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Parton distributions at low x and gluon- and quark average multiplicities

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We shown the general approach for Q^2 evolution of parton densities and fragmentation functions at low x based on the diagonalization. The diagonalization leads to the two components in the Q^2 evolution, each of which contains a nonperturbative parameter. The values of the parameters can be found by fits of the experimental data for the deep-inelastic scattering structure function F_2 and for average jet multiplicities. One of the components contains the all large logarithms $\ln(1/x)$ and produce the basic contribution at small x region. The second one is regular at low x but its contribution is very important to have a good agreement with experimental data.

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1. Introduction

The evaluation of the cross-sections for hadron-hadron iteractions needs the sufficiently precise knowledge of parton distribution functions (PDFs) and parton fragmentation functions (FFs), which are the important part of any cross-section. The properties of PDFs and FFs can be taken from processes of the deep-inelatic scattering (DIS) and e^+e^--collisions, respectively. In this report we will concentrate only for the high-energy limits of PDFs and FFs, which are needed for modern experiments studied on LHC collider.

1.1 PDFs

The experimental data from HERA on the DIS structure function (SF) F_2 [1]-[3], its derivative $\partial \ln F_2/\partial \ln(1/x)$ [4] and the heavy quark parts F_2^{cc} and F_2^{bb} [5] enable us to enter into a very interesting kinematical range for testing the theoretical ideas on the behavior of quarks and gluons carrying a very low fraction of momentum of the proton, the so-called small-x region. In this limit one expects that the conventional treatment based on the Dokshitzer–Gribov–Lipatov–Altarelli–Parisi (DGLAP) equations [6] does not account for contributions to the cross section which are leading in $\alpha_s \ln(1/x)$ and, moreover, the parton densities are becoming large and need to develop a high density formulation of QCD. However, the reasonable agreement between HERA data and the next-to-leading-order (NLO) and next-to-next-to-leading-order (NNLO) approximations of perturbative QCD has been observed for $Q^2 \geq 2$ GeV^2 (see reviews in [7] and references therein) and, thus, perturbative QCD could describe the evolution of F_2 and its derivatives up to very low Q^2 values, traditionally explained by soft processes.

The standard program to study the x behavior of quarks and gluons is carried out by comparison of data with the numerical solution of the DGLAP equation [6][1] by fitting the parameters of the PDF x-profile at some initial Q_0^2 and the QCD energy scale Λ [9]-[12]. However, for analyzing exclusively the low-x region, there is the alternative of doing a simpler analysis by using some of the existing analytical solutions of DGLAP evolution in the low-x limit [13]–[16]. This was done so in [13] where it was pointed out that the HERA small-x data can be interpreted in terms of the so-called doubled asymptotic scaling (DAS) phenomenon related to the asymptotic behavior of the DGLAP evolution discovered many years ago [17].

The study of [13] was extended in [14, 15, 16] to include the finite parts of anomalous dimensions of Wilson operators [2]. This has led to predictions [15, 16] of the small-x asymptotic PDF form in the framework of the DGLAP dynamics starting at some Q_0^2 with the flat function

$$f_a(Q_0^2) \;=\; A_a \quad (\text{hereafter } a = q, g), \tag{1.1}$$

where f_a are the parton distributions multiplied by x and A_a are unknown parameters to be determined from the data.

[1]At small x there is another approach based on the Balitsky–Fadin–Kuraev–Lipatov (BFKL) equation [8], whose application is out of the scope of this work.

[2]In the standard DAS approximation [17] only the singular parts of the anomalous dimensions were used.

1.2 FFs and average jet multiplicities

Collisions of particles and nuclei at high energies usually produce many hadrons and their production is a typical process where nonperturbative phenomena are involved. However, for particular observables, this problem can be avoided. In particular, the *counting* of hadrons in a jet that is initiated at a certain scale Q belongs to this class of observables. Hence, if the scale Q is large enough, this would in principle allow perturbative QCD to be predictive without the need to consider phenomenological models of hadronization. Nevertheless, such processes are dominated by soft-gluon emissions, and it is a well-known fact that, in such kinematic regions of phase space, fixed-order perturbation theory fails, rendering the usage of resummation techniques indispensable. As we shall see, the computation of average jet multiplicities (AJMs) indeed requires small-x resummation, as was already realized a long time ago [18]. In Ref. [18], it was shown that the singularities for $x \sim 0$, which are encoded in large logarithms of the kind $1/x \ln^k(1/x)$, spoil perturbation theory, and also render integral observables in x ill-defined, disappear after resummation. Usually, resummation includes the singularities from all orders according to a certain logarithmic accuracy, for which it *restores* perturbation theory.

Small-x resummation has recently been carried out for timelike splitting fuctions in the $\overline{\text{MS}}$ factorization scheme, which is generally preferable to other schemes, yielding fully analytic expressions. In a first step, the next-to-leading-logarithmic (NLL) level of accuracy has been reached [19, 20]. In a second step, this has been pushed to the next-to-next-to-leading-logarithmic (NNLL), and partially even to the next-to-next-to-next-to-leading-logarithmic (N^{3}LL), level [21]. Thanks to these results, we were able in [22, 23] to analytically compute the NNLL contributions to the evolutions of the gluon and quark AJMs with normalization factors evaluated to NLO and approximately to next-to-next-to-next-to-order (N^{3}LO) in the $\sqrt{\alpha_s}$ expansion. The previous literature contains a NLL result on the small-x resummation of timelike splitting fuctions obtained in a massive-gluon scheme. Unfortunately, this is unsuitable for the combination with available fixed-order corrections, which are routinely evaluated in the $\overline{\text{MS}}$ scheme. A general discussion of the scheme choice and dependence in this context may be found in Refs. [24].

The gluon and quark AJMs, which we denote as $\langle n_h(Q^2)\rangle_g$ and $\langle n_h(Q^2)\rangle_q$, respectively, represent the average numbers of hadrons in a jet initiated by a gluon or a quark at scale Q. In the past, analytic predictions were obtained by solving the equations for the generating functionals in the modified leading-logarithmic approximation (MLLA) in Ref. [25] through N^{3}LO in the expansion parameter $\sqrt{\alpha_s}$, i.e. through $\mathcal{O}(\alpha_s^{3/2})$. However, the theoretical prediction for the ratio $r(Q^2) = \langle n_h(Q^2)\rangle_g / \langle n_h(Q^2)\rangle_q$ given in Ref. [25] is about 10% higher than the experimental data at the scale of the Z^0 boson. An alternative approach was proposed in Ref. [26], where a differential equation for the gluon-to-quark AJM ratio was obtained in the MLLA within the framework of the colour-dipole model, and the constant of integration, which is supposed to encode nonperturbative contributions, was fitted to experimental data. A constant offset to the gluon and quark AJMs jet multiplicities was also introduced in Ref. [27].

Recently, we proposed a new formalism [28, 22, 23] that solves the problem of the apparent good convergence of the perturbative series and does not require any ad-hoc offset, once the effects due to the mixing between quarks and gluons are fully included. Our result is a generalization of the result obtained in Ref. [25]. In our new approach, the nonperturbative informations to the gluon-to-quark AJM ratio are encoded in the initial conditions of the evolution equations.

This contribution is organized as follows. Section 2 contains general formulae for the Q^2-evolution of PDFs and FFs. The generalized DAS approach is presented in Section 3. Sections 4 and 5 contain basic formulae of Q^2-dependence of FFs at low x and the AJMs, respectively. In Section 6 we compare our formulae with the experimental data for the DIS SF F_2 and the AJMs and present the obtained results. Some discussions can be found in the conclusions. The procedure of the diagonalization and its results for PDF and SF Mellin moments can be found in AppendixA.

2. Approach

Here we breifly touch on some points concerning theoretical part of our analysis.

2.1 Strong coupling constant

The strong coupling constant is determined from the renormalization group equation. Moreover, the perturbative coupling constant $a_s(Q^2)$ is different at the leading-order (LO), NLO and NNLO approximations. Indeed, from the renormalization group equation we can obtain the following equations for the coupling constant

$$\frac{1}{a_s^{\rm LO}(Q^2)} - \frac{1}{a_s^{\rm LO}(M_Z^2)} = \beta_0 \ln\left(\frac{Q^2}{M_Z^2}\right) \tag{2.1}$$

at the LO approximationm and

$$\frac{1}{a_s^{\rm NLO}(Q^2)} - \frac{1}{a_s^{\rm NLO}(M_Z^2)} + b_1 \ln\left[\frac{a_s^{\rm NLO}(Q^2)(1+b_1 a_s^{\rm NLO}(M_Z^2))}{a_s^{\rm NLO}(M_Z^2)(1+b_1 a_s^{\rm NLO}(Q^2))}\right] = \beta_0 \ln\left(\frac{Q^2}{M_Z^2}\right) \tag{2.2}$$

at the NLO approximation.

At NNLO level $a_s^{\rm NNLO}(Q^2) \equiv a_s(Q^2)$ is more complicated and it is given by

$$\frac{1}{a_s(Q^2)} - \frac{1}{a_s(M_Z^2)} + b_1 \ln\left[\frac{a_s(Q^2)}{a_s(M_Z^2)}\sqrt{\frac{1+b_1 a_s(M_Z^2)+b_2 a_s^2(M_Z^2)}{1+b_1 a_s(Q^2)+b_2 a_s^2(Q^2)}}\right] + \left(b_2 - \frac{b_1^2}{2}\right)\cdot I = \beta_0 \ln\left(\frac{Q^2}{M_Z^2}\right). \tag{2.3}$$

The expression for I looks:

$$I = \begin{cases} \dfrac{2}{\sqrt{\Delta}}\left(\arctan\dfrac{b_1+2b_2 a_s(Q^2)}{\sqrt{\Delta}} - \arctan\dfrac{b_1+2b_2 a_s(M_Z^2)}{\sqrt{\Delta}}\right) & \text{for } f=3,4,5;\Delta>0, \\ \dfrac{1}{\sqrt{-\Delta}}\ln\left[\dfrac{b_1+2b_2 a_s(Q^2)-\sqrt{-\Delta}}{b_1+2b_2 a_s(Q^2)+\sqrt{-\Delta}}\cdot\dfrac{b_1+2b_2 a_s(M_Z^2)+\sqrt{-\Delta}}{b_1+2b_2 a_s(M_Z^2)-\sqrt{-\Delta}}\right] & \text{for } f=6; \quad \Delta<0, \end{cases}$$

where $\Delta = 4b_2 - b_1^2$ and $b_i = \beta_i/\beta_0$ are read off from the QCD β-function:

$$\beta(a_s) = -\beta_0 a_s^2 - \beta_1 a_s^3 - \beta_2 a_s^4 + \ldots = -\beta_0 a_s^2\left(1 + b_1 a_s + b_2 a_s^2 + \ldots\right), \tag{2.4}$$

where

$$\beta_0 = 11 - \frac{2}{3}f, \quad \beta_1 = 102 - \frac{38}{3}f, \quad \beta_2 = \frac{2857}{2} - \frac{5033}{18}f + \frac{325}{54}f^2, \tag{2.5}$$

with f being the number of active quark flavours.

2.2 PDFs and DIS SF F_2

The DIS SF can be represented as a sum of two terms:

$$F_2(x,Q^2) = F_2^{NS}(x,Q^2) + F_2^{S}(x,Q^2)\,, \tag{2.6}$$

the nonsinglet (NS) and singlet (S) parts. At this point let's introduce PDFs, the gluon distribution function $f_g(x,Q^2)$ and the singlet and nonsinglet quark distribution functions $f_S(x,Q^2)$ and $f_{NS}(x,Q^2)$ [3]:

$$\begin{aligned} \mathbf{f}_S(x,Q^2) &\equiv \sum_i^f \mathbf{f}_i(x,Q^2) = V(x,Q^2) + S(x,Q^2)\,, \\ \mathbf{f}_{NS}(x,Q^2) &\equiv \sum_i^f e_i^2\,\mathbf{f}_i^{NS}(x,Q^2),\;\; \mathbf{f}_i^{NS}(x,Q^2) = \mathbf{f}_i(x,Q^2) - \frac{1}{f}\,\mathbf{f}_S(x,Q^2)\,, \end{aligned} \tag{2.7}$$

where $V(x,Q^2) = \mathbf{u}_v(x,Q^2) + \mathbf{d}_v(x,Q^2)$ is the distribution of valence quarks and $S(x,Q^2)$ is a sum of sea parton distributions set equal to each other.

There is a direct relation between SF moments $M_n(Q^2)$ and those of PDFs

$$M^j(n,Q^2) \;=\; \int_0^1 dx x^{n-2} F_2^j(x,Q^2),\;\; \mathbf{f}_j(n,Q^2) \;=\; \int_0^1 dx x^{n-2}\mathbf{f}_j(x,Q^2), \quad (j = NS,S,G), \tag{2.8}$$

which has the following form

$$M_n^{NS}(Q^2) \;=\; C_{NS}(n,\overline{a}_s(Q^2))\cdot \mathbf{f}_{NS}(n,Q^2)\,, \tag{2.9}$$

$$M_n^{S}(Q^2) \;=\; e\left(C_S(n,\overline{a}_s(Q^2))\cdot \mathbf{f}_S(n,Q^2) + C_g(n,\overline{a}_s(Q^2))\cdot \mathbf{f}_G(n,Q^2)\right), \tag{2.10}$$

with $C_j(n,\overline{a}_s(Q^2))$ $(j = NS,S,G)$ are the Wilson coefficient functions. The constant e depends on weak and electromagnetic charges and is fixed for electromagnetic charges to

$$e \;=\; \frac{1}{f}\sum_q^f e_q^2\,. \tag{2.11}$$

Note that the NS and valence quark parts are negledgible at low x and, thus, $F_2^S(x,Q^2) = F_2(x,Q^2)$ and $S(x,Q^2) = \mathbf{f}_S(n,Q^2) \equiv \mathbf{f}_q(n,Q^2)$.

2.3 Q^2-dependence of SF moments

The coefficient functions $C_q(n,a_s(Q^2)) \equiv C_S(n,a_s(Q^2))$ and $C_g(n,a_s(Q^2))$ are further expressed through the functions $B_q^{(i)}(n)$ and $B_g^{(i)}(n)$, respectively, which are known exactly [29, 30] [4]

$$C_a(n,a_s(Q^2)) = 1 - \delta_a^g + a_s\cdot B_a^{(1)}(n) + a_s^2\cdot B_a^{(2)}(n) + \mathscr{O}(a_s^3)\,, \quad (a = (q,g)) \tag{2.12}$$

where δ_a^g is the Kroneker symbol.

[3]Unlike the standard case, here PDFs are multiplied by x.

[4]For the integral and even complex n values, the coefficients $B_a^{(i)}(n)$ and $Z_{a,b}^{(i)}(n)$ $(a,b = q,g)$ can be obtained using the analytic continuation [31].

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The Q^2-evolution of the PDF moments can be calculated within a framework of perturbative QCD (see e.g. [30, 32]). After diagonalization (see Appendix A), we see that the quark and gluon densities contain the so called "$+$"- and "$-$"-components

$$\mathbf{f}_a(n,Q^2) = \mathbf{f}_a^{+}(n,Q^2) + \mathbf{f}_a^{-}(n,Q^2) \quad (a=q,g), \tag{2.13}$$

which in-turn evaluated already independently:

$$\mathbf{f}_a^{\pm}(n,Q^2) = \tilde{\mathbf{f}}_a^{\pm}(n,Q_0^2) \cdot \left[\frac{a_s(Q^2)}{a_s(Q_0^2)}\right]^{\frac{\gamma_{\pm}^{(0)}(n)}{2\beta_0}} \cdot H_a^{\pm}(n,Q^2), \tag{2.14}$$

where

$$\gamma_{\pm}^{(0)}(n) \;=\; \frac{1}{2}\left(\gamma_{qq}^{(0)}(n) + \gamma_{gg}^{(0)}(n) \pm \sqrt{\left(\gamma_{qq}^{(0)}(n) - \gamma_{gg}^{(0)}(n)\right) + 4\gamma_{qg}^{(0)}(n)\gamma_{gq}^{(0)}(n)}\right) \tag{2.15}$$

is the anomalous dimensions of the "$+$"- and "$-$"-components, which are obtained from the elements of the martix of the LO anomalous quark and gluon anomalous dimensions.

At LO, the normalization coefficients $\tilde{\mathbf{f}}_a^{\pm}(n,Q_0^2)$ have the form

$$\tilde{\mathbf{f}}_a^{\pm}(n,Q_0^2) \;=\; \mathbf{f}_a^{\pm}(n,Q_0^2), \tag{2.16}$$

where [5]

$$\begin{aligned}
\mathbf{f}_q^{-}(n,Q_0^2) \;&=\; \mathbf{f}_S(n,Q_0^2)\cdot\alpha_n + \mathbf{f}_g(n,Q_0^2)\cdot\beta_n, \;\; \mathbf{f}_g^{-}(n,Q_0^2) \;=\; \mathbf{f}_g(n,Q_0^2)\cdot(1-\alpha_n) + \mathbf{f}_q(n,Q_0^2)\cdot\varepsilon_n,\\
\mathbf{f}_a^{+}(n,Q_0^2) \;&=\; \mathbf{f}_a(n,Q_0^2) - \mathbf{f}_a^{-}(n,Q_0^2)
\end{aligned} \tag{2.17}$$

and

$$\alpha_n = \frac{\gamma_{qq}^{(0)}(n) - \gamma_{+}^{(0)}(n)}{\gamma_{-}^{(0)}(n) - \gamma_{+}^{(0)}(n)}, \;\; \beta_n = \frac{\gamma_{qg}^{(0)}(n)}{\gamma_{-}^{(0)}(n) - \gamma_{+}^{(0)}(n)}, \;\; \varepsilon_n = \frac{\gamma_{gq}^{(0)}(n)}{\gamma_{-}^{(0)}(n) - \gamma_{+}^{(0)}(n)}. \tag{2.18}$$

Above LO, the normalization factors $\tilde{\mathbf{f}}_a^{\pm}(n,Q_0^2)$ become to be

$$\begin{aligned}
\tilde{\mathbf{f}}_a^{\pm}(n,Q_0^2) \;&=\; \mathbf{f}_a^{\pm}(n,Q_0^2)\cdot\left(1 - a_s(Q_0^2)Z_{\pm\pm}^{(1)}(n) - a_s^2(Q_0^2)Z_{\pm\pm}^{(2)}(n)\right)\\
&\quad + \mathbf{f}_a^{\mp}(n,Q_0^2)\cdot a_s(Q_0^2)\left(Z_{\mp\pm,a}^{(1)}(n) + a_s(Q_0^2)Z_{\mp\pm,a}^{(2)}(n)\right),
\end{aligned} \tag{2.19}$$

where (see Appendix A, where $Z_{\pm\pm}^{(i)} = -V_{\pm\pm}^{(i)}$, $Z_{\mp\pm,a}^{(i)} = -V_{\mp\pm,a}^{(i)}$, $(i=1,2)$)

$$\begin{aligned}
Z_{\pm\pm}^{(1)}(n) \;&=\; \frac{1}{2\beta_0}\left[\gamma_{\pm\pm}^{(1)}(n) - \gamma_{\pm}^{(0)}(n)\,b_1\right], \;\; Z_{\pm\mp,q}^{(1)}(n) \;=\; \frac{1}{2\beta_0 + \gamma_{\pm}^{(0)}(n) - \gamma_{\mp}^{(0)}(n)}\,\gamma_{\pm\mp}^{(1)}(n), \\
Z_{\pm\pm}^{(2)}(n) \;&=\; \frac{1}{4\beta_0}\left[\gamma_{\pm\pm}^{(2)}(n) - \left(\gamma_{\pm\pm}^{(1)}(n) - \gamma_{\pm}^{(0)}(n)Z_{\pm\pm}^{(1)}(n)\right)b_1 + \gamma_{\pm}^{(0)}(n)(b_1^2 - b_2) - \sum_{i=\pm}\gamma_{\pm i}^{(1)}(n)Z_{i\pm,q}^{(1)}\right],\\
Z_{\pm\mp,q}^{(2)}(n) \;&=\; \frac{1}{4\beta_0 + \gamma_{\pm}^{(0)}(n) - \gamma_{\mp}^{(0)}(n)}\left[\gamma_{\pm\mp}^{(2)}(n) - \left(\gamma_{\pm\mp}^{(1)}(n) - \gamma_{\pm}^{(0)}(n)Z_{\pm\mp}^{(1)}(n)\right)b_1 - \sum_{i=\pm}\gamma_{\pm i}^{(1)}(n)Z_{i\mp,q}^{(1)}\right]
\end{aligned} \tag{2.20}$$

[5] To conrary [30] we replace $\tilde{\alpha}_n$ by β_n. Another expressions for the projectors α_n, β_n and ε_n can be found in [33].

