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$E_{6(6)}$ Exceptional Field Theory: Review and Embedding of Type IIB

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We review $E_{6(6)}$ exceptional field theory with a particular emphasis on the embedding of type IIB supergravity, which is obtained by picking the $\mathrm{GL}(5) \times \mathrm{SL}(2)$ invariant solution of the section constraint. We work out the precise decomposition of the $E_{6(6)}$ covariant fields on the one hand and the Kaluza-Klein-like decomposition of type IIB supergravity on the other. Matching the symmetries, this allows to establish the precise dictionary between both sets of fields. Finally, we establish on-shell equivalence. In particular, we show how the self-duality constraint for the four-form potential in type IIB is reconstructed from the duality relations in the off-shell formulation of the $E_{6(6)}$ exceptional field theory.

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1. Introduction

One of the most intriguing aspects of maximal supergravity is the emergence of exceptional symmetry groups upon compactification on tori [1]. For instance, compactifying 11-dimensional or type II supergravity to $D = 5$ one obtains a rigid (continuous) $\mathrm{E}_{6(6)}$ symmetry [2]. Although these symmetries are understood as the supergravity manifestations of the (discrete) U-dualities of string-/M-theory [3], from the point of view of conventional Riemannian geometry they are deeply mysterious. In fact, except for certain 'geometric subgroups', the exceptional groups cannot be understood from the symmetries present in the conventional formulation of supergravity, although there is a reformulation of $D = 11$ supergravity due to de Wit and Nicolai in which the compact subgroup $\mathrm{SU}(8) \subset \mathrm{E}_{7(7)}$ is manifest [4]. Over the decades this has led to various proposals of how to extend or embed the higher-dimensional theories in a way that explains the emergence of exceptional symmetries [5–11], but the complete formulation of such a theory, in the following called 'exceptional field theory', was only found quite recently [12–15], using insights from 'double field theory' [16–21], subsequent generalizations to U-duality groups [22–25], and extended geometry [26–29], an extension of the 'generalized geometry' of [30, 31] to the case of exceptional duality groups. Here we will review the $\mathrm{E}_{6(6)}$ exceptional field theory with a particular emphasis on the explicit embedding of type IIB supergravity. The exceptional field theories in higher dimensions have been constructed in [32–34] and the supersymmetric completions have been given in [35, 36].

The formulation of exceptional field theory (EFT) is based on an extended spacetime that 'geometrizes' the exceptional U-duality group. Specifically, in the $\mathrm{E}_{6(6)}$ EFT all fields depend on $5+27$ coordinates (x^μ, Y^M), where $\mu, \nu = 0, \ldots, 4$, while lower and upper indices $M, N = 1, \ldots, 27$ label the (inequivalent) fundamental representations $\mathbf{27}$ and $\bar{\mathbf{27}}$ of $\mathrm{E}_{6(6)}$, respectively. All functions on this extended space are subject to a covariant 'section constraint' or 'strong constraint' that implies that locally the fields only live on a 'physical slice' of the extended space. In the present case this constraint can be written in terms of the invariant symmetric d-symbol d^{MNK} that $\mathrm{E}_{6(6)}$ admits as

$$d^{MNK}\partial_N\partial_K A \;=\; 0\;, \qquad d^{MNK}\partial_N A\,\partial_K B \;=\; 0\;, \tag{1.1}$$

for arbitrary functions A, B on the extended space. In particular, this constraint holds for all fields and gauge parameters. It was shown in [12] that this constraint allows for (at least) two inequivalent solutions, in analogy to the type II double field theory [37, 38]. First, breaking $\mathrm{E}_{6(6)}$ to $\mathrm{GL}(6)$ the constraint is solved by fields depending on 6 internal coordinates, and we recover the spacetime of 11-dimensional supergravity. Second, breaking $\mathrm{E}_{6(6)}$ to $\mathrm{GL}(5) \times \mathrm{SL}(2)$ the constraint is solved by fields depending on 5 internal coordinates, and we recover the spacetime of type IIB supergravity. Indeed, upon picking one of these solutions one obtains a theory with the field content and symmetries of $D = 11$ or type IIB supergravity, respectively, but in a non-standard formulation. These formulations are obtained from the standard ones by splitting the coordinates and tensor fields a la Kaluza-Klein, however, without truncating the coordinate dependence, as pioneered by de Wit and Nicolai [4]. The full embedding of $D = 11$ supergravity into EFT has been given in detail in [12]. In this article we provide all the details for the embedding of the type IIB theory.

In order to illustrate this formulation, an instructive analogy is the ADM formulation of, say, four-dimensional gravity, in which one singles out a 'time direction', i.e., performs a $1+3$

split, and realizes spacetime as a one-dimensional foliation of three-geometries. One can similarly view the generalized spacetime of the $E_{6(6)}$ EFT as a five-dimensional foliation of a (generalized and extended) 27-dimensional geometry. However, an important difference is that the total 32-dimensional space cannot be viewed as a conventional manifold, because the gauge symmetries of EFT are governed by *generalized* external and internal diffeomorphisms satisfying an algebra that differs from the standard diffeomorphism algebra. Although the total space does not have a conventional geometrical interpretation, for the physical slices corresponding to the $D = 11$ or type IIB solutions of the section constraint, describing *inequivalent* subspaces of the extended space, the generalized diffeomorphisms of EFT reduce to conventional 10 or 11-dimensional diffeomorphisms plus tensor gauge transformations, thereby reconstructing the physical spacetimes in terms of five-dimensional foliations.

Concretely, the $E_{6(6)}$ EFT has the following field content, with all fields depending on the $5+27$ coordinates (x^μ, Y^M),

$$g_{\mu\nu}\,, \quad \mathcal{M}_{MN}\,, \quad \mathcal{A}_\mu{}^M\,, \quad \mathcal{B}_{\mu\nu M}\,. \tag{1.2}$$

Here $g_{\mu\nu}$ is the external, five-dimensional metric, $\mathcal{M}_{MN}$ is the *generalized* internal metric, while the tensor fields $\mathcal{A}_\mu{}^M$ and $\mathcal{B}_{\mu\nu M}$ describe off-diagonal field components that encode, in particular, the interconnection between the external and internal generalized geometries. Upon breaking the $E_{6(6)}$ covariance by solving the section constraint, imposing that all fields depend only on a particular subset of the internal coordinates Y^M, one can decompose the above fields in terms of their components. Modulo field redefinitions, these can then be interpreted as tensor fields with conventional gauge transformations. In this regime, and *truncated* to the purely 'internal' fields encoded in $\mathcal{M}_{MN}$, this formulation can be thought of as implementing what is sometimes referred to as extended or exceptional generalized geometry, which formally combines conventional tensors of different types into larger objects viewed as sections of extended tangent bundles [26,27]. For each solution of the section constraint we may thus reinterpret EFT as realizing a generalized geometry (enlarged, however, by including all 'external' and 'off-diagonal' fields in (1.2) and dependence on external coordinates x^μ), without additional unphysical coordinates. Why, then, do we insist on introducing seemingly unphysical coordinates, together with a constraint that eliminates most of them, as opposed to simply picking a solution from the start? Let us summarize several reasons why it is beneficial to work on such an extended space.

- The theory is manifestly $E_{d(d)}$ covariant provided it is written with the extended derivatives ∂_M properly transforming in the fundamental representation. For instance, the fields couple to the derivatives as in $\mathcal{A}_\mu{}^M\partial_M$. Thus, only this framework makes manifest the emergence of the $E_{d(d)}$ symmetry upon toroidal reduction by simply setting $\partial_M = 0$.
- By defining EFT on the extended space we simultaneously cover $D = 11$ supergravity and type IIB supergravity (and all of their Kaluza-Klein descendants). These are obtained by putting different solutions of the section constraint, which then determines, for instance, which field components in $\mathcal{A}_\mu{}^M\partial_M$ survive for which set of coordinates. In this way it is possible to describe in one single framework $D = 11$ and type IIB supergravity, which are inequivalent theories and so would correspond to two different generalized geometries.

- Although the coordinates beyond those of supergravity are unphysical, at least in the currently understood formulation due to the strong form of the section constraint, in the full string theory they are actually physical and real. More precisely, at least for the T-duality subgroup $\mathrm{O}(d-1,d-1) \subset \mathrm{E}_{d(d)}$ we known from closed string field theory on toroidal backgrounds that the string field depends on momentum and winding coordinates, subject to the level-matching constraint that allows for a simultaneous dependence on all coordinates. It is thus unavoidable that eventually we come to terms with such extended spaces, and so it appears highly significant that much of this extended geometry is already visible at the level of the presently known EFT that essentially encodes supergravity.

Other than of conceptional interest, the manifestly covariant formulation of EFT has proven a rather powerful tool in order to describe consistent truncations of the standard supergravities, in particular for sphere and hyperboloid compactifications in terms of generalized Scherk-Schwarz reductions [39], see [25, 40–48] for earlier related work. This is remarkable, for although in these backgrounds there is no longer a physical $\mathrm{E}_{d(d)}$ symmetry, the corresponding compactifications can be encoded very efficiently in terms of $\mathrm{E}_{d(d)}$-valued twist matrices. The twist matrices take a universal form that is applicable to both solutions of the section constraint, so that, for instance, one covers in one stroke the sphere compactifications of $D = 11$ supergravity, such as $\mathrm{AdS}_4 \times \mathrm{S}^7$ [49] and $\mathrm{AdS}_7 \times \mathrm{S}^4$ [50], the $\mathrm{AdS}_5 \times \mathrm{S}^5$ compactification of type IIB, together with all their non-compact cousins, predicted in [51]. In terms of the conventional formulation, this consistency requires a number of seemingly miraculous identities, suggesting the presence of an underlying larger structure — the extended geometry of EFT. Combining the expressions for the $\mathrm{E}_{6(6)}$-valued twist matrices together with the explicit dictionary of the type IIB embedding into $\mathrm{E}_{6(6)}$ EFT that we provide in this paper, allows to straightforwardly derive the non-linear reduction formulas for the full set of IIB fields on the sphere S^5 and hyperboloid $H^{p,q}$ backgrounds. We give that result in [52].

This review article is organizes as follows. In sec. 2 we briefly review the manifestly $\mathrm{E}_{6(6)}$ covariant formulation, introducing generalized diffeomorphisms and the tensor hierarchy governing one- and two-forms. This construction is completely rigid in that the theory is uniquely determined by invariance under the bosonic gauge symmetries, i.e., internal and external generalized diffeomorphisms. In particular, nowhere is it necessary to refer to 11-dimensional or type IIB supergravity. The latter only emerge upon choosing a solution of the section constraint. In [13] it was shown in detail how $D = 11$ supergravity, in a $5+6$ split of coordinates and tensor fields, is embedded in the $\mathrm{E}_{6(6)}$ EFT. For the IIB theory, one can argue on general grounds that its embedding into EFT is guaranteed by the match of symmetries. In fact, it is easy to see that EFT yields the same field content as type IIB in the $5+5$ splitting, and we will show explicitly in sec. 3 that the EFT gauge algebra contains the full 10-dimensional diffeomorphism algebra. Together with the fact that both theories can be supersymmetrized and reduce to the same 5-dimensional theory, it follows that EFT reduces to type IIB for the appropriate solution of the section constraint. To be very explicit, in this article we work out the precise embedding formulas for IIB into EFT. To this end, we perform the Kaluza-Klein decomposition of type IIB without truncation in sec. 4 and then establish the full dictionary with EFT in sec. 5. In particular, we will show how the duality constraints in EFT allow to reconstruct the 3- and 4-forms that are not among the fundamental fields of EFT but of course are present in type IIB. Finally, we review the generalized Scherk-Schwarz compactifications in sec. 6, which reduces the consistent embedding of five-dimensional gauged supergravities into EFT to a set of consistency equations for the $E_{6(6)}$-valued twist matrices that capture the dependence on the internal coordinates. By means of the explicit dictionary between EFT and IIB and $D = 11$ supergravity, respectively, this gives rise to the full reduction ansaetze for the consistent embedding into standard higher-dimensional supergravity.

Summary of conventions and notation

The EFT fields are denoted by calligraphic letters, as in (1.2). We keep the same letters for these fields after decomposing the $E_{6(6)}$ indices down to $\mathrm{GL}(5)\times\mathrm{SL}(2)$ in accordance with the IIB solution of the section constraint (1.1). The two-forms require further redefinition which will be denoted by $\tilde{\mathscr{B}}_{\mu\nu}$.

The original type IIB fields and space-time indices in $D = 10$ on the other hand are indicated by hats, and the forms are called C:

$$\hat{G}_{\hat{\mu}\hat{\nu}}\ , \qquad \hat{C}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}\ , \quad \text{etc.} \tag{1.3}$$

Upon Kaluza-Klein decomposition of the IIB fields, the new variables obtained by a standard procedure of flattening and unflattening of indices are denoted by a bar. The presence of Chern-Simons terms in the IIB field strengths requires yet another redefinition to bring the gauge structure into canonical form, which we denote without any hat. Thus we have the series of field redefinitions

$$\hat{C} \rightarrow \overline{C} \rightarrow C\,. \tag{1.4}$$

These fields will eventually be identified with the various components of the EFT fields.

In section 6, we describe Scherk-Schwarz reduction of EFT, parametrizing all EFT fields in terms of Y-dependent $E_{6(6)}$-valued twist matrices and the corresponding x-dependent fields of five-dimensional supergravity, which we denote by straight letters:

$$\begin{aligned} g_{\mu\nu}(x,Y) &\rightarrow \mathbf{g}_{\mu\nu}(x)\,, \qquad & \mathscr{M}_{MN}(x,Y) &\rightarrow M_{MN}(x)\,, \\ \mathscr{A}_\mu{}^M(x,Y) &\rightarrow A_\mu{}^M(x)\,, \qquad & \mathscr{B}_{\mu\nu M}(x,Y) &\rightarrow B_{\mu\nu M}(x)\,. \end{aligned} \tag{1.5}$$

2. Review of $E_{6(6)}$ Exceptional Field Theory

Here we present a brief review of the $E_{6(6)}$ EFT, starting with the generalized Lie derivatives and their gauge algebra (the ‘E-bracket’), which govern the internal (generalized) diffeomorphisms. We then introduce the tensor hierarchy and define the full gauge transformations, including generalized external diffeomorphisms, in order to construct the complete theory.

2.1 Generalized diffeomorphisms and tensor hierarchy

We start by collecting the relevant facts about $E_{6(6)}$. Its dimension is 78 and we denote the generators by t_α, with Cartan-Killing form $\kappa_{\alpha\beta}$. As recalled in the introduction, $E_{6(6)}$ admits two inequivalent fundamental representations of dimension 27, denoted by $\mathbf{27}$ and $\bar{\mathbf{27}}$ and labelled by indices $M,N = 1,\ldots,27$. In these fundamental representations, there are two cubic $E_{6(6)}$-invariant tensors, the fully symmetric d-symbols d^{MNK} and d_{MNK}, which we normalize as $d_{MPQ}d^{NPQ} = \delta_M{}^N$. The d-symbols define the manifestly $\mathrm{E}_{6(6)}$ covariant section constraint [28]

$$d^{MNK}\,\partial_N\partial_K A \;=\; 0\,, \quad d^{MNK}\,\partial_N A\,\partial_K B \;=\; 0\,, \tag{2.1}$$

and also satisfy the following cubic identities

$$\begin{aligned} d_{S(MN}\,d_{PQ)T}\,d^{STR} &= \frac{2}{15}\delta_{(M}{}^R\,d_{NPQ)}\,, \\ d_{STR}\,d^{S(MN}\,d^{PQ)T} &= \frac{2}{15}\delta_R{}^{(M}\,d^{NPQ)}\,. \end{aligned} \tag{2.2}$$

In order to define the generalized Lie derivatives below we need the projector onto the adjoint representation in the tensor product $\mathbf{27}\otimes\bar{\mathbf{27}} = \mathbf{78}+\cdots$, which reads

$$\mathbb{P}^M{}_N{}^K{}_L \equiv (t_\alpha)_N{}^M(t^\alpha)_L{}^K \;=\; \frac{1}{18}\,\delta_N{}^M\delta_L{}^K + \frac{1}{6}\,\delta_N{}^K\delta_L{}^M - \frac{5}{3}\,d_{NLR}d^{MKR}\,. \tag{2.3}$$

With respect to a vector like parameter Λ^M one would naively define the Lie derivative as in standard geometry, acting on, say, a vector as

$$\mathscr{L}_\Lambda V^M \;\equiv\; \Lambda^K\partial_K V^M - \partial_K\Lambda^M V^K\,. \tag{2.4}$$

The problem with applying this definition to EFT is that some fields are subject to further constraints, for instance the generalized metric $\mathscr{M}_{MN}$ is an $\mathrm{E}_{6(6)}$-valued matrix, and this condition is not preserved under (2.4). This is fixed by simply projecting the tensor $\partial_K\Lambda^M$ living in $\mathbf{27}\otimes\bar{\mathbf{27}}$ onto the adjoint by means of the projector (2.3). Gauge transformations w.r.t. the internal diffeomorphism parameter Λ^M for a vector with upper or lower indices in terms of the *generalized* Lie derivative, denoted by $\mathbb{L}_\Lambda$ in the following, are thus defined as [28]

$$\begin{aligned} \delta V^M &= \mathbb{L}_\Lambda V^M \;\equiv\; \Lambda^K\partial_K V^M - 6\,\mathbb{P}^M{}_N{}^K{}_L\,\partial_K\Lambda^L\,V^N + \lambda\,\partial_P\Lambda^P\,V^M\,, \\ \delta W_M &= \mathbb{L}_\Lambda W_M \;\equiv\; \Lambda^K\partial_K W_M + 6\,\mathbb{P}^N{}_M{}^K{}_L\,\partial_K\Lambda^L\,W_N + \lambda'\,\partial_P\Lambda^P\,W_M\,. \end{aligned} \tag{2.5}$$

Here we also included a density term proportional to $\lambda\,\partial_P\Lambda^P$. The generalized Lie derivatives are consistent for arbitrary density weights λ, and indeed in formulating EFT it is crucial to assign particular non-trivial weights to the fields. Writing out the projector (2.3), the gauge transformations are given by

$$\begin{aligned} \delta_\Lambda V^M &= \Lambda^K\partial_K V^M - \partial_K\Lambda^M V^K + \left(\lambda - \frac{1}{3}\right)\partial_P\Lambda^P\,V^M + 10\,d_{NLR}\,d^{MKR}\partial_K\Lambda^L V^N\,, \\ \delta_\Lambda W_M &= \Lambda^K\partial_K W_M + \partial_M\Lambda^K W_K + \left(\lambda + \frac{1}{3}\right)\partial_P\Lambda^P\,W_M - 10\,d_{MLR}\,d^{NKR}\partial_K\Lambda^L W_N. \end{aligned} \tag{2.6}$$

The generalized Lie derivatives can similarly be defined for $\mathrm{E}_{6(6)}$ tensors with an arbitrary number of upper and lower fundamental indices. In particular, the gauge transformations for the generalized metric take the form $\delta_\Lambda\mathscr{M}_{MN} = \mathbb{L}_\Lambda\mathscr{M}_{MN}$, with the generalized Lie derivative for density weight $\lambda = 0$. In this form the condition $\mathscr{M}\in\mathrm{E}_{6(6)}$ is indeed preserved.

