an extended Higgs sector.

Thus, after the gauge symmetry of the GUT theory is broken we are left with the MSSM, with the boundary conditions for the third family given by the finiteness conditions, while the other two families are not restricted.

We will now examine the phenomenology of such all-loop Finite Unified Theories with $SU(5)$ gauge group and, for the reasons expressed above, we will concentrate only on the third generation of quarks and leptons.

4. Predictions of low energy parameters

Since the gauge symmetry is spontaneously broken below M_{GUT}, the finiteness conditions do not restrict the renormalization properties at low energies, and all it remains are boundary conditions on the gauge and Yukawa couplings (11) or (14), the $h = -MC$ (6) relation, and the soft scalar-mass sum rule at M_{GUT}, as applied in the two models, Eq. (12) or (15). Thus we examine the evolution of these parameters according to their RGEs up to two-loops for dimensionless parameters and at one-loop for dimensionful ones with the relevant boundary conditions. Below M_{GUT} their evolution is assumed to be governed by the MSSM. We further assume a unique supersymmetry breaking scale M_s (which we define as the geometric mean of the stop masses) and therefore below that scale the effective theory is just the SM.

We now briefly review the comparison of the predictions of the two models (**FUTA**, **FUTB**) with the experimental data, starting with the heavy quark masses see Ref. [16] for more details.

We use for the top quark the value for the pole mass [33]

$$m_t^{\mathrm{exp}} = (173.2 \pm 0.9)\ \mathrm{GeV}, \tag{16}$$

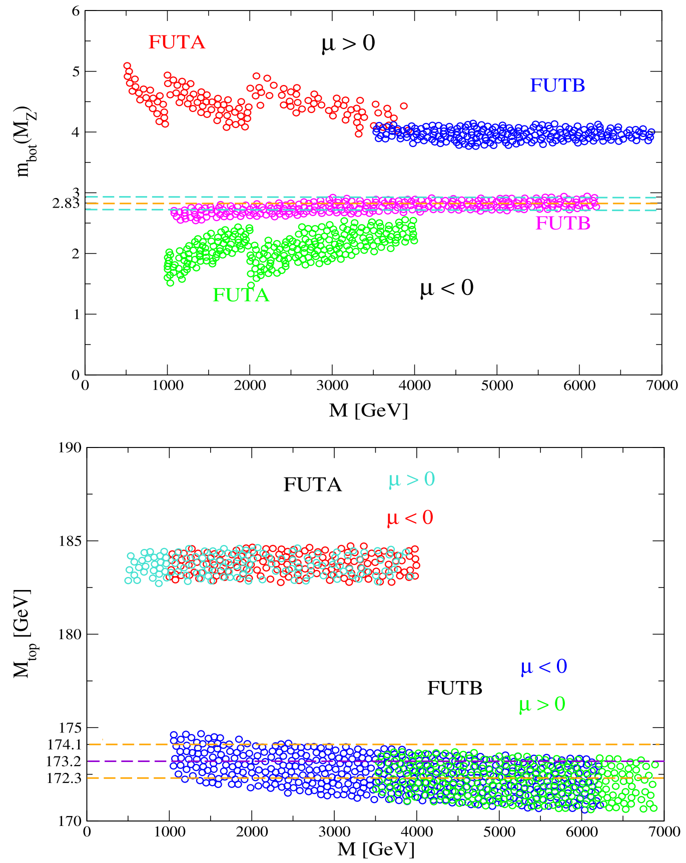

Fig. 1. The bottom quark mass at the Z boson scale (upper) and top quark pole mass (lower plot) are shown as function of M, the unified gaugino mass, for both models.

and we recall that the theoretical prediction for m_t of the present framework may suffer from a correction of $\sim 4\%$ [9,12,34,35]. For the bottom quark mass we use the value at M_Z [36]

$$m_b(M_Z) = (2.83 \pm 0.10)\ \text{GeV}, \tag{17}$$

to avoid uncertainties that come from the further running from the M_Z to the m_b mass.

In Fig. 1 we show the **FUTA** and **FUTB** predictions for m_t and $m_b(M_Z)$ as a function of the unified gaugino mass M, for the two cases $\mu < 0$ and $\mu > 0$. In the value of the bottom mass m_b, we have included the corrections coming from bottom squark-gluino loops and top squark-chargino loops [37], known usually as the Δ_b effects. The bounds on the $m_b(M_Z)$ and the m_t mass clearly single out **FUTB** with $\mu < 0$, as the solution most compatible with this experimental constraints. Although $\mu < 0$ is already challenged by present data of the anomalous magnetic moment of the muon a_μ [38,39], a heavy SUSY spectrum as the one we have here (see below) gives results for a_μ very close to the SM result, and thus cannot be excluded.

We now analyze the impact of further low-energy observables on the model **FUTB** with $\mu < 0$. As additional constraints we consider the following observables: the rare b decays $\text{BR}(b \to s\gamma)$ and $\text{BR}(B_s \to \mu^+\mu^-)$.

For the branching ratio $\text{BR}(b \to s\gamma)$, we take the value given by the Heavy Flavour Averaging Group (HFAG) is [40]

$$\text{BR}(b \to s\gamma) = \left(3.55 \pm 0.24^{+0.09}_{-0.10} \pm 0.03\right) \times 10^{-4}. \tag{18}$$

For the branching ratio $\text{BR}(B_s \to \mu^+\mu^-)$, the SM prediction is at the level of 10^{-9}, while the present experimental upper limit is

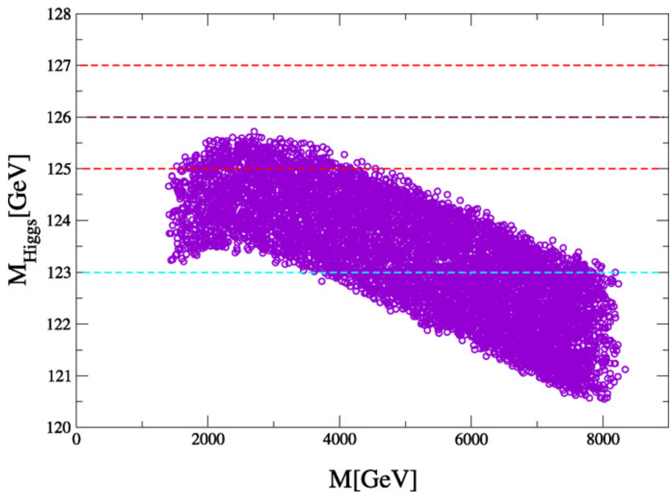

Fig. 2. The lightest Higgs mass, M_h, as function of M for the model **FUTB** with $\mu < 0$, see text. (For interpretation of the references to colour, the reader is referred to the web version of this Letter.)

$$\text{BR}\left(B_s \to \mu^+\mu^-\right) = 4.5 \times 10^{-9} \tag{19}$$

at the 95% C.L. [41].[2]

For the lightest Higgs mass prediction we use the code FeynHiggs [43–45]. The prediction for M_h of **FUTB** with $\mu < 0$ is shown in Fig. 2, where the constraints from the two B-physics observables are taken into account. The lightest Higgs mass ranges in

$$M_h \sim 121\text{–}126\ \text{GeV}, \tag{20}$$

where the uncertainty comes from variations of the soft scalar masses. To this value one has to add at least ± 2 GeV coming from unknown higher order corrections [44]. We have also included a small variation, due to threshold corrections at the GUT scale, of up to 5% of the FUT boundary conditions. The masses of the heavier Higgs bosons are found at higher values in comparison with our previous analyses [16,46]. This is due to the more stringent bound on $\text{BR}(B_s \to \mu^+\mu^-)$, which pushes the heavy Higgs masses beyond ~ 1 TeV, excluding their discovery at the LHC. We furthermore find in our analysis that the lightest observable SUSY particle (LOSP) is either the stau or the second lightest neutralino, with mass starting around ~ 500 GeV.