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and

$$Z^{(i)}_{\pm\pm,g}(n) \;=\; Z^{(i)}_{\pm\pm,q}(n) \;=\; Z^{(i)}_{\pm\pm}(n),\;\; Z^{(i)}_{\pm\mp,g}(n) \;=\; Z^{(i)}_{\pm\mp,q}(n)\cdot\frac{\gamma^{(0)}_{qq}(n)-\gamma^{(0)}_{\mp}(n)}{\gamma^{(0)}_{qq}(n)-\gamma^{(0)}_{\pm}(n)},\;\; (i=1,2) \quad (2.21)$$

Here $\gamma^{(k)}_{\pm\pm}(n)$ and $\gamma^{(k)}_{\pm\mp}(n)$ are the elemens of matrices of anomalous dimensions, which have been obtained after diagonalization from $\gamma^{(k)}_{ab}(n)$ $(a,b=q,g)$ (latter taken in the exact form from [34]):

$$\begin{aligned}
\gamma^{(k)}_{--}(n) &= \gamma^{(k)}_{qq}(n)\cdot\alpha_n+\gamma^{(k)}_{qg}(n)\cdot\varepsilon_n+\gamma^{(k)}_{gq}(n)\cdot\beta_n+\gamma^{(k)}_{gg}(n)\cdot(1-\alpha_n),\\
\gamma^{(k)}_{-+}(n) &= \gamma^{(k)}_{--}(n)-\left(\gamma^{(k)}_{qq}(n)+\frac{1-\alpha_n}{\beta_n}\gamma^{(k)}_{qg}(n)\right),\;\; \gamma^{(k)}_{++}(n)=\gamma^{(k)}_{qq}(n)+\gamma^{(k)}_{gg}(n)-\gamma^{(k)}_{--}(n),\\
\gamma^{(k)}_{+-}(n) &= \gamma^{(k)}_{++}(n)-\left(\gamma^{(k)}_{qq}(n)-\frac{\alpha_n}{\beta_n}\gamma^{(k)}_{qg}(n)\right).
\end{aligned} \quad (2.22)$$

The function $H^{\pm}_a(n,Q^2)$ up to NNLO may be represented as

$$H^{\pm}_a(n,Q^2) \;=\; 1+a_s(Q^2)\left(Z^{(1)}_{\pm\pm}(n)-Z^{(1)}_{\pm\mp,a}(n)\right)+a_s^2(Q^2)\left(\tilde{Z}^{(2)}_{\pm\pm}(n)-\tilde{Z}^{(2)}_{\pm\mp,a}(n)\right)+\mathscr{O}\left(a_s^3(Q^2)\right), \quad (2.23)$$

where (see [30] and Appendix A)

$$\begin{aligned}
\tilde{Z}^{(2)}_{\pm\pm}(n) &= Z^{(2)}_{\pm\pm}(n)+\sum_{i=\pm}Z^{(1)}_{\pm i,q}Z^{(1)}_{i\pm,q},\;\; \tilde{Z}^{(2)}_{\pm\mp,q}(n) \;=\; Z^{(2)}_{\pm\mp,q}(n)+\sum_{i=\pm}Z^{(1)}_{\pm i,q}Z^{(1)}_{i\mp,q},\\
\tilde{Z}^{(2)}_{\pm\mp.g}(n) &= \tilde{Z}^{(2)}_{\pm\mp,q}(n)\cdot\frac{\gamma^{(0)}_{qq}(n)-\gamma^{(0)}_{\mp}(n)}{\gamma^{(0)}_{qq}(n)-\gamma^{(0)}_{\pm}(n)}.
\end{aligned} \quad (2.24)$$

2.4 Fragmentation functions and their evolution

When one considers AJM observables, the basic equation is the one governing the evolution of FFs $D_a(x,\mu^2)$ for the gluon–quark-singlet system $a=g,q$. In Mellin space, it reads:

$$\mu^2\frac{\partial}{\partial\mu^2}\begin{pmatrix}D_q(\omega,\mu^2)\\ D_g(\omega,\mu^2)\end{pmatrix}=\begin{pmatrix}P_{qq}(\omega,a_s) & P_{gq}(\omega,a_s)\\ P_{qg}(\omega,a_s) & P_{gg}(\omega,a_s)\end{pmatrix}\begin{pmatrix}D_q(\omega,\mu^2)\\ D_g(\omega,\mu^2)\end{pmatrix}, \quad (2.25)$$

where $P_{ab}(\omega,a_s)$, with $a,b=g,q$, are the timelike splitting functions, [6] $\omega=n-1$, with n being the standard Mellin moments with respect to x. The standard definition of the hadron AJMs corresponds to the first Mellin moment, with $\omega=0$ (see, e.g., Ref. [35]):

$$\langle n_h(Q^2)\rangle_a \equiv \left[\int_0^1 dx\,x^{\omega}D_a(x,Q^2)\right]_{\omega=0}=D_a(\omega=0,Q^2),\quad (a=g,q). \quad (2.26)$$

The timelike splitting functions $P_{ab}(\omega,a_s)$ in Eq. (2.25) may be computed perturbatively in a_s,

$$P_{a,b}(\omega,a_s)=\sum_{k=0}^{\infty}a_s^{k+1}P^{(k)}_{ab}(\omega). \quad (2.27)$$

The functions $P^{(k)}_{ab}(\omega)$ for $k=0,1,2$ in the $\overline{\text{MS}}$ scheme may be found in Refs. [36, 37, 38] through NNLO and in Refs. [19, 20, 21] with small-x resummation through NNLL accuracy.

[6] $P_{a,b}=-\gamma^{\tau}_{a,b}/2$, where $\gamma^{\tau}_{a,b}$ are the timelike anomalous dimensions.

2.5 Diagonalization of FFs

As it was in the spacelike case (see subsection 2.3 and Appendix A), it is not in general possible to diagonalize Eq. (2.25) because the contributions to the timelike-splitting-function matrix do not commute at different orders. The usual approach is then to write a series expansion about the LO solution, which can in turn be diagonalized. One thus starts by choosing a basis in which the timelike-splitting-function matrix is diagonal at LO (see, e.g., Ref. [30] and Appendix A),

$$P(\omega,a_s)=\begin{pmatrix} P_{++}(\omega,a_s) & P_{-+}(\omega,a_s) \\ P_{+-}(\omega,a_s) & P_{--}(\omega,a_s)\end{pmatrix}=a_s\begin{pmatrix} P_{++}^{(0)}(\omega) & 0 \\ 0 & P_{--}^{(0)}(\omega)\end{pmatrix}+a_s^2P^{(1)}(\omega)+\mathscr{O}(a_s^3), \quad (2.28)$$

with eigenvalues $P_{\pm\pm}^{(0)}(\omega)$. In one important simplification of QCD, namely $\mathscr{N}=4$ super Yang-Mills theory, this basis is actually more natural than the (g,q) basis because the diagonal splitting functions $P_{\pm\pm}^{(k)\mathscr{N}=4}(\omega)$ may there be expressed in all orders of perturbation theory as one universal function $P_{\rm uni}^{(k)}(\omega)$ with shifted arguments [39].

It is convenient to represent the change of FF basis order by order for $k\geq 0$ as [30] [7] :

$$D^+(\omega,\mu_0^2)=(1-\alpha_\omega)D_s(\omega,\mu_0^2)-\varepsilon_\omega D_g(\omega,\mu_0^2),\;\; D^-(\omega,\mu_0^2)=\alpha_\omega D_s(\omega,\mu_0^2)+\varepsilon_\omega D_g(\omega,\mu_0^2). \quad (2.29)$$

This implies for the components of the timelike-splitting-function matrix that

$$\begin{aligned}
P_{--}^{(k)}(\omega) &= \alpha_\omega P_{qq}^{(k)}(\omega)+\varepsilon_\omega P_{qg}^{(k)}(\omega)+\beta_\omega P_{gq}^{(k)}(\omega)+(1-\alpha_\omega)P_{gg}^{(k)}(\omega),\\
P_{-+}^{(k)}(\omega) &= P_{--}^{(k)}(\omega)-\left(P_{qq}^{(k)}(\omega)+\frac{1-\alpha_\omega}{\varepsilon_\omega}P_{gq}^{(k)}(\omega)\right),\;\; P_{++}^{(k)}(\omega)P_{qq}^{(k)}(\omega)+P_{gg}^{(k)}(\omega)-P_{--}^{(k)}(\omega),\\
P_{+-}^{(k)}(\omega) &= P_{++}^{(k)}(\omega)-\left(P_{qq}^{(k)}(\omega)-\frac{\alpha_\omega}{\varepsilon_\omega}P_{gq}^{(k)}(\omega)\right)=P_{gg}^{(k)}(\omega)-\left(P_{--}^{(k)}(\omega)-\frac{\alpha_\omega}{\varepsilon_\omega}P_{gq}^{(k)}(\omega)\right), \quad (2.30)
\end{aligned}$$

where α_ω, β_ω and ε_ω are given in Eq. (2.18).

Note, howerver, that the approach (2.28) is not so conveninet in FF case, because we would like to keep the diagonal part of $P(\omega,a_s)$ matrix without an expansion on a_s. So. below our approach to solve Eq. (2.25) differs from the usual one (see [30]) We write the solution expanding about the diagonal part of the all-order timelike-splitting-function matrix in the plus-minus basis, instead of its LO contribution. For this purpose, we rewrite Eq. (2.28) in the following way:

$$P(\omega,a_s)=\begin{pmatrix} P_{++}(\omega,a_s) & 0 \\ 0 & P_{--}(\omega,a_s)\end{pmatrix}+a_s^2\begin{pmatrix} 0 & P_{-+}^{(1)}(\omega) \\ P_{+-}^{(1)}(\omega) & 0\end{pmatrix}+\begin{pmatrix} 0 & \mathscr{O}(a_s^3) \\ \mathscr{O}(a_s^3) & 0\end{pmatrix}. \quad (2.31)$$

In general, the solution to Eq. (2.25) in the plus-minus basis can be formally written as

$$D(\mu^2)=T_{\mu^2}\left\{\exp\int_{\mu_0^2}^{\mu^2}\frac{d\overline{\mu}^2}{\overline{\mu}^2}P(\overline{\mu}^2)\right\}D(\mu_0^2), \quad (2.32)$$

where T_{μ^2} denotes the path ordering with respect to μ^2 and

$$D=\begin{pmatrix} D^+ \\ D^- \end{pmatrix}. \quad (2.33)$$

[7] The difference in the diagonalization to compare with the spacelike case considered above is following: $\gamma_{qg}^{(i)}\leftrightarrow -2P_{gq}^{(i)}$ and, thus, $\beta_\omega\leftrightarrow\varepsilon_\omega$.

As anticipated, we make the following ansatz to expand about the diagonal part of the timelike-splitting-function matrix in the plus-minus basis:

$$T_{\mu^2}\left\{\exp\int_{\mu_0^2}^{\mu^2}\frac{d\bar{\mu}^2}{\bar{\mu}^2}P(\bar{\mu}^2)\right\}=Z^{-1}(\mu^2)\exp\left[\int_{\mu_0^2}^{\mu^2}\frac{d\bar{\mu}^2}{\bar{\mu}^2}P^D(\bar{\mu}^2)\right]Z(\mu_0^2), \tag{2.34}$$

where

$$P^D(\omega)=\begin{pmatrix} P_{++}(\omega) & 0 \\ 0 & P_{--}(\omega)\end{pmatrix} \tag{2.35}$$

is the diagonal part of Eq. (2.31) and Z is a matrix in the plus-minus basis which has a perturbative expansion of the form

$$Z(\mu^2)=1+a_s(\mu^2)Z^{(1)}+\mathscr{O}(a_s^2). \tag{2.36}$$

Changing integration variable in Eq. (2.34), we obtain

$$T_{a_s}\left\{\exp\int_{a_s(\mu_0^2)}^{a_s(\mu^2)}\frac{d\bar{a}_s}{\beta(\bar{a}_s)}P(\bar{a}_s)\right\}=Z^{-1}(a_s(\mu^2))\exp\left[\int_{a_s(\mu_0^2)}^{a_s(\mu^2)}\frac{d\bar{a}_s}{\beta(\bar{a}_s)}P^D(\bar{a}_s)\right]Z(a_s(\mu_0^2)). \tag{2.37}$$

Substituting then Eq. (2.36) into Eq. (2.37), differentiating it with respect to a_s, and keeping only the first term in the a_s expansion, we obtain the following condition for the $Z^{(1)}$ matrix (see Section 8.1 for the similar procedure in the spacelike case):

$$Z^{(1)}+\left[\frac{P^{(0)D}}{\beta_0},Z^{(1)}\right]=\frac{P^{(1)OD}}{\beta_0},\quad P^{(1)OD}(\omega)=\begin{pmatrix} 0 & P^{(1)}_{-+}(\omega) \\ P^{(1)}_{+-}(\omega) & 0\end{pmatrix}. \tag{2.38}$$

Solving it, we find:

$$Z^{(1)}_{\pm\pm}(\omega)=0,\qquad Z^{(1)}_{\pm\mp}(\omega)=\frac{P^{(1)}_{\pm\mp}(\omega)}{\beta_0+P^{(0)}_{\pm\pm}(\omega)-P^{(0)}_{\mp\mp}(\omega)}. \tag{2.39}$$

At this point, an important comment is in order. In the conventional approach to solve Eq.(2.25), one expands about the diagonal LO matrix given in Eq. (2.28), while here we expand about the all-order diagonal part of the matrix given in Eq. (2.31). The motivation for us to do this arises from the fact that the functional dependence of $P_{\pm\pm}(\omega,a_s)$ on a_s is different after resummation.

Now reverting the change of basis specified in Eq. (2.29), we find the gluon and quark-singlet fragmentation functions to be given by

$$D_g(\omega,\mu^2)=-\frac{\alpha_\omega}{\varepsilon_\omega}D^+(\omega,\mu^2)+\left(\frac{1-\alpha_\omega}{\varepsilon_\omega}\right)D^-(\omega,\mu^2),\quad D_q(\omega,\mu^2)=D^+(\omega,\mu^2)+D^-(\omega,\mu^2). \tag{2.40}$$

As expected, this suggests to write the gluon and quark-singlet fragmentation functions in the following way:

$$D_a(\omega,\mu^2)\equiv D_a^+(\omega,\mu^2)+D_a^-(\omega,\mu^2),\qquad a=g,q, \tag{2.41}$$

where $D_a^+(\omega,\mu^2)$ evolves like a plus component and $D_a^-(\omega,\mu^2)$ like a minus component.

We now explicitly compute the functions $D^{\pm}_a(\omega,\mu^2)$ appearing in Eq. (2.41). To this end, we first substitute Eq. (2.34) into Eq. (2.32). Using Eqs. (2.35) and (2.39), we then obtain

$$\begin{aligned} D^{+}(\omega,\mu^2) &= \tilde{D}^{+}(\omega,\mu_0^2)\hat{T}_{+}(\omega,\mu^2,\mu_0^2) - a_s(\mu^2)Z^{(1)}_{-+}(\omega)\tilde{D}^{-}(\omega,\mu_0^2)\hat{T}_{-}(\omega,\mu^2,\mu_0^2), \\ D^{-}(\omega,\mu^2) &= \tilde{D}^{-}(\omega,\mu_0^2)\hat{T}_{-}(\omega,\mu^2,\mu_0^2) - a_s(\mu^2)Z^{(1)}_{+-}(\omega)\tilde{D}^{+}(\omega,\mu_0^2)\hat{T}_{+}(\omega,\mu^2,\mu_0^2), \end{aligned} \tag{2.42}$$

where

$$\tilde{D}^{\pm}(\omega,\mu_0^2) = D^{\pm}(\omega,\mu_0^2) + a_s(\mu_0^2)Z^{(1)}_{\mp\pm}(\omega)D^{\mp}(\omega,\mu_0^2), \tag{2.43}$$

and

$$\hat{T}_{\pm}(\omega,\mu^2,\mu_0^2) = \exp\left[\int_{a_s(\mu_0^2)}^{a_s(\mu^2)} \frac{d\overline{a}_s}{\beta(\overline{a}_s)} P_{\pm\pm}(\omega,\overline{a}_s)\right] \tag{2.44}$$

has a RG-type exponential form. Finally, inserting Eq. (2.42) into Eq. (2.40), we find by comparison with Eq. (2.41) that

$$D^{\pm}_a(\omega,\mu^2) = \tilde{D}^{\pm}_a(\omega,\mu_0^2)\hat{T}_{\pm}(\omega,\mu^2,\mu_0^2)H^{\pm}_a(\omega,\mu^2), \tag{2.45}$$

where

$$\begin{aligned} \tilde{D}^{+}_g(\omega,\mu_0^2) = -\frac{\alpha_\omega}{\varepsilon_\omega}\tilde{D}^{+}_q(\omega,\mu_0^2),\ \ \tilde{D}^{-}_g(\omega,\mu_0^2) = \frac{1-\alpha_\omega}{\varepsilon_\omega}\tilde{D}^{-}_q(\omega,\mu_0^2), \\ \tilde{D}^{+}_q(\omega,\mu_0^2) = \tilde{D}^{+}(\omega,\mu_0^2),\ \ \tilde{D}^{-}_q(\omega,\mu_0^2) = \tilde{D}^{-}(\omega,\mu_0^2), \end{aligned} \tag{2.46}$$

and $H^{\pm}_a(\omega,\mu^2)$ are perturbative functions given by

$$H^{\pm}_a(\omega,\mu^2) = 1 - a_s(\mu^2)Z^{(1)}_{\pm\mp,a}(\omega) + \mathscr{O}(a_s^2). \tag{2.47}$$

At $\mathscr{O}(\alpha_s)$, we have

$$Z^{(1)}_{\pm\mp,g}(\omega) = -Z^{(1)}_{\pm\mp}(\omega)\left(\frac{1-\alpha_\omega}{\alpha_\omega}\right)^{\pm 1}, \qquad Z^{(1)}_{\pm\mp,q}(\omega) = Z^{(1)}_{\pm\mp}(\omega), \tag{2.48}$$

where $Z^{(1)}_{\pm\mp}(\omega)$ is given by Eq. (2.39).