Given the modified form of generalized Lie derivatives, as opposed to the conventional Lie derivatives, it is no longer clear that they are consistent, in particular that they satisfy an algebra, i.e., that they lead to gauge transformations that close. Closure can, however, be established, but here it is crucial to employ the section constraint (2.1). An explicit computation then shows that the generalized Lie derivatives close according to

$$\left[\mathbb{L}_{\Lambda_1},\mathbb{L}_{\Lambda_2}\right] \;=\; \mathbb{L}_{[\Lambda_1,\Lambda_2]_{\rm E}}\;, \tag{2.7}$$

with the 'E-bracket'

$$\left[\Lambda_1,\Lambda_2\right]_{\rm E}^{M} \;\equiv\; 2\,\Lambda_{[1}^{K}\partial_K\Lambda_{2]}^{M}-10\,d^{MNP}\,d_{KLP}\,\Lambda_{[1}^{K}\partial_N\Lambda_{2]}^{L}\;. \tag{2.8}$$

The first term in here has the same form as the standard Lie bracket governing the algebra of standard diffeomorphisms. The second term explicitly involves the $E_{6(6)}$ structure in form of the d-symbols. Thus, the gauge algebra on this space differs from the diffeomorphism algebra. In particular, the Lie derivative of a generalized vector w.r.t. another generalized vector (both of weights $\frac{1}{3}$) does not coincide with their E-bracket. More precisely, the antisymmetric part coincides with the E-bracket, but there is a non-trivial symmetric part, given by

$$\left(\mathbb{L}_V W+\mathbb{L}_W V\right)^{M} \;=\; 10\,d^{MNK}d_{PQK}\partial_N\left(V^P W^Q\right)\;. \tag{2.9}$$

Moreover, the E-bracket does not define a Lie algebra in that the Jacobi identity is not satisfied. The non-trivial 'Jacobiator' as well as the 'anomalous' symmetric part in (2.9) are, however, of the form $\Lambda^M = d^{MNK}\partial_N\chi_K$, for some explicit function χ, and one can verify that due to the section constraint the Lie derivative vanishes for this parameter. Hence, the Jacobi identity does hold acting on fields satisfying the strong constraint (see [53] for more details), but the non-vanishing Jacobiator has important consequences, upon taking into account the external coordinate dependence.

So far we have defined the generalized internal diffeomorphisms by generalized Lie derivatives. We will refer to a tensor structure as transforming 'covariantly' iff its transformation is governed by the generalized Lie derivative (of some weight) and call such objects generalized tensors. Since all fields are functions of internal and external coordinates Y^M and x^μ, respectively, we now need to set up a calculus that allows us to differentiate w.r.t. x^μ. Indeed, as all fields and parameters in the full theory, $\Lambda^M = \Lambda^M(x,Y)$ depends on the external x^μ and therefore the derivative ∂_μ of any tensor field is not covariant in the above sense. In order to remedy this we introduce a gauge connection $\mathscr{A}_\mu{}^M$, of which we can think as taking values in the 'E-bracket algebra', and define the covariant derivatives

$$D_\mu \;\equiv\; \partial_\mu-\mathbb{L}_{\mathscr{A}_\mu}\;. \tag{2.10}$$

The covariant derivative of any generalized tensor then transforms covariantly provided the gauge vector transforms as $\delta_\Lambda\mathscr{A}_\mu{}^M = D_\mu\Lambda^M$, where the gauge parameter Λ^M carries weight $\lambda_\Lambda = \frac{1}{3}$. Next, we would like to define a field strength for $\mathscr{A}_\mu{}^M$. Naively, one would write the standard formula for the field strength or curvature of a gauge connection, but with the Lie bracket replaced by the E-bracket (2.8). However, since the E-bracket does not satisfy the Jacobi identity the resulting object does not transform covariantly and also does not satisfy a Bianchi identity. Since the failure of the E-bracket to satisfy the Jacobi identity is of the form $d^{MNK}\partial_N\chi_K$ we can repair this by introducing two-forms $\mathscr{B}_{\mu\nu M}$ with appropriate gauge transformations and adding the term $d^{MNK}\,\partial_K\mathscr{B}_{\mu\nu N}$ to the field strength. This defines (the beginning of) the so-called tensor hierarchy, originally introduced in gauged supergravity [54, 55]. Using (2.8) we thus obtain the field strength

$$\begin{aligned}\mathscr{F}_{\mu\nu}{}^{M} &= 2\,\partial_{[\mu}\mathscr{A}_{\nu]}{}^{M} - 2\,\mathscr{A}_{[\mu}{}^{K}\partial_{K}\mathscr{A}_{\nu]}{}^{M} + 10\,d^{MKR}d_{NLR}\,\mathscr{A}_{[\mu}{}^{N}\,\partial_{K}\mathscr{A}_{\nu]}{}^{L} \\ &\quad + 10\,d^{MNK}\,\partial_{K}\mathscr{B}_{\mu\nu N}\,.\end{aligned} \tag{2.11}$$

This tensor transforms covariantly under the appropriate gauge transformations of $\mathscr{A}$ and $\mathscr{B}$ given in (2.15) below. The presence of the 2-form in (2.11) also ensures that this field strength satisfies a modified covariant Bianchi identity

$$3\,D_{[\mu}\mathscr{F}_{\nu\rho]}{}^{M} \;=\; 10\,d^{MNK}\partial_{K}\mathscr{H}_{\mu\nu\rho N}\,, \tag{2.12}$$

giving rise to the 3-form curvature of the 2-form. The 3-form field strength $\mathscr{H}_{\mu\nu\rho M}$ is defined by this equations as

$$\begin{aligned}\mathscr{H}_{\mu\nu\rho M} &= 3\,D_{[\mu}\mathscr{B}_{\nu\rho]M} - 3\,d_{MKL}\,\mathscr{A}_{[\mu}{}^{K}\,\partial_{\nu}\mathscr{A}_{\rho]}{}^{L} + 2\,d_{MKL}\,\mathscr{A}_{[\mu}{}^{K}\mathscr{A}_{\nu}{}^{P}\partial_{P}\mathscr{A}_{\rho]}{}^{L} \\ &\quad - 10\,d_{MKL}d^{LPR}d_{RNQ}\,\mathscr{A}_{[\mu}{}^{K}\mathscr{A}_{\nu}{}^{N}\,\partial_{P}\mathscr{A}_{\rho]}{}^{Q} + \cdots\,,\end{aligned} \tag{2.13}$$

where $\mathscr{B}$ carries weight $\lambda_{\mathscr{B}} = \frac{2}{3}$, up to terms that vanish under the projection with $d^{MNK}\partial_{K}$. Now in turn we can establish a Bianchi identity for $\mathscr{H}$, which reads

$$4\,D_{[\mu}\mathscr{H}_{\nu\rho\sigma]M} \;=\; -3\,d_{MPQ}\mathscr{F}_{[\mu\nu}{}^{P}\mathscr{F}_{\rho\sigma]}{}^{Q} + \ldots\,, \tag{2.14}$$

again up to terms annihilated by the projection with $d^{MNK}\partial_{K}$.

We close this section by collecting the complete bosonic gauge transformations. The external and internal metric $g_{\mu\nu}$ and $\mathscr{M}_{MN}$ transform under internal generalized diffeomorphisms as a scalar density of weight $\frac{2}{3}$ and a symmetric 2-tensor of weight zero, respectively. Recalling that $\mathscr{A}$ carries weight $\lambda = \frac{1}{3}$ and noting that $\mathscr{B}$ carries weight $\lambda = \frac{2}{3}$, the gauge transformations then read

$$\begin{aligned}\delta\mathscr{A}_{\mu}{}^{M} &= D_{\mu}\Lambda^{M} - 10\,d^{MNK}\partial_{K}\Xi_{\mu N}\,, \\ \Delta\mathscr{B}_{\mu\nu M} &= 2D_{[\mu}\Xi_{\nu]M} + d_{MKL}\Lambda^{K}\mathscr{F}_{\mu\nu}{}^{L} + \mathscr{O}_{\mu\nu M}\,,\end{aligned} \tag{2.15}$$

where we defined

$$\Delta\mathscr{B}_{\mu\nu N} \equiv \delta\mathscr{B}_{\mu\nu N} + d_{NKL}\,\mathscr{A}_{[\mu}{}^{K}\,\delta\mathscr{A}_{\nu]}{}^{L}\,. \tag{2.16}$$

Here we also specified the gauge transformations under the new parameter $\Xi_{\mu M}$ of weight $\frac{2}{3}$ associated to the 2-form, and we note that the gauge transformations are so far only determined up to yet unspecified terms $\mathscr{O}_{\mu\nu M}$ satisfying

$$d^{MNK}\partial_{K}\mathscr{O}_{\mu\nu N} \;=\; 0\,. \tag{2.17}$$

This corresponds to the gauge redundancy of the next form in the tensor hierarchy, but it turns out that this ambiguity drops out of all terms in the action and equations of motion.

We finally give the form of the *external* diffeomorphisms of the x^μ, which are generated by a parameter $\xi^\mu = \xi^\mu(x,Y)$,

$$\begin{aligned}
\delta_\xi e_\mu{}^a &= \xi^\nu D_\nu e_\mu{}^a + D_\mu \xi^\nu e_\nu{}^a \,, \\
\delta_\xi \mathcal{M}_{MN} &= \xi^\mu D_\mu \mathcal{M}_{MN} \,, \\
\delta_\xi \mathcal{A}_\mu{}^M &= \xi^\nu \mathcal{F}_{\nu\mu}{}^M + \mathcal{M}^{MN} g_{\mu\nu} \partial_N \xi^\nu \,, \\
\Delta_\xi \mathcal{B}_{\mu\nu M} &= \tfrac{1}{2\sqrt{10}} \xi^\rho \, e\varepsilon_{\mu\nu\rho\sigma\tau} \mathcal{F}^{\sigma\tau N} \mathcal{M}_{MN} \,.
\end{aligned} \tag{2.18}$$

Let us note that they take the same form as standard diffeomorphisms generated by conventional Lie derivatives, except that all partial derivatives are replaced by gauge covariant derivatives. Moreover, in $\delta\mathcal{A}_\mu$ there is an additional $\mathcal{M}$-dependent term and in $\Delta\mathcal{B}_{\mu\nu}$ the naively covariant form $\xi^\rho \mathcal{H}_{\mu\nu\rho}$ has been replaced according to a duality relation to be discussed momentarily. We will discuss these external diffeomorphisms, in particular their gauge algebra, in more detail in sec. 2.4 below.

2.2 $\mathrm{E}_{6(6)}$ covariant dynamics

Let us now define the dynamics of the $\mathrm{E}_{6(6)}$ EFT by giving the unique action principle on the extended space, which decomposes into the five terms

$$S_{\rm EFT} = S_{\rm EH} + S_{\rm sc} + S_{\rm VT} + S_{\rm top} - V \,. \tag{2.19}$$

The first term formally takes the same form as the standard Einstein-Hilbert term,

$$S_{\rm EH} = \int d^5x\, d^{27}Y \, e\widehat{R} = \int d^5x\, d^{27}Y \, e\, e_a{}^\mu e_b{}^\nu \widehat{\mathcal{R}}_{\mu\nu}{}^{ab} \,, \tag{2.20}$$

except that in the definition of the Riemann tensor all partial derivatives are replaced by $\mathcal{A}_\mu$ covariant derivatives and one adds an additional term to make it properly local Lorentz invariant, $\widehat{R}_{\mu\nu}{}^{ab} \equiv R_{\mu\nu}{}^{ab} + \mathcal{F}_{\mu\nu}{}^M e^{\rho[a}\partial_M e_\rho{}^{b]}$. The second term is the 'scalar-kinetic' term defined by

$$\mathcal{L}_{\rm sc} = \frac{1}{24} e\, g^{\mu\nu} D_\mu \mathcal{M}_{MN} D_\nu \mathcal{M}^{MN} \,, \tag{2.21}$$

with $e \equiv \sqrt{|g|}$. The third term in (2.19) is the kinetic term for the gauge-vectors, written in terms of the gauge covariant curvature (2.11),

$$\mathcal{L}_{\rm VT} \equiv -\frac{1}{4} e\, \mathcal{F}_{\mu\nu}{}^M \mathcal{F}^{\mu\nu N} \mathcal{M}_{MN} \,. \tag{2.22}$$

The fourth term is a Chern-Simons-type topological term, which is only gauge invariant up to boundary terns. It is most conveniently defined by writing it as a manifestly gauge invariant action in one higher dimension, where it reduces to a total derivative, reducing it to the boundary integral in one dimension lower. Using form notation it reads

$$\begin{aligned}
S_{\rm top} &= \int d^5x\, d^{27}Y \, \mathcal{L}_{\rm top} \\
&= \tfrac{1}{6}\sqrt{10} \int d^{27}Y \int_{\mathcal{M}_6} \left(d_{MNK}\, \mathcal{F}^M \wedge \mathcal{F}^N \wedge \mathcal{F}^K - 40\, d^{MNK} \mathcal{H}_M \wedge \partial_N \mathcal{H}_K \right) .
\end{aligned} \tag{2.23}$$

Under a general variation of $\mathscr{A}$ and $\mathscr{B}$ the topological Lagrangian varies as

$$\delta\mathscr{L}_{\rm top} = \tfrac{1}{8}\sqrt{10}\,\varepsilon^{\mu\nu\rho\sigma\tau}\Big(d_{MNK}\,\mathscr{F}_{\mu\nu}{}^{M}\mathscr{F}_{\rho\sigma}{}^{N}\delta\mathscr{A}_{\tau}{}^{K}+\tfrac{20}{3}\,d^{MNK}\,\partial_N\mathscr{H}_{\mu\nu\rho M}\Delta\mathscr{B}_{\sigma\tau K}\Big)\,. \tag{2.24}$$

The final term in the action is the 'scalar potential' that involves only internal derivatives ∂_M and reads

$$\begin{aligned} V \;=\; & -\frac{1}{24}\mathscr{M}^{MN}\partial_M\mathscr{M}^{KL}\,\partial_N\mathscr{M}_{KL}+\frac{1}{2}\mathscr{M}^{MN}\partial_M\mathscr{M}^{KL}\partial_L\mathscr{M}_{NK} \\ & -\frac{1}{2}g^{-1}\partial_M g\,\partial_N\mathscr{M}^{MN}-\frac{1}{4}\mathscr{M}^{MN}g^{-1}\partial_M g\,g^{-1}\partial_N g-\frac{1}{4}\mathscr{M}^{MN}\partial_M g^{\mu\nu}\partial_N g_{\mu\nu}\,. \end{aligned} \tag{2.25}$$

Its form is uniquely determined by the internal generalized diffeomorphism invariance (up to the relative coefficient between the last two terms in the second line that is, however, universal for all EFTs).

The field equations of the $\mathrm{E}_{6(6)}$ EFT follow by varying (2.19) naively w.r.t. all fields. For now we focus on the field equations for the two-form only, because they will be significant below. The 2-form $\mathscr{B}_{\mu\nu M}$ does not enter with a kinetic term, but appears inside the Yang-Mills-type kinetic term, c.f. the definition (2.11), and the topological term (2.23). Therefore, its field equations are first order and read

$$d^{MNK}\partial_K\left(e\mathscr{M}_{NL}\mathscr{F}^{\mu\nu L}+\frac{1}{6}\sqrt{10}\,\varepsilon^{\mu\nu\rho\sigma\tau}\,\mathscr{H}_{\rho\sigma\tau N}\right) \;=\; 0\,. \tag{2.26}$$

These equations take the same form as the standard duality relations in five dimensions between vectors and two-forms. However, here they appear only under a differential operator, which thus leads to different sets of duality relations for different solutions of the section constraint.

2.3 Fermions and Supersymmetry

The bosonic sector of exceptional field theory is uniquely determined upon imposing invariance under generalized diffeomorphisms in the internal and external space-time. Supersymmetry has not been imposed in order to determine the interactions; however, as expected the bosonic action (2.19) can be embedded into a supersymmetric theory [36]. The fermions of the theory are those of the maximal five-dimensional theory [2], however, living now on the full $(5+27)$-dimensional space-time (subject to the section constraint). In particular, they are $\mathrm{SO}(1,4)$ symplectic Majorana spinors spinors and we refer to [56] for our spinor conventions.[1] With respect to the R-symmetry group (or generalized internal Lorentz group) $\mathrm{USp}(8)$, the fermion fields fall into irreducible representations with the gravitino fields ψ_μ^i transforming in the fundamental 8, and the spin-$\frac{1}{2}$ fermions χ^{ijk} transforming in the totally anti-symmetric, Ω-traceless 42

$$\chi^{ijk}=\chi^{[\![ijk]\!]}\;\equiv\;\chi^{ijk}-\frac{1}{2}\Omega^{[ij}\chi^{k]mn}\Omega_{mn}\,, \tag{2.27}$$

where $\Omega_{ij}=\Omega_{[ij]}$ denotes the symplectic invariant tensor. Here and in the following we use the notation of double brackets $[\![\ldots]\!]$ to denote the projection of an $\mathrm{USp}(8)$ tensor onto the Ω-traceless

[1] Just for the conventions for the Levi-Civita density we follow [13, 36], with the two conventions related by $\varepsilon^{[1312.0614]}_{\mu\nu\rho\sigma\tau}=-i\varepsilon^{[\rm hep-th/0412173]}_{\mu\nu\rho\sigma\tau}$. Accordingly, γ-matrices satisfy $\gamma^{abcde}=i\varepsilon^{abcde}$.

part. With respect to generalized internal diffeomorphisms (2.5) the fermionic fields transform as weighted scalars of weight $\lambda_\psi = \frac{1}{6}$, $\lambda_\chi = -\frac{1}{6}$.

