PROCEEDINGS OF SCIENCE

Causality and Noncommutative Geometry

Fabien Besnard*
École Polytechnique Féminine, Sceaux, France
E-mail: fabien.besnard@epf.fr

Motivated by the introduction of causality in noncommutative geometry, we define the notion of isocone. An isocone is a closed convex cone in a C^*-algebra, containing the unit, which separates the states and is stable by non-decreasing continuous functional calculus. This definition corresponds exactly to the structure of non-decreasing real functions on a (compact) topological ordered set satisfying a natural compatibility condition between the topology and the partial order, when the C^*-algebra is commutative. We give the complete classification of isocones in finite dimensional algebras, corresponding to finite noncommutative ordered spaces, and give some examples in infinite dimension. Finally we show that the existence of an isocone on an almost commutative algebra $\mathscr{C}(M)\otimes A_f$ of the kind which appears in the NCG formulation of the Standard Model forces the causal order relation on M to disappear in the neighbourhood of every point.

Frontiers of Fundamental Physics 14
15-18 July 2014
Aix Marseille University (AMU) Saint-Charles Campus, Marseille, France

*Speaker.

1. Motivations

There are at least two roads which lead to problem of defining a noncommutative ordered space. The first is the search for a quantum theory of gravity. In such a theory, it is expected that classical observables are replaced by non-commuting operators. In particular time functions, and more generally causal functions, will become such operators. Let us denote the set of quantized causal functions by I. This set will have a structure which we can try to guess on physical principles. It can be argued for instance that I is stable by sum and functional calculus $a \mapsto f(a)$ where f is a non-decreasing real-valued continuous function (see [4] p 6 for more details). However we will see that we are more easily led to the axioms we seek by following the second road, which is noncommutative geometry. There already have been some efforts put in the extension of noncommutative geometry to the semi-Riemannian setting [8, 11, 12], and more specifically the Lorentzian setting (see the talk of Michal Eckstein in this conference). However these existing approaches are 'top-down' in the sense that one starts with a suitably generalized spectral triple and define causality out of it in the same way that one defines causality out of the metric in the classical case. On the contrary, our approach is 'bottom-up' : we start from causality and we try to render it noncommutative as a first move. The rationale for doing this is that causality is a strictly Lorentzian phenomenon which we believe plays a fundamental role. Moreover the information given by the metric can be quite naturally decomposed into the causal ordering and the conformal factor. Briefly said, we choose to make this decomposition right from the start in a spirit similar to the causal set program [5]. Causality being a partial order relation, we know that the first step (the only one about which we will talk here) is to find an C^*-algebraic counterpart to partially ordered sets.

2. A Gelfand-Naimark theorem for topological ordered sets

Let us begin by giving some definitions.

Definition 1.
1. *Let $(M, \preceq)$, $(N, \leq)$ be two posets. A map $f : M \to N$ is* isotone *iff $x \preceq y \Rightarrow f(x) \leq f(y)$.*
2. *Let $(M, \preceq, \tau)$ be a poset equipped with a topology τ. We set $I(M) := \{f : M \to \mathbb{R} | f$ is isotone and continuous$\}$.*
3. *$(M, \preceq, \tau)$ is called a* toposet *iff for all $x, y \in M$, one has $x \preceq y \Leftrightarrow \forall f \in I(M), f(x) \leq f(y)$.*

Let us note that a toposet is usually called a completely separated topological ordered set. We choose the former name for the sake of brevity. Let us also remark that the toposet condition is tautologically the weakest condition one can ask in order to hope for a duality result of the Gelfand-Naimark sort.

An important class of examples of toposets is given by causally simple spacetimes. This is the second strongest causality condition on Loretzian manifolds after global hyperbolicity. It is in fact equivalent to the toposet property (see [4] for details and references).

The toposet property evidently implies that $\preceq$ is a closed subset of $M \times M$. Conversely if M is compact and $\preceq$ is closed, the toposet property holds.

Let us now introduce the dual object. We write $\mathrm{Re}\,A$ for the set of self-adjoint elements of a C^*-algebra A and $\sigma(a)$ for the spectrum of $a \in A$.

Definition 2. *Let A be a unital C^*-algebra. A subset $I \subset \mathrm{Re}\,A$ is called an isocone if the following holds:*

1. $1 \in I$,
2. $\forall a, a' \in I$, $a + a' \in I$,
3. $\forall a \in I$, $\forall f$ *isotone and continuous on* $\sigma(a)$, $f(a) \in I$,
4. *I is closed,*
5. $I - I = \{a - b | a, b \in I\}$ *is dense in* $\mathrm{Re}\,A$.

The couple (I,A) is called an I^*-algebra.