3. Generalized DAS approach

The flat initial condition (1.1) corresponds to the case when parton density tend to some constant value at $x\to 0$ and at some initial value Q_0^2. The main ingredients of the results [15, 16], are:

A. Both, the gluon and quark singlet densities are presented in terms of two components ("$+$" and "$-$") which are obtained from the analytic Q^2-dependent expressions of the corresponding ("$+$" and "$-$") PDF moments.

B. The twist-two part of the "$-$" component is constant at small x at any values of Q^2, whereas the one of the "$+$" component grows at $Q^2 \geq Q_0^2$ as

$$\sim e^{\sigma_{\rm NLO}}, \quad \sigma_{\rm NLO} = 2\sqrt{\left[\left|\hat{d}_{+}\right| s_{\rm NLO} - \left(\hat{d}_{++} + \left|\hat{d}_{+}\right| b_1\right) p_{\rm NLO}\right] \ln\left(\frac{1}{x}\right)}\,, \quad \rho_{\rm NLO} = \frac{\sigma_{\rm NLO}}{2\ln(1/x)}\,, \tag{3.1}$$

where σ and ρ are the generalized Ball–Forte variables,

$$s_{\rm NLO} = \ln\left(\frac{a_s^{\rm NLO}(Q_0^2)}{a_s^{\rm NLO}(Q^2)}\right), \quad p_{\rm NLO} = a_s^{\rm NLO}(Q_0^2) - a_s^{\rm NLO}(Q^2), \quad \hat{d}_+ = -\frac{12}{\beta_0}, \quad \hat{d}_{++} = \frac{412}{27\beta_0}. \tag{3.2}$$

and β_0 and β_1 are given in Eq. (2.5)

3.1 Parton distributions and the structure function F_2

The results for parton densities and F_2 are following:

- The structure function F_2 has the form:

$$F_2^{\rm LO}(x,Q^2) = e\, f_{q,,\rm LO}(x,Q^2), \quad f_{a,\rm LO}(x,Q^2) = f^+_{a,\rm LO}(x,Q^2) + f^-_{a,\rm LO}(x,Q^2) \tag{3.3}$$

at the LO approximation, where e is the average charge square (2.11), and

$$\begin{aligned} F_2^{\rm NLO}(x,Q^2) &= e\left(f_{q,\rm NLO}(x,Q^2) + \frac{2}{3} f a_s^{\rm NLO}(Q^2)\, f_{g,\rm NLO}(x,Q^2)\right), \\ f_{a,\rm NLO}(x,Q^2) &= f^+_{a,\rm NLO}(x,Q^2) + f^-_{a,\rm NLO}(x,Q^2) \end{aligned} \tag{3.4}$$

at the NLO approximation.

- The small-x asymptotic results for the LO parton densities $f^{\pm}_{a,\rm LO}$ are

$$f^+_{g,\rm LO}(x,Q^2) = \left(A_g + \frac{4}{9}A_q\right)\tilde{I}_0(\sigma_{\rm LO})\, e^{-\overline{d}_+ s_{\rm LO}} + O(\rho_{\rm LO}), \tag{3.5}$$

$$f^+_{q,\rm LO}(x,Q^2) = \frac{f}{9}\left(A_g + \frac{4}{9}A_q\right)\rho_{\rm LO}\tilde{I}_1(\sigma_{\rm LO})\, e^{-\overline{d}_+ s_{\rm LO}} + O(\rho_{\rm LO}), \tag{3.6}$$

$$f^-_{g,\rm LO}(x,Q^2) = -\frac{4}{9}A_q e^{-d_- s_{\rm LO}} + O(x), \quad f^-_{q,\rm LO}(x,Q^2) = A_q e^{-d_- s_{\rm LO}} + O(x), \tag{3.7}$$

where [8]

$$\overline{d}_+ = 1 + \frac{20f}{27\beta_0}, \quad d_- = \frac{16f}{27\beta_0} \tag{3.8}$$

are the regular parts of the anomalous dimensions $d_+(n)$ and $d_-(n)$, respectively, in the limit $n \to 1$[9]. Here n is the variable in Mellin space. The functions $\tilde{I}_\nu$ ($\nu = 0,1$) are related to the modified Bessel function I_ν and to the Bessel function J_ν by:

$$\tilde{I}_\nu(\sigma) = \begin{cases} I_\nu(\sigma), & \text{if } s \geq 0 \\ i^{-\nu} J_\nu(i\sigma),\ i^2 = -1, & \text{if } s \leq 0 \end{cases}. \tag{3.9}$$

At the LO, the variables $\sigma_{\rm LO}$ and $\rho_{\rm LO}$ are given by Eq. (3.1) when $p = 0$, i.e.

$$\sigma_{\rm LO} = 2\sqrt{\left|\hat{d}_+\right| s_{\rm LO} \ln\left(\frac{1}{x}\right)}, \quad \rho_{\rm LO} = \frac{\sigma_{\rm LO}}{2\ln(1/x)}, \tag{3.10}$$

and the variable $s_{\rm LO}$ is given by Eq. (3.2) with $a_s^{\rm LO}(Q^2)$ as in Eq. (2.1).

[8] The dependence on the colour factors $C_A = 3$, $C_F = 4/3$ in Eqs. (3.2), (3.8) and (3.17) can be found in [16].

[9] We denote the singular and regular parts of a given quantity $k(n)$ in the limit $n \to 1$ by $\hat{k}/(n-1)$ and $\overline{k}$, respectively.

- The small-x asymptotic results for the NLO parton densities $f_a^{\pm}$ are

$$
\begin{aligned}
f^{+}_{g,\mathrm{NLO}}(x,Q^2) &= A^{+}_{g,\mathrm{NLO}}(Q^2,Q_0^2)\tilde{I}_0(\sigma_{\mathrm{NLO}})\, e^{-\overline{d}_{+}s_{\mathrm{NLO}}-\overline{D}_{+}p_{\mathrm{NLO}}} \,+\, O(\rho_{\mathrm{NLO}}), && (3.11)\\
f^{+}_{q,\mathrm{NLO}}(x,Q^2) &= A^{+}_{q,\mathrm{NLO}}\Bigg[\Big(1-\overline{d}^{q}_{+-}a_s^{\mathrm{NLO}}(Q^2)\Big)\,\rho_{\mathrm{NLO}}\tilde{I}_1(\sigma_{\mathrm{NLO}}) \\
&\qquad +20a_s^{\mathrm{NLO}}(Q^2)I_0(\sigma_{\mathrm{NLO}})\Bigg]\, e^{-\overline{d}_{+}(1)s_{\mathrm{NLO}}-\overline{D}_{+}p_{\mathrm{NLO}}} \,+\, O(\rho_{\mathrm{NLO}}), && (3.12)\\
f^{-}_{g,\mathrm{NLO}}(x,Q^2) &= A^{-}_{g,\mathrm{NLO}}(Q^2,Q_0^2)e^{-d_{-}(1)s_{\mathrm{NLO}}-D_{-}p_{\mathrm{NLO}}} \,+\, O(x), && (3.13)\\
f^{-}_{q,\mathrm{NLO}}(x,Q^2) &= A^{-}_{q,\mathrm{NLO}}e^{-d_{-}(1)s_{\mathrm{NLO}}-D_{-}p_{\mathrm{NLO}}} \,+\, O(x), && (3.14)
\end{aligned}
$$

where $(b_1=\beta_1/\beta_0)$

$$D_{\pm} = d_{\pm\pm} - d_{\pm}b_1 \tag{3.15}$$

and similar for $\hat{D}_{+}$ and $\overline{D}_{+}$,

$$
\begin{aligned}
A^{+}_{g,\mathrm{NLO}}(Q^2,Q_0^2) &= \left(1-\frac{80f}{81}a_s^{\mathrm{NLO}}(Q)\right)A_g \\
&\quad +\frac{4}{9}\left(1+\left(3+\frac{f}{27}\right)a_s^{\mathrm{NLO}}(Q_0)-\frac{80f}{81}a_s^{\mathrm{NLO}}(Q)\right)A_q, \\
A^{-}_{g,\mathrm{NLO}}(Q^2,Q_0^2) &= A_g - A^{+}_{g,\mathrm{NLO}}(Q_0^2,Q^2)\,. && (3.16)
\end{aligned}
$$

The coupling constant $a_s(Q^2)$ is introduced in Eq. (2.2). The variables $\hat{d}_{+}$, $\hat{d}_{++}$ $\overline{d}_{+}$ and d_{-} are diven in Eqs. (3.2) and (3.8), respectively. The nonzero variables $\overline{d}_{++}$, d_{--} and d^{a}_{+-} $(a=q,g)$ have the form

$$
\begin{aligned}
\overline{d}_{++} &= \frac{8}{\beta_0}\left(36\zeta_3+33\zeta_2-\frac{1643}{12}+\frac{2f}{9}\left[\frac{68}{9}-4\zeta_2-\frac{13f}{243}\right]\right),\ \overline{d}^{g}_{+-}=\frac{80f}{61},\ \overline{d}^{g}_{-+}=-3-\frac{f}{27}, \\
d_{--} &= \frac{16}{9\beta_0}\left(2\zeta_3-3\zeta_2+\frac{13}{4}+f\left[4\zeta_2-\frac{23}{18}+\frac{13f}{243}\right]\right),\ d^{q}_{+-}=23-12\zeta_2-\frac{13f}{81}, && (3.17)
\end{aligned}
$$

with ζ_3 and ζ_2 are Eller functions.

4. Resummation in FFs

As already mentioned in Introduction, reliable computations of AJMs require resummed analytic expressions for the splitting functions because one has to evaluate the first Mellin moment (corresponding to $\omega=0$), which is a divergent quantity in the fixed-order perturbative approach. As is well known, resummation overcomes this problem, as demonstrated in the pioneering works by Mueller [18] and others [40].

In particular, as we shall see in previous subsection, resummed expressions for the first Mellin moments of the timelike splitting functions in the plus-minus basis appearing in Eq. (2.28) are required in our approach. Up to the NNLL level in the $\overline{\mathrm{MS}}$ scheme, these may be extracted from the available literature [18, 19, 20, 21] in closed analytic form using the relations in Eq. (2.30).

For future considerations, we remind the reader of an assumpion already made in Ref. [20] according to which the splitting functions $P^{(k)}_{--}(\omega)$ and $P^{(k)}_{+-}(\omega)$ are supposed to be free of singularities in the limit $\omega \to 0$. In fact, this is expected to be true to all orders. This is certainly true at the LL and NLL levels for the timelike splitting functions, as was verified in [20]. This is also true at the NNLL level, as may be explicitly checked by inserting the results of Ref. [21] in Eq. (2.30). Moreover, this is true through NLO in the spacelike case [15] and holds for the LO and NLO singularities [41, 42, 39] to all orders in the framework of the BFKL dynamics [8], a fact that was exploited in various approaches (see, e.g., Refs. [43] and references cited therein). We also note that the timelike splitting functions share a number of simple properties with their spacelike counterparts. In particular, the LO splitting functions are the same, and the diagonal splitting functions grow like $\ln\omega$ for $\omega \to \infty$ at all orders. This suggests the conjecture that the double-logarithm resummation in the timelike case and the BFKL resummation in the spacelike case are only related via the plus components. The minus components are devoid of singularities as $\omega \to 0$ and thus are not resummed. Now that this is known to be true for the first three orders of resummation, one has reason to expect this to remain true for all orders.

Using the relationships between the components of the splitting functions in the two bases given in Eq. (2.30), we find that the absence of singularities for $\omega = 0$ in $P_{--}(\omega, a_s)$ and $P_{+-}(\omega, a_s)$ implies that the singular terms are related as

$$P^{\rm sing}_{gq}(\omega,a_s) = -\frac{\varepsilon_\omega}{\alpha_\omega} P^{\rm sing}_{gg}(\omega,a_s), \quad P^{\rm sing}_{qg}(\omega,a_s) = -\frac{\alpha_\omega}{\varepsilon_\omega} P^{\rm sing}_{qq}(\omega,a_s), \tag{4.1}$$

where, through the NLL level, [10]

$$-\frac{\alpha_\omega}{\varepsilon_\omega} = \frac{C_A}{C_F}\left[1 - \frac{\omega}{6}\left(1 + 2\frac{T_F}{C_A} - 4\frac{C_F T_F}{C_A^2}\right)\right] + \mathscr{O}(\omega^2). \tag{4.2}$$

An explicit check of the applicability of the relationships in Eqs. (4.1) for $P_{ij}(\omega, a_s)$ with $i, j = g, g$ themselves is performed in the Appendix of Ref. [23]. Of course, the relationships in Eqs. (4.1) may be used to fix the singular terms of the off-diagonal timelike splitting functions $P_{qg}(\omega, a_s)$ and $P_{gq}(\omega, a_s)$ using known results for the diagonal timelike splitting functions $P_{qq}(\omega, a_s)$ and $P_{gg}(\omega, a_s)$. Since Refs. [19, 38] became available during the preparation of Ref. [20], the relations in Eqs. (4.1) provided an important independent check rather than a prediction.

We take here the opportunity to point out that Eqs. (2.45) and (2.46) together with Eq. (4.2) support the motivations for the numerical effective approach that we used in Ref. [28, 23] to study the gluon-to-quark AJM ratio. In fact, according to the findings of Ref. [28, 23], substituting $\omega = \omega_{\rm eff}$, where

$$\omega_{\rm eff} = 2\sqrt{2C_A a_s}, \tag{4.3}$$

into Eq. (4.2) exactly reproduces the result for the average gluon-to-quark jet multiplicity ratio $r(Q^2)$ obtained in Ref. [44]. In the next section, we shall obtain improved analytic formulae for the ratio $r(Q^2)$ and also for the average gluon and quark jet multiplicities.

[10] To have a possibility to compare different approximations, it is convenient to keep the general forms of the colour factors $C_A = 3$, $C_F = 4/3$ in the present and the next sections.

Here we would also like to note that, at first sight, the substitution $\omega = \omega_{\rm eff}$ should induce a Q^2 dependence to the diagonalization matrix. This is not the case, however, because to double-logarithmic accuracy the Q^2 dependence of $a_s(Q^2)$ can be neglected, so that the factor $\alpha_\omega/\varepsilon_\omega$ does not recieve any Q^2 dependence upon the substitution $\omega = \omega_{\rm eff}$. This supports the possibility to use this substitution in our analysis and gives an explanation of the good agreement with other approaches, e.g. that of Ref. [44]. Nevertheless, this substitution only carries a phenomenological meaning. It should only be done in the factor $\alpha_\omega/\varepsilon_\omega$, but not in the RG exponents of Eq. (2.44), where it would lead to a double-counting problem. In fact, the dangerous terms are already resummed in Eq. (2.44).

In order to be able to obtain the AJMs, we have to first evaluate the first Mellin momoments of the timelike splitting functions in the plus-minus basis. According to Eq. (2.30) together with the results given in Refs. [18, 21], we have

$$P_{++}^{\rm NNLL}(\omega=0) = \gamma_0(1-K_1\gamma_0+K_2\gamma_0^2), \tag{4.4}$$

where

$$\gamma_0 \;=\; P_{++}^{\rm LL}(\omega=0) = \sqrt{2C_Aa_s}, \quad K_1 \;=\; \frac{1}{12}\left[11+4\frac{T_F}{C_A}\left(1-\frac{2C_F}{C_A}\right)\right], \tag{4.5}$$

$$K_2 \;=\; \frac{1}{288}\left[1193-576\zeta_2-56\frac{T_F}{C_A}\left(5+2\frac{C_F}{C_A}\right)\right]+16\frac{T_F^2}{C_A^2}\left(1+4\frac{C_F}{C_A}-12\frac{C_F^2}{C_A^2}\right), \tag{4.6}$$

and

$$P_{-+}^{\rm NNLL}(\omega=0) = -\frac{C_F}{C_A}P_{qg}^{NNLL}(\omega=0), \tag{4.7}$$

where

$$P_{qg}^{\rm NNLL}(\omega=0) = \frac{16}{3}T_Fa_s-\frac{2}{3}T_F\left[17-4\frac{T_F}{C_A}\left(1-\frac{2C_F}{C_A}\right)\right]\left(2C_Aa_s^3\right)^{1/2}. \tag{4.8}$$

For the P_{+-} component, we obtain

$$P_{+-}^{\rm NNLL}(\omega=0) = \mathscr{O}(a_s^2). \tag{4.9}$$

Finally, as for the P_{--} component, we note that its LO expression produces a finite, nonvanishing term for $\omega = 0$ that is of the same order in a_s as the NLL-resummed results in Eq. (4.4), which leads us to use the following expression for the P_{--} component:

$$P_{--}^{\rm NNLL}(\omega=0) = -\frac{8T_FC_F}{3C_A}a_s+\mathscr{O}(a_s^2), \tag{4.10}$$

at NNLL accuracy.

We can now perform the integration in Eq. (2.44) through the NNLL level, which yields

$$\hat{T}_\pm^{\rm NNLL}(0,Q^2,Q_0^2) \;=\; \frac{T_\pm^{\rm NNLL}(Q^2)}{T_\pm^{\rm NNLL}(Q_0^2)}, \;\; T_-^{\rm NNLL}(Q^2) = T_-^{\rm NLL}(Q^2) = \left(a_s(Q^2)\right)^{d_-}, \tag{4.11}$$

$$T_+^{\rm NNLL}(Q^2) \;=\; \exp\left\{\frac{4C_A}{\beta_0\gamma^0(Q^2)}\left[1+(b_1-2C_AK_2)\,a_s(Q^2)\right]\right\}\left(a_s(Q^2)\right)^{d_+}, \tag{4.12}$$

where

$$b_1 = \frac{\beta_1}{\beta_0}, \qquad d_- = \frac{8T_FC_F}{3C_A\beta_0}, \qquad d_+ = \frac{2C_AK_1}{\beta_0}. \tag{4.13}$$

5. Multiplicities

According to Eqs. (2.44) and (2.45), the $\pm\mp$ components are not involved in the AJM Q^2 evolution, which is performed at $\omega = 0$ using the resummed expressions for the plus and minus components given in Eq. (4.4) and (4.10), respectively. We are now ready to define the average gluon and quark jet multiplicities in our formalism, namely

$$\langle n_h(Q^2)\rangle_a \equiv D_a(0,Q^2) = D_a^+(0,Q^2) + D_a^-(0,Q^2), \quad (a = g,q). \tag{5.1}$$

On the other hand, from Eqs. (2.45) and (2.46), it follows that

$$r_+(Q^2) \equiv \frac{D_g^+(0,Q^2)}{D_q^+(0,Q^2)} = -\lim_{\omega\to 0} \frac{\alpha_\omega}{\varepsilon_\omega} \frac{H_g^+(\omega,Q^2)}{H_q^+(\omega,Q^2)}, \quad r_-(Q^2) \equiv \frac{D_g^-(0,Q^2)}{D_q^-(0,Q^2)} = \lim_{\omega\to 0} \frac{1-\alpha_\omega}{\varepsilon_\omega} \frac{H_g^-(\omega,Q^2)}{H_q^-(\omega,Q^2)}. \tag{5.2}$$

Using these definitions and again Eq. (2.45), we may write general expressions for the gluon and quark AJMs:

$$\begin{aligned}
\langle n_h(Q^2)\rangle_g &= \tilde{D}_g^+(0,Q_0^2)\hat{T}_+^{\text{res}}(0,Q^2,Q_0^2)H_g^+(0,Q^2) + \tilde{D}_q^-(0,Q_0^2)r_-(Q^2)\hat{T}_-^{\text{res}}(0,Q^2,Q_0^2)H_q^-(0,Q^2), \\
\langle n_h(Q^2)\rangle_s &= \frac{\tilde{D}_g^+(0,Q_0^2)}{r_+(Q^2)}\hat{T}_+^{\text{res}}(0,Q^2,Q_0^2)H_g^+(0,Q^2) + \tilde{D}_q^-(0,Q_0^2)\hat{T}_-^{\text{res}}(0,Q^2,Q_0^2)H_q^-(0,Q^2).
\end{aligned} \tag{5.3}$$

At the LO in a_s, the coefficients of the RG exponents are given by

$$r_+(Q^2) = \frac{C_A}{C_F}, \qquad r_-(Q^2) = 0, \qquad H_s^{\pm}(0,Q^2) = 1, \qquad \tilde{D}_a^{\pm}(0,Q_0^2) = D_a^{\pm}(0,Q_0^2). \tag{5.4}$$

It would, of course, be desirable to include higher-order corrections in Eqs. (5.4). However, this is highly nontrivial because the general perturbative structures of the functions $H_a^{\pm}(\omega,\mu^2)$ and $Z_{\pm\mp,a}(\omega,a_s)$, which would allow us to resum those higher-order corrections, are presently unknown. Fortunatly, some approximations can be made. On the one hand, it is well-known that the plus components by themselves represent the dominant contributions to both the gluon and quark AJMs (see, e.g., Ref. [45] for the gluon case and Ref. [46] for the quark case). On the other hand, Eq. (5.2) tells us that $D_g^-(0,Q^2)$ is suppressed with respect to $D_q^-(0,Q^2)$ because $\alpha_\omega \sim 1 + \mathscr{O}(\omega)$. These two observations suggest that keeping $r_-(Q^2) = 0$ also beyond LO should represent a good approximation. Nevertheless, we shall explain below how to obtain the first nonvanishing contribution to $r_-(Q^2)$. Furthermore, we notice that higher-order corrections to $H_a^{\pm}(0,Q^2)$ and $\tilde{D}_a^{\pm}(0,Q_0^2)$ just represent redefinitions of $D_a^{\pm}(0,Q_0^2)$ by constant factors apart from running-coupling effects. Therefore, we assume that these corrections can be neglected.