Coupling of the fermions requires the introduction of frame fields underlying the external and internal metric,

$$g_{\mu\nu} \;=\; e_\mu{}^a e_\nu{}^b \eta_{ab}\;, \qquad \mathcal{M}_{MN} \;=\; \mathcal{V}_M{}^{ij}\mathcal{V}_{N\,ij}\;, \tag{2.28}$$

with the fünfbein $e_\mu{}^a$, and the pseudo-real 27-bein

$$\Big\{\mathcal{V}_M{}^{ij}, \mathcal{V}_{M\,ij} = (\mathcal{V}_M{}^{ij})^* = \mathcal{V}_M{}^{kl}\Omega_{ki}\Omega_{lj}\Big\}\;, \tag{2.29}$$

satisfying $\mathcal{V}_M{}^{ij} = \mathcal{V}_M{}^{[\![ij]\!]}$. The inverse 27-bein is defined as

$$\mathcal{V}_M{}^{ij}\mathcal{V}_{ij}{}^N \;=\; \delta_M{}^N\;, \qquad \mathcal{V}_M{}^{kl}\mathcal{V}_{ij}{}^M \;=\; \delta^{kl}_{ij} - \frac{1}{8}\Omega_{ij}\Omega^{kl}\;, \tag{2.30}$$

with conventions $\delta^{ij}_{kl} = \frac{1}{2}(\delta^i_k\delta^j_l - \delta^i_l\delta^j_k)$ and $\Omega_{ik}\Omega^{jk} = \delta^j_i$. The 27-bein is an $\mathrm{E}_{6(6)}$ group-valued matrix, which is encoded in the structure of its infinitesimal variation,

$$\delta\mathcal{V}_M{}^{ij} \;=\; -2\,\delta q_k{}^{[i}\,\mathcal{V}_M{}^{j]k} + \delta p^{ijkl}\,\mathcal{V}_{M\,kl}\;, \tag{2.31}$$

with $\delta q_i{}^j$ and p^{ijkl} spanning the **36** and **42** of USp(8), respectively, i.e.

$$\delta q_i{}^j \;=\; -\delta q_l{}^k\Omega_{ik}\Omega^{jl}\;, \qquad \delta p^{ijkl} \;=\; \delta p^{[\![ijkl]\!]}\;, \tag{2.32}$$

and corresponding to the compact and non-compact generators of $\mathfrak{e}_{6(6)}$, respectively.

The full $\mathrm{SO}(1,4)\times\mathrm{USp}(8)$ covariant derivatives are then defined as

$$\begin{aligned}
\mathcal{D}_\mu\psi^i &\equiv \partial_\mu\psi^i + \frac{1}{4}\,\omega_\mu{}^{ab}\gamma_{ab}\,\psi^i - \mathcal{Q}_{\mu\,j}{}^i\psi^j - \mathbb{L}_{\mathcal{A}_\mu}\,\psi^i\;,\\
\mathcal{D}_M\psi^i &\equiv \partial_M\psi^i + \frac{1}{4}\,\omega_M{}^{ab}\gamma_{ab}\,\psi^i - \mathcal{Q}_{M\,j}{}^i\psi^j\;,
\end{aligned} \tag{2.33}$$

with spin connections ω, $\mathcal{Q}$ defined in terms of the bosonic frame fields and the Lie derivative $\mathbb{L}$ taking care of the weight of ψ^i under generalized diffeomorphisms. From the spin connections the Christoffel connections $\Gamma_{\mu\nu}{}^\rho$, $\Gamma_{MN}{}^K$, can be defined by the generalized vielbein postulates

$$\begin{aligned}
0 &\equiv \nabla_\mu e_\nu{}^a = \mathcal{D}_\mu e_\nu{}^a - \Gamma_{\mu\nu}{}^\rho\, e_\rho{}^a = D_\mu e_\nu{}^a + \omega_\mu{}^{ab}e_{\nu\,b} - \Gamma_{\mu\nu}{}^\rho\, e_\rho{}^a\;,\\
0 &\equiv \nabla_M\mathcal{V}_N{}^{ij} = \mathcal{D}_M\mathcal{V}_N{}^{ij} - \Gamma_{MN}{}^K\,\mathcal{V}_K{}^{ij} = \partial_M\mathcal{V}_N{}^{ij} + 2\,\mathcal{Q}_{M\,k}{}^{[i}\mathcal{V}_N{}^{j]k} - \Gamma_{MN}{}^K\,\mathcal{V}_K{}^{ij}\;,
\end{aligned} \tag{2.34}$$

declaring covariant constancy of the frame fields. In turn, the spin connections are defined by properly generalized vanishing torsion conditions. For the SO(1,4) connection $\omega_\mu{}^{ab}$ the absence of torsion takes the familiar form

$$\mathcal{D}_{[\mu}e_{\nu]}{}^a \equiv D_{[\mu}e_{\nu]}{}^a + \omega_{[\mu}{}^{ab}e_{\nu]b} \;\stackrel{!}{=}\; 0 \qquad\Longleftrightarrow\qquad \Gamma_{[\mu\nu]}{}^\rho \;=\; 0\,, \tag{2.35}$$

describing a deformation of Riemannian geometry by the fact that the derivative D_μ is covariantized w.r.t. internal generalized diffeomorphisms (2.10), under which the fünfbein $e_\mu{}^a$ transforms as a weighted scalar. For the internal sector on the other hand, vanishing torsion translates into the projection condition [28]

$$\Gamma_{MN}{}^{K}\Big|_{\mathbf{351}} = 0\,, \tag{2.36}$$

for the generalized Christoffel connection, decomposed into irreducible $E_{6(6)}$ representations. More precisely, by its definition (2.34) the Christoffel connection $\Gamma_{MN}{}^{K}$ is algebra valued in its last two indices

$$\Gamma_{MN}{}^{K} = \Gamma_{M}{}^{\alpha}\,(t_{\alpha})_{N}{}^{K}\,, \qquad \Gamma_{M}{}^{\alpha} \sim \mathbf{27}\otimes\mathbf{78} = \mathbf{27}\oplus\mathbf{351}\oplus\mathbf{1728}\,, \tag{2.37}$$

and (2.36) indicates that $\Gamma_{M}{}^{\alpha}$ only has non-vanishing components in the $\mathbf{27}\oplus\mathbf{1728}$. Explicitly, parametrizing the USp(8) connection as

$$\mathcal{Q}_{M\,j}{}^{i} = q_{M\,j}{}^{i} + \mathcal{V}_{M}{}^{kl}\Omega^{im}\,q_{kl,jm}\,, \tag{2.38}$$

with $q_{kl,ij} = q_{[\![kl]\!],(ij)}$, equations (2.36) translate into

$$\begin{aligned} q_{kl,mn} &= -p_{M\,klp(m}\mathcal{V}_{n)q}{}^{M}\,\Omega^{pq} - \frac{1}{4}\mathcal{V}^{pqM}\left(p_{M\,pqk(m}\Omega_{n)l} - p_{M\,pql(m}\Omega_{n)k}\right) \\ &\quad + \frac{1}{4}\Gamma_{KM}{}^{K}\left(\mathcal{V}_{k(m}{}^{M}\Omega_{n)l} - \mathcal{V}_{l(m}{}^{M}\Omega_{n)k}\right) + u_{kl,mn}\,, \end{aligned} \tag{2.39}$$

with

$$q_{M\,i}{}^{j} \equiv \frac{1}{3}\mathcal{V}_{ik}{}^{N}\partial_{M}\mathcal{V}_{N}{}^{jk}\,, \qquad p_{M}{}^{ijkl} \equiv \partial_{M}\mathcal{V}_{N}{}^{[ij}\mathcal{V}^{kl]N}\,, \tag{2.40}$$

and $u_{kl,mn}$ satisfying

$$u_{kl,jm} = u_{[\![kl]\!],(jm)}\,, \quad u_{[kl,m]n} = 0\,, \quad u_{kl,jm}\,\Omega^{lj} = 0\,, \tag{2.41}$$

dropping out from equations (2.36). Vanishing torsion thus determines the USp(8) connection (and thereby the Christoffel connection) up to a block $u_{kl,mn}$ transforming in the **594** of USp(8), which drops out of all field equations and supersymmetry variations [28,29,36,57]. The Christoffel connection gives rise to covariant derivatives

$$\nabla_{M}X_{N} \equiv \partial_{M}X_{N} - \Gamma_{MN}{}^{K}X_{K} - \frac{3}{4}\lambda_{X}\Gamma_{KM}{}^{K}X_{N}\,, \tag{2.42}$$

where λ_X denotes the weight of X_N under generalized diffeomorphisms, and the trace part in the Christoffel connection is fixed by demanding

$$\nabla_{M}e \stackrel{!}{=} 0 \qquad \Longrightarrow \qquad \Gamma_{NM}{}^{N} = \frac{4}{5}\,e^{-1}\,\partial_{M}e\,. \tag{2.43}$$

The remaining connections in (2.33) finally are determined by demanding that the $\mathfrak{gl}(5)\oplus\mathfrak{e}_{6(6)}$ algebra-valued currents

$$\mathcal{J}_{M}{}^{ab} \equiv e^{a\mu}\,\mathcal{D}[\omega]_{M}e_{\mu}{}^{b}\,, \qquad \mathcal{J}_{\mu\,kl}{}^{ij} \equiv \mathcal{V}_{kl}{}^{M}\mathcal{D}[\mathcal{A},\mathcal{Q}]_{\mu}\mathcal{V}_{M}{}^{ij}\,, \tag{2.44}$$

of the frame fields live in the complement of the Lorentz algebra $\mathfrak{so}(1,4)\oplus\mathfrak{usp}(8)$, specifically

$$\mathscr{J}_M{}^{ab}\Big|_{\mathfrak{so}(1,4)} = 0\,, \qquad \mathscr{J}_{\mu\,kl}{}^{ij}\Big|_{\mathfrak{usp}(8)} = 0\,. \tag{2.45}$$

They give the explicit form

$$\omega_M{}^{ab} = e^{\mu[a}\partial_M e_\mu{}^{b]}\,, \qquad \mathscr{Q}_{\mu\,i}{}^j = \frac{1}{3}\mathscr{V}_{ik}{}^M D_\mu \mathscr{V}_M{}^{jk}\,, \tag{2.46}$$

of the respective spin connections and give rise to the definition of the coset currents

$$\mathscr{J}_M{}^{ab} \equiv \pi_M{}^{ab} = \pi_M{}^{(ab)}\,, \qquad \mathscr{J}_{\mu\,mn}{}^{ij}\Omega^{km}\Omega^{ln} \equiv \mathscr{P}_\mu{}^{ijkl} = \mathscr{P}_\mu{}^{[\![ijkl]\!]}\,. \tag{2.47}$$

Moreover, it turns out that the Lagrangian and supersymmetry transformation rules are conveniently given in terms of the modified internal spin connections

$$\omega_M^{\pm ab} \equiv \omega_M{}^{ab} \pm \frac{1}{2}\mathscr{M}_{MN}\mathscr{F}_{\mu\nu}{}^N e^{\mu a}e^{\nu b}\,, \tag{2.48}$$

shifted by the non-abelian field strength (2.11), and we denote the corresponding covariant derivatives by $\mathscr{D}^\pm$.

The different curvatures of these spin connections are the building blocks for the bosonic Lagrangian and field equations [25, 29, 36], once projected onto the components such that the undetermined part (2.41) drops out. Some of the relevant curvatures are obtained from the commutators

$$\begin{aligned}
\left[\mathscr{D}_\mu,\mathscr{D}_\nu\right]\varepsilon^i &= \frac{1}{4}\widehat{\mathscr{R}}_{\mu\nu}{}^{ab}\gamma_{ab}\,\varepsilon^i + \frac{2}{3}\mathscr{P}_{[\mu\,jklm}\mathscr{P}_{\nu]}{}^{iklm}\,\varepsilon^j - \mathscr{F}_{\mu\nu}{}^M\,\nabla_M\varepsilon^i \\
&\quad + \nabla_M\mathscr{F}_{\mu\nu}{}^N\left(\mathscr{V}_N{}^{jk}\mathscr{V}_{ik}{}^M - \mathscr{V}_{N\,ik}\mathscr{V}^{jkM}\right)\varepsilon^j - \frac{1}{6}\nabla_M\mathscr{F}_{\mu\nu}{}^M\,\varepsilon^i\,, \\
\mathscr{V}_{ij}{}^M\left[\nabla_M^-,\mathscr{D}_\mu\right]\varepsilon^j &= \frac{1}{2}\mathscr{V}^{jkM}\mathscr{D}_M\mathscr{P}_{\mu\,ijkn}\varepsilon^n + \frac{1}{4}\mathscr{R}^-_{M\mu}{}^{ab}\gamma_{ab}\,\varepsilon^j\,,
\end{aligned} \tag{2.49}$$

$$\begin{aligned}
\mathscr{V}^{ikM}\mathscr{V}_{kj}{}^N\left[\nabla_M,\nabla_N\right]\varepsilon^j + \left(4\mathscr{V}^{ikM}\mathscr{V}_{kj}{}^N + \frac{1}{2}\mathscr{M}^{MN}\delta^i_j\right)\nabla_{(M}\nabla_{N)}\varepsilon^j \\
= \frac{1}{4}\mathscr{V}^{ikM}\mathscr{V}_{kj}{}^N\mathscr{R}_{MN}{}^{ab}\gamma_{ab}\,\varepsilon^j - \frac{1}{16}\mathscr{R}\,\varepsilon^i\,.
\end{aligned}$$

Explicitly, the curvature tensors read

$$\begin{aligned}
\widehat{\mathscr{R}}_{\mu\nu}{}^{ab} &= 2D_{[\mu}\omega_{\nu]}{}^{ab} + 2\,\omega_{[\mu}{}^{ac}\,\omega_{\nu]c}{}^b + \mathscr{F}_{\mu\nu}{}^M\,\omega_M{}^{ab}\,, \\
\mathscr{R}^-_{M\mu}{}^{ab} &\equiv \partial_M\,\omega_\mu{}^{ab} - \mathscr{D}_\mu\,\omega_M^{-ab}\,, \\
\mathscr{R}_{MN}{}^{ab} &= -\frac{1}{2}e^{\mu[a}e^{b]\nu}g^{\sigma\tau}\nabla_M g_{\mu\sigma}\nabla_N g_{\nu\tau}\,,
\end{aligned} \tag{2.50}$$

of which the first two enter the Einstein and the vector field equations, respectively. The curvature scalar $\mathscr{R}$ is related to the scalar potential from (2.25) as

$$\mathscr{R} = V + \frac{1}{4}\mathscr{M}^{MN}\nabla_M g_{\mu\nu}\nabla_N g^{\mu\nu} + \nabla_M I^M\,, \tag{2.51}$$

up to boundary terms $\nabla_M I^M$.

The full supersymmetric extension of the bosonic action (2.19) can be given in very compact form in terms of the above spin connections. It reads

$$\begin{aligned} e^{-1}\mathscr{L} \;=\; & \mathscr{L}_{\rm bos} - \bar{\psi}_{\mu i}\gamma^{\mu\nu\rho}\mathscr{D}_\nu\psi^i_\rho + 2\sqrt{2}\,i\,\mathscr{V}_{ij}{}^M\Omega^{ik}\bar{\psi}_{\mu k}\gamma^{[\mu}\nabla^+_M\left(\gamma^{\nu]}\psi_\nu{}^j\right) \\ & -\frac{4}{3}\bar{\chi}_{ijk}\gamma^\mu\mathscr{D}_\mu\chi^{ijk} + 8\sqrt{2}\,i\,\mathscr{V}_{mn}{}^M\Omega^{np}\bar{\chi}_{pkl}\nabla^+_M\chi^{mkl} \\ & +\frac{4i}{3}\mathscr{P}_\mu{}^{ijkl}\bar{\chi}_{ijk}\gamma^\nu\gamma^\mu\psi_\nu{}^m\Omega_{lm} + 4\sqrt{2}\,\mathscr{V}^{ijM}\,\bar{\chi}_{ijk}\gamma^\mu\nabla^-_M\psi_\mu{}^k\,, \end{aligned} \tag{2.52}$$

up to quartic fermion terms. The latter are expected to coincide with the quartic terms of the $D=5$ theory [2]. The Lagrangian (2.52) is invariant up to total derivatives under the following set of supersymmetry transformation rules

$$\begin{aligned} \delta_\varepsilon\psi^i_\mu \;&=\; \mathscr{D}_\mu\varepsilon^i - i\sqrt{2}\,\mathscr{V}^{ijM}\left(\nabla^-_M(\gamma_\mu\varepsilon^k) - \frac{1}{3}\gamma_\mu\nabla^-_M\varepsilon^k\right)\Omega_{jk}\,, \\ \delta_\varepsilon\chi^{ijk} \;&=\; \frac{i}{2}\mathscr{P}_\mu{}^{ijkl}\Omega_{lm}\gamma^\mu\varepsilon^m + \frac{3}{\sqrt{2}}\mathscr{V}^{[\![ijM}\,\nabla^-_M\varepsilon^{k]\!]}\,, \end{aligned} \tag{2.53}$$

for the fermionic fields, and

$$\begin{aligned} \delta_\varepsilon e^a_\mu \;&=\; \frac{1}{2}\bar{\varepsilon}_i\gamma^a\psi^i_\mu\,, \qquad \delta_\varepsilon\mathscr{V}_M{}^{ij} \;=\; 4i\,\Omega^{im}\Omega^{jn}\mathscr{V}_M{}^{kl}\,\Omega_{p[\![k}\bar{\chi}_{lmn]\!]}\varepsilon^p\,, \\ \delta_\varepsilon\mathscr{A}_\mu{}^M \;&=\; \sqrt{2}\left(i\,\Omega^{ik}\bar{\varepsilon}_k\psi_\mu{}^j + \bar{\varepsilon}_k\gamma_\mu\chi^{ijk}\right)\mathscr{V}_{ij}{}^M\,, \\ \Delta_\varepsilon\mathscr{B}_{\mu\nu M} \;&=\; -\frac{1}{\sqrt{5}}\mathscr{V}_M{}^{ij}\left(2\,\bar{\psi}_{i[\mu}\gamma_{\nu]}\varepsilon^k\Omega_{jk} + i\bar{\chi}_{ijk}\gamma_{\mu\nu}\varepsilon^k\right)\,, \end{aligned} \tag{2.54}$$

for the bosonic fields. Equations (2.53) depict the Killing spinor equations of the theory. It is remarkable, that in the supersymmetry transformation rules all explicit appearance of the field strength $\mathscr{F}_{\mu\nu}{}^M$ can be entirely absorbed into the shifted spin connection ω^- form (2.48) whereas the Lagrangian (2.52) carries both ω^+ and ω^- .