As the crucial new ingredient we take into account the recent observations of a Higgs-like state discovered at LHC. We impose a constraint on our results to the Higgs mass of

$$M_h \sim 126.0 \pm 1 \pm 2\ \text{GeV}, \tag{21}$$

where ± 1 comes from the experimental error and ± 2 corresponds to the theoretical error, and see how this affects the SUSY spectrum. Constraining the allowed values of the Higgs mass this way puts a limit on the allowed values of the unified gaugino mass, as can be seen from Fig. 2. The red lines correspond to the pure experimental uncertainty and restrict $2\ \text{TeV} \lesssim M \lesssim 5\ \text{TeV}$. The blue line includes the additional theory uncertainty of ± 2 GeV. Taking this uncertainty into account no bound on M can be placed. However, a substantial part of the formerly allowed parameter points

[2] While we were finalizing this Letter, a first measurement at the $\sim 3\sigma$ level of $\text{BR}(B_s \to \mu^+\mu^-)$ was published by the LHCb Collaboration [42]. The value is given as $\text{BR}(B_s \to \mu^+\mu^-) = (3.2^{+1.4}_{-1.2}(\text{stat})^{+0.5}_{-0.3}(\text{syst})) \times 10^{-9}$, i.e. the upper limit at the 95% C.L. is slightly higher than what we used as an upper limit. Furthermore, no combination of this new result with the existing limits exists yet. Consequently, as we do not expect a sizable impact of the very new measurement on our results, we stick for this analysis to the simple upper limit.

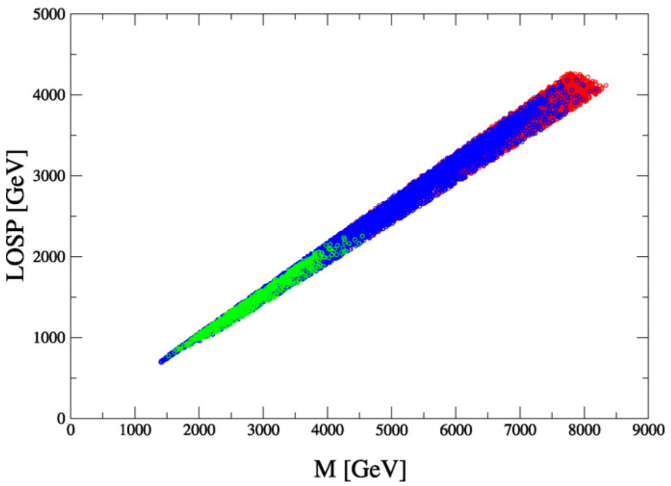

Fig. 3. The mass of the LOSP is presented as a function of M. Shown are only points that fulfill the B-physics constraints. The green (light shaded) points correspond to $M_h = 126 \pm 1$ GeV, the blue (dark shaded) points have $M_h = 126 \pm 3$ GeV, and the red points have no M_h restriction. (For interpretation of the references to colour, the reader is referred to the web version of this Letter.)

are now excluded. This in turn restricts the lightest observable SUSY particle (LOSP), which turns out to be the light scalar tau. In Fig. 3 the effects on the mass of the LOSP are demonstrated. Without any Higgs mass constraint all coloured points are allowed. Imposing $M_h = 126 \pm 1$ GeV only the green (light shaded) points are allowed, restricting the mass to be between about 500 GeV and 2500 GeV. The lower values might be experimentally accessible at the ILC with 1000 GeV centre-of-mass energy or at CLIC with an energy up to ~ 3 TeV. Taking into account the theory uncertainty on M_h also the blue (dark shaded) points are allowed, permitting the LOSP mass up to ~ 4 TeV. If the upper end of the parameter space were realized the light scalar tau would remain unobservable even at CLIC.

The full particle spectrum of model **FUTB** with $\mu < 0$, compliant with quark mass constraints and the B-physics observables is shown in Fig. 4. In the upper (lower) plot we impose $M_h = 126 \pm 3(1)$ GeV. Without any M_h restrictions the coloured SUSY particles have masses above ~ 1.8 TeV in agreement with the non-observation of those particles at the LHC [47]. Including the Higgs mass constraints in general favours the lower part of the SUSY particle mass spectra, but also cuts away the very low values. Neglecting the theory uncertainties of M_h (as shown in the lower plot of Fig. 4) permits SUSY masses which would remain unobservable at the LHC, the ILC or CLIC. On the other hand, large parts of the allowed spectrum of the lighter scalar tau or the lighter neutralinos might be accessible at CLIC with $\sqrt{s} = 3$ TeV. Including the theory uncertainties, even higher masses are permitted, further weakening the discovery potential of the LHC and future e^+e^- colliders. A numerical example of the lighter part of the spectrum is shown in Table 1. If such a spectrum were realized, the coloured particles are at the border of the discovery region at the LHC. Some uncoloured particles like the scalar taus, the light chargino or the lighter neutralinos would be in the reach of a high-energy Linear Collider.

5. Conclusions

We examined the predictions of two $SU(5)$ Finite Unified Theories in light of the recent discovery of a Higgs-like state at the LHC and on the new bounds on the branching ratio $\mathrm{BR}(B_s \to \mu^+\mu^-)$. Only one model is consistent with all the phenomenological constraints. Compared to our previous analysis [16], the new bound on $\mathrm{BR}(B_s \to \mu^+\mu^-)$ excludes values for the heavy Higgs bosons masses below $1 \sim$ TeV, and in general allows only a very heavy SUSY spectrum. The Higgs mass constraint favours the lower part of this spectrum, with SUSY masses ranging from ~ 500 GeV up to the multi-TeV level, where the lower part of the spectrum could be accessible at the ILC or CLIC.

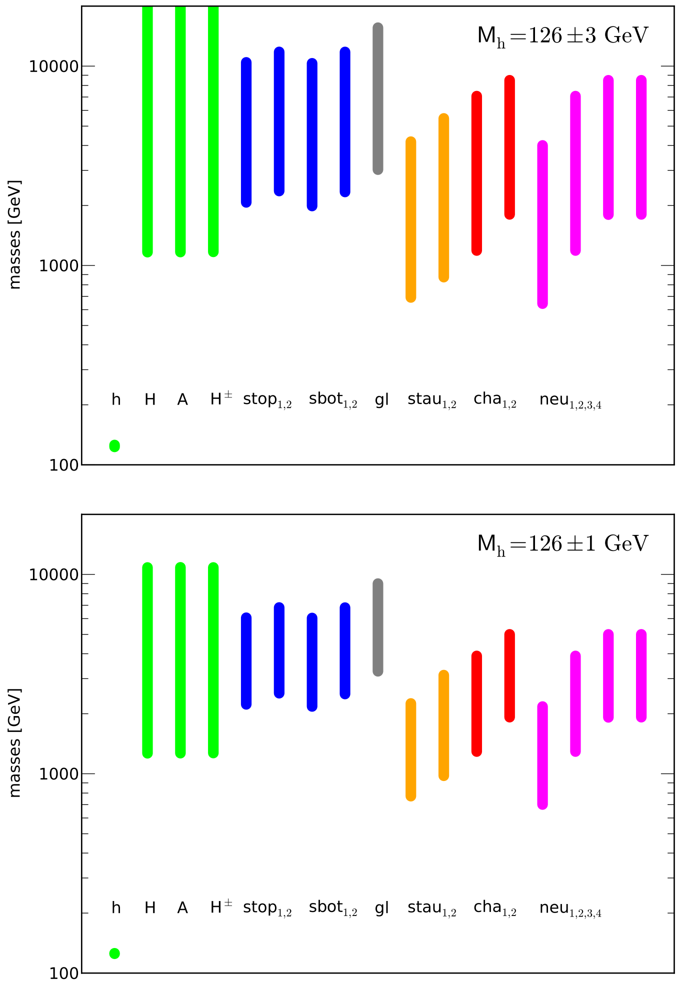

Fig. 4. The upper (lower) plot shows the spectrum after imposing the constraint $M_h = 126 \pm 3(1)$ GeV. The particle spectrum of model **FUTB** with $\mu < 0$, where the points shown are in agreement with the quark mass constraints and the B-physics observables. The light (green) points on the left are the various Higgs boson masses. The dark (blue) points following are the two scalar top and bottom masses, followed by the lighter (grey) gluino mass. Next come the lighter (beige) scalar tau masses. The darker (red) points to the right are the two chargino masses followed by the lighter shaded (pink) points indicating the neutralino masses. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this Letter.)