Note that the resummation of the $\pm\pm$ components was performed similarly to Eq. (2.44) for the case of parton distribution functions in Ref. [15]. Such resummations are very important because they reduce the Q^2 dependences of the considered results at fixed order in perturbation theory by properly taking into account terms that are potentially large in the limit $\omega \to 0$ [47, 48]. We anticipate similar properties in the considered case, too, which is in line with our approximations. Some additional support for this may be obtained from $\mathscr{N} = 4$ super Yang-Mills theory, where the diagonalization can be performed exactly in any order of perturbation theory because the coupling constant and the corresponding martices for the diagonalization do not depended on Q^2. Consequently, there are no $Z_{\pm\mp,a}^{(k)}(\omega)$ terms, and only $P_{\pm\pm}^{(k)}(\omega)$ terms contribute to the integrand of the RG exponent. Looking at the r.h.s. of Eqs. (2.43) and (2.47), we indeed observe that the corrections of $\mathscr{O}(a_s)$ would cancel each other if the coupling constant were scale independent.

We now discuss higher-order corrections to $r_+(Q^2)$. As already mentioned above, we introduced in Ref. [28] an effective approach to perform the resummation of the first Mellin moment of the plus component of the anomalous dimension. In that approach, resummation is performed by taking the fixed-order plus component and substituting $\omega = \omega_{\rm eff}$, where $\omega_{\rm eff}$ is given in Eq. (4.3). We now show that this approach is exact to $\mathscr{O}(\sqrt{a_s})$. We indeed recover Eq. (4.5) by substituting $\omega = \omega_{\rm eff}$ in the leading singular term of the LO splitting function $P_{++}(\omega, a_s)$,

$$P_{++}^{\rm LO}(\omega) = \frac{4C_A a_s}{\omega} + \mathscr{O}(\omega^0). \tag{5.5}$$

We may then also substitute $\omega = \omega_{\rm eff}$ in Eq. (5.2) before taking the limit in $\omega = 0$. Using also Eq. (4.2), we thus find

$$r_+(Q^2) = \frac{C_A}{C_F}\left[1 - \frac{\sqrt{2a_s(Q^2)C_A}}{3}\left(1 + 2\frac{T_F}{C_A} - 4\frac{C_F T_F}{C_A^2}\right)\right] + \mathscr{O}(a_s), \tag{5.6}$$

which coincides with the result obtained by Mueller in Ref. [44]. For this reason and because, in Ref. [49], the gluon and quark AJMs evolve with only one RG exponent, we interprete the result in Eq. (5) of Ref. [25] as higher-order corrections to Eq. (5.6). Complete analytic expressions for all the coefficients of the expansion through $\mathscr{O}(a_s^{3/2})$ may be found in Appendix 1 of Ref. [25]. This interpretation is also explicitely confirmed in Chapter 7 of Ref. [50] through $\mathscr{O}(a_s)$.

Since we showed that our approach reproduces exact analytic results at $\mathscr{O}(\sqrt{a_s})$, we may safely apply it to predict the first non-vanishing correction to $r_-(Q^2)$ defined in Eq. (5.2), which yields

$$r_-(Q^2) = -\frac{4T_F}{3}\sqrt{\frac{2a_s(Q^2)}{C_A}} + \mathscr{O}(a_s). \tag{5.7}$$

However, contributions beyond $\mathscr{O}(\sqrt{\alpha_s})$ obtained in this way cannot be trusted, and further investigation is required. Therefore, we refrain from considering such contributions here.

For the reader's convenience, we list here expressions with numerical coefficients for $r_+(Q^2)$ through $\mathscr{O}(a_s^{3/2})$ and for $r_-(Q^2)$ through $\mathscr{O}(\sqrt{a_s})$ in QCD with $n_f = 5$:

$$r_+(Q^2) \;=\; 2.25 - 2.18249\sqrt{a_s(Q^2)} - 27.54\,a_s(Q^2) + 10.8462\,a_s^{3/2}(Q^2) + \mathscr{O}(a_s^2), \tag{5.8}$$

$$r_-(Q^2) \;=\; -2.72166\sqrt{a_s(Q^2)} + \mathscr{O}(a_s). \tag{5.9}$$

We denote the approximation in which Eqs. (4.11)–(4.12) and (5.4) are used as LO + NNLL, the improved approximation in which the expression for $r_+(Q^2)$ in Eq. (5.4) is replaced by Eq. (5.8), i.e. Eq. (5) in Ref. [25], as $\rm N^3LO_{approx}$ + NNLL, and our best approximation in which, on top of that, the expression for $r_-(Q^2)$ in Eq. (5.4) is replaced by Eq. (5.9) as $\rm N^3LO_{approx}$ + NLO + NNLL. We shall see in the next Section, where we compare with the experimental data and extract the strong-coupling constant, that the latter two approximations are actually very good and that the last one yields the best results, as expected.

In all the approximations considered here, we may summarize our main theoretical results for the gluon and quark AJMs in the following way:

$$\begin{aligned}\langle n_h(Q^2)\rangle_g &= n_1(Q_0^2)\hat{T}_+^{\rm res}(0,Q^2,Q_0^2)+n_2(Q_0^2)\,r_-(Q^2)\hat{T}_-^{\rm res}(0,Q^2,Q_0^2),\\ \langle n_h(Q^2)\rangle_s &= n_1(Q_0^2)\frac{\hat{T}_+^{\rm res}(0,Q^2,Q_0^2)}{r_+(Q^2)}+n_2(Q_0^2)\,\hat{T}_-^{\rm res}(0,Q^2,Q_0^2),\end{aligned} \tag{5.10}$$

where

$$n_1(Q_0^2)=r_+(Q_0^2)\frac{D_g(0,Q_0^2)-r_-(Q_0^2)D_s(0,Q_0^2)}{r_+(Q_0^2)-r_-(Q_0^2)},\;\; n_2(Q_0^2)=\frac{r_+(Q_0^2)D_s(0,Q_0^2)-D_g(0,Q_0^2)}{r_+(Q_0^2)-r_-(Q_0^2)}. \tag{5.11}$$

The gluon-to-quark AJM ratio may thus be written as

$$r(Q^2)\equiv\frac{\langle n_h(Q^2)\rangle_g}{\langle n_h(Q^2)\rangle_s}=r_+(Q^2)\left[\frac{1+r_-(Q^2)R(Q_0^2)\hat{T}_-^{\rm res}(0,Q^2,Q_0^2)/\hat{T}_+^{\rm res}(0,Q^2,Q_0^2)}{1+r_+(Q^2)R(Q_0^2)\hat{T}_-^{\rm res}(0,Q^2,Q_0^2)/\hat{T}_+^{\rm res}(0,Q^2,Q_0^2)}\right], \tag{5.12}$$

where

$$R(Q_0^2)=\frac{n_2(Q_0^2)}{n_1(Q_0^2)}. \tag{5.13}$$

It follows from the definition of $\hat{T}_\pm^{\rm res}(0,Q^2,Q_0^2)$ in Eq. (4.11) and from Eq. (5.11) that, for $Q^2=Q_0^2$, Eqs. (5.10) and (5.12) become

$$\langle n_h(Q_0^2)\rangle_g=D_g(0,Q_0^2),\qquad \langle n_h(Q_0^2)\rangle_q=D_s(0,Q_0^2),\qquad r(Q_0^2)=\frac{D_g(0,Q_0^2)}{D_s(0,Q_0^2)}. \tag{5.14}$$

These represent the initial conditions for the Q^2 evolution at an arbitrary initial scale Q_0. In fact, Eq. (5.10) is independ of Q_0^2, as may be observed by noticing that

$$\hat{T}_\pm^{\rm res}(0,Q^2,Q_0^2)=\hat{T}_\pm^{\rm res}(0,Q^2,Q_1^2)\hat{T}_\pm^{\rm res}(0,Q_1^2,Q_0^2), \tag{5.15}$$

for an arbitrary scale Q_1 (see also Ref. [51] for a detailed discussion of this point).

In the approximations with $r_-(Q^2)=0$ [22], i.e. the LO + NNLL and $\mathrm{N^3LO_{approx}}$ + NNLL ones, our general results in Eqs. (5.10), and (5.12) collapse to

$$\begin{aligned}\langle n_h(Q^2)\rangle_g &= D_g(0,Q_0^2)\hat{T}_+^{\rm res}(0,Q^2,Q_0^2),\\ \langle n_h(Q^2)\rangle_s &= D_g(0,Q_0^2)\frac{\hat{T}_+^{\rm res}(0,Q^2,Q_0^2)}{r_+(Q^2)}+\left[D_s(0,Q_0^2)-\frac{D_g(0,Q_0^2)}{r_+(Q_0^2)}\right]\hat{T}_-^{\rm res}(0,Q^2,Q_0^2),\\ r(Q^2) &= \frac{r_+(Q^2)}{\left[1+\frac{r_+(Q^2)}{r_+(Q_0^2)}\left(\frac{D_s(0,Q_0^2)r_+(Q_0^2)}{D_g(0,Q_0^2)}-1\right)\frac{\hat{T}_-^{\rm res}(0,Q^2,Q_0^2)}{\hat{T}_+^{\rm res}(0,Q^2,Q_0^2)}\right]}.\end{aligned} \tag{5.16}$$

The NNLL-resummed expressions for the gluon and quark AJMs given by Eq. (5.10) only depend on two nonperturbative constants, namely $D_g(0,Q_0^2)$ and $D_s(0,Q_0^2)$. These allow for a simple physical interpretation. In fact, according to Eq. (5.14), they are the average gluon and quark jet multiplicities at the arbitrary scale Q_0. We should also mention that identifying the quantity $r_+(Q^2)$ with the one computed in Ref. [25], we assume the scheme dependence to be negligible. This should be justified because of the scheme independence through NLL established in Ref. [20].

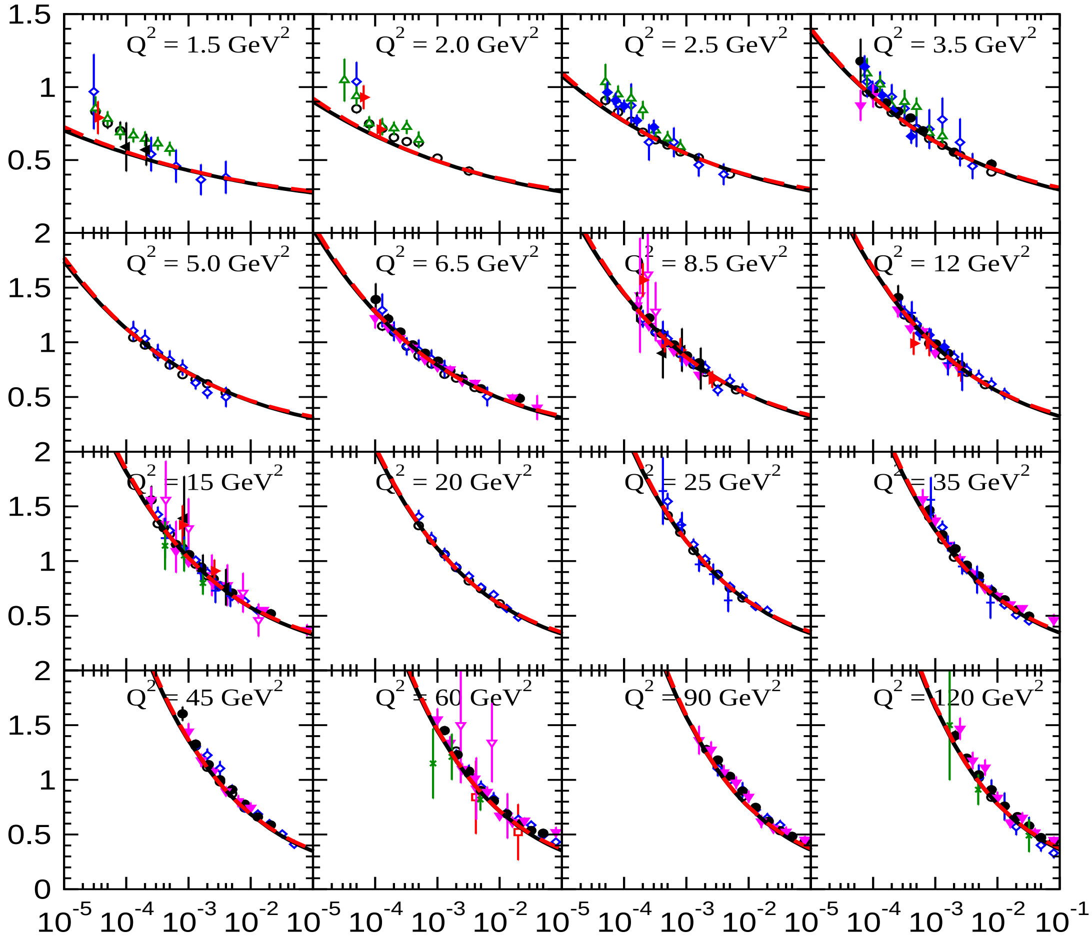

Figure 1: $F_2(x, Q^2)$ as a function of x for different Q^2 bins. The experimental points are from H1 [1] (open points) and ZEUS [2] (solid points) at $Q^2 \geq 1.5$ GeV^2. The solid curve represents the NLO fit. The dashed curve (hardly distinguishable from the solid one) represents the LO fit.

We note that the Q^2 dependence of our results is always generated via $a_s(Q^2)$ according to Eq. (2.4). This allows us to express Eq. (4.11) entirely in terms of $\alpha_s(Q^2)$. In fact, substituting the QCD values for the color factors and choosing $n_f = 5$ in the formulae given in Refs. [22, 23], we may write at NNLL

$$\hat{T}^{\rm res}_{\pm}(Q^2, Q_0^2) = \frac{T^{\rm res}_{\pm}(Q^2)}{T^{\rm res}_{\pm}(Q_0^2)}, \quad T^{\rm res}_{-}(Q^2) = \alpha_s^{d_1}(Q^2), \quad T^{\rm res}_{+}(Q^2) = \exp\left[\frac{d_2 + d_3\alpha_s(Q^2)}{\sqrt{\alpha_s(Q^2)}}\right]\alpha_s^{d_4}(Q^2), \tag{5.17}$$

where

$$d_1 = 0.38647, \qquad d_2 = 2.65187, \qquad d_3 = -3.87674, \qquad d_4 = 0.97771. \tag{5.18}$$

Table 1: The result of the LO and NLO fits to H1 and ZEUS data for different low Q^2 cuts. In the fits f is fixed to 4 flavors.

	A_g	A_q	Q_0^2 [GeV^2]	$\chi^2/n.o.p.$
$Q^2 \geq 1.5\text{GeV}^2$				
LO	0.784±.016	0.801±.019	0.304±.003	754/609
LO&an.	0.932±.017	0.707±.020	0.339±.003	632/609
LO&fr.	1.022±.018	0.650±.020	0.356±.003	547/609
NLO	-0.200±.011	0.903±.021	0.495±.006	798/609
NLO&an.	0.310±.013	0.640±.022	0.702±.008	655/609
NLO&fr.	0.180±.012	0.780±.022	0.661±.007	669/609
$Q^2 \geq 0.5\text{GeV}^2$				
LO	0.641±.010	0.937±.012	0.295±.003	1090/662
LO&an.	0.846±.010	0.771±.013	0.328±.003	803/662
LO&fr.	1.127±.011	0.534±.015	0.358±.003	679/662
NLO	-0.192±.006	1.087±.012	0.478±.006	1229/662
NLO&an.	0.281±.008	0.634±.016	0.680±.007	633/662
NLO&fr.	0.205±.007	0.650±.016	0.589±.006	670/662

6. Comparison with experimental data

Here we compare our formulae with experimental data for DIS SF $F_2(x,Q^2)$ and for the AJMs. In the DIS case, we limite ourselves by consideration only the SF $F_2(x,Q^2)$. The comparison of the generalized DAS approach predictions with the data for the slope $\partial \ln F_2/\partial \ln(1/x)$ [4] and for the heavy parts of F_2 [5] can be found in Refs. [52, 48] and [53], respectively (see also the review [54]). An estimation of the cross-sections of very high-energy neutrino and nucleon scattering has been found in [55].

6.1 DIS SF F_2

Using the results of section 3 we have analyzed HERA data for F_2 at small x from the H1 and ZEUS Collaborations [1, 2, 3].

In order to keep the analysis as simple as possible, we fix $f = 4$ and $\alpha_s(M_Z^2) = 0.1166$ (i.e., $\Lambda^{(4)} = 284$ MeV) in agreement with the more recent ZEUS results [2].

As it is possible to see in Fig. 1 (see also [15, 16]), the twist-two approximation is reasonable at $Q^2 \geq 2$ GeV^2. At smaller Q^2, some modification of the approximation should be considered. In Ref. [16] we have added the higher twist corrections. For renormalon model of higher twists, we have found a good agreement with experimental data at essentially lower Q^2 values: $Q^2 \geq 0.5$ GeV^2 (see Figs. 4 and 5 in Ref. [16]), but we have added 4 additional parameters: amplitudes of twist-4 and twist-6 corrections to quark and gluon densities.