2.4 Algebra of external and internal generalized diffeomorphisms

The algebra of internal generalized diffeomorphisms is governed by the E-bracket and has been discussed extensively in the literature. The algebra of the external diffeomorphisms, which acts in a more subtle way due to the field-dependent modifications in (2.18) compared to standard diffeomorphisms, has been determined in [32] (for the $\mathrm{SL}(3)\times\mathrm{SL}(2)$ EFT, but the results generalize immediately). Here we use the opportunity to complete the literature by discussing the off-diagonal part of the total gauge algebra, i.e., the algebra of external and internal generalized diffeomorphisms. This will be important below, when we show that, upon solving the section constraint, the internal and external *conventional* diffeomorphisms indeed close according to the 10- or 11-dimensional diffeomorphism algebra, implying the full diffeomorphism invariance of the resulting supergravities.

For simplicity, let us first consider a pure $\mathrm{E}_{6(6)}$ tensor T (whose indices we suppress) that is an external scalar, i.e., does not carry external indices $\mu,\nu,\ldots$ (an example is the generalized metric

$\mathcal{M}_{MN}$). We then compute for the gauge algebra

$$\begin{aligned}\big[\delta_\Lambda,\delta_\xi\big]T &= \delta_\Lambda(\xi^\mu D_\mu T)-\delta_\xi(\mathbb{L}_\Lambda T)\\ &= \xi^\mu\mathbb{L}_\Lambda(D_\mu T)-\mathbb{L}_\Lambda(\xi^\mu D_\mu T)\\ &= -\mathbb{L}_\Lambda\xi^\mu\, D_\mu T\;.\end{aligned}\tag{2.55}$$

Here we used in the second line that the covariant derivative transforms covariantly. Moreover, recall that the gauge parameter ξ^μ is not to be varied in the gauge algebra. Thus, we have closure

$$\big[\delta_\Lambda,\delta_\xi\big] = \delta_{\xi'}\;,\qquad \xi'^\mu = -\mathbb{L}_\Lambda\xi^\mu = -\Lambda^N\partial_N\xi^\mu\;,\tag{2.56}$$

which defines the effective (Λ-transformed) ξ^μ parameter. Next, we inspect the (external) vielbein ${e_\mu}^a$, which is slightly more involved because it carries a vector index. With (2.18) we compute

$$\begin{aligned}\big[\delta_\Lambda,\delta_\xi\big]{e_\mu}^a &= \delta_\Lambda(\xi^\nu D_\nu {e_\mu}^a+D_\mu\xi^\nu {e_\nu}^a)-\delta_\xi(\mathbb{L}_\Lambda {e_\mu}^a)\\ &= \delta_\Lambda(\xi^\nu D_\nu {e_\mu}^a+D_\mu\xi^\nu {e_\nu}^a)-\mathbb{L}_\Lambda(\xi^\nu D_\nu {e_\mu}^a+D_\mu\xi^\nu {e_\nu}^a)\\ &= -(\mathbb{L}_{\delta_\Lambda\mathcal{A}_\mu}\xi^\nu){e_\nu}^a-\mathbb{L}_\Lambda\xi^\nu\, D_\nu {e_\mu}^a-\mathbb{L}_\Lambda(D_\mu\xi^\nu){e_\nu}^a\\ &= -D_\mu\Lambda^N\partial_N\xi^\nu {e_\nu}^a-\mathbb{L}_\Lambda\xi^\nu\, D_\nu {e_\mu}^a-\Lambda^N D_\mu(\partial_N\xi^\nu){e_\nu}^a\\ &= -\mathbb{L}_\Lambda\xi^\nu\, D_\nu {e_\mu}^a-D_\mu(\Lambda^N\partial_N\xi^\nu){e_\nu}^a\\ &= \xi'^\nu D_\nu {e_\mu}^a+D_\mu\xi'^\nu {e_\nu}^a\;.\end{aligned}\tag{2.57}$$

Here we used again, in the third line, the covariance of the covariant derivative, due to which various terms cancelled and that ξ^μ is a scalar with respect to internal diffeomorphisms. We thus established closure according to the same parameter as in (2.56).

Let us now turn to the gauge vector ${\mathcal{A}_\mu}^M$, whose transformation in (2.18) is $\mathcal{M}$-dependent. In order to simplify the discussion, we first consider the minimal variation without this term,

$$\delta^0_\xi{\mathcal{A}_\mu}^M \equiv \xi^\nu{\mathcal{F}_{\nu\mu}}^M\;.\tag{2.58}$$

Although this transformation rule is insufficient for the complete gauge invariance of EFT, it does lead to a consistent gauge algebra, as we discuss now. In order to prove closure of the gauge algebra we have to compute

$$\begin{aligned}\big[\delta_\Lambda,\delta^0_\xi\big]{\mathcal{A}_\mu}^M &= \delta_\Lambda(\xi^\nu{\mathcal{F}_{\nu\mu}}^M)-\delta^0_\xi(D_\mu\Lambda^M)\\ &= \xi^\nu\mathbb{L}_\Lambda{\mathcal{F}_{\nu\mu}}^M+\mathbb{L}_{\delta^0_\xi\mathcal{A}_\mu}\Lambda^M\;.\end{aligned}\tag{2.59}$$

For the second term we find

$$\mathbb{L}_{\delta^0_\xi\mathcal{A}_\mu}\Lambda^M = \mathbb{L}_{\xi^\nu\mathcal{F}_{\nu\mu}}\Lambda^M = \xi^\nu\mathbb{L}_{\mathcal{F}_{\nu\mu}}\Lambda^M-\partial_K\xi^\nu{\mathcal{F}_{\nu\mu}}^M\Lambda^K+10\,d_{NLR}\,d^{MKR}\partial_K\xi^\nu{\mathcal{F}_{\nu\mu}}^L\Lambda^N\;,\tag{2.60}$$

which follows by writing out the Lie derivative and collecting the terms where the derivative ∂_M hits the gauge parameter ξ^μ. The second term in here is the required ξ' transformation, so that we have shown

$$\big[\delta_\Lambda,\delta^0_\xi\big]{\mathcal{A}_\mu}^M = \delta^0_{\xi'}{\mathcal{A}_\mu}^M+\xi^\nu(\mathbb{L}_\Lambda{\mathcal{F}_{\nu\mu}}^M+\mathbb{L}_{\mathcal{F}_{\nu\mu}}\Lambda^M)+10\,d_{NLR}\,d^{MKR}\partial_K\xi^\nu{\mathcal{F}_{\nu\mu}}^L\Lambda^N\;.\tag{2.61}$$

The second term in here is the *symmetrized* Lie derivative that in turn is 'trivial' and given by (2.9),

$$\mathbb{L}_\Lambda \mathscr{F}_{\nu\mu}{}^M + \mathbb{L}_{\mathscr{F}_{\nu\mu}} \Lambda^M \;=\; 10\, d_{LNR}\, d^{MKR} \partial_K (\mathscr{F}_{\nu\mu}{}^L \Lambda^N)\,. \tag{2.62}$$

Inserting this in (2.61) above, we finally obtain

$$\big[\delta_\Lambda, \delta_\xi^0\big] \mathscr{A}_\mu{}^M \;=\; \delta^0_{\xi'} \mathscr{A}_\mu{}^M + 10\, d_{NLR}\, d^{MKR} \partial_K \big(\xi^\nu \mathscr{F}_{\nu\mu}{}^L \Lambda^N\big)\,. \tag{2.63}$$

Comparing with the general gauge transformations of $\mathscr{A}_\mu{}^M$ in (2.15) we infer that the additional term on the right-hand side can be interpreted as a field-dependent gauge transformation for the one-form parameter Ξ_μ corresponding to the two-form potential in the hierarchy. We thus established closure according to

$$\big[\delta_\Lambda, \delta_\xi^0\big] \mathscr{A}_\mu{}^M \;=\; \big(\delta^0_{\xi'} + \delta_{\Xi'}\big) \mathscr{A}_\mu{}^M\,, \qquad \Xi'_{\mu N} \;=\; -d_{NKL} \xi^\nu \mathscr{F}_{\nu\mu}{}^K \Lambda^L\,. \tag{2.64}$$

We see once more that the higher forms of the tensor hierarchy and their associated gauge symmetries are essential for the consistency of EFT.

Let us now return to the full gauge transformations of $\mathscr{A}_\mu{}^M$ w.r.t. ξ^μ, including the extra term that we denote in the following by $\delta'_\xi \mathscr{A}_\mu{}^M \equiv \mathscr{M}^{MN} g_{\mu\nu} \partial_N \xi^\nu$. We collect the additional contributions in the gauge algebra and find

$$\begin{aligned}\big[\delta_\Lambda, \delta_\xi\big] \mathscr{A}_\mu{}^M \;&=\; \cdots + \delta_\Lambda(\mathscr{M}^{MN} g_{\mu\nu}) \partial_N \xi^\nu + \mathbb{L}_{\delta'_\xi \mathscr{A}_\mu} \Lambda^M \\ &=\; \cdots + \mathbb{L}_\Lambda(\mathscr{M}^{MN} g_{\mu\nu}) \partial_N \xi^\nu + \mathbb{L}_{\mathscr{M}^{\bullet N} g_{\mu\nu} \partial_N \xi^\nu} \Lambda^M\,,\end{aligned} \tag{2.65}$$

where the dots indicate the terms already computed in the previous paragraph. The second term on the right-hand side can be written as

$$\mathbb{L}_{\mathscr{M}^{\bullet N} g_{\mu\nu} \partial_N \xi^\nu} \Lambda^M \;=\; -\mathbb{L}_\Lambda(\mathscr{M}^{MN} g_{\mu\nu} \partial_N \xi^\nu) + 10\, d^{MKR} d_{PLR} \partial_K (\mathscr{M}^{PQ} g_{\mu\nu} \partial_Q \xi^\nu \Lambda^L)\,, \tag{2.66}$$

where we used again the identity (2.9). Using this in (2.65) we obtain

$$\big[\delta_\Lambda, \delta_\xi\big] \mathscr{A}_\mu{}^M \;=\; -\mathscr{M}^{MN} g_{\mu\nu} \mathbb{L}_\Lambda(\partial_N \xi^\nu) + 10\, d^{MKR} d_{PLR} \partial_K (\mathscr{M}^{PQ} g_{\mu\nu} \partial_Q \xi^\nu \Lambda^L)\,. \tag{2.67}$$

Recalling that ξ^ν is a scalar, $\mathbb{L}_\Lambda(\partial_N \xi^\nu) = \partial_N(\mathbb{L}_\Lambda \xi^\nu)$ and so the first term becomes the ξ' transformation defined in (2.56). The second term can be interpreted as an additional field-dependent contribution to the effective one-form parameter Ξ'. Thus, in total we learned

$$\big[\delta_\Lambda, \delta_\xi\big] \;=\; \delta_{\xi'} + \delta_{\Xi'}\,, \tag{2.68}$$

where

$$\xi'^\mu \;=\; -\Lambda^N \partial_N \xi^\mu\,, \quad \Xi'_{\mu M} \;=\; -d_{MNK} (\xi^\nu \mathscr{F}_{\nu\mu}{}^N + \mathscr{M}^{KL} g_{\mu\nu} \partial_L \xi^\nu) \Lambda^K\,. \tag{2.69}$$

We leave it as an exercise for the reader to verify closure on the two-form, which can only be established up to unknown terms corresponding to the gauge symmetry of the three-form.

For completeness we record here that the algebra of external generalized diffeomorphisms is given by

$$\big[\delta_{\xi_1}, \delta_{\xi_1}\big] \;=\; \delta_{\xi_{12}} + \delta_{\Lambda_{12}} + \cdots\,, \tag{2.70}$$

with effective parameters

$$\begin{aligned} \xi_{12}^{\mu} &\equiv \xi_2^{\nu}\mathscr{D}_\nu\xi_1^{\mu} - \xi_1^{\nu}\mathscr{D}_\nu\xi_2^{\mu}\,, \\ \Lambda_{12}^{M} &\equiv \xi_2^{\mu}\xi_1^{\nu}\mathscr{F}_{\mu\nu}{}^{M} - 2\,\mathscr{M}^{MN}g_{\mu\nu}\,\xi_{[2}^{\mu}\partial_N\xi_{1]}^{\nu}\,. \end{aligned} \tag{2.71}$$

The dots in (2.70) indicate possible gauge transformations corresponding to higher forms entering the tensor hierarchy, see [32] for more details.

3. Type IIB solution and embedding of diffeomorphisms

In this section we will show, for the type IIB solution of the section constraint, how the fields and symmetries of EFT are related to those of the standard formulation of supergravity in which ten-dimensional diffeomorphism invariance is manifest.[2] To this end we show in the first subsection how, upon solving the section constraint, the standard diffeomorphism algebra is generically embedded in the gauge algebra of EFT (in particular the E-bracket), illustrating this with a simple toy model. In the second and third subsection we turn to the specific solution of the section constraint for type IIB and show how the coordinates and tensor fields decompose. In the final subsection we return to the external diffeomorphisms of EFT and supergravity (that, we recall, are not manifest symmetries), which in the following section will be shown to match precisely, thereby proving that EFT leads to a 10-dimensional theory with full diffeomorphism invariance.

3.1 Embedding of standard diffeomorphisms into E-bracket algebra

We now discuss how to embed the standard diffeomorphisms into the E-bracket algebra of EFT. More precisely, we will show that the external and internal diffeomorphisms in EFT close in the same way as those of a $D=10$ gravity theory, implying that there is a 'hidden' 10-dimensional diffeomorphism symmetry in EFT upon choosing a $D=10$ solution of the section constraint.

Before focusing on type IIB supergravity, let us start from a generic theory of Einstein gravity, coupled to some matter, and inspect the action of the diffeomorphism group under a Kaluza-Klein-type decomposition. To this end we split the ten-dimensional world and tangent space indices, here and in the following indicated by a hat, according to $\hat{\mu}=(\mu,m)$ and $\hat{a}=(a,\alpha)$, respectively, where $\mu=0,\dots n-1$, and $m=1,\dots,d$, with $n+d=10$, and similarly for the flat indices. Correspondingly, we decompose the tensor fields and symmetry parameters of the theory according to this $n+d$ split. For instance, the ten-dimensional frame field encoding the metric is written as

$$E_{\hat{\mu}}{}^{\hat{a}} = \begin{pmatrix} \phi^{-\gamma}e_\mu{}^a & A_\mu{}^m\phi_m{}^\alpha \\ 0 & \phi_m{}^\alpha \end{pmatrix}, \tag{3.1}$$

where $\phi=\det(\phi_m{}^\alpha)$ and $\gamma=\frac{1}{n-2}$. Here we employed a gauge fixing of the ten-dimensional Lorentz group $\mathrm{SO}(1,9)$ to $\mathrm{SO}(1,n-1)\times\mathrm{SO}(d)$. We next perform an analogous decomposition of the remaining gauge symmetries, i.e., of the ten-dimensional diffeomorphisms $x^{\hat{\mu}}\to x^{\hat{\mu}}-\xi^{\hat{\mu}}$ and local Lorentz transformations parametrized by $\lambda^{\hat{a}}{}_{\hat{b}}$, acting on the vielbein as

$$\delta E_{\hat{\mu}}{}^{\hat{a}} = \xi^{\hat{\nu}}\partial_{\hat{\nu}}E_{\hat{\mu}}{}^{\hat{a}} + \partial_{\hat{\mu}}\xi^{\hat{\nu}}E_{\hat{\nu}}{}^{\hat{a}} + \lambda^{\hat{a}}{}_{\hat{b}}E_{\hat{\mu}}{}^{\hat{b}}\,. \tag{3.2}$$

[2]For the M-theory solution we refer the reader to [13].

