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LQC on curved FRW space time

Linda Linsefors*
Laboratoire de Physique Subatomique et de Cosmologie, Universite Grenoble-Alpes, CNRS-IN2P3
E-mail: linsefors@lpsc.in2p3.fr

Aurelien Barrau
Laboratoire de Physique Subatomique et de Cosmologie, Universite Grenoble-Alpes, CNRS-IN2P3
E-mail: Aurelien.Barrau@cern.ch

This study uses very simple symmetry and consistency considerations to put constraints on possible Friedmann equations for modified gravity models in curved spaces. As an example, it is applied to loop quantum cosmology.

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*Speaker.

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1. Introduction

From the mathematical perspective, the equations governing the background cosmological evolution can be seen as a symmetry reduced version of the gravity field equations. As well as being successful in describing the evolution of the universe, cosmology can be seen as an interesting testing ground for new theories of gravity, in particular motivated for being effective models (or low-energy limits) of quantum gravity.

In this study we investigate a general class of modified cosmologies that will be defined by a number of assumptions. We will find how these theories are constrained by the coordinate freedom that is fundamentally encoded in the metric, whatever the considered theory. Our study is rooted in the symmetries of de Sitter and Minkowski spaces. Intuitively speaking, the idea is to consider a de Sitter phase and use its maximal symmetry.

As a fruitful example, the conclusions previously derived will be applied to loop quantum cosmology (LQC), see [1] for general introductions. In itself, LQC is a symmetry reduced version of loop quantum gravity, see [2] for introductory reviews.

A more complete version of this can be found in [3].

2. FLRW metric

The FLRW metric reads as

$$ds^2 = -dt^2 + a(t)^2\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right). \tag{2.1}$$

This is the most general homogenous and isotropic metric one can write down. More precisely, this is the interval written in a coordinate system where the symmetries of the Universe are clearly manifest. The only way to preserve the homogeneity and isotropy of space and yet incorporate time evolution is to allow the curvature scale, characterized by a, to be time-dependent. At this stage, only symmetries are involved and nothing is assumed about the details of the considered gravitational theory. In this expression, k is a constant and $a(t)$ is the scale factor. The evolution of $a(t)$ is determined by Einstein's equations or, alternatively, by some modified gravity or modified cosmology theory.

In general, there are two possible coordinate transformation witch leave the FLRW formalism invariant. The first one is a re-scaling of the radial coordinate by a constant $b > 0$. Such a transformation affects both a and k, but keeps the FLRW expression unchanged:

$$r' = r/b, \quad a'(t) = b\,a(t), \quad k' = b^2 k. \tag{2.2}$$

The other possibility is a time translation, which is of no interest in this study.

It is common in the literature to fix this coordinate freedom by choosing $k \pm 1$ whenever $k \neq 0$. We will not do so in this article.

3. (Modified) Friedman Equation

Classically, the evolution of $a(t)$ is given by the first Friedmann equation,

$$H^2 = -\frac{k}{a^2} + \frac{\kappa}{3}\rho + \frac{\Lambda}{3}, \tag{3.1}$$

where $H := \frac{\dot{a}}{a}$ is the Hubble parameter, $\kappa = 8\pi G$, ρ is the matter energy density, and Λ is the cosmological constant. This equation is correct for any type of homogenous matter. By homogenous we mean that ρ is constant over space-like slices defined by a constant value to the time variable t.

The first Friedmann equation is directly derived from general relativity (GR) field equations, or alternatively from the Hamiltionan constraint. A modified theory of gravity (that may or may not come out from some version of quantum gravity) will most probably give rise to a modified Friedmann equation.

It should be noticed that the Friedmann Eq. (3.1) is invariant under the rescalings given by Eq. (2.2). Any modified Friedmann equation must have this property. Otherwise, the theory would be inconsistent, or alternatively Eq. (2.1) would not describe a metric.