Moreover, the results of fits in [16] have an important property: they are very similar in LO and NLO approximations of perturbation theory. The similarity is related to the fact that the small-x

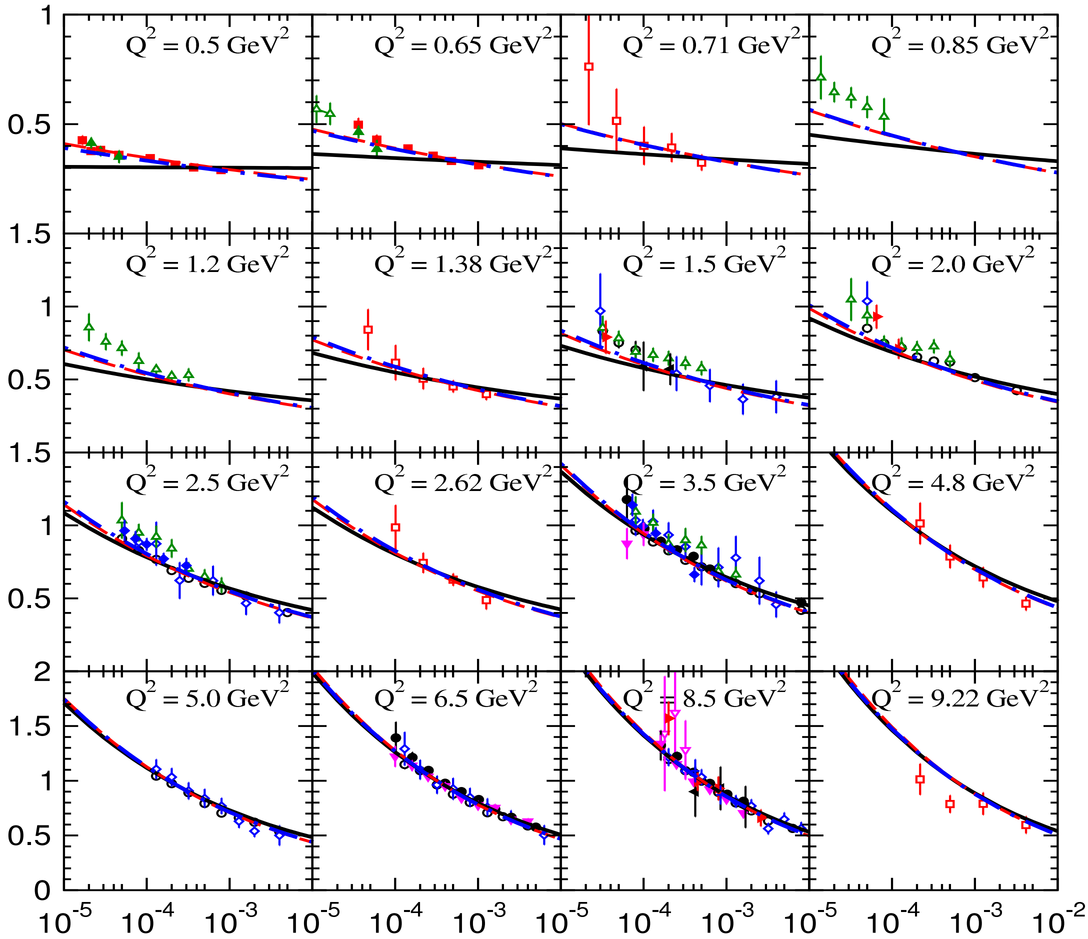

Figure 2: x dependence of $F_2(x,Q^2)$ in bins of Q^2. The experimental data from H1 (open points) and ZEUS (solid points) are compared with the NLO fits for $Q^2 \geq 0.5$ GeV^2 implemented with the canonical (solid lines), frozen (dot-dashed lines), and analytic (dashed lines) versions of the strong-coupling constant. For comparison, also the results obtained in Ref. [16] through a fit based on the renormalon model of higher-twist terms are shown (dotted lines).

asymptotics of the NLO corrections are usually large and negative (see, for example, α_s-corrections [41, 42] to BFKL kernel [8][11]). Then, the LO form $\sim \alpha_s(Q^2)$ for some observable and the NLO one $\sim \alpha_s(Q^2)(1-K\alpha_s(Q^2))$ with a large value of K are similar, because $\Lambda_{\rm NLO} \gg \Lambda_{\rm LO}$[12] and, thus, $\alpha_s(Q^2)$ at LO is considerably smaller then $\alpha_s(Q^2)$ at NLO for HERA Q^2 values.

In other words, performing some resummation procedure (such as Grunberg's effective-charge method [56]), one can see that the results up to NLO approximation may be represented as $\sim \alpha_s(Q^2_{\rm eff})$, where $Q^2_{\rm eff} \gg Q^2$. Indeed, from different studies [57, 58, 59], it is well known that at

[11] It seems that it is a property of any processes in which gluons, but not quarks play a basic role.

[12] The equality of $\alpha_s(M_Z^2)$ at LO and NLO approximations, where M_Z is the Z-boson mass, relates $\Lambda_{\rm NLO}$ and $\Lambda_{\rm LO}$: $\Lambda^{(4)}_{\rm NLO} = 284$ MeV (as in [2]) corresponds to $\Lambda_{\rm LO} = 112$ MeV (see [16]).

small-x values the effective argument of the coupling constant is higher then Q^2. As it was shown in [60], the usage of the effective scale in the generalized DAS approach improves the agreement with data for SF $F_2(x,Q^2)$.

Here, to improve the agreement at small Q^2 values without additional parameters, we modify the QCD coupling constant. We consider two modifications, which effectively increase the argument of the coupling constant at small Q^2 values (in agreement with [57, 58, 59]).

In one case, which is more phenomenological, we introduce freezing of the coupling constant by changing its argument $Q^2 \to Q^2 + M_\rho^2$, where M_ρ is the ρ-meson mass (see [61]). Thus, in the formulae of the Section 2 we should do the following replacement:

$$a_s(Q^2) \to a_{\rm fr}(Q^2) \equiv a_s(Q^2 + M_\rho^2) \tag{6.1}$$

The second possibility incorporates the Shirkov–Solovtsov idea [62]-[65] about analyticity of the coupling constant that leads to the additional its power dependence. Then, in the formulae of the previous section the coupling constant $a_s(Q^2)$ should be replaced as follows:

$$a^{\rm LO}_{\rm an}(Q^2) = a_s(Q^2) - \frac{1}{\beta_0} \frac{\Lambda^2_{\rm LO}}{Q^2 - \Lambda^2_{\rm LO}} \tag{6.2}$$

at the LO approximation and

$$a_{\rm an}(Q^2) = a_s(Q^2) - \frac{1}{2\beta_0} \frac{\Lambda^2}{Q^2 - \Lambda^2} + \dots, \tag{6.3}$$

at the NLO approximation, where the symbol ... stands for terms which have negligible contributions at $Q \geq 1$ GeV [62][13].

Figure 2 and Table 1 show a strong improvement of the agreement with experimental data for F_2 (χ^2 values decreased almost 2 times!).

6.1.1 H1&ZEUS data

Here we have analyzed the very precise H1&ZEUS data for F_2 [3]. As can be seen from Fig. 3 and Table 2, the twist-two approximation is reasonable for $Q^2 \geq 4$ GeV^2. At lower Q^2 we observe that the fits in the cases with "frozen" and analytic strong coupling constants are very similar (see also [66]) and describe the data in the low Q^2 region significantly better than the standard fit (χ^2 values decreased 2÷ 3 times!) Nevertheless, for $Q^2 \leq 1.5$ GeV^2 there is still some disagreement with the data, which needs to be additionally studied. In particular, the BFKL resummation [8] may be important here [67]. It can be added in the generalized DAS approach according to the discussion in Ref. [54].

6.2 Average multiplicity and experimendal data

Now we show the results in [23] obtained from a global fit to the available experimental data of our formulas in Eq. (5.10) in the LO + NNLL, N^{3}LO$_{\rm approx}$ + NNLL, and N^{3}LO$_{\rm approx}$ + NLO +

[13] Note that in [63, 65] more accurate, but essentially more cumbersome approximations of $a_{an}(Q^2)$ have been proposed. We limit ourselves by above simple form (6.2), (6.3) and plan to add the other modifications in our future investigations.

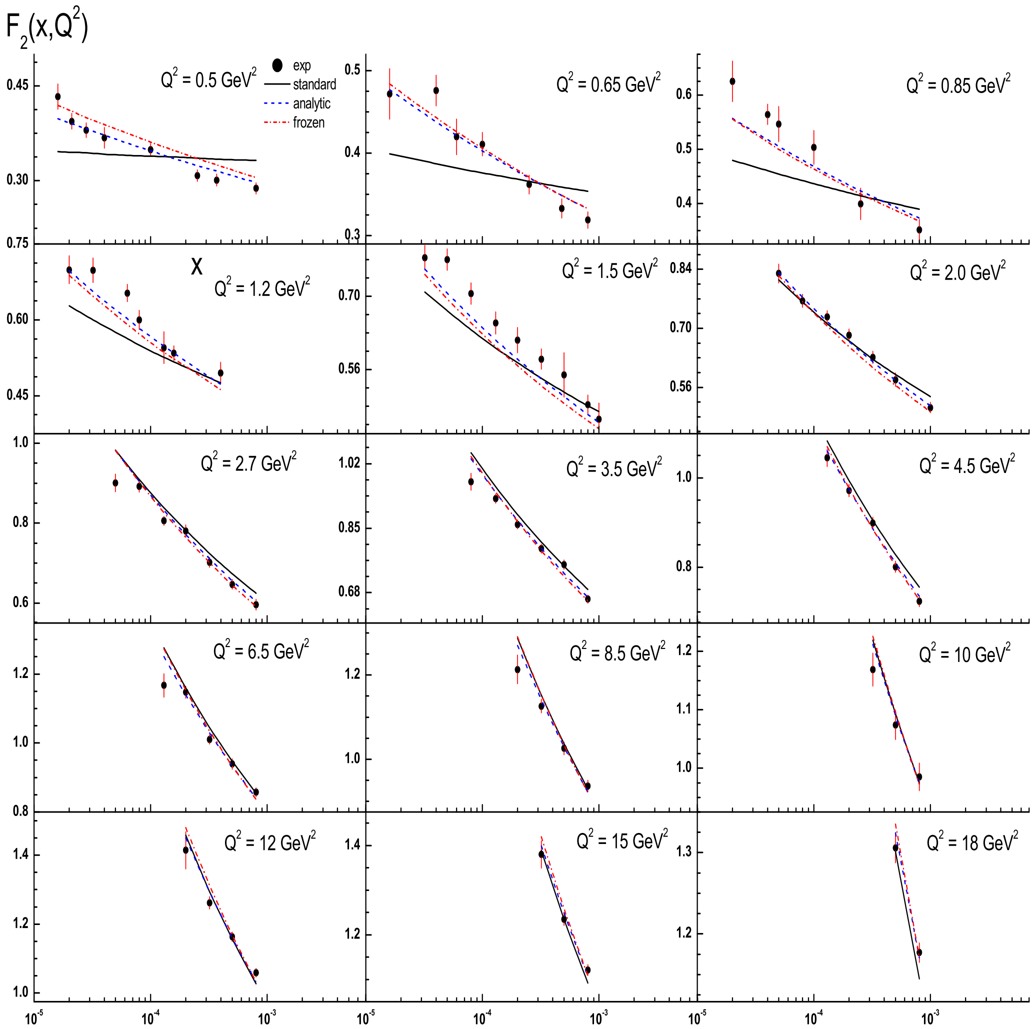

Figure 3: x dependence of $F_2(x,Q^2)$ in bins of Q^2. The combined experimental data from H1 and ZEUS Collaborations [3] are compared with the NLO fits for $Q^2 \geq 0.5\,\text{GeV}^2$ implemented with the standard (solid lines), frozen (dot-dashed lines), and analytic (dashed lines) versions of the strong coupling constant.

NNLL approximations, so as to extract the nonperturbative constants $D_g(0,Q_0^2)$ and $D_s(0,Q_0^2)$. We have to make a choice for the scale Q_0, which, in principle, is arbitrary. In [23], we adopted $Q_0 = 50$ GeV.

The gluon and quark AJMs extracted from experimental data strongly depend on the choice of jet algorithm. We adopt the selection of experimental data from Ref. [68] performed in such a way that they correspond to compatible jet algorithms. Specifically, these include the gluon AJM measurements in Refs. [68]-[72] and quark ones in Refs. [69, 73], which include 27 and 51 experimental data points, respectively. The results for $\langle n_h(Q_0^2)\rangle_g$ and $\langle n_h(Q_0^2)\rangle_q$ at $Q_0 = 50$ GeV

Table 2: The results of LO and NLO fits to H1 & ZEUS data [3], with various lower cuts on Q^2; in the fits the number of flavors f is fixed to 4.

	A_g	A_q	Q_0^2 [GeV^2]	$\chi^2/n.d.f.$
$Q^2 \geq 5$GeV^2				
LO	0.623±0.055	1.204±0.093	0.437±0.022	1.00
LO&an.	0.796±0.059	1.103±0.095	0.494±0.024	0.85
LO&fr.	0.782±0.058	1.110±0.094	0.485±0.024	0.82
NLO	-0.252±0.041	1.335±0.100	0.700±0.044	1.05
NLO&an.	0.102±0.046	1.029±0.106	1.017±0.060	0.74
NLO&fr.	-0.132±0.043	1.219±0.102	0.793±0.049	0.86
$Q^2 \geq 3.5$GeV^2				
LO	0.542±0.028	1.089±0.055	0.369±0.011	1.73
LO&an.	0.758±0.031	0.962±0.056	0.433±0.013	1.32
LO&fr.	0.775±0.031	0.950±0.056	0.432±0.013	1.23
NLO	-0.310±0.021	1.246±0.058	0.556±0.023	1.82
NLO&an.	0.116±0.024	0.867±0.064	0.909±0.330	1.04
NLO&fr.	-0.135±0.022	1.067±0.061	0.678±0.026	1.27
$Q^2 \geq 2.5$GeV^2				
LO	0.526±0.023	1.049±0.045	0.352±0.009	1.87
LO&an.	0.761±0.025	0.919±0.046	0.422±0.010	1.38
LO&fr.	0.794±0.025	0.900±0.047	0.425±0.010	1.30
NLO	-0.322±0.017	1.212±0.048	0.517±0.018	2.00
NLO&an.	0.132±0.020	0.825±0.053	0.898±0.026	1.09
NLO&fr.	-0.123±0.018	1.016±0.051	0.658±0.021	1.31
$Q^2 \geq 0.5$GeV^2				
LO	0.366±0.011	1.052±0.016	0.295±0.005	5.74
LO&an.	0.665±0.012	0.804±0.019	0.356±0.006	3.13
LO&fr.	0.874±0.012	0.575±0.021	0.368±0.006	2.96
NLO	-0.443±0.008	1.260±0.012	0.387±0.010	6.62
NLO&an.	0.121±0.008	0.656±0.024	0.764±0.015	1.84
NLO&fr.	-0.071±0.007	0.712±0.023	0.529±0.011	2.79

	LO + NNLL	N^{3}LO$_{\text{approx}}$ + NNLL	N^{3}LO$_{\text{approx}}$ + NLO + NNLL
$\langle n_h(Q_0^2)\rangle_g$	24.31 ± 0.85	24.02 ± 0.36	24.17 ± 0.36
$\langle n_h(Q_0^2)\rangle_q$	15.49 ± 0.90	15.83 ± 0.37	15.89 ± 0.33
χ^2_{dof}	18.09	3.71	2.92

Table 3: Fit results for $\langle n_h(Q_0^2)\rangle_g$ and $\langle n_h(Q_0^2)\rangle_q$ at $Q_0 = 50$ GeV with 90% CL errors and minimum values of χ^2_{dof} achieved in the LO + NNLL, N^{3}LO$_{\text{approx}}$ + NNLL, and N^{3}LO$_{\text{approx}}$ + NLO + NNLL approximations.

together with the χ^2_{dof} values obtained in our LO + NNLL, N^{3}LO$_{\text{approx}}$ + NNLL, and N^{3}LO$_{\text{approx}}$ + NLO + NNLL fits are listed in Table 3. The errors correspond to 90% CL as explained above. All these fit results are in agreement with the experimental data. Looking at the χ^2_{dof} values, we observe that the qualities of the fits improve as we go to higher orders, as they should. The improvement is most dramatic in the step from LO + NNLL to N^{3}LO$_{\text{approx}}$ + NNLL, where the errors on $\langle n_h(Q_0^2)\rangle_g$ and $\langle n_h(Q_0^2)\rangle_q$ are more than halved. The improvement in the step from N^{3}LO$_{\text{approx}}$ + NNLL to N^{3}LO$_{\text{approx}}$ + NLO + NNLL, albeit less pronounced, indicates that the inclusion of the first

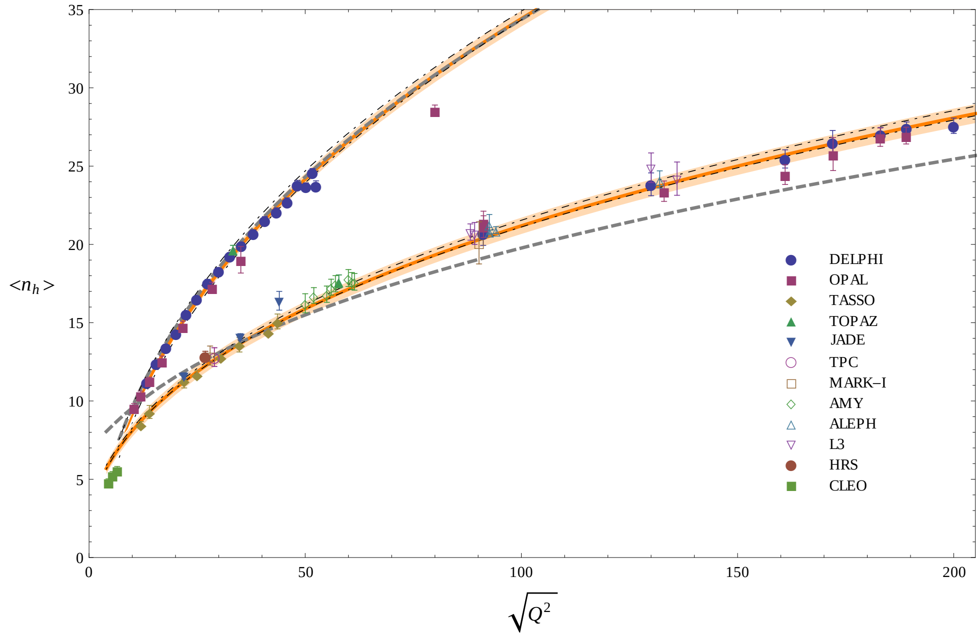

Figure 4: The average gluon (upper curves) and quark (lower curves) jet multiplicities evaluated from Eq. (5.10), respectively, in the LO + NNLL (dashed/gray lines) and N^{3}LO$_{\rm approx}$ + NLO + NNLL (solid/orange lines) approximations using the corresponding fit results for $\langle n_h(Q_0^2)\rangle_g$ and $\langle n_h(Q_0^2)\rangle_q$ from Table 3 are compared with the experimental data included in the fits. The experimental and theoretical uncertainties in the N^{3}LO$_{\rm approx}$ + NLO + NNLL results are indicated by the shaded/orange bands and the bands enclosed between the dot-dashed curves, respectively.

correction to $r_-(Q^2)$ as given in Eq. (5.7) is favored by the experimental data. We have verified that the values of $\chi^2_{\rm dof}$ are insensitive to the choice of Q_0, as they should. Furthermore, the central values converge in the sense that the shifts in the step from N^{3}LO$_{\rm approx}$ + NNLL to N^{3}LO$_{\rm approx}$ + NLO + NNLL are considerably smaller than those in the step from LO + NNLL to N^{3}LO$_{\rm approx}$ + NNLL and that, at the same time, the central values after each step are contained within error bars before that step. In the fits presented so far, the strong-coupling constant was taken to be the central value of the world avarage, $\alpha_s^{(5)}(m_Z^2) = 0.1184$ [74]. In the next Section, we shall include $\alpha_s^{(5)}(m_Z^2)$ among the fit parameters.