Specifically, we decompose the diffeomorphism parameter as

$$\xi^{\hat{\mu}} \;=\; (\xi^{\mu}\,,\,\Lambda^{m})\,, \tag{3.3}$$

and refer to the diffeomorphisms generated by ξ^{μ} as 'external' and those generated by Λ^{m} as 'internal'. Inserting (3.1) into (3.2) we read off the following action of the internal diffeomorphisms,

$$\begin{aligned}
\delta_{\Lambda} e_{\mu}{}^{a} &= \Lambda^{m}\partial_{m} e_{\mu}{}^{a} + \gamma\,\partial_{m}\Lambda^{m}\, e_{\mu}{}^{a}\,,\\
\delta_{\Lambda} \phi_{m}{}^{\alpha} &= \Lambda^{n}\partial_{n}\phi_{m}{}^{\alpha} + \partial_{m}\Lambda^{n}\,\phi_{n}{}^{\alpha}\,,\\
\delta_{\Lambda} A_{\mu}{}^{m} &= \partial_{\mu}\Lambda^{m} - A_{\mu}{}^{n}\partial_{n}\Lambda^{m} + \Lambda^{n}\partial_{n}A_{\mu}{}^{m}\,.
\end{aligned} \tag{3.4}$$

We will also use the notation $\mathscr{L}_{\Lambda}$ for the conventional Lie derivative of the purely internal space, acting in the standard fashion on tensors (of weight zero). Thus, the above transformations read

$$\begin{aligned}
\delta_{\Lambda} e_{\mu}{}^{a} &= \mathscr{L}_{\Lambda} e_{\mu}{}^{a} + \gamma\,\partial_{m}\Lambda^{m}\, e_{\mu}{}^{a}\,, \qquad \delta_{\Lambda}\phi_{m}{}^{\alpha} \;=\; \mathscr{L}_{\Lambda}\phi_{m}{}^{\alpha}\,,\\
\delta_{\Lambda} A_{\mu}{}^{m} &= \partial_{\mu}\Lambda^{m} - \mathscr{L}_{A_{\mu}}\Lambda^{m} \;\equiv\; \partial_{\mu}\Lambda^{m} + \mathscr{L}_{\Lambda}A_{\mu}{}^{m}\,.
\end{aligned} \tag{3.5}$$

Note that here we employ the convention in which the density term is not part of the Lie derivative. Analogously to the discussion in EFT, we can define derivatives and non-abelian field strengths that are covariant under these transformations,

$$\mathscr{D}^{\rm KK}_{\mu} \;\equiv\; \partial_{\mu} - \mathscr{L}_{A_{\mu}} - \lambda\,\partial_{m}A_{\mu}{}^{m}\,, \qquad F_{\mu\nu} \;\equiv\; 2\,\partial_{[\mu}A_{\nu]} - [A_{\mu},A_{\nu}]\,, \tag{3.6}$$

where λ is the density weight, e.g., $\lambda=\gamma$ for the external vielbein, and $[\,,\,]$ the conventional Lie bracket. Sometimes we will use the notation $D^{\rm KK}_{\mu} = \partial_{\mu} - \mathscr{L}_{A_{\mu}}$ for the part of the covariant derivative without the density term.[3] Specifically, for (3.4) we have

$$\begin{aligned}
\mathscr{D}^{\rm KK}_{\mu}\, e_{\nu}{}^{a} &= \partial_{\mu}e_{\nu}{}^{a} - A_{\mu}{}^{m}\partial_{m}e_{\nu}{}^{a} - \gamma\,\partial_{n}A_{\mu}{}^{n}\, e_{\nu}{}^{a}\,,\\
\mathscr{D}^{\rm KK}_{\mu}\, \phi_{m}{}^{\alpha} &= \partial_{\mu}\phi_{m}{}^{\alpha} - A_{\mu}{}^{n}\partial_{n}\phi_{m}{}^{\alpha} - \partial_{m}A_{\mu}{}^{n}\phi_{n}{}^{\alpha}\,,\\
F_{\mu\nu}{}^{m} &= \partial_{\mu}A_{\nu}{}^{m} - \partial_{\nu}A_{\mu}{}^{m} - A_{\mu}{}^{n}\partial_{n}A_{\nu}{}^{m} + A_{\nu}{}^{n}\partial_{n}A_{\mu}{}^{m}\,.
\end{aligned} \tag{3.7}$$

Let us now turn to the external diffeomorphisms. These are obtained from (3.2) by inserting (3.1), switching on only the ξ^{μ} component, and adding the compensating Lorentz transformation with parameter $\lambda^{a}{}_{\beta} = -\phi^{\gamma}\phi_{\beta}{}^{m}\partial_{m}\xi^{\nu}\, e_{\nu}{}^{a}$, which is necessary in order to preserve the gauge choice in (3.1). For instance, on the Kaluza-Klein vectors this yields

$$\delta^{\circ}_{\xi}A_{\mu}{}^{m} \;=\; \xi^{\nu}\partial_{\nu}A_{\mu}{}^{m} + \partial_{\mu}\xi^{\nu}A_{\nu}{}^{m} - A_{\mu}{}^{n}\partial_{n}\xi^{\nu}A_{\nu}{}^{m} + \phi^{-\frac{2}{3}}G^{mn}g_{\mu\nu}\partial_{n}\xi^{\nu}\,, \tag{3.8}$$

where $G^{mn}\equiv\phi_{\alpha}{}^{m}\phi^{\alpha n}$, and we specialized to $n=5$, corresponding to the $5+5$ split of type IIB that we will analyze momentarily. This gauge transformation can more conveniently be written in the form of 'improved' or 'covariant' diffeomorphisms by adding an *internal* diffeomorphism (3.4) with field-dependent parameter $\Lambda^{m} = -\xi^{\nu}A_{\nu}{}^{m}$. The gauge-field-dependent terms then organize into the covariant field strength in (3.7),

$$\delta_{\xi}A_{\mu}{}^{m} \;=\; \xi^{\nu}F_{\nu\mu}{}^{m} + \phi^{-\frac{2}{3}}G^{mn}g_{\mu\nu}\partial_{n}\xi^{\nu}\,. \tag{3.9}$$

[3]We emphasize that this is introduced for purely notational convenience. In general, acting with $D^{\rm KK}_{\mu}$ is not a covariant operation.

We infer that this is of the same structural form as the external diffeomorphism transformation of the EFT gauge vector in (2.18), and we will verify below that they can be matched precisely upon picking the type IIB solution of the section constraint. Similarly, these improved external diffeomorphisms act on the internal and external vielbein as

$$
\begin{aligned}
\delta_\xi e_\mu{}^a &= \xi^\nu \mathscr{D}_\nu^{\rm KK} e_\mu{}^a + \mathscr{D}_\mu^{\rm KK} \xi^\nu e_\nu{}^a , \\
\delta_\xi \phi_m{}^\alpha &= \xi^\nu \mathscr{D}_\nu^{\rm KK} \phi_m{}^\alpha ,
\end{aligned} \tag{3.10}
$$

again in structural agreement with the corresponding transformations (2.18) in EFT.

Next, we inspect the algebra of diffeomorphisms under this decomposition. Since the internal diffeomorphisms (five-dimensional in the case we are interested in) act on the fields via standard Lie derivatives w.r.t. the internal space, see (3.4), they close according to the standard Lie bracket,

$$
\big[\delta_{\Lambda_1}, \delta_{\Lambda_2}\big] = \delta_{\Lambda_{12}} , \qquad \Lambda_{12}^m \equiv [\Lambda_2, \Lambda_1]^m \equiv \Lambda_2^k \partial_k \Lambda_1^m - \Lambda_1^k \partial_k \Lambda_2^m . \tag{3.11}
$$

This is embedded in the E-bracket algebra (2.8) by solving the section constraint and restricting to the five 'lowest components' of the generalized diffeomorphism parameter.

The mixed algebra between internal and external diffeomorphisms is straightforwardly computed in the form of improved diffeomorphisms (3.9), (3.10). In fact, in this form every term on the right-hand side of the gauge variation is covariant w.r.t. the Lie derivative $\mathscr{L}_\Lambda$, with all derivatives entering via covariant derivatives or field strengths.[4] We thus compute, for instance, on the vector

$$
\begin{aligned}
\big[\delta_\Lambda, \delta_\xi\big] A_\mu{}^m &= \delta_\Lambda\big(\xi^\nu F_{\nu\mu}{}^m + \phi^{-\frac{2}{3}} G^{mn} g_{\mu\nu} \partial_n \xi^\nu\big) - \delta_\xi \left(\partial_\mu \Lambda^m + \mathscr{L}_\Lambda A_\mu{}^m\right) \\
&= \xi^\nu \mathscr{L}_\Lambda F_{\nu\mu}{}^m + \mathscr{L}_\Lambda\big(\phi^{-\frac{2}{3}} G^{mn} g_{\mu\nu}\big) \partial_n \xi^\nu - \mathscr{L}_\Lambda\big(\delta_\xi A_\mu{}^m\big) .
\end{aligned} \tag{3.12}
$$

Here we used the covariance of the expressions in $\delta_\xi A_\mu{}^m$. Thus, the terms in $\delta_\Lambda \delta_\xi A$ agree precisely with those in $\delta_\xi \delta_\Lambda A$, except that ξ, being a parameter and not a field, is not varied in the former but appears under the Lie derivative in the latter. These correspond to the left-over terms that do not cancel and that can in turn be interpreted as *external* diffeomorphisms with a parameter ξ that is 'rotated' (with the opposite sign) by the *internal* diffeomorphisms. Hence, the gauge algebra is given by

$$
\big[\delta_\Lambda, \delta_\xi\big] = \delta_{\xi'} , \qquad \xi'^\mu = -\mathscr{L}_\Lambda \xi^\mu = -\Lambda^m \partial_m \xi^\mu . \tag{3.13}
$$

The same conclusion follows for the external and internal vielbein. This algebra is embedded in the corresponding part of the gauge algebra of EFT, see (2.56).

Finally, we turn to the gauge algebra of external diffeomorphisms with themselves. Using again the improved diffeomorphisms (3.9), (3.10), an explicit computation shows

$$
\big[\delta_{\xi_1}, \delta_{\xi_2}\big] = \delta_{\xi_{12}} + \delta_{\Lambda_{12}} , \tag{3.14}
$$

where

$$
\xi_{12}^\mu = 2\, \xi_{[2}^\nu \mathscr{D}_\nu^{\rm KK} \xi_{1]}^\mu , \qquad \Lambda_{12}^m = \xi_2^\mu \xi_1^\nu F_{\mu\nu}{}^m - 2\, \phi^{-\frac{2}{3}} G^{mn} g_{\mu\nu}\, \xi_{[2}^\mu \partial_n \xi_{1]}^\nu . \tag{3.15}
$$

[4] The variation of the gauge vectors in (3.9) contains the partial derivative term $\partial_n \xi^\nu$, but ξ^ν has to be viewed as a scalar w.r.t. internal diffeomorphisms, hence its partial derivative is a covariant vector.

This is of the same structural form as the corresponding part of the gauge algebra (2.71)[5] and, together with our results below, implies that the full ten-dimensional diffeomorphism algebra is embedded in the gauge algebra of EFT.

So far we discussed the decomposition of fields and symmetries for pure (Einstein) gravity, but in supergravity there are additional matter fields, typically with associated gauge symmetries, which have to be decomposed similarly. Before turning to the specific field content of type IIB, let us consider a toy model, which exhibits already all essential features. We consider an abelian gauge vector $\hat{B}_{\hat{\mu}}$ (such as the RR one-form in type IIA) with gauge symmetries

$$\delta\hat{B}_{\hat{\mu}} \;=\; \partial_{\hat{\mu}}\chi+\xi^{\hat{\nu}}\partial_{\hat{\nu}}\hat{B}_{\hat{\mu}}+\partial_{\hat{\mu}}\xi^{\hat{\nu}}\hat{B}_{\hat{\nu}}\;, \tag{3.16}$$

for abelian parameter χ. Next we decompose the components as in (3.1) and redefine

$$\begin{aligned} B_m \;&=\; \hat{B}_m\;, \\ B_\mu \;&=\; \hat{B}_\mu-A_\mu{}^m\hat{B}_m\;. \end{aligned} \tag{3.17}$$

(In terms of the notation introduced in sec. 4 this corresponds to the action with the 'bar operator', $B\rightarrow\overline{B}$.) For these redefined fields the abelian gauge symmetry becomes

$$\begin{aligned} \delta_\chi B_\mu \;&=\; \mathscr{D}^{\rm KK}_\mu\chi \;=\; \partial_\mu\chi-A_\mu{}^m\partial_m\chi\;, \\ \delta_\chi B_m \;&=\; \partial_m\chi\;, \end{aligned} \tag{3.18}$$

and for the diffeomorphisms

$$\begin{aligned} \delta B_m \;&=\; \xi^\nu\partial_\nu B_m+\partial_m\xi^\nu\hat{B}_\nu+\mathscr{L}_\Lambda B_m\;, \\ \delta B_\mu \;&=\; \mathscr{L}_\Lambda B_\mu+\mathscr{L}_\xi B_\mu-A_\mu{}^m\partial_m\xi^\nu B_\nu-\phi^{-\frac{2}{3}}G^{mn}B_m g_{\mu\nu}\partial_n\xi^\nu\;, \end{aligned} \tag{3.19}$$

where $\mathscr{L}_\xi$ denotes the standard Lie derivative w.r.t. ξ^μ (with partial derivatives). Adding now field-dependent gauge transformations as above, with $\Lambda^m=-\xi^\nu A_\nu{}^m$ and $\chi=-\xi^\nu B_\nu$, this can be written more covariantly as

$$\delta_\xi B_m \;=\; \xi^\nu\widehat{\mathscr{D}}_\nu B_m \;\equiv\; \xi^\nu\big(\partial_\nu B_m-\mathscr{L}_{A_\nu}B_m-\partial_m B_\nu\big)\;, \tag{3.20}$$

for the internal components, and as

$$\delta_\xi B_\mu \;=\; \xi^\nu G_{\nu\mu}-\phi^{-\frac{2}{3}}G^{mn}g_{\mu\nu}\partial_m\xi^\nu\;, \tag{3.21}$$

where

$$G_{\mu\nu} \;\equiv\; \mathscr{D}^{\rm KK}_\mu B_\nu-\mathscr{D}^{\rm KK}_\nu B_\mu\;. \tag{3.22}$$

Note that due to the non-commutativity of covariant derivatives this is *not* an invariant field strength. Rather, G transforms as

$$\delta_{\Lambda,\chi}G_{\mu\nu} \;=\; \Lambda^m\partial_m G_{\mu\nu}-\partial_m\chi F_{\mu\nu}{}^m\;. \tag{3.23}$$

[5]It should be noted that, in general, in EFT there are higher-form transformations on the right-hand side of the gauge algebra, corresponding to the higher forms in the tensor hierarchy, which are not present here. As these are needed because of the anomalous 'Jacobiator' of the E-bracket, which vanishes on solutions of the section constraint, this is perfectly consistent with the embedding of the conventional diffeomorphism algebra.

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We could define a fully χ-invariant field strength by setting $\bar{G}_{\mu\nu} \equiv G_{\mu\nu} + F_{\mu\nu}{}^m B_m$, but it turns out that the match with EFT requires the (analogue of the) above form. In fact, in EFT a slightly more general notion of covariance is appropriate: the gauge parameters analogous to Λ^m and χ will be components of the generalized diffeomorphism parameter Λ^M, and the field strengths $G_{\mu\nu}$ and $F_{\mu\nu}{}^m$ correspond to components of the EFT field strength $\mathscr{F}_{\mu\nu}{}^M$, so that transformations such as (3.23) originate from the covariant transformation governed by the full generalized Lie derivative of EFT, $\delta_\Lambda \mathscr{F}_{\mu\nu}{}^M = \mathbb{L}_\Lambda \mathscr{F}_{\mu\nu}{}^M$.

We finally note that it is straightforward to verify that the transformations (3.21) and (3.23) close according to the gauge algebras (3.13) and (3.15), encoding the full diffeomorphism algebra. Conversely, starting with component fields B_μ, B_m, and gauge symmetries closing according to the above algebra (3.13), (3.15), we can reconstruct the form with manifest (say ten-dimensional) diffeomorphism invariance.

3.2 Type IIB solution of section constraint

We now turn to the specific solution of the section constraint that will be shown to lead to a formulation that is on-shell equivalent to type IIB supergravity. To this end we have to break $\mathrm{E}_{6(6)}$ to $\mathrm{GL}(5) \times \mathrm{SL}(2)$, embedding the residual group according to

$$\mathrm{GL}(5) \times \mathrm{SL}(2) \;\subset\; \mathrm{SL}(6) \times \mathrm{SL}(2) \;\subset\; \mathrm{E}_{6(6)}\,. \tag{3.24}$$

In this case, the fundamental and the adjoint representation of $\mathrm{E}_{6(6)}$ break as

$$\bar{\mathbf{27}} \;\rightarrow\; (5,1)_{+4} + (5',2)_{+1} + (10,1)_{-2} + (1,2)_{-5}\,, \tag{3.25}$$

$$\mathbf{78} \;\rightarrow\; (5,1)_{-6} + (10',2)_{-3} + (1+15+20)_0 + (10,2)_{+3} + (5',1)_{+6}\,, \tag{3.26}$$

with the subscripts referring to the charges under $\mathrm{GL}(1) \subset \mathrm{GL}(5)$. An explicit solution to the section condition (2.1) is given by restricting the Y^M dependence of all fields to the five coordinates in the $(5,1)_{+4}$. Explicitly, splitting the coordinates Y^M and the fundamental indices according to (3.25) into

$$\{Y^M\} \;\rightarrow\; \{\,y^m\,,\, y_{m\alpha}\,,\, y^{mn}\,,\, y_\alpha\,\}\,, \tag{3.27}$$

with internal indices $m,n = 1,\dots,5$ and $\mathrm{SL}(2)$ indices $\alpha = 1,2$, the non-vanishing components of the d-symbol are given by

$$\begin{aligned}
d^{MNK}\;:\; d^m{}_{n\alpha,\beta} &= \tfrac{1}{\sqrt{10}}\,\delta^m_n\,\varepsilon_{\alpha\beta}\,, \quad & d^{mn}{}_{k\alpha,l\beta} &= \tfrac{1}{\sqrt{5}}\,\delta^{mn}_{kl}\,\varepsilon_{\alpha\beta}\,, \quad & d^{mn,kl,p} &= \tfrac{1}{\sqrt{40}}\,\varepsilon^{mnklp}\,,\\
d_{MNK}\;:\; d_m{}^{n\alpha,\beta} &= \tfrac{1}{\sqrt{10}}\,\delta^n_m\varepsilon^{\alpha\beta}\,, \quad & d_{mn}{}^{k\alpha,l\beta} &= \tfrac{1}{\sqrt{5}}\,\delta^{kl}_{mn}\,\varepsilon^{\alpha\beta}\,, \quad & d_{mn,kl,p} &= \tfrac{1}{\sqrt{40}}\,\varepsilon_{mnklp}\,,
\end{aligned} \tag{3.28}$$

and all those related by symmetry, $d^{MNK} = d^{(MNK)}$. In particular, the $\mathrm{GL}(1)$ grading guarantees that all components d^{mnk} vanish. It follows that the section condition (2.1) indeed is solved by restricting the coordinate dependence of all fields according to

$$\{\partial^{m\alpha} A = 0\,,\; \partial_{mn} A = 0\,,\; \partial^\alpha A = 0\} \qquad \Longleftrightarrow \qquad A(x^\mu, Y^M) \;\longrightarrow\; A(x^\mu, y^m)\,. \tag{3.29}$$

Indeed, the section constraint then reduces to $d^{Mnk}\partial_n \otimes \partial_k = 0$, for which all relevant components of the d-symbol simply vanish.

3.3 Decomposition of EFT fields

In this subsection we analyze various objects of EFT, e.g., the generalized metric and the gauge covariant curvatures, in terms of the component fields originating under the above decomposition of $E_{6(6)}$, together with their gauge symmetries. This sets the stage for our analysis in sec. 4, where we start from type IIB supergravity and perform the complete Kaluza-Klein decomposition in order to match it to the fields and symmetries discussed here. Thus, here we split tensor fields and indices according to (3.25)–(3.28), assuming the explicit solution (3.29) of the section condition.