Table 1
A representative spectrum of a light **FUTB**, $\mu < 0$ spectrum, compliant with the B-physics constraints. All masses are in GeV.

Mbot(M_Z)	2.74	Mtop	174.1
Mh	125.0	MA	1517
MH	1515	MH$^\pm$	1518
Stop1	2483	Stop2	2808
Sbot1	2403	Sbot2	2786
Mstau1	892	Mstau2	1089
Char1	1453	Char2	2127
Neu1	790	Neu2	1453
Neu3	2123	Neu4	2127
Mgluino	3632		

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5.10 Reduction of Couplings in the MSSM

Title: Reduction of Couplings in the MSSM
Authors: M. Mondragon, N.D. Tracas, G. Zoupanos
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Comment (*Myriam Mondragón, George Zoupanos*)
This paper is of particular importance in the examination of realistic models in which the *reduction of couplings* can be achieved. It is of equal theoretical importance as the paper discussed in subsection 3.1, but more successful so far in the comparison with the known experimental facts. Moreover, contrary to the case in Finite Unified Theories, it realises the old dream of Zimmermann with asymptotic freedom at work in the reduction of the relevant couplings, as a fundamental requirement according to the original theorem.

More specifically the most important observation in this paper is that there exist RGI relations among the top, bottom Yukawa and the gauge colour couplings in the minimal supersymmetric SM, i.e. in the MSSM. This result was found by solving the reduction equations and using the power series ansatz for the solutions. The reduced system comprises the top and bottom Yukawa couplings reduced in terms of the strong coupling, whereas the tau Yukawa coupling is left as a free parameter. It was found that it is possible to have solutions for certain values of the tau Yukawa coupling and negative values of the μ parameter, which are consistent with the experimental results for the top and bottom quark masses simultaneously at the level of one sigma. Therefore the reduction of these couplings is a fact in the MSSM. Then, based on this observation and using the tools described in the subsection 5.6 it was possible to make further predictions. Assuming the existence of a RGI relation among the trilinear couplings in the superpotential and the SSB sector of the theory, it was possible to obtain predictions for the Higgs masses and the supersymmetric spectrum. It was found that the lightest Higgs mass is in the range 123.7 - 126.3 GeV, in striking agreement with the measurements at LHC [14, 15]. The rest of the spectrum was found to be generally very heavy. Specifically, it was found that the masses of the heavier Higgses have values above the TeV scale. In addition the supersymmetric spectrum starts with a neutralino as LSP at ~ 500 GeV, which allows for a comfortable agreement with the LHC bounds due to the non-observation of coloured supersymmetric particles [29, 30, 31]. The plan is to extend the present analysis by examining the restrictions that will be imposed in the spectrum by the B-physics as well as the CDM constraints, given that the LSP in this model is in principle a candidate for CDM.

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Reduction of couplings in the MSSM ☆

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A B S T R A C T

In this Letter, we first demonstrate the existence of renormalization group invariant relations among the top, bottom Yukawa and the gauge colour couplings in the minimal supersymmetric SM. Based on this observation and assuming furthermore the existence of a renormalization group invariant relation among the trilinear couplings in the superpotential and the soft supersymmetry breaking sector, we obtain predictions for the Higgs masses and the supersymmetric spectrum.

1. Introduction

With the recent discovery of the Higgs-like boson at the LHC [1], the new bounds on supersymmetric particles which place supersymmetry at least at the TeV scale [2], and the new data on B physics [3], the search for theoretical scenarios beyond the Standard Model in which all these experimental facts can be accommodated becomes more pressing.

Frameworks such as Superstrings and Noncommutative Theories were developed aiming to provide a unified description of all interactions, including gravity. However, the main goal from a unified description of interactions should be the understanding of the present day free parameters of the Standard Model (SM) in terms of a few fundamental ones, or in other words to achieve *reduction of couplings* at a more fundamental level. Unfortunately, the above theoretical frameworks have not provided yet an understanding of the free parameters of the SM.

We have developed a complementary strategy in searching for a more fundamental theory, possibly realized near the Planck scale, whose basic ingredients are Grand Unified Theories (GUTs) and supersymmetry (SUSY), but its consequences certainly go beyond the known ones [4–6]. The method consists in searching for renormalization group invariant (RGI) relations holding below the Planck scale, which in turn are preserved down to the GUT scale. An impressive aspect of the RGI relations is that one can guarantee their validity to all-orders in perturbation theory by studying the uniqueness of the resulting relations at one-loop, as was proven in the early days of the programme of *reduction of couplings* [7]. Even more remarkable is the fact that it is possible to find RGI relations among couplings that guarantee finiteness to all-orders in perturbation theory [8]. This programme, called Gauge–Yukawa unification (GYU) scheme, has been applied to the dimensionless couplings of supersymmetric GUTs, such as gauge and Yukawa couplings, with remarkable successes since it predicted correctly the top quark and the Higgs masses in finite $N = 1$ supersymmetric SU(5) GUTs [4–6,9].

Supersymmetry seems to be an essential feature of the GYU programme and understanding its breaking becomes crucial, since the programme has the ambition to supply the SM with predictions for several of its free parameters. Indeed, the search for RGI relations was extended to the soft supersymmetry breaking (SSB) sector of these theories [6,10], which involves parameters of dimension one and two. Based conceptually and technically on the work of Ref. [11], considerable progress was made concerning the renormalization properties of the SSB parameters [12–17]. In Ref. [11] the powerful supergraph method [18,19] was applied to softly broken SUSY theories using the "spurion" external space–time independent superfields [20,21].

In the spurion method, a softly broken supersymmetric gauge theory is considered as a supersymmetric one in which the various parameters such as couplings and masses have been promoted to external superfields that acquire "vacuum expectation values". Thus, the β-functions of the parameters of the softly broken theory are expressed in terms of partial differential operators involving the dimensionless parameters of the unbroken theory. By transforming the partial differential operators involved into total derivative operators it is possible to express all parameters in a RGI way

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[16,17], and in particular on the RGI surface which is defined by the solution of the reduction equations. Crucial to the success of this programme is that the soft scalar masses obey a sum rule [22,23], which is RGI to all orders in perturbation theory, both for the general GYU as for the particular finite case [17]. Based on the above tools and results we would like to apply the above programme in the case of MSSM.