In Fig. 4, we show as functions of Q the gluon and quark AJMs evaluated from Eq. (5.10) at LO + NNLL and N^{3}LO$_{\rm approx}$ + NLO + NNLL using the corresponding fit results for $\langle n_h(Q_0^2)\rangle_g$ and $\langle n_h(Q_0^2)\rangle_q$ at $Q_0 = 50$ GeV from Table 3. For clarity, we refrain from including in Fig. 4 the N^{3}LO$_{\rm approx}$ + NNLL results, which are very similar to the N^{3}LO$_{\rm approx}$ + NLO + NNLL ones already presented in Ref. [22]. In the N^{3}LO$_{\rm approx}$ + NLO + NNLL case, Fig. 4 also displays two error bands, namely the experimental one induced by the 90% CL errors on the respective fit parameters in Table 3 and the theoretical one, which is evaluated by varying the scale parameter between $Q/2$ and $2Q$.

While our fits rely on individual measurements of the gluon and quark AJMs, the experimental literature also reports determinations of their ratio; see Refs. [27, 68, 70, 72, 75], which essentially cover all the available measurements. In order to find out how well our fits describe the latter and thus to test the global consistency of the individual measurements, we compare in Fig. 5 the

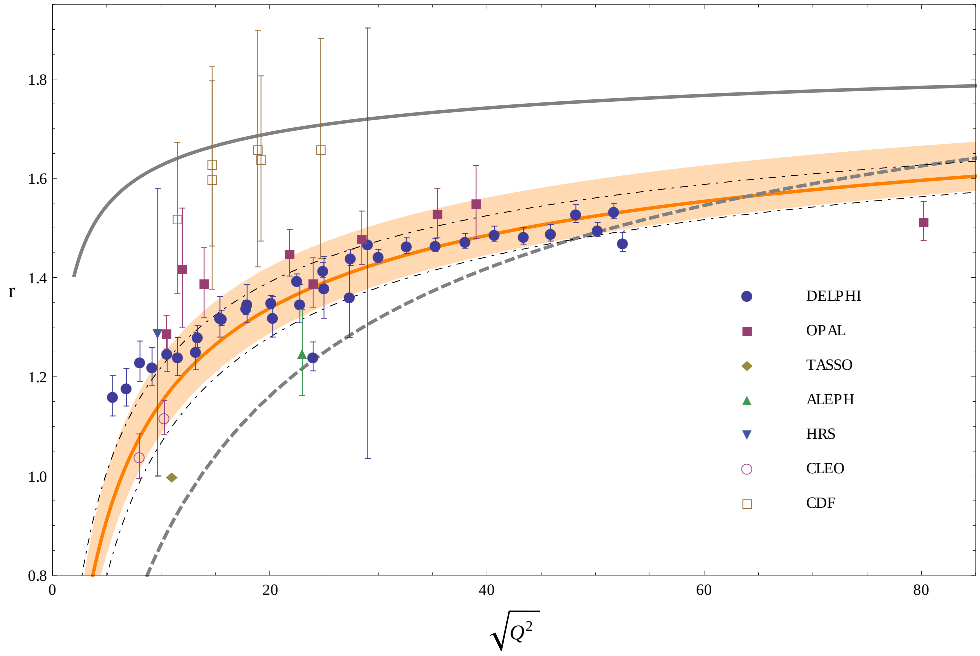

Figure 5: The average gluon-to-quark jet multiplicity ratio evaluated from Eq. (5.12) in the LO + NNLL (dashed/gray lines) and N^{3}LO$_{\rm approx}$ + NLO + NNLL (solid/orange lines) approximations using the corresponding fit results for $\langle n_h(Q_0^2)\rangle_g$ and $\langle n_h(Q_0^2)\rangle_q$ from Table 3 are compared with experimental data. The experimental and theoretical uncertainties in the N^{3}LO$_{\rm approx}$ + NLO + NNLL result are indicated by the shaded/orange bands and the bands enclosed between the dot-dashed curves, respectively. The prediction given by Eq. (5.8) [25] is indicated by the continuous/gray line.

experimental data on the gluon-to-quark AJM ratio with our evaluations of Eq. (5.12) in the LO + NNLL and N^{3}LO$_{\rm approx}$ + NLO + NNLL approximations using the corresponding fit results from Table 3. As in Fig. 4, we present in Fig. 5 also the experimental and theoretical uncertainties in the N^{3}LO$_{\rm approx}$ + NLO + NNLL result. For comparison, we include in Fig. 5 also the prediction of Ref. [25] given by Eq. (5.8).

Looking at Fig. 5, we observe that the experimental data are very well described by the N^{3}LO$_{\rm approx}$ + NLO + NNLL result for Q values above 10 GeV, while they somewhat overshoot it below. This discrepancy is likely to be due to the fact that, following Ref. [68], we excluded the older data from Ref. [27] from our fits because they are inconsistent with the experimental data sample compiled in Ref. [68].

The Monte Carlo analysis of Ref. [26] suggests that the average gluon and quark jet multiplicities should coincide at about $Q = 4$ GeV. As is evident from Fig. 5, this agrees with our N^{3}LO$_{\rm approx}$ + NLO + NNLL result reasonably well given the considerable uncertainties in the small-Q^2 range discussed above.

As is obvious from Fig. 5, the approximation of $r(Q^2)$ by $r_+(Q^2)$ given in Eq. (5.8) [25] leads to a poor approximation of the experimental data, which reach up to Q values of about 50 GeV. It is, therefore, interesting to study the high-Q^2 asymptotic behavior of the average gluon-to-quark jet ratio. This is done in Fig. 6, where the N^{3}LO$_{\rm approx}$ + NLO + NNLL result including its experimental and theoretical uncertainties is compared with the approximation by Eq. (5.8) way up to

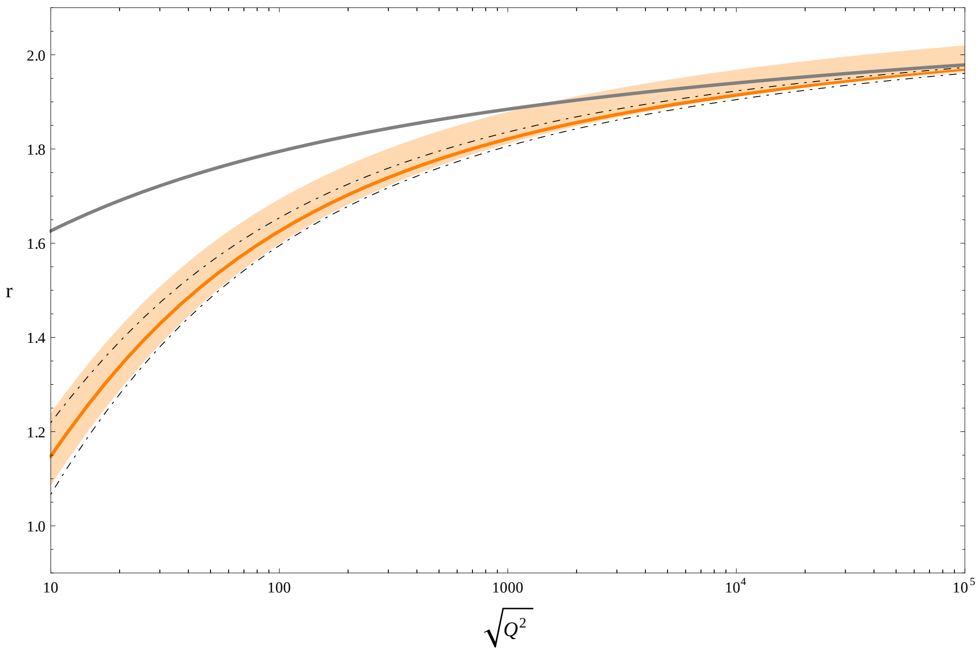

Figure 6: High-Q extension of Fig. 5.

	$\mathrm{N^3LO_{approx}+NNLL}$	$\mathrm{N^3LO_{approx}+NLO+NNLL}$
$\langle n_h(Q_0^2)\rangle_g$	24.18 ± 0.32	24.22 ± 0.33
$\langle n_h(Q_0^2)\rangle_q$	15.86 ± 0.37	15.88 ± 0.35
$\alpha_s^{(5)}(m_Z^2)$	0.1242 ± 0.0046	0.1199 ± 0.0044
χ^2_{dof}	2.84	2.85

Table 4: Fit results for $\langle n_h(Q_0^2)\rangle_g$ and $\langle n_h(Q_0^2)\rangle_q$ at $Q_0 = 50$ GeV and for $\alpha_s^{(5)}(m_Z^2)$ with 90% CL errors and minimum values of χ^2_{dof} achieved in the $\mathrm{N^3LO_{approx}+NNLL}$ and $\mathrm{N^3LO_{approx}+NLO+NNLL}$ approximations.

$Q = 100$ TeV. We observe from Fig. 6 that the approximation approaches the $\mathrm{N^3LO_{approx}+NLO+}$ NNLL result rather slowly. Both predictions agree within theoretical errors at $Q = 100$ TeV, which is one order of magnitude beyond LHC energies, where they are still about 10% below the asymptotic value $C_A/C_F = 2.25$.

6.2.1 Determination of strong-coupling constant from average multiplicity

In the previous Section, we took $\alpha_s^{(5)}(m_Z^2)$ to be a fixed input parameter for our fits. Motivated by the excellent goodness of our $\mathrm{N^3LO_{approx}+NNLL}$ and $\mathrm{N^3LO_{approx}+NLO+NNLL}$ fits, we now include it among the fit parameters, the more so as the fits should be sufficiently sensitive to it in view of the wide Q^2 range populated by the experimental data fitted to. We fit to the same experimental data as before and again put $Q_0 = 50$ GeV. The fit results are summarized in Table 4. We observe from Table 4 that the results of the $\mathrm{N^3LO_{approx}+NNLL}$ [51] and $\mathrm{N^3LO_{approx}+}$ NLO + NNLL fits for $\langle n_h(Q_0^2)\rangle_g$ and $\langle n_h(Q_0^2)\rangle_q$ are mutually consistent. They are also consistent with the respective fit results in Table 3. As expected, the values of χ^2_{dof} are reduced by relasing $\alpha_s^{(5)}(m_Z^2)$ in the fits, from 3.71 to 2.84 in the $\mathrm{N^3LO_{approx}+NNLL}$ approximation and from 2.95 to

2.85 in the $N^3LO_{approx} + NLO + NNLL$ one. The three-parameter fits strongly confine $\alpha_s^{(5)}(m_Z^2)$, within an error of 3.7% at 90% CL in both approximations. The inclusion of the $r_-(Q^2)$ term has the beneficial effect of shifting $\alpha_s^{(5)}(m_Z^2)$ closer to the world average, 0.1184 ± 0.0007 [74]. In fact, our $N^3LO_{approx} + NLO + NNLL$ value, 0.1199 ± 0.0044 at 90% CL, which corresponds to 0.1199 ± 0.0026 at 68% CL, is in excellent agreement with the former. Note that similar $\alpha_s^{(5)}(m_Z^2)$ valu has been otained recently [76] in an extension of the MLLA approach.

7. Conclusions

We have shown the Q^2-dependences of the SF F_2 at small-x values and of AJMs in the framework of perturbative QCD. We would like to stress that a good agreement wit the experimental data for the variables cannot be obtained without a proper consideration of the cotributions of both the "$+$" and "$-$" components.

The "$+$" components contain all large logarithms $\ln(1/x)$ as far as DIS SF F_2 and also for the average jet multiplicities. The large logarithms are resummed using famous BFKL approach [8] in the PDF case and another famous MLLA approach [50] in the FF case. [14] Nevertheless, the contributions of the "$-$" components are very important to have a good agreement with experimental data: they come with the additional free parameters. Moreover, the "$-$" components have other shapes to compare with the "$+$" ones. For example, in the AJM case the "$-$" component is responsable for the difference in the Q^2-dependences of quark and gluon multiplicities. Indeed, the "$-$" component gives essential contribution to the quark AJM but not to the gluon one.

In the case of DIS SF F_2, our results are in very good agreement with precise HERA data at $Q^2 \geq 2 \div 3$ GeV^2, where perturbative theory can be applicable. The application of the "frozen" and analytic coupling constants $\alpha_{\rm fr}(Q^2)$ and $\alpha_{\rm an}(Q^2)$ improves the agreement with the recent HERA data [3] for small Q^2 values, $Q^2 \geq 0.5$ GeV^2.

Prior to our analysis in Ref. [22, 23], experimental data on the gluon and quark AJMs could not be simultaneously described in a satisfactory way mainly because the theoretical formalism failed to account for the difference in hadronic contents between gluon and quark jets, although the convergence of perturbation theory seemed to be well under control [25]. This problem was solved by including the "$-$" components governed by $\hat{T}_-^{\rm res}(0, Q^2, Q_0^2)$ in Eqs. (5.10) and (5.12). This was done for the first time in Ref. [22]. The quark-singlet "$-$" component comes with an arbitrary normalization and has a slow Q^2 dependence. Consequently, its numerical contribution may be approximately mimicked by a constant introduced to the average quark jet multiplicity as in Ref. [27].

Motivated by the goodness of our fits in [22, 23] with fixed value of $\alpha_s^{(5)}(m_Z^2)$, we then included $\alpha_s^{(5)}(m_Z^2)$ among the fit parameters, which yielded a further reduction of $\chi^2_{\rm dof}$. The fit results are listed in Table 4.

[14] Note, however, that in the case of DIS SF F_2 we use obly the first two orders of the perturbation theory and our "$+$" component resum by DGLAP equation [6]. The resummation leads to the Bessel-like form of the "$+$" component. Including all orders of the perturbation theory should lead to a power-like form as it was predicted in the framework of BFKL approach [8] (see discussion in [54]).

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8. Appendix A

For diagonalization of quark and gluon interaction it is neseccary to introduce the corresponding matrix U, which diagonalize exactly the LO AD

$$\hat{U}^{-1}\begin{pmatrix} \gamma_{qq}^{(0)}(n) & \gamma_{gq}^{(0)}(n) \\ \gamma_{qg}^{(0)}(n) & \gamma_{gg}^{(0)}(n) \end{pmatrix}\hat{U} \;=\; \begin{pmatrix} \gamma_{-}^{(0)}(n) & 0 \\ 0 & \gamma_{+}^{(0)}(n) \end{pmatrix} \qquad \left(\hat{U}^{-1}\hat{U}=1\right), \tag{A1}$$

where

$$\hat{U} \;=\; \begin{pmatrix} \alpha_n & \alpha_n-1 \\ \beta_n & \beta_n \end{pmatrix}, \;\; \hat{U}^{-1} \;=\; \begin{pmatrix} 1 & \frac{1-\alpha_n}{\beta_n} \\ -1 & \frac{1\alpha_n}{\beta_n} \end{pmatrix} \tag{A2}$$

and α_n and β_n have been defined in the main text, in Eq. (2.18).

At higher orders the anomalous dimensions are transformed as follows

$$\hat{U}^{-1}\begin{pmatrix} \gamma_{qq}^{(i)}(n) & \gamma_{gq}^{(i)}(n) \\ \gamma_{qg}^{(i)}(n) & \gamma_{gg}^{(i)}(n) \end{pmatrix}\hat{U} \;=\; \begin{pmatrix} \gamma_{--}^{(n)}(j) & \gamma_{-+}^{(i)}(n) \\ \gamma_{+-}^{(i)}(n) & \gamma_{++}^{(i)}(n) \end{pmatrix}, \tag{A3}$$

where exact representations for $\gamma_{\pm\pm}^{(i)}(n)$ and $\gamma_{\pm\mp}^{(i)}(n)$ were given in the main text, in Eq. (2.22).

8.1 Diagonalization of the renormalization group exponent

Consider the renormalization group exponent (hereafter in the Appendix A $\overline{a}_s = a_s(Q^2)$ and $a_s = a_s(Q_0^2)$)

$$\hat{W}(a_s,\overline{a}_s) \equiv T_{a_s}\,\exp\left[\int_{\overline{a}_s}^{a_s}\frac{da'}{a'}\,\frac{\hat{\gamma}(a')}{2\beta(a')}\right], \tag{A4}$$

in the following form

$$\hat{W}(a_s,\overline{a}_s) = \hat{V}(a_s)\,\exp\left[\frac{\hat{\gamma}^{(0)}(n)}{2\beta_0}\ln\frac{\overline{a}_s}{a_s}\right]\hat{V}^{-1}(\overline{a}_s) \equiv \hat{V}(a_s)\,\hat{W}^{(0)}(a_s,\overline{a}_s)\,\hat{V}^{-1}(\overline{a}_s), \tag{A5}$$

where the matrix $\hat{V}(a_s)$ contains high order coefficients.

To find the matrix $\hat{V}(a_s)$, it is better to find the derivation

$$\frac{d}{da_s}\hat{W}(a_s,\overline{a}_s)\,. \tag{A6}$$

The l.h.s. of (A5) leads to

$$\frac{d}{da_s}\hat{W}(a_s,\overline{a}_s) = \frac{\hat{\gamma}(a_s)}{2\beta(a_s)}\hat{V}(a_s)\hat{W}^{(0)}(a_s,\overline{a}_s)\hat{V}^{-1}(\overline{a}_s). \tag{A7}$$

For the r.h.s. of (A5), we have

$$\frac{d}{da_s}\hat{W}(a_s,\overline{a}_s) = \left[\frac{d}{da_s}\hat{V}(a_s) - \hat{V}(a_s)\,\frac{\hat{\gamma}^{(0)}(n)}{2\beta_0}\,\frac{1}{a_s}\right]\hat{W}^{(0)}(a_s,\overline{a}_s)\hat{V}^{-1}(\overline{a}_s). \tag{A8}$$

Thus, the matrix $\hat{V}(a_s)$ obeys the following equation

$$\frac{d}{da_s}\hat{V}(a_s) + \frac{1}{a_s}\left[\frac{\hat{\gamma}^{(0)}(n)}{2\beta_0},\hat{V}(a_s)\right] = \left(\frac{\hat{\gamma}(a_s)}{2\beta(a_s)} + \frac{\hat{\gamma}^{(0)}(n)}{2\beta_0}\,\frac{1}{a_s}\right)\hat{V}(a_s)\,, \tag{A9}$$

where the second term in the l.h.s. is the commutator of the matrices $\hat{\gamma}^{(0)}(n)$ and $\hat{V}(a_s)$.

Now we consider LO, NLO and NNLO approximataions separately.