The set $I(M)$ where M is a compact toposet is easily seen to be an isocone of $\mathscr{C}(M)$. Some might worry that the unital case is too restrictive, since causal Lorentzian manifolds cannot be compact. On the other hand, one cannot avoid a unit in the algebra: even if we start with a locally compact manifold, the algebra $\mathscr{C}_0(M)$ of functions vanishing at infinity do not contain non zero isotone functions ! In fact there is no difficulty here. Causal Lorentzian manifolds can indeed be compact provided they have a boundary. Working with a causal compactification, hence adding a boundary, solves the problem.

Now observe that an isocone I induces a reflexive and transitive relation $\preceq_I$ on the pure state space $P(A)$ of the algebra by

$$\phi \preceq_I \psi \Longleftrightarrow \forall a \in I, \phi(a) \leq \psi(a)$$

This relation is a partial order thanks to axiom 5. By construction $(P(A), \preceq_I)$ is a toposet.

We can now state the duality result which justifies the above definition.

Theorem 1. (Gelfand-Naimark theorem for commutative I^*-algebras) *Let (I,A) be a commutative I^*-algebra. Then I is isomorphic to $I(P(A), \preceq_I)$ by the Gelfand transform.*

This theorem can be raised to the level of a dual equivalence of categories given the appropriate notion of morphisms. We refer to [1] for the proof and details (see also [2] and [3] for an update on the axioms).

This theorem can be applied to the C^*-algebra $C^*(a)$ generated by a self-adjoint element a in any I^*-algebra. It then shows the existence of a poset structure on $\sigma(a)$ induced by I.

3. Noncommutative examples

A trivial example of isocone is the set $\mathrm{Re}\,A$ in any unital C^*-algebra A. This induces the equality ordering on $P(A)$.

For a less trivial but still very simple example consider the algebra $M_2(\mathbb{C})$. In that case the action of an isotone function on a self-adoint matrix a can be emulated by an affine function with positive slope. Hence I is an isocone iff it is a closed convex cone with non-empty interior and containing the unit.

For an infinite-dimensional noncommutative example consider a compact toposet M with a regular Borel measure, and call $\pi : \mathscr{C}(M) \to B(H)$ the representation on $H = L^2(M)$ by multiplication. Let $A = \pi(\mathscr{C}(M))$ and $I = \pi(I(M))$. Then $I + \mathrm{Re}K$ is an isocone in the C^*-algebra $A+K$, where K denotes the ideal of compact operators. The intuition behind this example is that an infinitesimal perturbation of a non-decreasing function is also non-decreasing.

These examples can be used to build other ones thanks to the following construction.

Theorem 2. *Let $(P,\preceq)$ be a finite poset and for each $x \in P$ let (I_x, A_x) be an I^*-algebra. We set $I = \bigoplus_{x\in P} I_x$, $A = \bigoplus_{x\in P} A_x$. Then*

$$\operatorname*{Lex}_{x\in P} I_x := \{a \in I | \forall x, y \in P, x \prec y \Rightarrow \max \sigma(a_x) \leq \min \sigma(a_y)\}$$

is an isocone of A.

The isocone $L = \mathrm{Lex}_{x\in P} I_x$ is called the lexicographic sum of the family $(I_x)_{x\in P}$. The ordering induced by L is indeed lexicographic. Let us write an element of $P(A)$ in the form (x, ϕ), with $x \in P$ and $\phi \in P(A_x)$. Then

$$(x, \phi) \preceq_L (y, \psi) \iff x \prec y \text{ or } (x = y \text{ and } \phi \preceq_{I_x} \psi)$$

4. Finite-dimensional I^*-algebras

It turns out that the lexicographic sums of the two first examples of the previous section exhaust all the finite-dimensional I^*-algebras. More precisely, we have the following classification theorem.

Theorem 3. *Let I be an isocone in the finite-dimensional C^*-algebra $A = \bigoplus_{x\in P} M_{n_x}(\mathbb{C})$, with $P = \{1;\dots;k\}$, $k \in \mathbb{N}^*$, $n_x \in \mathbb{N}^*$. Then there exists a poset structure on P, and isocones $I_x \subset \mathrm{Re}M_{n_x}(\mathbb{C})$ such that $I = \mathrm{Lex}_{x\in P} I_x$.*

Moreover, if $n_x \neq 2$ then $I_x = \mathrm{Re}M_{n_x}(\mathbb{C})$.

The proof can be found in [2].

5. Isocones in almost-commutative algebras

This section describes the main result of [4] to which we refer for details and proofs. An almost-commutative algebra is one of the form $\mathscr{C}(M) \otimes A_f$ where M is a commutative space and A_f a noncommutative finite-dimensional algebra. For notational simplicity we consider here the case $A_f = M_n(\mathbb{C})$, but the results are easily generalized.