2. The reduction of couplings method

In this section we will briefly outline the reduction of couplings method. Any RGI relation among couplings (i.e. which does not depend on the renormalization scale μ explicitly) can be expressed, in the implicit form $\Phi(g_1,\dots,g_A)=\text{const.}$, which has to satisfy the partial differential equation (PDE)

$$\frac{d\Phi}{dt}=\sum_{a=1}^{A}\frac{\partial\Phi}{\partial g_a}\frac{dg_a}{dt}=\sum_{a=1}^{A}\frac{\partial\Phi}{\partial g_a}\beta_a=\vec{\nabla}\Phi\cdot\vec{\beta}=0, \tag{1}$$

where $t=\ln\mu$ (μ being the renormalization scale) and β_a is the β-function of g_a. This PDE is equivalent to a set of ordinary differential equations, the so-called reduction equations (REs) [7,24],

$$\beta_g\frac{dg_a}{dg}=\beta_a,\quad a=1,\dots,A, \tag{2}$$

where g and β_g are the primary coupling and its β-function, and the counting on a does not include g. Since maximally $(A-1)$ independent RGI "constraints" in the A-dimensional space of couplings can be imposed by the Φ_a's, one could in principle express all the couplings in terms of a single coupling g. The strongest requirement is to demand power series solutions to the REs,

$$g_a=\sum_{n=0}\rho_a^{(n)}g^{2n+1}, \tag{3}$$

which formally preserve perturbative renormalizability. Remarkably, the uniqueness of such power series solutions can be decided already at the one-loop level [7,24]. To illustrate this, let us assume that the β-functions have the form

$$\beta_a=\frac{1}{16\pi^2}\left[\sum_{b,c,d\neq g}\beta_a^{(1)\,bcd}g_bg_cg_d+\sum_{b\neq g}\beta_a^{(1)\,b}g_bg^2\right]+\cdots,$$

$$\beta_g=\frac{1}{16\pi^2}\beta_g^{(1)}g^3+\cdots, \tag{4}$$

where $\cdots$ stands for higher order terms, and $\beta_a^{(1)\,bcd}$'s are symmetric in b,c,d. We then assume that the $\rho_a^{(n)}$'s with $n\leqslant r$ have been uniquely determined. To obtain $\rho_a^{(r+1)}$'s, we insert the power series (3) into the REs (2) and collect terms of $O(g^{2r+3})$ and find

$$\sum_{d\neq g}M(r)_a^d\rho_d^{(r+1)}=\text{lower order quantities}, \tag{5}$$

where the r.h.s. is known by assumption, and

$$M(r)_a^d=3\sum_{b,c\neq g}\beta_a^{(1)\,bcd}\rho_b^{(1)}\rho_c^{(1)}+\beta_a^{(1)\,d}-(2r+1)\beta_g^{(1)}\delta_a^d, \tag{6}$$

$$0=\sum_{b,c,d\neq g}\beta_a^{(1)\,bcd}\rho_b^{(1)}\rho_c^{(1)}\rho_d^{(1)}+\sum_{d\neq g}\beta_a^{(1)\,d}\rho_d^{(1)}-\beta_g^{(1)}\rho_a^{(1)}. \tag{7}$$

Therefore, the $\rho_a^{(n)}$'s for all $n>1$ for a given set of $\rho_a^{(1)}$'s can be uniquely determined if $\det M(n)_a^d\neq 0$ for all $n\geqslant 0$.

Our experience examining specific examples has taught us that the various couplings in supersymmetric theories could have the same asymptotic behaviour. Therefore, searching for a power series solution of the form (3) to the REs (2) is justified and moreover, one can rely that keeping only the first terms a good approximation is obtained in realistic applications.

3. Sum rule for soft breaking terms

The method of reducing the dimensionless couplings has been extended [6,10], as we have discussed in the Introduction, to the soft supersymmetry breaking (SSB) dimensionful parameters of $N=1$ supersymmetric theories. In addition it was found [22,23] that RGI SSB scalar masses in Gauge–Yukawa unified models satisfy a universal sum rule.

Consider the superpotential given by

$$W=\frac{1}{2}\mu^{ij}\Phi_i\Phi_j+\frac{1}{6}C^{ijk}\Phi_i\Phi_j\Phi_k, \tag{8}$$

along with the Lagrangian for SSB terms

$$-\mathcal{L}_{\mathrm{SSB}}=\frac{1}{6}h^{ijk}\phi_i\phi_j\phi_k+\frac{1}{2}b^{ij}\phi_i\phi_j+\frac{1}{2}(m^2)_i^j\phi^{*\,i}\phi_j$$
$$+\frac{1}{2}M\lambda\lambda+\text{H.c.}, \tag{9}$$

where the ϕ_i are the scalar parts of the chiral superfields Φ_i, λ are the gauginos and M their unified mass.

Let us recall that the one-loop β-function of the gauge coupling g is given by [25]

$$\beta_g^{(1)}=\frac{dg}{dt}=\frac{g^3}{16\pi^2}\left[\sum_i T(R_i)-3C_2(G)\right], \tag{10}$$

where $C_2(G)$ is the quadratic Casimir of the adjoint representation of the associated gauge group G. $T(R)$ is given by the relation $\mathrm{Tr}[T^aT^b]=T(R)\delta^{ab}$ where T^a is the generators of the group in the appropriate representation. Similarly the β-functions of C_{ijk}, by virtue of the non-renormalization theorem, are related to the anomalous dimension matrix γ_j^i of the chiral superfields as:

$$\beta_C^{ijk}=\frac{dC_{ijk}}{dt}=C_{ijl}\gamma_k^l+C_{ikl}\gamma_j^l+C_{jkl}\gamma_i^l. \tag{11}$$

At one-loop level the anomalous dimension, $\gamma^{(1)}{}_j^i$ of the chiral superfield is [25]

$$\gamma^{(1)}{}_j^i=\frac{1}{32\pi^2}\left[C^{ikl}C_{jkl}-2g^2C_2(R_i)\delta_{ij}\right], \tag{12}$$

where $C_2(R_i)$ is the quadratic Casimir of the representation R_i, and $C^{ijk}=C^*_{ijk}$. Then, the $N=1$ non-renormalization theorem [19,26] ensures there are no extra mass and cubic-interaction-term renormalizations, implying that the β-functions of C_{ijk} can be expressed as linear combinations of the anomalous dimensions γ_j^i.

Here we assume that the reduction equations admit power series solutions of the form

$$C^{ijk}=g\sum_{n=0}\rho_{(n)}^{ijk}g^{2n}. \tag{13}$$

In order to obtain higher-loop results instead of knowledge of explicit β-functions, which anyway are known only up to two-loops, relations among β-functions are required.

The progress made using the spurion technique [18–20] leads to the following all-loop relations among SSB β-functions (in an obvious notation) [12–14,16]

$$\beta_M=2\mathcal{O}\left(\frac{\beta_g}{g}\right), \tag{14}$$

$$\beta_h^{ijk}=\gamma^i{}_l h^{ljk}+\gamma^j{}_l h^{ilk}+\gamma^k{}_l h^{ijl}$$

$$-2\gamma_1^i{}_l C^{ljk} - 2\gamma_1^j{}_l C^{ilk} - 2\gamma_1^k{}_l C^{ijl}, \tag{15}$$

$$(\beta_{m^2})^i{}_j = \left[\Delta + X\frac{\partial}{\partial g}\right]\gamma^i{}_j, \tag{16}$$

where

$$\mathcal{O} = \left(Mg^2\frac{\partial}{\partial g^2} - h^{lmn}\frac{\partial}{\partial C^{lmn}}\right), \tag{17}$$

$$\Delta = 2\mathcal{O}\mathcal{O}^* + 2|M|^2 g^2\frac{\partial}{\partial g^2} + \tilde{C}_{lmn}\frac{\partial}{\partial C_{lmn}} + \tilde{C}^{lmn}\frac{\partial}{\partial C^{lmn}}, \tag{18}$$

$$(\gamma_1)^i{}_j = \mathcal{O}\gamma^i{}_j, \tag{19}$$

$$\tilde{C}^{ijk} = (m^2)^i{}_l C^{ljk} + (m^2)^j{}_l C^{ilk} + (m^2)^k{}_l C^{ijl}. \tag{20}$$