8.1.1 LO

At LO, the matrix $\hat{V}(a_s) = I$ and the renormalization group exponent have the form

$$\hat{W}(a_s,\overline{a}_s) = \hat{W}^{(0)}(a_s,\overline{a}_s) = \begin{pmatrix} \left(\frac{\overline{a}_s}{a_s}\right)^{d_-(n)} & 0 \\ 0 & \left(\frac{\overline{a}_s}{a_s}\right)^{d_+(n)} \end{pmatrix}, \tag{A10}$$

where

$$d_{\pm}(n) = \frac{\gamma_{\pm}^{(0)}(n)}{2\beta_0} \tag{A11}$$

8.1.2 NLO

At NLO, the matrices $\hat{V}(a_s)$ and $\hat{V}^{-1}(a_s)$ has the form

$$\hat{V}(a_s) = I + a_s\hat{V}^{(1)},\ \ \hat{V}^{-1}(a_s) = I - a_s\hat{V}^{(1)}, \tag{A12}$$

and the Eq. (A9) can be replaced by one

$$2\hat{V}^{(1)}(n) + \left[\frac{\hat{\gamma}^{(0)}(n)}{\beta_0},\hat{V}^{(1)}(n)\right] = -\frac{\hat{\gamma}^{(1)}(n)}{\beta_0} + \frac{\hat{\gamma}^{(0)}(n)\beta_1}{\beta_0^2} \tag{A13}$$

Applying the matrices $\hat{U}^{-1}$ and $\hat{U}$ to left and right sides of above equation, respectively, and using Eq. (A3) for $i=1$ and the representation

$$\hat{U}^{-1}\hat{V}^{(i)}(n)\hat{U} \;=\; \begin{pmatrix} V^{(i)}_{--}(n) & V^{(i)}_{-+}(n) \\ V^{(i)}_{+-}(n) & V^{(i)}_{++}(n) \end{pmatrix}, \tag{A14}$$

for $i=1$, we have the following matrix equation

$$\begin{pmatrix} 2V^{(1)}_{--}(n) & V^{(1)}_{-+}(n)\cdot\frac{2\beta_0+\gamma^{(0)}_{-}(n)-\gamma^{(0)}_{+}(n)}{\beta_0} \\ V^{(1)}_{+-}(n)\cdot\frac{2\beta_0+\gamma^{(0)}_{+}(n)-\gamma^{(0)}_{-}(n)}{\beta_0} & 2V^{(1)}_{++}(n) \end{pmatrix} \;=\; -\frac{1}{\beta_0}\begin{pmatrix} \Gamma^{(1)}_{--}(n) & \Gamma^{(1)}_{-+}(n) \\ \Gamma^{(1)}_{+-}(n) & \Gamma^{(1)}_{++}(n) \end{pmatrix}, \tag{A15}$$

where

$$\Gamma^{(1)}_{\pm\pm}(n) = \gamma^{(1)}_{\pm\pm}(n) - \gamma^{(0)}_{\pm}(n)b_1, \;\; \Gamma^{(1)}_{\pm\mp}(n) = \gamma^{(1)}_{\pm\mp}(n) \tag{A16}$$

The Eq. (A15) leads to the results

$$V^{(1)}_{\pm\pm}(n) \;=\; -\frac{\Gamma^{(1)}_{\pm\pm}(n)}{2\beta_0}, \;\;\; V^{(1)}_{\pm\mp}(n) \;=\; -\frac{\gamma^{(1)}_{\pm\mp}(n)}{2\beta_0+\gamma^{(0)}_{\pm}(n)-\gamma^{(0)}_{\mp}(n)} \tag{A17}$$

8.1.3 NNLO

At NNLO, the matrices $\hat{V}(a_s)$ and $\hat{V}^{-1}(a_s)$ has the form

$$\hat{V}(a_s) = I + a_s\hat{V}^{(1)} + a_s^2\hat{V}^{(2)}, \;\; \hat{V}^{-1}(a_s) = I - a_s\hat{V}^{(1)} - a_s^2\hat{\tilde{V}}^{(2)}, \;\; \hat{\tilde{V}}^{(2)} = \hat{V}^{(2)} - \hat{V}^{(1)}\cdot\hat{V}^{(1)} \tag{A18}$$

and the Eq. (A9) can be replaced by one

$$4\hat{V}^{(2)}(n) + \left[\frac{\hat{\gamma}^{(0)}(n)}{\beta_0}, \hat{V}^{(2)}(n)\right] = -\frac{1}{\beta_0}\left[\hat{\gamma}^{(2)}(n) + \hat{\gamma}^{(1)}(n)\left(\hat{V}^{(1)}(n) - b_1\right) - \hat{\gamma}^{(0)}(n)\left(b_1\hat{V}^{(1)}(n) + \tilde{b}_2\right)\right] \tag{A19}$$

Applying the matrices $\hat{U}^{-1}$ and $\hat{U}$ to left and right sides of above equation, respectively, and using Eqs. (A3) and (A14) for $i=1$ and $i=2$, we have the following matrix equation

$$\begin{pmatrix} 4V^{(2)}_{--}(n) & V^{(2)}_{-+}(n)\cdot\frac{4\beta_0+\gamma^{(0)}_{-}(n)-\gamma^{(0)}_{+}(n)}{\beta_0} \\ V^{(2)}_{+-}(n)\cdot\frac{4\beta_0+\gamma^{(0)}_{+}(n)-\gamma^{(0)}_{-}(n)}{\beta_0} & 4V^{(2)}_{++}(n) \end{pmatrix} \;=\; -\frac{1}{\beta_0}\begin{pmatrix} \Gamma^{(2)}_{--}(n) & \Gamma^{(2)}_{-+}(n) \\ \Gamma^{(1)}_{+-}(n) & \Gamma^{(2)}_{++}(n) \end{pmatrix}, \tag{A20}$$

where

$$\begin{aligned}
\Gamma^{(2)}_{\pm\pm}(n) \;&=\; \gamma^{(2)}_{\pm\pm}(n) + \sum_{i=\pm}\gamma^{(1)}_{\pm i}(n)V^{(1)}_{i\pm}(n) - b_1\left(\gamma^{(1)}_{\pm\pm}(n) + \gamma^{(0)}_{\pm}(n)V^{(1)}_{\pm\pm}(n)\right) - \left(b_2 - b_1^2\right)\gamma^{(0)}_{\pm}(n), \\
\Gamma^{(2)}_{\pm\mp}(n) \;&=\; \gamma^{(2)}_{\pm\mp}(n) + \sum_{i=\pm}\gamma^{(1)}_{\pm i}(n)V^{(1)}_{i\mp}(n) - b_1\left(\gamma^{(1)}_{\pm\mp}(n) + \gamma^{(0)}_{\pm}(n)V^{(1)}_{\pm\mp}(n)\right)
\end{aligned} \tag{A21}$$

The Eq. (A20) leads to the results

$$V^{(2)}_{\pm\pm}(n) \;=\; -\frac{\Gamma^{(2)}_{\pm\pm}(n)}{4\beta_0}, \;\;\; V^{(2)}_{\pm\mp}(n) \;=\; -\frac{\Gamma^{(2)}_{\pm\mp}(n)}{4\beta_0+\gamma^{(0)}_{\pm}(n)-\gamma^{(0)}_{\mp}(n)} \tag{A22}$$

Taking in brascets the relation between $\hat{\tilde{V}}^{(2)}$ and $\hat{V}^{(2)}$ given in the last relation of (A18), we have

$$\tilde{V}^{(2)}_{\pm\pm}(n) \;=\; V^{(2)}_{\pm\pm}(n) - \sum_{i=\pm}V^{(1)}_{\pm i}(n)V^{(1)}_{i\pm}(n), \;\;\; \tilde{V}^{(2)}_{\pm\mp}(n) \;=\; V^{(2)}_{\pm\mp}(n) - \sum_{i=\pm}V^{(1)}_{\pm i}(n)V^{(1)}_{i\mp}(n), \tag{A23}$$

8.2 Q^2 evolution of parton distributions

In the matrix form, the Q^2 evolution of parton distributions

$$\left[\mathbf{f}_q(Q^2),\mathbf{f}_g(Q^2)\right] = \left[\mathbf{f}_q(Q_0^2),\mathbf{f}_g(Q_0^2)\right] \cdot \hat{W}(a_s,\overline{a}_s) \tag{A24}$$

can be represented in the form

$$\left[\mathbf{f}_q(Q^2),\mathbf{f}_g(Q^2)\right] = \left[\mathbf{f}_q(Q_0^2),\mathbf{f}_g(Q_0^2)\right] \hat{U} \cdot \left(\hat{U}^{-1}\hat{W}(a_s,\overline{a}_s)\hat{U}\right) \hat{U}^{-1}. \tag{A25}$$

The first part in the r.h.s. is

$$\left[\mathbf{f}_q(Q_0^2),\mathbf{f}_g(Q_0^2)\right] \hat{U} = \left[\mathbf{f}_q(Q_0^2)\alpha_n + \mathbf{f}_g(Q_0^2)\beta_n, \mathbf{f}_q(Q_0^2)\,(\alpha_n-1) + \mathbf{f}_g(Q_0^2)\beta_n\right] \equiv \left[\mathbf{f}_S^-(Q_0^2), -\mathbf{f}_S^+(Q_0^2)\right], \tag{A26}$$

where (see also (2.17) in the main text)

$$\mathbf{f}_q^-(Q_0^2) \;=\; \mathbf{f}_q(Q_0^2)\alpha_n + \mathbf{f}_g(Q_0^2)\beta_n, \;\; \mathbf{f}_q^+(Q_0^2) \;=\; \mathbf{f}_q(Q_0^2)\,(1-\alpha_n) - \mathbf{f}_g(Q_0^2)\beta_n\,. \tag{A27}$$

8.2.1 LO

At the LO, the renormalization group exponent $\hat{U}^{-1}\hat{W}(a_s,\overline{a}_s)\hat{U}$ has the diagonal form (A10) and, thus, we have

$$\left[\mathbf{f}_q(Q_0^2),\mathbf{f}_g(Q_0^2)\right] \hat{U} \cdot \left(\hat{U}^{-1}\hat{W}(a_s,\overline{a}_s)\hat{U}\right) \;=\; \left[\mathbf{f}_q^-(Q_0^2)\left(\frac{\overline{a}_s}{a_s}\right)^{d_-(n)}, -\mathbf{f}_q^+(Q_0^2)\left(\frac{\overline{a}_s}{a_s}\right)^{d_+(n)}\right], \tag{A28}$$

Then, for the Q^2 evolution of parton distributions we have

$$\begin{aligned}
\left[\mathbf{f}_q(Q^2),\mathbf{f}_g(Q^2)\right] \;&=\; \left[\mathbf{f}_q^-(Q_0^2)\left(\frac{\overline{a}_s}{a_s}\right)^{d_-(n)}, -\mathbf{f}_q^+(Q_0^2)\left(\frac{\overline{a}_s}{a_s}\right)^{d_+(n)}\right] \cdot \begin{pmatrix} 1 & \frac{1-\alpha_n}{\beta_n} \\ -1 & \frac{\alpha_n}{\beta_n} \end{pmatrix} \\
&= \left[\mathbf{f}_q^-(Q_0^2)\left(\frac{\overline{a}_s}{a_s}\right)^{d_-(n)} + \mathbf{f}_q^+(Q_0^2)\left(\frac{\overline{a}_s}{a_s}\right)^{d_+(n)}, \mathbf{f}_q^-(Q_0^2)\frac{1-\alpha_n}{\beta_n}\left(\frac{\overline{a}_s}{a_s}\right)^{d_-(n)} \mathbf{f}_q^+(Q_0^2)\frac{\alpha_n}{\beta_n}\left(\frac{\overline{a}_s}{a_s}\right)^{d_+(n)}\right] \\
&\equiv \left[\sum_{i=\pm}\mathbf{f}_q^i(Q_0^2)\left(\frac{\overline{a}_s}{a_s}\right)^{d_i(n)}, \sum_{i=\pm}\mathbf{f}_q^i(Q_0^2)\left(\frac{\overline{a}_s}{a_s}\right)^{d_i(n)}\right],
\end{aligned} \tag{A29}$$

where (see also (2.17) in the main text)

$$\begin{aligned}
\mathbf{f}_g^-(Q_0^2) \;&=\; \mathbf{f}_q^-(Q_0^2)\,\frac{1-\alpha_n}{\beta_n} \;=\; \mathbf{f}_q(Q_0^2)\varepsilon_n + \mathbf{f}_g(Q_0^2)\,(1-\alpha_n)\,, \\
\mathbf{f}_g^+(Q_0^2) \;&=\; \mathbf{f}_q^+(Q_0^2)\,\frac{\alpha_n}{\beta_n} \;=\; -\,\mathbf{f}_q(Q_0^2)\varepsilon_n + \mathbf{f}_g(Q_0^2)\alpha_n\,,
\end{aligned} \tag{A30}$$

because

$$\varepsilon_n \;=\; \frac{\alpha_n(1-\alpha_n)}{\beta_n}\,. \tag{A31}$$

8.2.2 NLO

At the NLO, the renormgroup exponent $\hat{U}^{-1}\hat{W}(a_s,\overline{a}_s)\hat{U}$ has the form

$$\hat{U}^{-1}\hat{W}(a_s,\overline{a}_s)\hat{U} = \left(I + a_s\hat{V}^{(1)}\right)\cdot \hat{U}^{-1}\hat{W}^{(0)}(a_s,\overline{a}_s)\hat{U}\cdot \left(I - \overline{a}_s\hat{V}^{(1)}\right) \tag{A32}$$

Thus, it is convenient to consider firstly the part

$$\left[\mathbf{f}_q(Q_0^2),\mathbf{f}_g(Q_0^2)\right]\hat{U}\cdot\left(I + a_s\hat{V}^{(1)}\right) \tag{A33}$$

Following to (A26) we can rewrite it as

$$\left[\mathbf{f}_q(Q_0^2),\mathbf{f}_g(Q_0^2)\right]\hat{U}\cdot\left(I + a_s\hat{V}^{(1)}\right) = \left[\mathbf{f}_q^-(Q_0^2),-\mathbf{f}_q^+(Q_0^2)\right]\cdot\left(I + a_s\hat{V}^{(1)}\right) = \left[\tilde{\mathbf{f}}_S^-(Q_0^2),-\tilde{\mathbf{f}}_S^+(Q_0^2)\right], \tag{A34}$$

where (see also (2.19) in the main text)

$$\tilde{\mathbf{f}}_q^{\pm}(n,Q_0^2) \;=\; \mathbf{f}_q^{\pm}(n,Q_0^2)\left(1 + a_sV_{\pm\pm}^{(1)}(n)\right) - \mathbf{f}_q^{\mp}(n,Q_0^2)a_sV_{\mp\pm,S}^{(1)}(n) \tag{A35}$$

and

$$V_{\mp\pm,S}^{(i)} \;=\; V_{\mp\pm}^{(i)},\;\; \tilde{V}_{\mp\pm,S}^{(2)} \;=\; \tilde{V}_{\mp\pm}^{(2)} \tag{A36}$$

We introduce notations $V_{\mp\pm,S}^{(i)}$ and $\tilde{V}_{\mp\pm,S}^{(2)}$ in (A36), because the corresponding ones in the gluon case are different (see Eq.(A45) below).

The $\overline{a}_s$-part of (A32) has the form

$$\hat{U}^{-1}\hat{W}^{(0)}(a_s,\overline{a}_s)\hat{U}\left(I - \overline{a}_s\hat{V}^{(1)}\right) = \begin{pmatrix} \left(\frac{\overline{a}_s}{a_s}\right)^{d_-(n)}\left(1-\overline{a}_sV_{--}^{(1)}\right) & -\overline{a}_s\left(\frac{\overline{a}_s}{a_s}\right)^{d_-(n)}V_{-+}^{(1)} \\ -\overline{a}_s\left(\frac{\overline{a}_s}{a_s}\right)^{d_+(n)}V_{+-}^{(1)} & \left(\frac{\overline{a}_s}{a_s}\right)^{d_+(n)}\left(1-\overline{a}_sV_{++}^{(1)}\right) \end{pmatrix} \tag{A37}$$

Thus, we have

$$\begin{aligned} &\left[\mathbf{f}_q(Q_0^2),\mathbf{f}_g(Q_0^2)\right]\hat{U}\cdot\left(\hat{U}^{-1}\hat{W}(a_s,\overline{a}_s)\hat{U}\right) \\ &= \Bigg[\tilde{\mathbf{f}}_q^-(Q_0^2)\left(\frac{\overline{a}_s}{a_s}\right)^{d_-(n)}\left(1-\overline{a}_sV_{--}^{(1)}\right) + \overline{a}_s\tilde{\mathbf{f}}_q^+(Q_0^2)\left(\frac{\overline{a}_s}{a_s}\right)^{d_+(n)}V_{+-}^{(1)}, \\ &\quad -\left\{\tilde{\mathbf{f}}_q^+(Q_0^2)\left(\frac{\overline{a}_s}{a_s}\right)^{d_+(n)}\left(1-\overline{a}_sV_{++}^{(1)}\right) + \overline{a}_s\tilde{\mathbf{f}}_q^-(Q_0^2)\left(\frac{\overline{a}_s}{a_s}\right)^{d_-(n)}V_{-+}^{(1)}\right\}\Bigg]. \end{aligned} \tag{A38}$$

Then, to obtain the Q^2 evolution of parton distributions $\mathbf{f}_S(Q^2)$ and $\mathbf{f}_G(Q^2)$ we should product the r.h.s. of (A38) on the matrix $\hat{U}^{-1}$. By analogy with the calculations at LO, we have for quark density

$$\begin{aligned} \mathbf{f}_q(Q^2) \;=&\; \tilde{\mathbf{f}}_q^-(Q_0^2)\left(\frac{\overline{a}_s}{a_s}\right)^{d_-(n)}\left(1-\overline{a}_sV_{--}^{(1)}\right) + \overline{a}_s\tilde{\mathbf{f}}_q^+(Q_0^2)\left(\frac{\overline{a}_s}{a_s}\right)^{d_+(n)}V_{+-}^{(1)} \\ &+\tilde{\mathbf{f}}_q^+(Q_0^2)\left(\frac{\overline{a}_s}{a_s}\right)^{d_+(n)}\left(1-\overline{a}_sV_{++}^{(1)}\right) + \overline{a}_s\tilde{\mathbf{f}}_q^-(Q_0^2)\left(\frac{\overline{a}_s}{a_s}\right)^{d_-(n)}V_{-+}^{(1)}. \end{aligned} \tag{A39}$$

Taking together terms in the front of $(\overline{a}_s/a_s)^{d_\pm}$, we have

$$\mathbf{f}_q(Q^2) = \tilde{\mathbf{f}}_q^-(Q_0^2)\left(\frac{\overline{a}_s}{a_s}\right)^{d_-(n)} \left(1-\overline{a}_s\left[V_{--}^{(1)}-V_{-+}^{(1)}\right]\right) + \tilde{\mathbf{f}}_q^+(Q_0^2)\left(\frac{\overline{a}_s}{a_s}\right)^{d_+(n)} \left(1-\overline{a}_s\left[V_{++}^{(1)}-V_{+-}^{(1)}\right]\right), \tag{A40}$$

or in more compact form

$$\mathbf{f}_q(n,Q^2) = \sum_{i=\pm}\mathbf{f}_q^i(n,Q^2), \;\; \mathbf{f}_q^i(n,Q^2) = \tilde{\mathbf{f}}_q^i(n,Q_0^2)\left(\frac{\overline{a}_s}{a_s}\right)^{d_i(n)} H_q^i(n,Q^2),$$
$$H_q^\pm(n,Q^2) = 1-\overline{a}_s\left[V_{\pm\pm}^{(1)}(n)-V_{\pm\mp,q}^{(1)}(n)\right],. \tag{A41}$$