The first Friedmann equation (that we are interested in for this study) is a reformulation of the Hamiltionan constraint, this is why it only involves first order derivatives. We assume that this will also be the case for the the modified cosmologies considered here. Since we are restricted to first order derivatives of a in Eq (3.1), there are only three independent gravitational variables as far as this specific equation in concerned: a, $\dot{a}$ and k. From these, we can construct two independent gravitational quantities that are invariant under Eq. (2.2): H and $\frac{k}{a^2}$. The Hamiltionan constraint can in principle always be solved for H^2 and the result has to be a function of $\frac{k}{a^2}$ and matter variables.

3.1 Main assumptions and their consequences

The assumptions so far for the modified cosmology or modified gravity theory considered are:

1. If the universe starts out homogenous and isotropic, it remains homogenous and isotropic. This is certainly not true at all scales as any consistent theory should lead to a growth of inhomogeneities. But this is very reasonable at the background order.

2. The theory allows for a metric interpretation, *i.e.* all physical equations must be invariant under metric coordinate transformations.

3. Given the metric, Eq. (2.1), the equation of motion for the scale factor $a(t)$ is given by the first Friedmann equation or its analogous in the modified theory considered, which is first order in the time derivative of $a(t)$.

4. There are no hidden gravitational degrees of freedom apart from the metric.

Any theory of modified gravity or modified cosmology that fulfills the above assumptions will have a (modified) Friedmann equation of the form

$$H^2 = \tilde{f}\left(\frac{k}{a^2}, \text{matter}\right), \tag{3.2}$$

where $\tilde{f}$ is a function of $\frac{k}{a^2}$ and of any set of homogenous coordinate-independent matter variables. This is grounded in the symmetries.

It can be noticed that in the flat case, $k = 0$, the modified Friedmann equation is not allowed to depend explicitly on a. This is of course true in GR.

3.2 Additional assumptions

It is now necessary to add two more assumptions to go ahead in the study.

5. The total energy density is the only matter variable that enters the first modified Friedmann equation.

By combining the above assumption with Eq. (3.2), one gets

$$H^2 = f\left(\frac{k}{a^2}, \rho\right), \tag{3.3}$$

where f is a function of $\frac{k}{a^2}$ and ρ.

6. Given an arbitrary constant ρ_1 such that $f\left(\frac{k}{a^2}, \rho_1\right) \geq 0$, the theory allows $\rho = \rho_1$ for at least a non-vanishing amount of time.

A situation with a constant energy density could for example be realized by a scalar field temporarily trapped in a false vacuum, or by a vacuum quantum-fluctuations domination stage. It is important to stress that we don't need this specific stage to have been explicitly realized in the history of the Universe, we just need the theory to be able to account for such a stage. This is obviously the case for GR and for all the most discussed theories beyond GR.

In the analysis performed so far, the possibility of a cosmological constant and/or dark energy has not been left out. If the acceleration of the universe is due to some exotic matter content (dark energy), then this will be included in ρ. If, on the other hand, the acceleration of the universe is due to a true cosmological constant Λ, this will be included directly in the function f by the relation $\frac{\Lambda}{3} = f(0,0)$.

3.3 de Sitter / Minkovski space-time

Let us choose a situation where $k = 0$ and $\rho = \rho_1$ such that $f(0, \rho_1) \geq 0$ for some time. Then we have:

$$H^2 = f(0, \rho_1) = \text{constant}, \tag{3.4}$$

for a non-vanishing amount of time. The above equation together with the FLRW metric, Eq. (2.1), describes exactly the de Sitter space-time for $f(0, \rho_1) > 0$, and Minkowski space-time for $f(0, \rho_1) = 0$.

By choosing a specific situation where ρ is constant in time, we get an extra symmetry of the system. In the general case, Eq. (2.2) are the only coordinate transformations that preserve the FLRW formulation. However, due to the time symmetry of Minkowski and de Sitter space-times, more coordinate transformations are available still within the FLRW metric formulation.