The assumption, following [13], that the relation among couplings

$$h^{ijk} = -M(C^{ijk})' \equiv -M\frac{dC^{ijk}(g)}{d\ln g}, \tag{21}$$

is RGI and furthermore, the use of the all-loop gauge β-function of Novikov et al. [27] given by

$$\beta_g^{\mathrm{NSVZ}} = \frac{g^3}{16\pi^2}\left[\frac{\sum_l T(R_l)(1-\gamma_l/2) - 3C_2(G)}{1 - g^2C_2(G)/8\pi^2}\right], \tag{22}$$

lead to the all-loop RGI sum rule [17] (assuming $(m^2)^i{}_j = m_j^2\delta^i_j$),

$$\begin{aligned} &m_i^2 + m_j^2 + m_k^2 \\ &\quad = |M|^2\left\{\frac{1}{1 - g^2C_2(G)/(8\pi^2)}\frac{d\ln C^{ijk}}{d\ln g} + \frac{1}{2}\frac{d^2\ln C^{ijk}}{d(\ln g)^2}\right\} \\ &\qquad + \sum_l \frac{m_l^2 T(R_l)}{C_2(G) - 8\pi^2/g^2}\frac{d\ln C^{ijk}}{d\ln g}. \end{aligned} \tag{23}$$

Surprisingly enough, the all-loop result of Eq. (23) coincides with the superstring result for the finite case in a certain class of orbifold models [23,28] if

$$\frac{d\ln C^{ijk}}{d\ln g} = 1,$$

as discussed in Ref. [5].

4. All-loop RGI relations in the SSB sector

Let us now see how the all-loop results on the SSB β-functions, Eqs. (14)–(20), lead to all-loop RGI relations. We assume:

(a) the existence of a RGI surfaces on which $C = C(g)$, or equivalently that

$$\frac{dC^{ijk}}{dg} = \frac{\beta_C^{ijk}}{\beta_g} \tag{24}$$

holds, i.e. reduction of couplings is possible, and

(b) the existence of a RGI surface on which

$$h^{ijk} = -M\frac{dC(g)^{ijk}}{d\ln g} \tag{25}$$

holds too in all-orders.

Then one can prove [29,30], that the following relations are RGI to all-loops (note that in both (a) and (b) assumptions above we do not rely on specific solutions of these equations)

$$M = M_0\frac{\beta_g}{g}, \tag{26}$$

$$h^{ijk} = -M_0\beta_C^{ijk}, \tag{27}$$

$$b^{ij} = -M_0\beta_\mu^{ij}, \tag{28}$$

$$(m^2)^i{}_j = \frac{1}{2}|M_0|^2\mu\frac{d\gamma^i{}_j}{d\mu}, \tag{29}$$

where M_0 is an arbitrary reference mass scale to be specified shortly. The assumption that

$$C_a\frac{\partial}{\partial C_a} = C_a^*\frac{\partial}{\partial C_a^*} \tag{30}$$

for a RGI surface $F(g, C^{ijk}, C^{*ijk})$ leads to

$$\frac{d}{dg} = \left(\frac{\partial}{\partial g} + 2\frac{\partial}{\partial C}\frac{dC}{dg}\right) = \left(\frac{\partial}{\partial g} + 2\frac{\beta_C}{\beta_g}\frac{\partial}{\partial C}\right) \tag{31}$$

where Eq. (24) has been used. Now let us consider the partial differential operator $\mathcal{O}$ in Eq. (17) which, assuming Eq. (21), becomes

$$\mathcal{O} = \frac{1}{2}M\frac{d}{d\ln g}. \tag{32}$$

In turn, β_M given in Eq. (14), becomes

$$\beta_M = M\frac{d}{d\ln g}\left(\frac{\beta_g}{g}\right), \tag{33}$$

which by integration provides us [29,31] with the generalized, i.e. including Yukawa couplings, all-loop RGI Hisano–Shifman relation [12]

$$M = \frac{\beta_g}{g}M_0, \tag{34}$$

where M_0 is the integration constant and can be associated to the unification scale M_U in GUTs or to the gravitino mass $m_{3/2}$ in a supergravity framework. Therefore, Eq. (34) becomes the all-loop RGI Eq. (26). Note that β_M using Eqs. (33) and (34) can be written as

$$\beta_M = M_0\frac{d}{dt}(\beta_g/g). \tag{35}$$

Similarly

$$(\gamma_1)^i{}_j = \mathcal{O}\gamma^i{}_j = \frac{1}{2}M_0\frac{d\gamma^i{}_j}{dt}. \tag{36}$$

Next, from Eq. (21) and Eq. (34) we obtain

$$h^{ijk} = -M_0\beta_C^{ijk}, \tag{37}$$

while β_h^{ijk}, given in Eq. (15) and using Eq. (36), becomes [29]

$$\beta_h^{ijk} = -M_0\frac{d}{dt}\beta_C^{ijk}, \tag{38}$$

which shows that Eq. (37) is all-loop RGI. In a similar way Eq. (28) can be shown to be all-loop RGI.

Finally we would like to emphasize that under the same assumptions (a) and (b) the sum rule given in Eq. (23) has been proven [17] to be all-loop RGI, which (using Eq. (34)) gives us a generalization of Eq. (29) to be applied in considerations of non-universal soft scalar masses, which are necessary in many cases including the MSSM.

Having obtained Eqs. (26)–(29) from Eqs. (14)–(20) with the assumptions (a) and (b), we would like to conclude the present section with some remarks. First it is worth noting the difference, say in first order in g, among the possibilities to consider specific solution of the reduction equations or just assume the existence of a RGI surface, which is a weaker assumption. So in the case we consider the reduction equation (24) without relying on a specific solution, the sum rule (23) reads

$$m_i^2+m_j^2+m_k^2=|M|^2\frac{d\ln C^{ijk}}{d\ln g}, \tag{39}$$

and we find that

$$\frac{d\ln C^{ijk}}{d\ln g}=\frac{g}{C^{ijk}}\frac{dC^{ijk}}{dg}=\frac{g}{C^{ijk}}\frac{\beta_C^{ijk}}{\beta_g}, \tag{40}$$

which is clearly model dependent. However assuming a specific power series solution of the reduction equation, as in Eq. (3), which in first order in g is just a linear relation among C^{ijk} and g, we obtain that

$$\frac{d\ln C^{ijk}}{d\ln g}=1 \tag{41}$$

and therefore the sum rule (39) becomes model independent. We should also emphasize that in order to show [13] that the relation

$$\left(m^2\right)^i{}_j=\frac{1}{2}\frac{g^2}{\beta_g}|M|^2\frac{d\gamma^i{}_j}{dg}, \tag{42}$$

which using Eq. (34) becomes Eq. (29), is RGI to all-loops a specific solution of the reduction equations has to be required. As it has already been pointed out above such a requirement is not necessary in order to obtain the all-loop RG invariance of the sum rule (23).

As it was emphasized in Ref. [29] the set of the all-loop RGI relations (26)–(29) is the one obtained in the *Anomaly Mediated SB Scenario* [32], by fixing the M_0 to be $m_{3/2}$, which is the natural scale in the supergravity framework.

A final remark concerns the resolution of the fatal problem of the anomaly induced scenario in the supergravity framework, which is here solved thanks to the sum rule (23), as it will become clear in the next section. Other solutions have been provided by introducing Fayet–Iliopoulos terms [33].

5. MSSM and RGI relations

We would like now to apply the RGI relations to the SSB sector of the MSSM, assuming power series solutions of the reduction equations at the unification scale. According to the analysis presented in Section 4 the RGI relations in the SSB sector hold, assuming the existence of RGI surfaces where Eqs. (24) and (25) hold. We show first that Eq. (24) indeed holds in the MSSM, then we assume the validity of Eq. (25) and examine the consequences in the MSSM phenomenology.