For gluon density we have

$$\mathbf{f}_g(Q^2) = \tilde{\mathbf{f}}_q^-(Q_0^2)\,\frac{1-\alpha_n}{\beta_n}\left(\frac{\overline{a}_s}{a_s}\right)^{d_-(n)}\left(1-\overline{a}_s V_{--}^{(1)}\right)+\overline{a}_s\tilde{\mathbf{f}}_q^+(Q_0^2)\,\frac{1-\alpha_n}{\beta_n}\left(\frac{\overline{a}_s}{a_s}\right)^{d_+(n)} V_{+-}^{(1)}$$
$$-\tilde{\mathbf{f}}_q^+(Q_0^2)\,\frac{\alpha_n}{\beta_n}\left(\frac{\overline{a}_s}{a_s}\right)^{d_+(n)}\left(1-\overline{a}_s V_{++}^{(1)}\right)-\overline{a}_s\tilde{\mathbf{f}}_q^-(Q_0^2)\,\frac{\alpha_n}{\beta_n}\left(\frac{\overline{a}_s}{a_s}\right)^{d_-(n)} V_{-+}^{(1)}. \tag{A42}$$

Taking together terms in the front of $(\overline{a}_s/a_s)^{d_\pm}$, we have

$$\mathbf{f}_g(Q^2) = \tilde{\mathbf{f}}_q^-(Q_0^2)\,\frac{1-\alpha_n}{\beta_n}\left(\frac{\overline{a}_s}{a_s}\right)^{d_-(n)}\left(1-\overline{a}_s\left[V_{--}^{(1)}-\frac{\alpha_n}{\alpha_n-1}V_{-+}^{(1)}\right]\right)$$
$$-\tilde{\mathbf{f}}_q^+(Q_0^2)\,\frac{\alpha_n}{\beta_n}\left(\frac{\overline{a}_s}{a_s}\right)^{d_+(n)}\left(1-\overline{a}_s\left[V_{++}^{(1)}-\frac{\alpha_n-1}{\alpha_n}V_{+-}^{(1)}\right]\right), \tag{A43}$$

or in more compact form

$$\mathbf{f}_g(n,Q^2) = \sum_{i=\pm}\mathbf{f}_g^i(n,Q^2), \;\; \mathbf{f}_g^i(n,Q^2) = \tilde{\mathbf{f}}_g^i(n,Q_0^2)\left(\frac{\overline{a}_s}{a_s}\right)^{d_i(n)} H_g^i(n,Q^2),$$
$$H_g^\pm(n,Q^2) = 1-\overline{a}_s\left[V_{\pm\pm}^{(1)}(n)-V_{\pm\mp,g}^{(1)}\right](n), \tag{A44}$$

where

$$V_{-+,g}^{(1)}(n) = V_{-+}^{(1)}(n)\frac{\alpha_n}{\alpha_n-1},\; V_{+-,g}^{(1)}(n) = V_{+-}^{(1)}(n)\frac{\alpha_n-1}{\alpha_n} \tag{A45}$$

and (see also (2.19) in the main text)

$$\tilde{\mathbf{f}}_g^-(n,Q_0^2) = \tilde{\mathbf{f}}_q^-(n,Q_0^2)\,\frac{1-\alpha_n}{\beta_n} = \mathbf{f}_q^-(n,Q_0^2)\,\frac{1-\alpha_n}{\beta_n}\left(1+a_sV_{--}^{(1)}(n)\right)-\mathbf{f}_q^+(n,Q_0^2)a_sV_{+-}^{(1)}(n)\,\frac{1-\alpha_n}{\beta_n}$$
$$= \mathbf{f}_g^-(n,Q_0^2)\left(1+a_sV_{--}^{(1)}(n)\right)-\mathbf{f}_g^+(n,Q_0^2)a_sV_{+-}^{(1)}(n)\,\frac{\alpha_n-1}{\alpha_n}$$
$$= \mathbf{f}_g^-(n,Q_0^2)\left(1+a_sV_{--}^{(1)}(n)\right)-\mathbf{f}_g^+(Q_0^2)a_sV_{+-,g}^{(1)}(n),$$
$$\tilde{\mathbf{f}}_g^+(n,Q_0^2) = -\tilde{\mathbf{f}}_q^+(n,Q_0^2)\,\frac{\alpha_n}{\beta_n} = -\mathbf{f}_q^+(n,Q_0^2)\,\frac{\alpha_n}{\beta_n}\left(1+a_sV_{++}^{(1)}9n0\right)+\mathbf{f}_q^-(n,Q_0^2)a_sV_{-+}^{(1)}(n)\,\frac{\alpha_n}{\beta_n}$$
$$= \mathbf{f}_g^+(n,Q_0^2)\left(1+a_sV_{++}^{(1)}(n)\right)-\mathbf{f}_g^-(n,Q_0^2)a_sV_{-+}^{(1)}(n)\,\frac{\alpha_n}{\alpha_n-1}$$
$$= \mathbf{f}_g^+(n,Q_0^2)\left(1+a_sV_{++}^{(1)}(n)\right)-\mathbf{f}_g^-(n,Q_0^2)a_sV_{-+,g}^{(1)}(n), \tag{A46}$$

or by analogy with (A35), in the general form,

$$\tilde{\mathbf{f}}_g^\pm(n,Q_0^2) = \mathbf{f}_g^\pm(n,Q_0^2)\left(1+a_sV_{\pm\pm}^{(1)}(n)\right)-\mathbf{f}_g^\mp(n,Q_0^2)a_sV_{\mp\pm,G}^{(1)}(n), \tag{A47}$$

8.2.3 NNLO

At the NNLO, we can perform an analysis, which is very similar to the one in the previous subsection for the NLO approfimation. The one difference is the terms $\sim a_s^2$ and $\sim \overline{a}_s^2$ for the matrices $\hat{V}$ and $\hat{V}^{-1}$ (see Eq. (A18)).

So, the final resuls have the form $(a = q,g)$

$$\begin{aligned}
\mathbf{f}_a(n,Q^2) &= \sum_{i=\pm} \mathbf{f}_a^i(n,Q^2), \;\; \mathbf{f}_a^i(n,Q^2) \;=\; \tilde{\mathbf{f}}_a^i(n,Q_0^2)\left(\frac{\overline{a}_s}{a_s}\right)^{d_i(n)} H_a^i(n,Q^2), \qquad \text{(A48)}\\
H_a^{\pm}(n,Q^2) &= 1 - \overline{a}_s\left[V_{\pm\pm}^{(1)}(n) - V_{\pm\mp,a}^{(1)}(n)\right] - \overline{a}_s^2\left[\tilde{V}_{\pm\pm}^{(2)}(n) - \tilde{V}_{\pm\mp,a}^{(2)}(n)\right],\\
\tilde{\mathbf{f}}_a^{\pm}(n,Q_0^2) &= \mathbf{f}_a^{\pm}(n,Q_0^2)\left(1 + a_s V_{\pm\pm}^{(1)}(n) + a_s^2 V_{\pm\pm}^{(2)}(n)\right) - \mathbf{f}_a^{\mp}(n,Q_0^2)\left(a_s V_{\mp\pm,a}^{(1)}(n) + a_s^2 V_{\mp\pm,a}^{(2)}(n)\right).
\end{aligned}$$

8.3 Q^2-dependence of Mellin moments

The Q^2-dependence of the singlet part $M_n^S(Q^2)$ of the Mellin moments can be obtained using the PDF Q^2-dependence (see the previous subsection of the Appendix) and the relation between the parton densities and the (singlet part of) the Mellin moments given by Eqs. (2.9) and (2.10).

Sometimes, it is convenient to obtain directly the Q^2-dependence of the Mellin moments $M_n^S(Q^2)$. In the matrix form, it has the form (A25)

$$M_n^S(Q^2) = \left[\mathbf{f}_q(Q^2), \mathbf{f}_g(Q^2)\right]\begin{pmatrix} C_q(n,\overline{a}_s) \\ C_g(n,\overline{a}_s)\end{pmatrix} = \left[\mathbf{f}_q(Q_0^2), \mathbf{f}_g(Q_0^2)\right]\hat{U}\cdot\left(\hat{U}^{-1}\hat{W}(a_s,\overline{a}_s)\hat{U}\right)\hat{U}^{-1}\begin{pmatrix} C_q(n,\overline{a}_s) \\ C_g(n,\overline{a}_s)\end{pmatrix}, \qquad \text{(A49)}$$

where

$$\hat{U}^{-1}\begin{pmatrix} C_q(n,\overline{a}_s) \\ C_g(n,\overline{a}_s)\end{pmatrix} = \begin{pmatrix} C_-(n,\overline{a}_s) \\ -C_+(n,\overline{a}_s)\end{pmatrix} \qquad \text{(A50)}$$

and

$$C_{\pm}(n,\overline{a}_s) = 1 + \overline{a}_s B_{\pm}^{(1)}(n) + \overline{a}_s^2 B_{\pm}^{(2)}(n) \qquad \text{(A51)}$$

with

$$\begin{aligned}
B_+^{(i)}(n) &= B_q^{(i)}(n) - \frac{\alpha_n}{\beta_n} B_g^{(i)}(n), \quad (i = 1,2)\\
B_-^{(i)}(n) &= B_q^{(i)}(n) + \frac{1-\alpha_n}{\beta_n} B_g^{(i)}(n) \;=\; B_+^{(i)}(n) + \frac{1}{\beta_n} B_g^{(i)}(n); \qquad \text{(A52)}
\end{aligned}$$

The basic idea is to split the Q_0^2-dependence to the initial conditions $\mathbf{f}_q^{\pm}(Q_0^2)$ (A27) and above LO to $\tilde{\mathbf{f}}_q^{\pm}(n,Q_0^2)$ (A35) and (A48). The Q^2-dependence combines the PDF one from the previous subsection and the one in (A51) and (A52). As it was above, we will consider LO, NLO and NNLO cases separately.

Mote here that the Mellin moments $M_n^S(Q_0^2)$ can be easy extracted from above equation (A49)

$$\begin{aligned}
M_n^S(Q_0^2) &= \left[\mathbf{f}_q(Q_0^2), \mathbf{f}_g(Q_0^2)\right]\begin{pmatrix} C_q(n,a_s) \\ C_g(n,a_s)\end{pmatrix} \;=\; \left[\mathbf{f}_q(Q_0^2), \mathbf{f}_g(Q_0^2)\right]\hat{U}\cdot\hat{U}^{-1}\begin{pmatrix} C_q(n,a_s) \\ C_g(n,a_s)\end{pmatrix}\\
&= \left[\mathbf{f}_q^-(Q_0^2), -\mathbf{f}_q^+(Q_0^2)\right]\begin{pmatrix} C_-(n,a_s) \\ -C_+(n,a_s)\end{pmatrix} \;=\; \sum_{i=\pm} \mathbf{f}_q^i(Q_0^2) C_i(n,a_s), \qquad \text{(A53)}
\end{aligned}$$

where $C_i(n,a_s)$ are given by (A51).

8.3.1 LO

Here

$$M_n^S(Q^2) \;=\; \left[\mathbf{f}_q^-(Q^2),\mathbf{f}_q^+(Q^2)\right] \begin{pmatrix} \left(\frac{\overline{a}_s}{a_s}\right)^{d_-(n)} & 0 \\ 0 & \left(\frac{\overline{a}_s}{a_s}\right)^{d_+(n)} \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \;=\; \sum_{i=\pm} M_n^{(S,i)}(Q^2)\,, \tag{A54}$$

where

$$M_n^{(S,\pm)}(Q^2) \;=\; \mathbf{f}_q^i(Q_0^2)\left(\frac{\overline{a}_s}{a_s}\right)^{d_\pm(n)} \tag{A55}$$

8.3.2 NLO

Here we have Eq. (A32)

$$\hat{U}^{-1}\hat{W}(a_s,\overline{a}_s)\hat{U} = \left(I + a_s\hat{V}^{(1)}\right)\cdot \hat{U}^{-1}\hat{W}^{(0)}(a_s,\overline{a}_s)\hat{U}\cdot\left(I - \overline{a}_s\hat{V}^{(1)}\right)\,, \tag{A56}$$

which leads to Q_0^2-part (A34))

$$\left[\mathbf{f}_q(Q_0^2),\mathbf{f}_g(Q_0^2)\right]\hat{U}\cdot\left(I + a_s\hat{V}^{(1)}\right) \;=\; \left[\tilde{\mathbf{f}}_q^-(Q_0^2),-\tilde{\mathbf{f}}_q^+(Q_0^2)\right]\,, \tag{A57}$$

with $\tilde{\mathbf{f}}_q^\pm(n,Q_0^2)$ given by (A35).

The Q^2-dependent part consists from

$$\hat{U}^{-1}\hat{W}^{(0)}(a_s,\overline{a}_s)\hat{U}\left(I - \overline{a}_s\hat{V}^{(1)}\right) \tag{A58}$$

given by the r.h.s. of Eq. (A37) and $\hat{U}^{-1}\hat{C}(n,\overline{a}_s)$ give by the Eqs. (A50) with the NLO coefficints (A51).

So, we have

$$\begin{aligned} &\hat{U}^{-1}\hat{W}^{(0)}(a_s,\overline{a}_s)\hat{U}\left(I - \overline{a}_s\hat{V}^{(1)}\right)\hat{U}^{-1}\hat{C}(n,\overline{a}_s) \\ &= \begin{pmatrix} \left(\frac{\overline{a}_s}{a_s}\right)^{d_-(n)}\left(1-\overline{a}_sV_{--}^{(1)}\right) & -\overline{a}_s\left(\frac{\overline{a}_s}{a_s}\right)^{d_-(n)}V_{-+}^{(1)} \\ -\overline{a}_s\left(\frac{\overline{a}_s}{a_s}\right)^{d_+(n)}V_{+-}^{(1)} & \left(\frac{\overline{a}_s}{a_s}\right)^{d_+(n)}\left(1-\overline{a}_sV_{++}^{(1)}\right) \end{pmatrix} \begin{pmatrix} 1+\overline{a}_sB_-^{(1)} \\ -\left(1+\overline{a}_sB_+^{(1)}\right) \end{pmatrix} \\ &= \begin{pmatrix} \left(\frac{\overline{a}_s}{a_s}\right)^{d_-(n)}\left(1+\overline{a}_s\left[B_-^{(1)}-V_{--}^{(1)}+V_{-+}^{(1)}\right]\right) \\ -\left(\frac{\overline{a}_s}{a_s}\right)^{d_+(n)}\left(1+\overline{a}_s\left[B_+^{(1)}-V_{++}^{(1)}+V_{+-}^{(1)}\right]\right) \end{pmatrix} \end{aligned} \tag{A59}$$

For the Mellin moments $M_n^S(Q^2)$ we have product Q^2 and Q_0^2 parts:

$$M_n^S(Q^2) \;=\; \sum_{i=\pm} M_n^{(S,i)}(Q^2)\,, \tag{A60}$$

where

$$M_n^{(S,\pm)}(Q^2) \;=\; \tilde{\mathbf{f}}_q^i(Q_0^2)\left(\frac{\overline{a}_s}{a_s}\right)^{d_\pm(n)}\left(1+\overline{a}_sR_\pm^{(1)}\right) \tag{A61}$$

and

$$R^{(1)}_{\pm} \;=\; B^{(1)}_{\pm} - V^{(1)}_{\pm\pm} + V^{(1)}_{\pm\mp} \tag{A62}$$

Note that the eqs (A61)–(A62) can be rewriten as

$$M^{(S,\pm)}_n(Q^2) \;=\; M^{(S,\pm)}_n(Q^2_0)\left(\frac{\overline{a}_s}{a_s}\right)^{d_\pm(n)} \frac{\left(1+\overline{a}_s R^{(1)}_{\pm}\right)}{\left(1+a_s R^{(1)}_{\pm}\right)}, \tag{A63}$$

where

$$\begin{aligned} M^{(S,\pm)}_n(Q^2_0) \;&=\; \tilde{\mathbf{f}}^{\pm}_q(Q^2_0)\left(1+a_s\left[B^{(1)}_{\pm} - V^{(1)}_{\pm\pm} + V^{(1)}_{\pm\mp}\right]\right) \\ &=\; \mathbf{f}^{\pm}_q(Q^2_0)\left(1+a_s\left[B^{(1)}_{\pm} + V^{(1)}_{\pm\mp}\right]\right) + \mathbf{f}^{\mp}_S(Q^2_0)a_s V^{(1)}_{\mp\pm} \end{aligned} \tag{A64}$$

8.3.3 NNLO

Repeating the calculations in the previous case, we have

$$M^S_n(Q^2) \;=\; \sum_{i=\pm} M^{(S,i)}_n(Q^2), \tag{A65}$$

where

$$M^{(S,\pm)}_n(Q^2) \;=\; \tilde{\mathbf{f}}^{i}_q(Q^2_0)\left(\frac{\overline{a}_s}{a_s}\right)^{d_\pm(n)} \left(1+\overline{a}_s R^{(1)}_{\pm} + \overline{a}^2_s R^{(2)}_{\pm}\right), \tag{A66}$$

where $\tilde{\mathbf{f}}^{i}_q(Q^2_0)$ and $R^{(1)}_{\pm}$ are given by (A48) and (A62), respectively, and

$$R^{(2)}_{\pm} \;=\; B^{(2)}_{\pm} - B^{(1)}_{\pm}\left(V^{(1)}_{\pm\pm} - V^{(1)}_{\pm\mp}\right) - \tilde{V}^{(2)}_{\pm\pm} + \tilde{V}^{(2)}_{\pm\mp} \tag{A67}$$

with $\tilde{V}^{(2)}_{\pm\pm}$ and $\tilde{V}^{(2)}_{\pm\mp}$ given by Eqs. (A23).

The eqs (A66)–(A67) can be rewriten as

$$M^{(S,\pm)}_n(Q^2) \;=\; M^{(S,\pm)}_n(Q^2_0)\left(\frac{\overline{a}_s}{a_s}\right)^{d_\pm(n)} \frac{\left(1+\overline{a}_s R^{(1)}_{\pm} + \overline{a}^2_s R^{(2)}_{\pm}\right)}{\left(1+a_s R^{(1)}_{\pm} + a^2_s R^{(2)}_{\pm}\right)}, \tag{A68}$$

where

$$\begin{aligned} M^{(S,\pm)}_n(Q^2_0) &= \tilde{\mathbf{f}}^{\pm}_q(Q^2_0)\left(1+a_s R^{(1)}_{\pm} + a^2_s R^{(2)}_{\pm}\right) \\ &= \mathbf{f}^{\pm}_q(Q^2_0)\left(1+a_s\left[B^{(1)}_{\pm} + V^{(1)}_{\pm\mp}\right] + a^2_s\left[B^{(2)}_{\pm} + B^{(1)}_{\pm}V^{(1)}_{\pm\mp} + \tilde{V}^{(2)}_{\pm\mp}\right]\right) + \mathbf{f}^{\mp}_S(Q^2_0)\left[a_s V^{(1)}_{\mp\pm} + a^2_s \tilde{V}^{(2)}_{\pm\mp}\right] \end{aligned} \tag{A69}$$