To begin, let us consider the p-form field content of the $E_{6(6)}$ EFT under the split (3.25). This yields

$$\mathscr{A}_\mu{}^M \;:\; \left\{\mathscr{A}_\mu{}^m, \mathscr{A}_{\mu\, m\alpha}, \mathscr{A}_{\mu\, kmn}, \mathscr{A}_{\mu\, \alpha}\right\}\,, \qquad \mathscr{B}_{\mu\nu M} \;:\; \left\{\mathscr{B}_{\mu\nu}{}^\alpha, \mathscr{B}_{\mu\nu\, mn}, \mathscr{B}_{\mu\nu}{}^{m\alpha}, \mathscr{B}_{\mu\nu\, m}\right\} \tag{3.30}$$

where we have defined $\mathscr{A}_{\mu\, kmn} \equiv \frac{1}{2}\varepsilon_{kmnpq}\mathscr{A}_\mu{}^{pq}$. However, the EFT Lagrangian actually depends on the two-forms only under certain derivatives,

$$\left\{\, \partial_m \mathscr{B}_{\mu\nu}{}^\alpha\,,\; \partial_{[k}\mathscr{B}_{|\mu\nu|\, mn]}\,,\; \partial_m \mathscr{B}_{\mu\nu}{}^{m\alpha} \,\right\}\,, \tag{3.31}$$

introducing an additional redundancy in the two-form field content, which will be important for the match with type IIB. As discussed above, the vector fields $\mathscr{A}_\mu{}^m$ will be identified with the IIB Kaluza-Klein vector fields, which transform according to (3.4) and in particular close according to the standard Lie bracket of five-dimensional diffeomorphisms, see (3.11), embedded into the E-bracket (2.8).

Let us now work out the general formulas of the $E_{6(6)}$-covariant formulation with (3.28) and imposing the explicit solution of the section condition (3.29) on all fields. We then obtain, by inserting (3.28) into (2.11), the following covariant field strengths of the different vector fields in (3.30),

$$\begin{aligned}
\mathscr{F}_{\mu\nu}{}^m &= 2\partial_{[\mu}\mathscr{A}_{\nu]}{}^m - \mathscr{A}_\mu{}^n\partial_n\mathscr{A}_\nu{}^m + \mathscr{A}_\nu{}^n\partial_n\mathscr{A}_\mu{}^m\,,\\
\mathscr{F}_{\mu\nu\, m\alpha} &= 2D^{\rm KK}_{[\mu}\mathscr{A}_{\nu]\, m\alpha} + \varepsilon_{\alpha\beta}\,\partial_m\tilde{\mathscr{B}}_{\mu\nu}{}^\beta\,,\\
\mathscr{F}_{\mu\nu\, kmn} &= 2D^{\rm KK}_{[\mu}\mathscr{A}_{\nu]\, kmn} - 3\sqrt{2}\,\varepsilon^{\alpha\beta}\,\mathscr{A}_{[\mu\,[k|\alpha|}\partial_m\mathscr{A}_{\nu]\,n]\beta} + 3\,\partial_{[k}\tilde{\mathscr{B}}_{|\mu\nu|\, mn]}\,,\\
\mathscr{F}_{\mu\nu\,\alpha} &= 2D^{\rm KK}_{[\mu}\mathscr{A}_{\nu]\,\alpha} - 2(\partial_k\mathscr{A}_{[\mu}{}^k)\,\mathscr{A}_{\nu]\,\alpha} - \sqrt{2}\,\mathscr{A}_{[\mu}{}^{mn}\partial_n\mathscr{A}_{\nu]\, m\alpha}\\
&\quad - \sqrt{2}\,\mathscr{A}_{[\mu|m\alpha|}\partial_n\mathscr{A}_{\nu]}{}^{mn} - \varepsilon_{\alpha\beta}\,\partial_k\tilde{\mathscr{B}}_{\mu\nu}{}^{k\beta}\,,
\end{aligned} \tag{3.32}$$

with the redefined two-forms

$$\begin{aligned}
\tilde{\mathscr{B}}_{\mu\nu}{}^\alpha &\equiv \sqrt{10}\,\mathscr{B}_{\mu\nu}{}^\alpha - \varepsilon^{\alpha\beta}\,\mathscr{A}_{[\mu}{}^n\mathscr{A}_{\nu]\, n\beta}\,,\\
\tilde{\mathscr{B}}_{\mu\nu\, mn} &\equiv \sqrt{10}\,\mathscr{B}_{\mu\nu\, mn} + \mathscr{A}_{[\mu}{}^k\mathscr{A}_{\nu]\, kmn}\,,\\
\tilde{\mathscr{B}}_{\mu\nu}{}^{k\alpha} &\equiv \sqrt{10}\,\mathscr{B}_{\mu\nu}{}^{k\alpha} + \varepsilon^{\alpha\beta}\,\mathscr{A}_{[\mu}{}^k\mathscr{A}_{\nu]\beta}\,.
\end{aligned} \tag{3.33}$$

Here all covariant derivatives are $D^{\rm KK}_\mu \equiv \partial_\mu - \mathscr{L}_{\mathscr{A}_\mu}$, covariantized w.r.t. to the action of the five-dimensional internal diffeomorphisms reviewed above. The corresponding vector gauge transformations, obtained from (2.15), are given by

$$\begin{aligned}
\delta\mathscr{A}_\mu{}^m &= D^{\rm KK}_\mu\Lambda^m\,,\\
\delta\mathscr{A}_{\mu\, m\alpha} &= D^{\rm KK}_\mu\Lambda_{m\alpha} + \mathscr{L}_\Lambda\mathscr{A}_{\mu\, m\alpha} - \varepsilon_{\alpha\beta}\,\partial_m\tilde{\Xi}_\mu{}^\beta\,,\\
\delta\mathscr{A}_{\mu\, kmn} &= D^{\rm KK}_\mu\Lambda_{kmn} + \mathscr{L}_\Lambda\mathscr{A}_{\mu\, kmn} - 3\sqrt{2}\,\varepsilon^{\alpha\beta}\,\partial_{[k}\mathscr{A}_{|\mu|m|\alpha|}\Lambda_{n]\beta} - 3\,\partial_{[k}\tilde{\Xi}_{|\mu|\, mn]}\,,
\end{aligned} \tag{3.34}$$

with

$$\tilde{\Xi}_\mu{}^\alpha \equiv \sqrt{10}\,\Xi_\mu{}^\alpha - \varepsilon^{\alpha\beta}\,\Lambda^n \mathscr{A}_{\mu\,n\beta}\,, \qquad \tilde{\Xi}_{\mu\,mn} \equiv \sqrt{10}\,\Xi_{\mu\,mn} + \Lambda^k \mathscr{A}_{\mu\,kmn}\,. \tag{3.35}$$

For the vector fields $\mathscr{A}_{\mu\,\alpha}$ we observe that its gauge variation contains the contribution

$$\delta\mathscr{A}_{\mu\,\alpha} = \cdots + \varepsilon_{\alpha\beta}\,\partial_k \tilde{\Xi}_\mu{}^{k\beta}\,. \tag{3.36}$$

This implies that it can entirely be gauged away by the tensor gauge symmetry associated with the two-forms $\mathscr{B}_{\mu\nu}{}^{k\beta}$. Consequently, it will automatically disappear from the Lagrangian upon integrating out $\partial_k\mathscr{B}_{\mu\nu}{}^{k\beta}$. The remaining two-form field strengths in turn come with gauge transformations

$$\begin{aligned}
\delta\tilde{\mathscr{B}}_{\mu\nu}{}^\alpha &= 2D^{\text{KK}}_{[\mu}\,\tilde{\Xi}_{\nu]}{}^\alpha + \mathscr{L}_\Lambda \tilde{\mathscr{B}}_{\mu\nu}{}^\alpha - \varepsilon^{\alpha\beta}\,\Lambda_{n\beta}\mathscr{F}_{\mu\nu}{}^n + \tilde{\mathscr{O}}_{\mu\nu}{}^\alpha\,,\\
\delta\tilde{\mathscr{B}}_{\mu\nu\,mn} &= 2D^{\text{KK}}_\mu\left(\tilde{\Xi}_{\nu\,mn} + \frac{1}{\sqrt{2}}\,\varepsilon^{\alpha\beta}\,\mathscr{A}_{\nu\,m\alpha}\Lambda_{n\beta}\right) + \sqrt{2}\,\partial_m\mathscr{A}_{\mu\,n\alpha}\,\tilde{\Xi}_\nu{}^\alpha\\
&\quad + \mathscr{L}_\Lambda\tilde{\mathscr{B}}_{\mu\nu\,mn} - \frac{1}{\sqrt{2}}\,\Lambda_{[m|\alpha|}\,\partial_{n]}\tilde{\mathscr{B}}_{\mu\nu}{}^\alpha + \Lambda_{mnk}\,\mathscr{F}_{\mu\nu}{}^k\\
&\quad + \frac{1}{\sqrt{2}}\,\varepsilon^{\alpha\beta}\,\mathscr{F}_{\mu\nu\,m\alpha}\Lambda_{n\beta} + \tilde{\mathscr{O}}_{\mu\nu\,mn}\,,
\end{aligned} \tag{3.37}$$

where

$$\begin{aligned}
\tilde{\mathscr{O}}_{\mu\nu}{}^\alpha &\equiv \sqrt{10}\,\mathscr{O}_{\mu\nu}{}^\alpha\,, \qquad\qquad (3.38)\\
\tilde{\mathscr{O}}_{\mu\nu\,mn} &\equiv \sqrt{10}\,\mathscr{O}_{\mu\nu\,mn} + \partial_m\left(2\Lambda^k\,\tilde{\mathscr{B}}_{\mu\nu\,nk} + \sqrt{2}\mathscr{A}_{\mu\,n\alpha}\,\Xi_\nu{}^\alpha + \sqrt{2}\,\varepsilon^{\alpha\beta}\,\mathscr{A}_{\mu\,n\alpha}\mathscr{A}_{\nu\,k\beta}\right).
\end{aligned}$$

Finally, the associated three-form field strengths are obtained from (2.13) and read

$$\begin{aligned}
\tilde{\mathscr{H}}_{\mu\nu\rho}{}^\alpha &\equiv \sqrt{10}\,\mathscr{H}_{\mu\nu\rho}{}^\alpha = 3\,D^{\text{KK}}_{[\mu}\,\tilde{\mathscr{B}}_{\nu\rho]}{}^\alpha + 3\,\varepsilon^{\alpha\beta}\,\mathscr{F}_{[\mu\nu}{}^n\mathscr{A}_{\rho]\,n\beta}\,, \qquad\qquad (3.39)\\
\tilde{\mathscr{H}}_{\mu\nu\rho\,mn} &\equiv \sqrt{10}\,\mathscr{H}_{\mu\nu\rho\,mn}\\
&= 3\,D^{\text{KK}}_\mu\,\tilde{\mathscr{B}}_{\nu\rho\,mn} - 3\,\mathscr{F}_{\mu\nu}{}^k\mathscr{A}_{\rho\,kmn} - 3\sqrt{2}\,\varepsilon^{\alpha\beta}\,\mathscr{A}_{\mu m\alpha}D_\nu\mathscr{A}_{\rho n\beta} + 3\sqrt{2}\,\mathscr{A}_{\mu m\alpha}\partial_n\tilde{\mathscr{B}}_{\nu\rho}{}^\alpha\,.
\end{aligned}$$

More precisely, this holds up to terms that are projected out from the Lagrangian under y-derivatives. The expressions on the r.h.s. in (3.37)–(3.39) are understood to be projected onto the corresponding antisymmetrizations in their parameters, i.e. $[mn]$, $[\mu\nu]$, $[\mu\nu\rho]$, etc.

It is also instructive to give the component form of the Bianchi identities originating from (2.12) and (2.14). From the latter we obtain the components

$$4\,D^{\text{KK}}_{[\mu}\,\tilde{\mathscr{H}}_{\nu\rho\sigma]}{}^\alpha = 6\,\varepsilon^{\alpha\beta}\,\mathscr{F}_{[\mu\nu}{}^n\mathscr{F}_{\rho\sigma]\,n\beta}\,. \tag{3.40}$$

After a straightforward but somewhat tedious computation one finds

$$\begin{aligned}
4\,D^{\text{KK}}_{[\mu}\,\tilde{\mathscr{H}}_{\nu\rho\sigma]\,mn} + 4\sqrt{2}\,\mathscr{A}_{\mu m\alpha}\partial_n\tilde{\mathscr{H}}_{\nu\rho\sigma}{}^\alpha &= -6\,\mathscr{F}_{[\mu\nu}{}^k\mathscr{F}_{\rho\sigma]\,kmn} - 3\sqrt{2}\,\varepsilon^{\alpha\beta}\,\mathscr{F}_{[\mu\nu\,|m\alpha|}\mathscr{F}_{\rho\sigma]\,n\beta}\\
&\quad + 3\sqrt{2}\,\partial_m\left(\varepsilon_{\alpha\beta}\,\tilde{\mathscr{B}}_{\mu\nu}{}^\alpha\partial_n\tilde{\mathscr{B}}_{\rho\sigma}{}^\beta\right) - 12\,\partial_m\left(\mathscr{F}_{\mu\nu}{}^k\tilde{\mathscr{B}}_{\rho\sigma\,kn}\right)\\
&\quad - 6\sqrt{2}\,\partial_m\left(\mathscr{A}_{\mu n\alpha}\varepsilon^{\alpha\beta}\,\mathscr{F}_{\nu\rho}{}^k\mathscr{A}_{\sigma\,k\beta}\right)\,.
\end{aligned} \tag{3.41}$$

Again, the indices m,n and μ,ν,ρ,σ in here are totally antisymmetrized, which we did not indicate explicitly in order not to clutter the notation.

Let us now move to the scalar field content of the theory. In the EFT formulation, they parametrize the symmetric matrix $\mathcal{M}_{MN}$. We now need to choose a parametrization of this matrix in accordance with the decomposition (3.26). In standard fashion [58], we build the matrix as $\mathcal{M}_{MN} = (\mathcal{V}\mathcal{V}^T)_{MN}$ from a 'vielbein' $\mathcal{V} \in \mathrm{E}_{6(6)}$ in triangular gauge

$$\mathcal{V} \equiv \exp\left[\varepsilon^{klmnp}\, c_{klmn}\, t_{(+6)\,p}\right] \exp\left[b_{mn}{}^{\alpha}\, t_{(+3)}{}^{mn}_{\alpha}\right] \mathcal{V}_5\, \mathcal{V}_2 \exp\left[\Phi\, t_{(0)}\right] . \tag{3.42}$$

Here, $t_{(0)}$ is the $\mathrm{E}_{6(6)}$ generator associated to the GL(1) grading of (3.26), $\mathcal{V}_2$, $\mathcal{V}_5$ denote matrices in the SL(2) and SL(5) subgroup, respectively, parametrized by vielbeins v_2, v_5. The $t_{(+n)}$ refer to the $\mathrm{E}_{6(6)}$ generators of positive grading in (3.26), with non-trivial commutator

$$\left[t_{(+3)}{}^{kl}_{\alpha}, t_{(+3)}{}^{mn}_{\beta}\right] = \varepsilon_{\alpha\beta}\, \varepsilon^{klmnp}\, t_{(+6)\,p} \,. \tag{3.43}$$

All generators are evaluated in the fundamental **27** representation (3.25), such that the symmetric matrix $\mathcal{M}_{MN}$ takes the block form

$$\mathcal{M}_{KM} = \begin{pmatrix} \mathcal{M}_{km} & \mathcal{M}_k{}^{m\beta} & \mathcal{M}_{k,mn} & \mathcal{M}_k{}^{\beta} \\ \mathcal{M}^{k\alpha}{}_m & \mathcal{M}^{k\alpha,m\beta} & \mathcal{M}^{k\alpha}{}_{mn} & \mathcal{M}^{k\alpha,\beta} \\ \mathcal{M}_{kl,m} & \mathcal{M}_{kl}{}^{m\beta} & \mathcal{M}_{kl,mn} & \mathcal{M}_{kl}{}^{\beta} \\ \mathcal{M}^{\alpha}{}_m & \mathcal{M}^{\alpha,m\beta} & \mathcal{M}^{\alpha}{}_{mn} & \mathcal{M}^{\alpha\beta} \end{pmatrix} . \tag{3.44}$$

Explicit evaluation of (3.42) determines the various blocks in (3.44). For instance,

$$\mathcal{M}_{mn,kl} = e^{2\Phi/3}\, m_{m[k} m_{l]n} + 2e^{5\Phi/3}\, b_{mn}{}^{\alpha} b_{kl}{}^{\beta}\, m_{\alpha\beta} \,, \tag{3.45}$$

while the components in the last line are given by[6]

$$\begin{aligned} \mathcal{M}^{\alpha\beta} &= e^{5\Phi/3}\, m^{\alpha\beta} \,, \qquad \mathcal{M}^{\alpha}{}_{mn} = \sqrt{2}\, e^{5\Phi/3}\, m^{\alpha\beta}\, \varepsilon_{\beta\gamma} b_{mn}{}^{\gamma} \,, \\ \mathcal{M}^{\alpha,m\beta} &= \frac{1}{2} e^{5\Phi/3}\, m^{\alpha\gamma} \varepsilon_{\gamma\delta}\, \varepsilon^{mklpq}\, b_{kl}{}^{\beta} b_{pq}{}^{\delta} - \frac{1}{24} e^{5\Phi/3}\, m^{\alpha\beta}\, \varepsilon^{mklpq}\, c_{klpq} \,, \\ \mathcal{M}^{\alpha}{}_m &= \frac{2}{3} e^{5\Phi/3}\, m_{\beta\gamma} \varepsilon^{kpqrs} \left(b_{mk}{}^{[\alpha} b_{pq}{}^{\beta]} b_{rs}{}^{\gamma} + \frac{1}{8} \varepsilon^{\alpha\beta}\, b_{mk}{}^{\gamma} c_{pqrs} \right) , \end{aligned} \tag{3.46}$$

with the symmetric matrix $m^{\alpha\beta} = (v_2)^{\alpha}{}_u (v_2)^{\beta\, u}$ built from the SL(2) vielbein from (3.42). We will also need the following combinations of the matrix entries of $\mathcal{M}_{MN}$ (that emerge after integrating out some of the fields),

$$\tilde{\mathcal{M}}_{MN} \equiv \mathcal{M}_{MN} - \mathcal{M}_M{}^{\alpha} (\mathcal{M}^{\alpha\beta})^{-1} \mathcal{M}_N{}^{\beta} \,, \tag{3.47}$$

[6] The explicit expressions (3.46) and (3.48) for the matrix components of $\mathcal{M}_{MN}$ and $\tilde{\mathcal{M}}_{MN}$ correct some typos in equations (5.22) and (5.24), respectively, in the published version of [13].

for which we find

$$
\begin{aligned}
\tilde{\mathcal{M}}_{mn,kl} &= e^{2\Phi/3}\, m_{m[k}m_{l]n} \,, \\
\tilde{\mathcal{M}}_{mn}{}^{k\alpha} &= \frac{1}{\sqrt{2}}\, e^{2\Phi/3}\, \varepsilon_{mnpqr} m^{kp} m^{qu} m^{rv} b_{uv}{}^{\alpha} \,, \\
\tilde{\mathcal{M}}_{mn,k} &= -\frac{1}{6\sqrt{2}}\, e^{2\Phi/3}\, \varepsilon^{uvpqr}\, m_{mu} m_{nv} \left(c_{kpqr} - 6\varepsilon_{\alpha\beta}\, b_{kp}{}^{\alpha} b_{qr}{}^{\beta} \right) \\
\tilde{\mathcal{M}}^{m\alpha,n\beta} &= e^{-\Phi/3}\, m^{mn} m^{\alpha\beta} + 2\, e^{2\Phi/3}\, m^{kp} \left(m^{mn} m^{lq} - 2\, m^{ml} m^{nq} \right) b_{kl}{}^{\alpha} b_{pq}{}^{\beta} \,,
\end{aligned}
\tag{3.48}
$$

etc., with $m_{mn} = (\nu_5)_m{}^a (\nu_5)_n{}^a$.