Using a perturbative ansatz concerning the solutions of Eqs. (24) and (25), the set of Eqs. (26)–(28) and Eq. (39) together with Eq. (41), clearly hold. Then one easily finds that Eq. (25) with (the first order) perturbative ansatz at the unification scale leads to the condition

$$h^{ijk}=-M_UC^{ijk}, \tag{43}$$

where M_U is the gaugino mass and C^{ijk} are the Yukawa couplings, both at the unification scale. Therefore, this assumption leads to Eqs. (43) as boundary conditions at the unification scale.

In a similar way, starting from Eq. (28) and assuming that μ^{ij} are reduced in favour of g, i.e. that the reduction equation holds

$$\beta_\mu^{ij}=\beta_g d\mu^{ij}/dg \tag{44}$$

and moreover has power series type solutions, we obtain

$$b^{ij}=-M_U\mu^{ij} \tag{45}$$

as boundary conditions at the unification scale.

Finally the sum rule (39) also holds at the unification scale in the form,

$$m_i^2+m_j^2+m_k^2=M_U^2. \tag{46}$$

Therefore, the above Eqs. (43), (45) and (46) have to be imposed as boundary conditions at the unification scale in the renormalization group equations that govern the evolution of the SSB parameters.

Let us now consider more specifically the MSSM, which is defined by the superpotential,

$$W=Y_tH_2Qt^c+Y_bH_1Qb^c+Y_\tau H_1L\tau^c+\mu H_1H_2, \tag{47}$$

with soft breaking terms,

$$\begin{aligned}-\mathcal{L}_{SSB}={}&\sum_\phi m_\phi^2\phi^*\phi+\left[m_3^2H_1H_2+\sum_{i=1}^3\frac{1}{2}M_i\lambda_i\lambda_i+\text{h.c.}\right]\\&+\left[h_tH_2Qt^c+h_bH_1Qb^c+h_\tau H_1L\tau^c+\text{h.c.}\right],\end{aligned} \tag{48}$$

where the last line refers to the scalar components of the corresponding superfield. In general $Y_{t,b,\tau}$ and $h_{t,b,\tau}$ are 3×3 matrices, but we work throughout in the approximation that the matrices are diagonal, and neglect the couplings of the first two generations.

5.1. Reduction of couplings

Assuming perturbative expansion of all three Yukawa couplings in favour of α_3 satisfying the reduction equations

$$\beta_{Y_{t,b,\tau}}=\beta_{g_3}\frac{dY_{t,b,\tau}}{dg_3}, \tag{49}$$

we run into trouble since the coefficients of the Y_τ coupling turn imaginary. Therefore, we take Y_τ at the GUT scale to be an independent variable. In that case, the coefficients of the expansions (again at the GUT scale)

$$\frac{Y_t^2}{4\pi}=c_1\frac{g_3^2}{4\pi}+c_2\left(\frac{g_3^2}{4\pi}\right)^2, \tag{50}$$

$$\frac{Y_b^2}{4\pi}=p_1\frac{g_3^2}{4\pi}+p_2\left(\frac{g_3^2}{4\pi}\right)^2 \tag{51}$$

are given by

$$\begin{aligned}c_1&=\frac{157}{175}+\frac{1}{35}K_\tau=0.897+0.029K_\tau,\\p_1&=\frac{143}{175}-\frac{6}{35}K_\tau=0.817-0.171K_\tau,\\c_2&=\frac{1}{4\pi}\frac{1457.55-84.491K_\tau-9.66181K_\tau^2-0.174927K_\tau^3}{818.943-89.2143K_\tau-2.14286K_\tau^2},\\p_2&=\frac{1}{4\pi}\frac{1402.52-223.777K_\tau-13.9475K_\tau^2-0.174927K_\tau^3}{818.943-89.2143K_\tau-2.14286K_\tau^2}\end{aligned} \tag{52}$$

where

$$K_\tau=Y_\tau^2/g_3^2. \tag{53}$$

The important new observation is that the couplings Y_t, Y_b and g_3 are not only reduced, but they provide predictions consistent with the observed experimental values (as it will be explained later in the discussion of Fig. 3).

Given the above solutions of the reduction equations

$$\beta_{Y_{t,b}}=\beta_{g_3}\frac{dY_{t,b}}{dg_3}, \tag{54}$$

and assuming the validity of Eq. (25) then, according to our earlier discussion, the following relations are RGI

$$M = \frac{\beta_{g_3}}{g_3} M_U, \tag{55}$$

$$h_{t,b} = -M g_3 \frac{dY_{t,b}}{dg_3}, \tag{56}$$

$$m_3^2 = -M g_3 \frac{d\mu}{dg_3}, \tag{57}$$

$$m_i^2 + m_j^2 + m_k^2 = M^2, \tag{58}$$

where i, j, k refer to the superfields appearing in the trilinear terms in the superpotential (47).[2]

Note that in the application of the reduction of couplings in the MSSM that we examine here, in the first stage we neglect the Yukawa couplings of the first two generations, while we keep Y_τ and the gauge couplings g_2 and g_1, which cannot be reduced consistently, as corrections. Therefore, strictly speaking, when we say above that Eqs. (55)–(58) are RGI we refer to the case that not only the first two generations but also the Y_τ, g_2 and g_1 are switched off.

In turn, since all gauge couplings in the MSSM meet at the unification point, we are led to the following boundary conditions at the GUT scale:

$$Y_t^2 = c_1 g_U^2 + c_2 g_U^4/(4\pi) \quad \text{and} \quad Y_b^2 = p_1 g_U^2 + p_2 g_U^4/(4\pi), \tag{59}$$

$$h_{t,b} = -M_U Y_{t,b}, \tag{60}$$

$$m_3^2 = -M_U \mu, \tag{61}$$

where $c_{1,2}$ and $p_{1,2}$ are the solutions of the algebraic system of the two reduction equations (49) taken at the GUT scale (while keeping only the first term[3] of the perturbative expansion of the Yukawas in favour of g_3 for Eqs. (60) and (61)), and a set of equations resulting from the application of the sum rule (46)

$$m_{H_2}^2 + m_Q^2 + m_{t^c}^2 = M_U^2, \tag{62}$$

$$m_{H_1}^2 + m_Q^2 + m_{b^c}^2 = M_U^2, \tag{63}$$

noting that the sum rule introduces four free parameters.

6. Discussion and conclusions

In the present Letter we have made a new important observation, that the Y_t, Y_b and α_3 obey RGI relations within the MSSM. Therefore, they can be reduced and can be considered as parameters dependent among themselves. This "reduced" system holds at all scales, and thus serve as boundary conditions of the RGEs of the MSSM at the unification scale, where we assume that the gauge couplings meet. With these boundary conditions we run the MSSM RGEs down to the SUSY scale, which we take to be the geometrical average of the stop masses, and then run the SM RGEs down to the electroweak scale (M_Z), where we compare with the experimental values of the third generation quark masses. The RGEs are taken at two-loops for the gauge and Yukawa couplings and at one-loop for the soft breaking parameters. We let M_U and $|\mu|$ at the unification scale to vary between ~ 1 TeV ~ 11 TeV, for the two possible signs of μ. In evaluating the τ and bottom masses we have taken into account the one-loop radiative corrections that come from the SUSY breaking [34]. These corrections have a dependence on the soft breaking parameters, in particular for large $\tan\beta$ they can give sizeable contributions to the bottom quark mass.