It is strait forward to chek that

$$H^2 = -\frac{k}{a^2} + f(0, \rho_1)\,, \quad \forall k \leq a^2 f(0, \rho_1), \tag{3.5}$$

together with the FLRS metric, describes exactly the same space-time as Eq. (3.4). Therefore, if Eq. (3.4) is correct then Eq. (3.5) must be correct too.

For any theory of modified gravity or cosmology that fulfill Assumptions (1) - (6), the modified Friedmann equation must therefore be of the form:

$$H^2 = -\frac{k}{a^2} + f_0(\rho), \tag{3.6}$$

where f_0 is a function of ρ related to previous expressions by $f_0(\rho) = f(0,\rho)$.

3.4 Preliminary conclusion

For a wide large class of modified cosmology models, it was shown that the modified Friedman equation for curved (i.e. $k \neq 0$) FLRW space-times, can be immediately derived from the modified Friedman equation for a flat (i.e. $k = 0$) FLRW space-time by Eq. (3.6). This basically relies on the symmetries and should be considered as a ground before going ahead.

4. Effective LQC

We now focus on effective loop quantum cosmology as an example of modified cosmology grounded in quantum gravity consideration. In LQC, for $k = 0$, the Friedmann equation is known to be [1]:

$$H^2 = \frac{\kappa}{3}\rho\left(1 - \frac{\rho}{\rho_c}\right). \tag{4.1}$$

This is the effective description of the bounce that replaces the Big Bang: the density is bounded from above at the value $\rho_c \sim \rho_{Pl}$ and the Hubble parameter vanishes when this density is reached. According to the previously given arguments, the Friedmann equation for a general k must be

$$H^2 = -\frac{k}{a^2} + \frac{\kappa}{3}\rho\left(1 - \frac{\rho}{\rho_c}\right). \tag{4.2}$$

This is in conflict with earlier precious results on LQC on specially curved space [4].

4.1 Hamiltionan

To avoid infinities we consider a finite region of space defined by a fiducial volume $\mathscr{V}$, given by some fixed region in coordinate space. It follows from the metric that $\mathscr{V}$ has the volume $V = vV_0$, where $v := a^3$ and V_0 is a constant.

The Hamiltonian constraints that leads to Eq. (4.2) is

$$\mathscr{H} = -vV_0\frac{\rho_c}{2}\left(1 - \sqrt{1 - \frac{12}{\kappa\rho_c}\frac{k}{v^{2/3}}}\cos\left(\sqrt{\frac{3\kappa}{\rho_c}}[\alpha - \alpha_1(v)]\right)\right) + vV_0\rho, \tag{4.3}$$

where α is defined by the Poisson bracket $\{\alpha, v\} = \frac{1}{V_0}$, and α_1 is an integration 'constant'. Since v was kept fixed during the integration, α_1 can be any function of v.

In this study we have chosen to work with the variables v and α for simplicity, and to clarify the dependence upon ρ_c which, together with the coupling constant $\kappa = 8\pi G$, is the only parameter entering the dynamics. However, Eq. (4.3) can be re-expressed using more familiar variables often used in the literature. In the effective formulation, the choice of canonical variables is just a matter of taste. The Hamiltonian can as well be expressed as

$$\mathscr{H} = -vV_0\frac{\rho_c}{2}\left(1-\sqrt{1-\frac{12}{\kappa\rho_c}\frac{k}{v^{2/3}}}\cos\left(2\lambda\left[\beta-\beta_1(v)\right]\right)\right)+vV_0\rho\,, \tag{4.4}$$

where $\{\beta,v\}=\frac{\gamma\kappa}{2V_0}$, or

$$\mathscr{H} = -p^{3/2}V_0\frac{\rho_c}{2}\left(1-\sqrt{1-\frac{12}{\kappa\rho_c}\frac{k}{p}}\cos\left(2\frac{\lambda}{\sqrt{p}}[c-c_1(p)]\right)\right)+V_0p^{3/2}\rho\,, \tag{4.5}$$

where $p=a^2=v^{2/3}$, and $\{c,p\}=\frac{\gamma\kappa}{3V_0}$.

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