Next, we can work out the covariant derivatives of the various 'scalar components' of the generalized metric. Using (3.28) we find for the covariant derivatives of the matrix parameters in (3.44)

$$
\begin{aligned}
\mathcal{D}_\mu \Phi &= D_\mu^{\rm KK}\, \Phi + \tfrac{4}{5}\, \partial_k \mathcal{A}_\mu{}^k \,, \\
\mathcal{D}_\mu m_{mn} &= D_\mu^{\rm KK}\, m_{mn} + \tfrac{2}{5}\, \partial_k \mathcal{A}_\mu{}^k\, m_{mn} \,, \\
\mathcal{D}_\mu b_{mn}{}^{\alpha} &= D_\mu^{\rm KK}\, b_{mn}{}^{\alpha} - \varepsilon^{\alpha\beta}\, \partial_{[m} \mathcal{A}_{n]\beta\,\mu} \,, \\
\mathcal{D}_\mu c_{klmn} &= D_\mu^{\rm KK}\, c_{klmn} + 4\sqrt{2}\, \partial_{[k} \mathcal{A}_{lmn]\mu} + 12\, b_{[kl}{}^{\alpha}\, \partial_m \mathcal{A}_{n]\,\alpha\,\mu} \,,
\end{aligned}
\tag{3.49}
$$

where we recall that $D_\mu^{\rm KK}$ denotes the covariant derivatives w.r.t. $\mathcal{A}_\mu{}^m$ (that below will be identified with the Kaluza-Klein vector $A_\mu{}^m$) without the density terms, which here have been indicated explicitly, thereby defining the weight of all fields. The form of these covariant derivatives implies in particular that we have the following gauge symmetries on these fields,

$$
\begin{aligned}
\delta \Phi &= \mathcal{L}_\Lambda \Phi - \tfrac{4}{5}\, \partial_k \Lambda^k \,, \\
\delta m_{mn} &= \mathcal{L}_\Lambda m_{mn} - \tfrac{2}{5}\, \partial_k \Lambda^k\, m_{mn} \,, \\
\delta b_{mn}{}^{\alpha} &= \mathcal{L}_\Lambda b_{mn}{}^{\alpha} + \varepsilon^{\alpha\beta}\, \partial_{[m} \Lambda_{n]\beta} \,, \\
\delta c_{klmn} &= \mathcal{L}_\Lambda c_{klmn} - 4\sqrt{2}\, \partial_{[k} \Lambda_{lmn]} - 12\, b_{[kl}{}^{\alpha}\, \partial_m \Lambda_{n]\,\alpha} \,.
\end{aligned}
\tag{3.50}
$$

We close this section by giving some relevant formulas for the decompositions of various terms in the action upon putting the solution of the section constraint. The scalar kinetic term (2.21) yields

$$
\begin{aligned}
\frac{1}{24}\, D_\mu \mathcal{M}_{MN} D^\mu \mathcal{M}^{MN} &= -\frac{5}{6}\, \mathcal{D}_\mu \Phi \mathcal{D}^\mu \Phi + \frac{1}{4}\, \mathcal{D}_\mu m_{\alpha\beta} \mathcal{D}^\mu m^{\alpha\beta} + \frac{1}{4}\, \mathcal{D}_\mu m_{mn} \mathcal{D}^\mu m^{mn} \\
&\quad - e^{\Phi}\, \mathcal{D}_\mu b_{mn}{}^{\alpha} \mathcal{D}^\mu b_{kl}{}^{\beta}\, m^{mk} m^{nl} m_{\alpha\beta} \\
&\quad - \frac{1}{48}\, e^{2\Phi}\, \widehat{\mathcal{D}}_\mu c_{klmn} \widehat{\mathcal{D}}^\mu c_{pqrs} m^{kp} m^{lq} m^{mr} m^{ns} \,,
\end{aligned}
\tag{3.51}
$$

where we defined

$$
\widehat{\mathcal{D}}_\mu c_{klmn} \equiv \mathcal{D}_\mu c_{klmn} + 12\varepsilon_{\alpha\beta}\, b_{kl}{}^{\alpha} \mathcal{D}_\mu b_{mn}{}^{\beta} \,.
\tag{3.52}
$$

The 'scalar potential' (2.25) takes the form

$$
\begin{aligned}
V &= 3\, e^{7\Phi/3}\, \partial_{[k} b_{mn]}{}^{\alpha}\, \partial_l b_{pq}{}^{\beta}\, m^{kl} m^{mp} m^{nq} m_{\alpha\beta} \\
&\quad + \frac{5}{48}\, e^{10\Phi/3}\, X_{klmnp} X_{qrstu} m^{kq} m^{lr} m^{ms} m^{nt} m^{pu} + V_\Phi(\partial_k \Phi, \partial_k m_{mn}) \,,
\end{aligned}
\tag{3.53}
$$

where the last term combines all contributions with the internal derivative acting on Φ and m_{mn}, and

$$X_{klmnp} \equiv \partial_{[k}c_{lmnp]} + 12\,\varepsilon_{\alpha\beta}\,b_{[kl}{}^{\alpha}\partial_m b_{np]}{}^{\beta}\,. \tag{3.54}$$

Finally, we give the topological term (2.23) in this parametrization,

$$\begin{aligned}
\mathscr{L}_{\rm top} = \frac{1}{8}\,\varepsilon^{\mu\nu\rho\sigma\tau}\varepsilon^{klmnp}\Big(&\frac{\sqrt{2}}{6}\,\varepsilon^{\alpha\beta}\,\mathscr{F}_{\mu\nu m\alpha}\mathscr{F}_{\rho\sigma n\beta}A_{\tau pkl} + \frac{1}{6}\,\mathscr{F}_{\mu\nu mnq}F_{\rho\sigma}{}^{q}A_{\tau klp} \\
&-\frac{\sqrt{2}}{2}\,\varepsilon^{\alpha\beta}\,A_{\mu m\alpha}\partial_n A_{\nu p\beta}F_{\rho\sigma}{}^{q}A_{\tau klq} + \frac{1}{2}\,\partial_p\tilde{B}_{\mu\nu mn}F_{\rho\sigma}{}^{q}A_{\tau klq} \\
&+\sqrt{2}\,\varepsilon^{\alpha\beta}\,A_{\mu m\alpha}D_\nu A_{\rho n\beta}\,\partial_p\tilde{B}_{\sigma\tau kl} - \sqrt{2}A_{\mu m\alpha}\partial_n\tilde{B}_{\nu\rho}{}^{\alpha}\,\partial_p\tilde{B}_{\sigma\tau kl} \\
&+\frac{2}{3}\,\varepsilon^{\alpha\beta}\,A_{\mu m\alpha}\partial_n A_{\nu k\beta}A_{\rho l\gamma}\partial_p\tilde{B}_{\sigma\tau}{}^{\gamma} - \varepsilon^{\alpha\beta}\,\varepsilon^{\gamma\delta}A_{\mu m\alpha}\partial_n A_{\nu k\beta}A_{\rho l\gamma}D_\sigma A_{\tau p\delta} \\
&+\frac{\sqrt{2}}{9}\,\partial_m\mathscr{H}_{\mu\nu\rho}{}^{\alpha}A_{\sigma n\alpha}A_{\tau klp} - D_\mu\tilde{B}_{\nu\rho mn}\partial_p\tilde{B}_{\sigma\tau kl} - \frac{2}{3}\,\varepsilon_{\alpha\beta}\,\mathscr{H}_{\mu\nu\rho}{}^{\beta}\,\partial_k\tilde{B}_{\sigma\tau}{}^{k\alpha} \\
&+\mathscr{O}(A_{\mu\,\alpha})\Big)\,.
\end{aligned} \tag{3.55}$$

3.4 External diffeomorphisms

Let us finally turn to the action of the external diffeomorphisms (2.18) under the type IIB decomposition. On the external vielbein $e_\mu{}^a$ this symmetry reduces to that found in the Kaluza-Klein decomposition in (3.10), because on scalar-densities such as $e_\mu{}^a$ and ξ^μ the gauge-covariant derivative of EFT simply reduces to the Kaluza-Klein covariant derivative w.r.t. $\mathscr{A}_\mu{}^m$. For the internal generalized metric $\mathscr{M}_{MN}$ the external diffeomorphism transformations on the various components in (3.44) are read off from (2.18), with the EFT covariant derivatives written out in (3.49).

Next, we consider the external diffeomorphism transformations of the vector fields, which are more subtle due to the presence of the term involving the inverse of the generalized metric $\mathscr{M}$. From (3.46) we determine the relevant components of the matrix $\mathscr{M}^{MN}$,

$$\begin{aligned}
\mathscr{M}^{m,n} &= e^{4\Phi/3}\,m^{mn}\,, \\
\mathscr{M}_{m\alpha,}{}^{n} &= 2\,e^{4\Phi/3}\,\varepsilon_{\alpha\beta}\,m^{nk}b_{km}{}^{\beta}\,, \\
\mathscr{M}^{mn,k} &= -\frac{\sqrt{2}}{12}\,e^{4\Phi/3}\,\varepsilon^{mnpqr}\,m^{ks}\left(c_{pqrs} - 6\,\varepsilon_{\alpha\beta}\,b_{pq}{}^{\alpha}b_{rs}{}^{\beta}\right)\,.
\end{aligned} \tag{3.56}$$

This in turn determines the following gauge variations of the vector field components in (3.30),

$$\begin{aligned}
\delta_\xi\mathscr{A}_\mu{}^m &= \xi^\nu\mathscr{F}_{\nu\mu}{}^m + \mathscr{M}^{m,n}g_{\mu\nu}\partial_n\xi^\nu\,, \\
\delta_\xi\mathscr{A}_{\mu\,m\alpha} &= \xi^\nu\mathscr{F}_{\nu\mu m\alpha} + \mathscr{M}_{m\alpha,}{}^{n}g_{\mu\nu}\partial_n\xi^\nu\,, \\
\delta_\xi\mathscr{A}_{\mu\,mnk} &= \tfrac{1}{2}\varepsilon_{mnkpq}\xi^\nu\mathscr{F}_{\nu\mu}{}^{pq} + \tfrac{1}{2}\varepsilon_{mnkpq}\mathscr{M}^{pq,n}\partial_n\xi^\nu\,,
\end{aligned} \tag{3.57}$$

with the field strengths given in (3.32). As a first check that EFT subjected to this solution of the section constraint is equivalent to type IIB supergravity, we infer from the first variation in here that $\mathscr{A}_\mu{}^m$ has the same external diffeomorphism variation as the Kaluza-Klein vector, c.f. (3.9),

$$\delta_\xi\mathscr{A}_\mu{}^m = \xi^\nu\mathscr{F}_{\nu\mu}{}^m + \phi^{-\frac{2}{3}}G^{mn}g_{\mu\nu}\partial_n\xi^\nu\,, \tag{3.58}$$

therefore justifying the identification of both fields. Indeed, the fields strength components $\mathscr{F}_{\mu\nu}{}^{m}$ reduce to the Kaluza-Klein components $F_{\mu\nu}{}^{m}$, see (3.32) and (3.7), and the metric-dependent terms coincide upon identifying

$$e^{4\Phi/3}\, m^{mn} \;=\; \phi^{-2/3} G^{mn} \,, \tag{3.59}$$

which relates the matrix $m^{mn} \in \mathrm{SL}(5)$ and the scale factor Φ to the metric G^{mn} with dynamical determinant ϕ^2. (This relation can be fixed, for instance, by noting with (3.50) that both sides transform in the same way under internal diffeomorphisms.) The precise match for the remaining vector field components will be the subject of the following sections.

4. Type IIB supergravity and its Kaluza-Klein decomposition

In this section, we review ten-dimensional IIB supergravity and bring it into the form that allows a convenient translation of its field content into the various components of the EFT fields identified above.

4.1 Type IIB supergravity

Denoting ten-dimensional curved indices by $\hat{\mu}, \hat{\nu}, \ldots$, the type IIB field content is given by

$$E_{\hat{\mu}}{}^{\hat{a}}\,, \quad m_{\alpha\beta}\,, \quad \hat{C}_{\hat{\mu}\hat{\nu}}{}^{\alpha}\,, \quad \hat{C}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}\,, \quad \alpha,\beta = 1,2\,, \tag{4.1}$$

i.e., the zehnbein, the two $\mathrm{SL}(2)/\mathrm{SO}(2)$ coset scalars parametrizing the symmetric $\mathrm{SL}(2)$ matrix $m_{\alpha\beta}$, a doublet of 2-forms and a 4-form. The 2-forms combine RR 2-form and the NS B-field, with the abelian field strengths given by

$$\hat{F}_{\hat{\mu}\hat{\nu}\hat{\rho}}{}^{\alpha} \;=\; 3\,\partial_{[\hat{\mu}}\hat{C}_{\hat{\nu}\hat{\rho}]}{}^{\alpha}\,. \tag{4.2}$$

The Chern-Simons (CS)-modified curvature of the 4-form is given in components by

$$\hat{F}_{\hat{\mu}_1\ldots\hat{\mu}_5} \;\equiv\; 5\,\partial_{[\hat{\mu}_1}\hat{C}_{\hat{\mu}_2\ldots\hat{\mu}_5]} - \frac{5}{4}\,\varepsilon_{\alpha\beta}\,\hat{C}_{[\hat{\mu}_1\hat{\mu}_2}{}^{\alpha}\hat{F}_{\hat{\mu}_3\hat{\mu}_4\hat{\mu}_5]}{}^{\beta}\,, \tag{4.3}$$

such that they satisfy the Bianchi identities

$$6\,\partial_{[\hat{\mu}_1}\hat{F}_{\hat{\mu}_2\hat{\mu}_3\hat{\mu}_4\hat{\mu}_5\hat{\mu}_6]} = -\frac{5}{2}\,\varepsilon_{\alpha\beta}\hat{F}_{[\hat{\mu}_1\hat{\mu}_2\hat{\mu}_3}{}^{\alpha}\hat{F}_{\hat{\mu}_4\hat{\mu}_5\hat{\mu}_6]}{}^{\beta}\,, \tag{4.4}$$

and transform as

$$\begin{aligned}
\delta\hat{C}_{\hat{\mu}\hat{\nu}}{}^{\alpha} \;&=\; 2\,\partial_{[\hat{\mu}}\hat{\lambda}_{\hat{\nu}]}{}^{\alpha}\,,\\
\delta\hat{C}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} \;&=\; 4\,\partial_{[\hat{\mu}}\hat{\lambda}_{\hat{\nu}\hat{\rho}\hat{\sigma}]} + \frac{1}{2}\,\varepsilon_{\alpha\beta}\,\hat{\lambda}_{[\hat{\mu}}{}^{\alpha}\hat{F}_{\hat{\nu}\hat{\rho}\hat{\sigma}]}{}^{\beta}\,,
\end{aligned} \tag{4.5}$$

under tensor gauge transformations. The IIB field equations have been constructed in [59–61]. They can be described by a pseudo-action which in our conventions is given by

$$\begin{aligned}
S \;=\; \int d^{10}\hat{x}\sqrt{|G|}\,\Big(&\hat{R} + \frac{1}{4}\partial_{\hat{\mu}}m_{\alpha\beta}\partial^{\hat{\mu}}m^{\alpha\beta} - \frac{1}{12}\hat{F}_{\hat{\mu}_1\hat{\mu}_2\hat{\mu}_3}{}^{\alpha}\hat{F}^{\hat{\mu}_1\hat{\mu}_2\hat{\mu}_3\,\beta}m_{\alpha\beta}\\
&- \frac{1}{30}\hat{F}_{\hat{\mu}_1\hat{\mu}_2\hat{\mu}_3\hat{\mu}_4\hat{\mu}_5}\hat{F}^{\hat{\mu}_1\hat{\mu}_2\hat{\mu}_3\hat{\mu}_4\hat{\mu}_5}\Big)\\
&- \frac{1}{864}\int d^{10}\hat{x}\,\varepsilon_{\alpha\beta}\,\varepsilon^{\hat{\mu}_1\ldots\hat{\mu}_{10}}C_{\hat{\mu}_1\hat{\mu}_2\hat{\mu}_3\hat{\mu}_4}\hat{F}_{\hat{\mu}_6\hat{\mu}_7\hat{\mu}_8}{}^{\alpha}\hat{F}_{\hat{\mu}_8\hat{\mu}_9\hat{\mu}_{10}}{}^{\beta}\,,
\end{aligned} \tag{4.6}$$

and which after variation of the fields has to be supplemented with the standard self-duality equations for the 5-form field strength

$$\hat{F}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\tau}} = \frac{1}{5!}\sqrt{|G|}\,\varepsilon_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\tau}\hat{\mu}_1\hat{\mu}_2\hat{\mu}_3\hat{\mu}_4\hat{\mu}_5}\,\hat{F}^{\hat{\mu}_1\hat{\mu}_2\hat{\mu}_3\hat{\mu}_4\hat{\mu}_5}\,, \tag{4.7}$$

with $|G| \equiv |\det G_{\hat{\mu}\hat{\nu}}| = |\det E_{\hat{\mu}}{}^{\hat{a}}|^2$. It is straightforward to verify that the integrability conditions of the self-duality equations together with the Bianchi identities (4.4) coincide with the second-order field equations obtained by variation of (4.6). Our SL(2) conventions can be translated into the $\mathrm{SU}(1,1)/\mathrm{U}(1)$ conventions of [60], by combining the real components of the doublet $\hat{F}_{\hat{\mu}\hat{\nu}\hat{\rho}}{}^{\alpha}$ into a complex F

$$F_{\hat{\mu}\hat{\nu}\hat{\rho}} \equiv \hat{F}_{\hat{\mu}\hat{\nu}\hat{\rho}}{}^{1} + i\,\hat{F}_{\hat{\mu}\hat{\nu}\hat{\rho}}{}^{2}\,, \tag{4.8}$$

and parametrizing the symmetric SL(2) matrix $m_{\alpha\beta}$ in terms of a complex scalar B as

$$m_{\alpha\beta} \equiv (1-BB^*)^{-1}\begin{pmatrix} (1-B)(1-B^*) & i(B-B^*) \\ i(B-B^*) & (1+B)(1+B^*) \end{pmatrix}. \tag{4.9}$$

In terms of the complex combinations

$$G_{\hat{\mu}\hat{\nu}\hat{\rho}} \equiv f(F_{\hat{\mu}\hat{\nu}\hat{\rho}} - BF^*_{\hat{\mu}\hat{\nu}\hat{\rho}})\,, \quad P_{\hat{\mu}} \equiv f^2\partial_{\hat{\mu}}B\,, \qquad \text{with } f = (1-BB^*)^{-1/2}\,, \tag{4.10}$$

charged under the $\mathrm{U}(1) \subset \mathrm{SU}(1,1)$, the kinetic terms of (4.6) translate into those of [60] with

$$\begin{aligned} m_{\alpha\beta}\hat{F}_{\hat{\mu}\hat{\nu}\hat{\rho}}{}^{\alpha}\hat{F}^{\hat{\mu}\hat{\nu}\hat{\rho}\,\beta} &= G^*_{\hat{\mu}\hat{\nu}\hat{\rho}}G^{\hat{\mu}\hat{\nu}\hat{\rho}}\,, \\ \frac{1}{4}\partial_{\hat{\mu}}m_{\alpha\beta}\partial^{\hat{\mu}}m^{\alpha\beta} &= -2P^*_{\hat{\mu}}P^{\hat{\mu}}\,. \end{aligned} \tag{4.11}$$

In the following, we will perform the standard $5+5$ Kaluza-Klein redefinitions of the IIB fields but keeping the dependence on all ten coordinates.