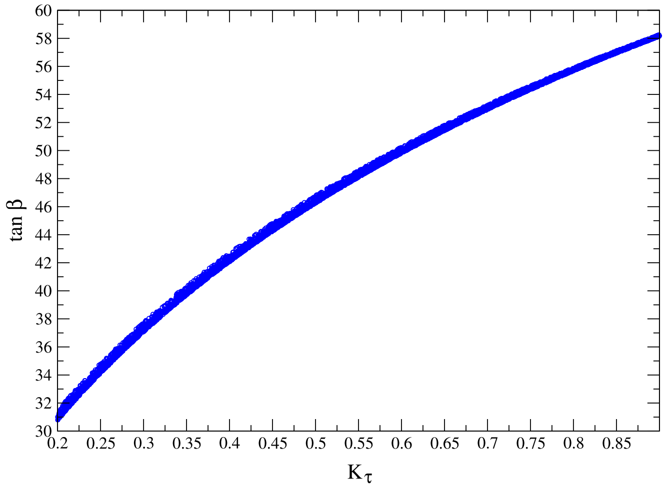

Fig. 1. Required values of $\tan\beta$ as a function of $K_\tau = Y_\tau^2/g_3^2$ in order to get the experimentally accepted tau mass.

The observation that Y_t, Y_b and α_3 are a reduced system is best demonstrated in Fig. 3, where we plot the predictions for the top quark mass, M_t, and the bottom quark mass, M_b, as they result from Eqs. (50) and (51) with $c_{1,2}$ and $p_{1,2}$ given in Eq. (52), for $\mathrm{sign}(\mu) = -$. As one can see the predicted values agree comfortably with the corresponding experimental values within 1σ. Recall that Y_τ is not reduced and is a free parameter in this analysis. In Fig. 1 we present a plot relating the values of $\tan\beta$ and $K_\tau = Y_\tau^2/g_3^2$ which are compatible with the observed experimental value of the tau mass M_τ (fixed at its experimental central value). In the case that $\mathrm{sign}(\mu) = +$, there is no value for K_τ where both the top and the bottom quark masses agree simultaneously with their experimental value, therefore we only consider the negative sign of μ from now on.

The parameter K_τ is further constrained by allowing only the values that are also compatible with the top and bottom quark masses within 1 and 2σ of their central experimental value. We use the experimental value of the top quark pole mass as [35]

$$M_t^{\exp} = (173.2 \pm 0.9)\ \text{GeV}. \tag{64}$$

The bottom mass is calculated at M_Z to avoid uncertainties that come from running down to the pole mass and, as previously mentioned, the SUSY radiative corrections both to the tau and the bottom quark masses have been taken into account [36]

$$M_b(M_Z) = (2.83 \pm 0.10)\ \text{GeV}. \tag{65}$$

In Fig. 2, we show these constrained K_τ values plotted against M_t (its central value corresponds to the purple dashed line), within 1σ (orange dashed lines), and 2σ (upper border of the graph), where also M_b is constrained to be within 1 and 2σ of its experimental value. We can do the same for M_b but we prefer to present in Fig. 3 the values of M_t vs M_b for the constrained K_τ values. From Fig. 3 it can be clearly seen that there is a set of values for the parameter K_τ where both M_t and M_b agree simultaneously within 1σ of their experimental values, for the boundary conditions given by the reduced system Y_t, Y_b and α_3.

Finally, assuming the validity of Eq. (25) for the corresponding couplings to those that have been reduced before, we calculate the Higgs mass as well as the whole Higgs and sparticle spectrum using Eqs. (59)–(63), and we present them in Figs. 4 and 5. The Higgs

[2] There is another RGI term in the form of the b-parameter that could be included in Eq. (28) as was suggested in Ref. [33]. This term would turn m_3^2 in Eqs. (57) in a free parameter to be determined by the minimization of the electroweak potential. Although we omit this term here, following other treatments in the literature, we plan to include this possibility in a future examination.

[3] The second term can be determined once the first term is known.

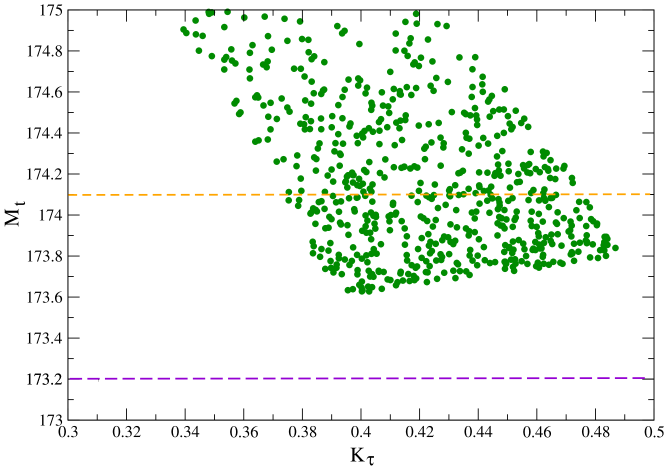

Fig. 2. The top mass as a function of $K_\tau = Y_\tau^2/g_3^2$, the purple dashed line is the experimental central value and the orange one is the 1σ value. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

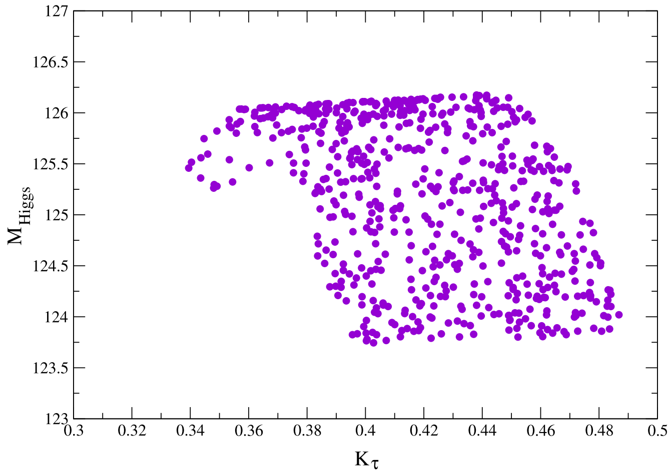

Fig. 4. The Higgs mass as a function of $K_\tau = Y_\tau^2/g_3^2$.

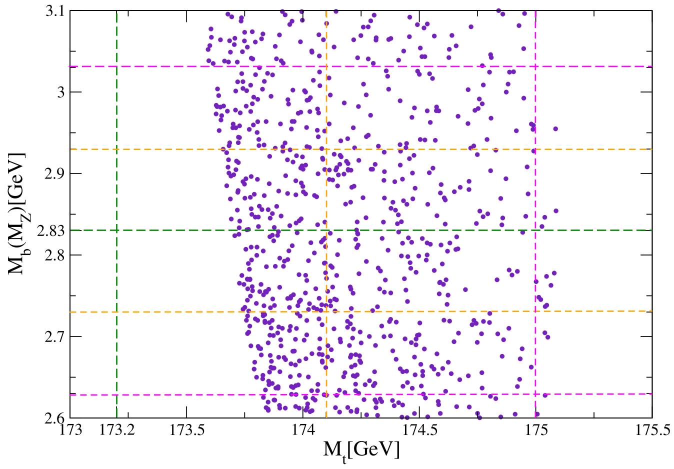

Fig. 3. Using the regions of values for $K_\tau = Y_\tau^2/g_3^2$ and $\tan\beta$ which give experimentally accepted tau mass, this figure shows the resulted points in the (M_t, M_b) phase space. The central value (green dashed lines), as well as the 1 and 2σ deviation (orange and magenta lines respectively), for the top and bottom masses is also drawn. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

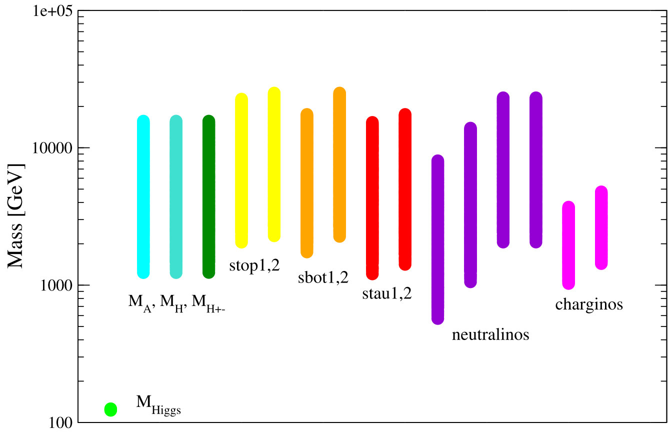

Fig. 5. The Higgs mass and s-spectrum for values of $M_U \sim 1.3$ TeV to ~ 11 TeV.

mass was calculated using a "mixed-scale" one-loop RG approach, which is known to be a very good approximation to the full diagrammatic calculation [37].