Let $\mathscr{A} = \mathscr{C}(M) \otimes M_n(\mathbb{C})$. We first obtain a necessary condition for the existence of an isocone

Theorem 4. *Let I be an isocone of $\mathscr{A}$. Then the order induced by I on $P(\mathscr{A}) = M \times \mathbb{P}(\mathbb{C}^n)$ is lexicographic: there exists an ordering $\preceq_M$ on M and for each $x \in M$ there exists an isocone $I_x \subset A_x$ such that : $(x,\xi) \preceq_I (y,\eta) \Leftrightarrow x \prec_M y$ or $(x = y$ and $\xi \preceq_{I_x} \eta)$.*

We conclude with a necessary and sufficient condition for the existence of an isocone in $\mathscr{A}$ inducing a particular lexicographic order on $M \times \mathbb{P}(\mathbb{C}^n)$, under the additional assumption that M is metrizable.

Theorem 5. *Assume M is metrizable. Let $\preceq$ be a lexicographic order on $M \times \mathbb{P}(\mathbb{C}^n)$ associated to a partial order $\preceq_M$ on M and local isocones $(I_x)_{x\in M}$. Then there is an isocone I in A inducing $\preceq$ iff*

- *$\prec_M$ is closed.*
- *$L : x \mapsto I_x$ is lower hemi-continuous.*

This theorem would be a classification theorem of isocones in almost-commutative algebras if one could prove that the isocone inducing $\preceq$ as above is unique. We conjecture that this is true.

6. Conclusion

The condition that $\prec_M$ is closed can be rephrased in a somewhat emphatic way by saying that causality, if induced by an isocone, must disappear in the neighbourhood of every point, that is at small scale, or if one wants, high-energy. This is in complete agreement with the results of [10] that high-energy bosons do not propagate, despite a completely different approach. This could indicate that the isocone approach is not misguided since it seems to be compatible with the properties of the spectral action.

On a more hypothetical note, let us note that the scale at which causality disappears, though unspecified by our very general mathematical result, only depends on the noncommutativity of the algebra A_f, hence on particle physics, and might thus be expected to be much larger than the Planck scale and to possibly have left an observable imprint on the cosmic microwave background. Hence this is in principle a testable prediction, though only a qualitative one at this stage.

Also, even more hypothetically, it is very tempting to relate the classification theorem of finite-dimensional isocones, which singles out $M_2(\mathbb{C})$ as a particular case, with the strong CP problem, that is our current inability to understand why the weak interaction appear to be the only ones to break CP symmetry, or, equivalently, T symmetry.

References

[1] F. Besnard, *A noncommutative view on topology and order*, J. Geom. Phys. **59**, pp. 861–875 (2009) abs/0804.3551

[2] F. Besnard, *An approximation theorem for non-decreasing functions on compact posets*, J. Approx. Theory **172** pp. 1–3 (2013), abs/1208.2616

[3] F. Besnard, *Noncommutative ordered spaces: examples and counterexamples*, arXiv:1312.2442

[4] N. Bizi, F. Besnard, *The disappearance of causality at small scale in noncommutative manifolds*, arXiv:1411.0878

[5] L. Bombelli, J. Lee, D. Meyer, and R. D. Sorkin, *Space-time as a causal set*, Phys. Rev. Lett. **59**, 521–524 (1987)

[6] A. H. Chamseddine, A. Connes, *Resilience of the Spectral Standard Model*, JHEP 1209 (2012) 104

[7] A. H. Chamseddine, A. Connes, W. van Suijlekom, *Beyond the Spectral Standard Model: Emergence of Pati-Salam Unification*, JHEP 1311 (2013) 132, arXiv:1304.8050

[8] K. van den Dungen, M. Paschke, A. Rennie, *Pseudo-Riemannian spectral triples and the harmonic oscillator*, J. Geom. Phys. **73** (2013), 37-55, arXiv:1207.2112.

[9] N. Franco and M. Eckstein, *Exploring the Causal Structures of Almost Commutative Geometries*, abs/1310.8225 (2013)

[10] M.A. Kurkov, F. Lizzi, D. Vassilevich, *High energy bosons do not propagate*, Phys. Lett. B, **731**, pp 311-315 (2014) arXiv:1312.2235

[11] M. Paschke , A. Sitarz, *Equivariant Lorentzian spectral triples*, math-ph/0611029

[12] A. Strohmaier, *On noncommutative and pseudo-Riemannian geometry*, J. Geom. Phys. **56** (2006), 175-195. math-ph/0110001