From Fig. 4 we notice that the lightest Higgs mass is in the range 123.7–126.3 GeV, where the uncertainty is due to the variation of K_τ, the gaugino mass M_U and the variation of the scalar soft masses, which are however constrained by the sum rules (62) and (63). The gaugino mass M_U is in the range ~ 1.3 TeV ~ 11 TeV, the lower values having been discarded since they do not allow for radiative electroweak symmetry breaking. The variation of K_τ is in the range $\sim 0.37 \sim 0.49$ in order to agree with the experimental values of the bottom and top masses at 1σ, and $\sim 0.34 \sim 0.49$ if the agreement is at the 2σ level. To the lightest Higgs mass value one has to add at least ± 2 GeV coming from unknown higher order corrections [38]. Therefore it is in excellent agreement with the experimental results of ATLAS and CMS [1].

From Fig. 5 we find that the masses of the heavier Higgses have relatively high values, above the TeV scale. In addition we find a generally heavy supersymmetric spectrum starting with a neutralino as LSP at ~ 500 GeV and comfortable agreement with the LHC bounds due to the non-observation of coloured supersymmetric particles [2]. Finally note that although the $\mu < 0$ found in our analysis would disfavour the model in connection with the anomalous magnetic moment of the muon, such a heavy spectrum gives only a negligible correction to the SM prediction. We plan to extend our analysis by examining the restrictions that will be imposed in the spectrum by the B-physics and CDM constraints.

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5.11 Conclusions to Section 5

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A number of proposals and ideas have matured with time and have survived after careful theoretical studies and confrontation with experimental data. These include part of the original GUTs ideas, mainly the unification of gauge couplings and, separately, the unification of the Yukawa couplings, a version of fixed point behaviour of couplings, and certainly the necessity of SUSY as a way to take care of the technical part of the hierarchy problem. On the other hand, a very serious theoretical problem, namely the presence of divergencies in Quantum Field Theories (QFT), although challenged by the founders of QFT [32, 33, 34], was mostly forgotten in the course of developments of the field partly due to the spectacular successes of renormalizable field theories, in particular of the SM. However, fundamental developments in theoretical particle physics are based in reconsiderations of the problem of divergencies and serious attempts to solve it. These include the motivation and construction of string and non-commutative theories, as well as $N = 4$ supersymmetric field theories [35, 36], $N = 8$ supergravity [37, 38, 39, 40, 41] and the AdS/CFT correspondence [42]. It is a thoroughly fascinating fact that many interesting ideas that have survived various theoretical and phenomenological tests, as well as the solution to the UV divergencies problem, find a common ground in the framework of $N = 1$ Finite Unified Theories, which we have described in the previous sections. From the theoretical side they solve the problem of UV divergencies in a minimal way. On the phenomenological side, since they are based on the principle of reduction of couplings (expressed via RGI relations among couplings and masses), they provide strict selection rules in choosing realistic models which lead to testable predictions.

Currently we are still examining the predictions of the best so far $SU(5)$ Finite Unified Theory, including the restrictions of third generation quark masses and B-physics observables. The model is consistent with all the phenomenological constraints. Compared to our previous analysis (see subsect. 5.8) the new bound on BR($B_s \to \mu^+\mu^-$) prefers a heavier (Higgs) spectrum and thus in general allows only a very heavy SUSY spectrum. The Higgs mass constraint, on the other hand, taking into account the improved M_h prediction for heavy scalar tops, favours the lower part of this spectrum, with SUSY masses ranging from ~ 600 GeV up to the multi-TeV level, where the lower part of the spectrum could be accessible at the ILC or CLIC. Taking into account the improved theory uncertainty evaluation some part of the electroweak spectrum should be accessible at future e^+e^- colliders. The coloured spectrum, on the other hand, could easily escape the LHC searches; also at the HL-LHC non-negligible parts of the spectrum remain beyond the discovery reach.

The celebrated success of predicting the top-quark mass (see subsects. 5.1, 5.2, 5.3 and [45, 21, 25]) has been extended to the predictions of the Higgs masses and the supersymmetric spectrum of the MSSM [43, 48]. Clear predictions for the discovery reach at current and future pp colliders as well as for future e^+e^- colliders result in somewhat more optimistic expectations compared to older analyses.

6 Discussion and Conclusions

In the above sections we presented the historical development of two notions: *reduction of couplings* and *finiteness* within $N = 1$ supersymmetric gauge theories and then how they have been applied to the standard model (SM) and extensions of it with the aim of forcasting or describing the experimental findings with as few parameters as possible. We selected those original papers in which the relevant results had been obtained. These papers should speak for themselves but by providing individual comments for them and by putting them in the appropriate context by introductory remarks at the beginning of the sections we tried to make the papers and the whole endavour easier accessible also to a reader who is not an expert in the field.
After having provided the machinery for reducing couplings in section 2 a first attempt to use it in particle physics has been presented in section 3, devoted to the SM. Its final outcome in the version with three families says that a top mass larger than roughly 111 GeV would not allow to realize asymptotic freedom of couplings in this theory. It also shows that the results are very sensitive to the details of the model. Already admitting a fourth generation would change drastically the predictions. Another warning feature came about when demanding cancellation of quadratical divergencies: it was not very well compatible with the bound obtained for the top mass.
An obvious candidate for guaranteeing absence of quadratical divergencies related to physical parameters is supersymmetry; a way of avoiding too many new parameters is provided by requiring finiteness. The basis for this is being given in section 4, together with the proof that reduction is a renormalization scheme independent concept.
The sequence of papers in section 5 then shows how one can reconcile supersymmetric models with phenomenology. The first interesting hint that this could be the right track came in the paper of subsection 5.1 (1992) with the prediction of 178.8 GeV for the top mass in two finite supersymmetric $SU(5)$ models. At that time this has been considered as a pretty large value.
Encouraged by the discovery of the top around this mass value a more systematic search has been initiated via grand unified supersymmetric models, unification of Yukawa couplings followed by a careful study of supersymmetry breaking through soft mass terms. As early as 2008 this analysis culminated eventually in the prediction of a Higgs mass value in the interval between 121...126 GeV (see subsect. 5.8). Once a Higgs-like particle had been found experimentally its mass value could be used for restricting further the supersymmetric spectrum. Eventually it was possible to reproduce the experimental value of this Higgs-like boson and to identify the lightest Higgs of the MSSM as *the* Higgs of the standard model by partial reduction (see subsect. 5.10).
Obviously this nice result prompts further questions. How can this model and its renormalization group relations be linked to the finite models which were so successful in pointing to the right value for the top mass? Is there the respective gauge group singled out by some specific, characterizing property? And, on top of this: Do not all these considerations point to supersymmetry as the relevant underlying symmetry?
These questions also imply that the search on the structure of matter goes on.

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