4.2 Kaluza-Klein decomposition and field redefinitions

We now split the the coordinates according to a $5+5$ Kaluza-Klein decomposition into

$$x^{\hat{\mu}} = (x^{\mu}, y^{m})\,, \tag{4.12}$$

and similarly for the flat indices $\hat{a} = (\underline{a}, \underline{\alpha})$. The μ and $\underline{a}$ indices range from $0,\ldots,4$ and respectively represent the curved and flat indices of what we will refer to as the external space. Similarly, the indices m and $\underline{\alpha}$ range from $1,\ldots,5$ and are associated with the internal space. After partial fixation of the Lorentz gauge symmetry, the vielbein may be brought into triangular form (3.1)

$$E_{\hat{\mu}}{}^{\hat{a}} = \begin{pmatrix} \phi^{-1/3}\,e_{\mu}{}^{\underline{a}} & A_{\mu}{}^{m}\phi_{m}{}^{\underline{\alpha}} \\ 0 & \phi_{m}{}^{\underline{\alpha}} \end{pmatrix}, \tag{4.13}$$

parametrized in terms of two 5 by 5 matrices $e_{\mu}{}^{\underline{a}}$ and $\phi_{m}{}^{\underline{\alpha}}$ with $\phi \equiv \det(\phi_{m}{}^{\alpha})$, and the Kaluza-Klein vectors $A_{\mu}{}^{m}$. We stress again that all fields depend on all ten coordinates, such that we are still describing the full IIB theory. The result of the ten-dimensional Einstein-Hilbert term in the parametrization (4.13) has been given in [13] and in particular features the non-abelian Kaluza-Klein field strength

$$F_{\mu\nu}{}^{m} \equiv 2\,\partial_{[\mu}A_{\nu]}{}^{m} - A_{\mu}{}^{n}\partial_{n}A_{\nu}{}^{m} + A_{\nu}{}^{n}\partial_{n}A_{\mu}{}^{m} \,. \tag{4.14}$$

In order to describe the Kaluza-Klein decomposition of the p-forms, we introduce in standard Kaluza-Klein manner the projector $P_{\mu}{}^{\hat{\nu}} = E_{\mu}{}^{a}E_{a}{}^{\hat{\nu}}$. It converts 10-dimensional curved indices into 5-dimensional ones such that the resulting fields transform covariantly (i.e. according to the structure of their internal indices) under internal diffeomorphisms. We denote its action by a bar on the corresponding p-form components,

$$\overline{C}_{\mu} \equiv P_{\mu}{}^{\hat{\nu}}\hat{C}_{\hat{\nu}}\,, \qquad \text{etc.}\,, \tag{4.15}$$

such that the IIB two- and four-form give rise to the components

$$\begin{aligned}
\overline{C}_{mn}{}^{\alpha} &= \hat{C}_{mn}{}^{\alpha}\,,\\
\overline{C}_{\mu\,m}{}^{\alpha} &= \hat{C}_{\mu m}{}^{\alpha} - A_{\mu}{}^{p}\hat{C}_{pm}{}^{\alpha}\,,\\
\overline{C}_{\mu\nu}{}^{\alpha} &= \hat{C}_{\mu\nu}{}^{\alpha} - 2A_{[\mu}{}^{p}\hat{C}_{|p|\nu]}{}^{\alpha} + A_{\mu}{}^{p}A_{\nu}{}^{q}\hat{C}_{pq}{}^{\alpha}\,,\\
\overline{C}_{mnkl} &= \hat{C}_{mnkl}\,,\\
\overline{C}_{\mu\,nkl} &= \hat{C}_{\mu nkl} - A_{\mu}{}^{p}\hat{C}_{pnkl}\,,\\
\overline{C}_{\mu\nu\,kl} &= \hat{C}_{\mu\nu kl} - 2A_{[\mu}{}^{p}\hat{C}_{|p|\nu]kl} + A_{\mu}{}^{p}A_{\nu}{}^{q}\hat{C}_{pqkl}\,,\\
\overline{C}_{\mu\nu\rho\,l} &= \hat{C}_{\mu\nu\rho\,l} - 3A_{[\mu}{}^{p}\hat{C}_{|p|\nu\rho]\,l} + 3A_{[\mu}{}^{p}A_{\nu}{}^{q}\hat{C}_{|pq|\rho]\,l} - A_{\mu}{}^{p}A_{\nu}{}^{q}A_{\rho}{}^{r}\hat{C}_{pqrl}\,,\\
\overline{C}_{\mu\nu\rho\sigma} &= \hat{C}_{\mu\nu\rho\sigma} - 4A_{[\mu}{}^{p}\hat{C}_{|p|\nu\rho\sigma]} + 6A_{[\mu}{}^{p}A_{\nu}{}^{q}\hat{C}_{|pq|\rho\sigma]} - 4A_{[\mu}{}^{p}A_{\nu}{}^{q}A_{\rho}{}^{r}\hat{C}_{|pqr|\sigma]}\\
&\quad + A_{\mu}{}^{p}A_{\nu}{}^{q}A_{\rho}{}^{r}A_{\sigma}{}^{s}\hat{C}_{pqrs}\,.
\end{aligned} \tag{4.16}$$

The same redefinition applies to field strengths and gauge parameters. The redefined fields now transform covariantly under internal diffeomorphisms. Indeed, separating ten-dimensional diffeomorphisms into $\xi^{\hat{\mu}} = (\xi^{\mu}, \Lambda^{m})$, we find together with (4.5)

$$\begin{aligned}
\delta\overline{C}_{mn}{}^{\alpha} &= 2\partial_{[m}\overline{\lambda}_{n]}{}^{\alpha} + \mathscr{L}_{\Lambda}\overline{C}_{mn}{}^{\alpha}\,,\\
\delta\overline{C}_{\mu\,m}{}^{\alpha} &= D^{\rm KK}_{\mu}\overline{\lambda}_{m}{}^{\alpha} - \partial_{m}\overline{\lambda}_{\mu}{}^{\alpha} + \mathscr{L}_{\Lambda}\overline{C}_{\mu\,m}{}^{\alpha}\,,\\
\delta\overline{C}_{\mu\nu}{}^{\alpha} &= 2D^{\rm KK}_{[\mu}\overline{\lambda}_{\nu]}{}^{\alpha} + F_{\mu\nu}{}^{k}\overline{\lambda}_{k}{}^{\alpha} + \mathscr{L}_{\Lambda}\overline{C}_{\mu\nu}{}^{\alpha}\,,
\end{aligned} \tag{4.17}$$

for the transformation behaviour of the redefined 2-forms under gauge transformations and internal diffeomorphisms. As in the previous section, derivatives $D^{\rm KK}_{\mu}$ are covariantized w.r.t. the action of internal diffeomorphisms, i.e.

$$D^{\rm KK}_{\mu}\overline{\lambda}_{m}{}^{\alpha} \equiv \partial_{\mu}\overline{\lambda}_{m}{}^{\alpha} - A_{\mu}{}^{n}\partial_{n}\overline{\lambda}_{m}{}^{\alpha} - \partial_{m}A_{\mu}{}^{n}\overline{\lambda}_{n}{}^{\alpha}\,, \qquad \text{etc.}. \tag{4.18}$$

In contrast to $D=11$ supergravity for which these redefinitions and covariant gauge transformations have been explicitly worked out in [13], the presence of Chern-Simons terms in the IIB field strengths (4.3) requires a further redefinition for the components of the 4-form in order to establish the dictionary to the fields of EFT. This is related to the fact that tensor gauge transformations for the EFT p-forms that we have discussed in the previous section do not mix these forms with the scalar fields of the theory. This motivates the following and final field redefinition[7]

$$
\begin{aligned}
C_{klmn} &\equiv \overline{C}_{klmn} \,, \\
C_{\mu\, kmn} &\equiv \overline{C}_{\mu\, kmn} - \frac{3}{8}\,\varepsilon_{\alpha\beta}\overline{C}_{\mu\,[k}{}^{\alpha}\overline{C}_{mn]}{}^{\beta} \,, \\
C_{\mu\nu\, mn} &\equiv \overline{C}_{\mu\nu\, mn} - \frac{1}{8}\,\varepsilon_{\alpha\beta}\overline{C}_{\mu\nu}{}^{\alpha}\overline{C}_{mn}{}^{\beta} \,, \\
C_{\mu\nu\rho\, m} &\equiv \overline{C}_{\mu\nu\rho\, m} - \frac{3}{8}\,\varepsilon_{\alpha\beta}\overline{C}_{[\mu\nu}{}^{\alpha}\overline{C}_{\rho]\, m}{}^{\beta} \,, \\
C_{\mu\nu\rho\sigma} &\equiv \overline{C}_{\mu\nu\rho\sigma} \,.
\end{aligned}
\tag{4.19}
$$

For the components of the two-form $\overline{C}_{\mu\nu}{}^{\alpha}$, etc., there is no further redefinition, so for simplicity of the notation, we will simply drop their bars in the following

$$
C_{mn}{}^{\alpha} \equiv \overline{C}_{mn}{}^{\alpha} \,, \qquad C_{\mu m}{}^{\alpha} \equiv \overline{C}_{\mu m}{}^{\alpha} \,, \qquad C_{\mu\nu}{}^{\alpha} \equiv \overline{C}_{\mu\nu}{}^{\alpha} \,. \tag{4.20}
$$

Although we have not seen the 3-form and the 4-form in the tensor hierarchy of the $E_{6(6)}$ EFT, we will show later that is possible to test their expressions by comparing the reduced $D = 10$ self duality equations (4.7) to the first order duality equations (2.26) from EFT. The redefined 4-forms (4.19) continue to transform covariantly under internal diffeomorphisms with their total gauge transformations given by

$$
\begin{aligned}
\delta C_{mnkl} &= 4\partial_{[m}\overline{\lambda}_{nkl]} + \frac{3}{2}\,\varepsilon_{\alpha\beta}\,\partial_{[m}\overline{\lambda}_{n}C_{kl]}{}^{\beta} + \mathscr{L}_{\Lambda}C_{mnkl} \,, \\
\delta C_{\mu\, kmn} &= D^{\rm KK}_{\mu}\overline{\lambda}_{kmn} - 3\partial_{[k}\overline{\lambda}_{|\mu|mn]} + \mathscr{L}_{\Lambda}C_{\mu\, kmn} \\
&\quad + \frac{3}{4}\,\varepsilon_{\alpha\beta}\left(\overline{\lambda}_{[k}{}^{\alpha}\partial_{m}C_{|\mu|n]}{}^{\beta} - \partial_{[m}\overline{\lambda}_{k}{}^{\alpha}C_{|\mu|n]}{}^{\beta}\right) \,, \\
\delta C_{\mu\nu\, mn} &= 2D^{\rm KK}_{[\mu}\overline{\lambda}_{\nu]\, mn} + 2\partial_{[m}\overline{\lambda}_{n]\mu\nu} + F_{\mu\nu}{}^{k}\overline{\lambda}_{kmn} + \mathscr{L}_{\Lambda}C_{\mu\nu mn} \\
&\quad + \frac{1}{4}\,\varepsilon_{\alpha\beta}\left(-2\partial_{[m}C_{|\mu|n]}{}^{\alpha}\overline{\lambda}_{\nu}{}^{\beta} + F_{\mu\nu\,[m}{}^{\alpha}\overline{\lambda}_{n]}{}^{\beta} - \overline{\lambda}_{[m}{}^{\alpha}\partial_{n]}C_{\mu\nu}{}^{\beta}\right) \,.
\end{aligned}
\tag{4.21}
$$

We see that after the redefinitions (4.19), the variation of $\delta C_{\mu\, kmn}$ and $\delta C_{\mu\nu\, mn}$ no longer carry any scalar fields $\overline{C}_{mn}{}^{\alpha}$ and are thus of the form to be matched with the fields and transformations of EFT. The field strengths appearing on the r.h.s. of (4.21) are the Kaluza-Klein field strength (4.14) and the modified three-form field strength

$$
\begin{aligned}
F_{\mu\nu\, n}{}^{\alpha} &\equiv \overline{F}_{\mu\nu\, n}{}^{\alpha} - F_{\mu\nu}{}^{k}C_{kn}{}^{\alpha} \,, \\
&= 2D_{[\mu}C_{\nu]m}{}^{\alpha} + \partial_{m}C_{\mu\nu}{}^{\alpha} \,,
\end{aligned}
\tag{4.22}
$$

again redefined such that the scalar contribution is split off. For completeness we also give the remaining components of the three-form field strength

$$
\begin{aligned}
F_{kmn}{}^{\alpha} &\equiv \overline{F}_{kmn}{}^{\alpha} = 3\partial_{[k}C_{mn]}{}^{\alpha} \,, \\
F_{\mu\, mn}{}^{\alpha} &\equiv \overline{F}_{\mu\, mn}{}^{\alpha} \;=\; D^{\rm KK}_{\mu}C_{mn}{}^{\alpha} - 2\partial_{[m}C_{|\mu|n]}{}^{\alpha} \,, \\
F_{\mu\nu\rho}{}^{\alpha} &\equiv \overline{F}_{\mu\nu\rho}{}^{\alpha} \;=\; 3D^{\rm KK}_{[\mu}C_{\nu\rho]}{}^{\alpha} - 3F_{[\mu\nu}{}^{k}C_{\rho]k}{}^{\alpha} \,,
\end{aligned}
\tag{4.23}
$$

[7]Similar redefinitions have been discussed in [62] in order to recover part of the $E_{6(6)}$ tensor hierarchy structure from the IIB supersymmetry variations.

as well as the properly redefined components of the five-form field strength, expressed in terms of the components (4.19) according to

$$
\begin{aligned}
F_{mpqrs} &\equiv \overline{F}_{mpqrs} = 5\,\partial_{[m}C_{pqrs]} - \frac{5}{4}\,\varepsilon_{\alpha\beta}\,C_{[mp}{}^{\alpha}\overline{F}_{qrs]}{}^{\beta}\;, \\
F_{\mu pqrs} &\equiv \overline{F}_{\mu pqrs} \\
&= D^{\rm KK}_{\mu}C_{pqrs} - 4\partial_{[p}C_{|\mu|qrs]} - \frac{3}{4}\,\varepsilon_{\alpha\beta}C_{[pq}{}^{\alpha}F_{|\mu|rs]}{}^{\beta} + \frac{3}{2}\,\varepsilon_{\alpha\beta}C_{[pq}{}^{\alpha}\partial_{r}C_{|\mu|s]}{}^{\beta}\;, \\
F_{\mu\nu kmn} &\equiv \overline{F}_{\mu\nu kmn} - \frac{3}{4}\,\varepsilon_{\alpha\beta}F_{\mu\nu[k}{}^{\alpha}C_{mn]}{}^{\beta} - F_{\mu\nu}{}^{p}(\overline{C}_{pkmn} - \frac{3}{8}\,\varepsilon_{\alpha\beta}C_{[km}{}^{\alpha}C_{|p|n]}{}^{\beta}) \\
&= 2D^{\rm KK}_{[\mu}C_{\nu]kmn} + 3\partial_{[k}C_{|\mu\nu|mn]} - \frac{3}{2}\,\varepsilon_{\alpha\beta}C_{\mu[k}{}^{\alpha}\partial_{m}C_{|\nu|n]}{}^{\beta}\;, \\
F_{\mu\nu\rho mn} &\equiv \overline{F}_{\mu\nu\rho mn} - \frac{1}{4}\varepsilon_{\alpha\beta}\overline{F}_{\mu\nu\rho}{}^{\alpha}C_{mn}{}^{\beta} \\
&= 3D^{\rm KK}_{[\mu}C_{\nu\rho]mn} - 2\partial_{[m}C_{|\mu\nu\rho|n]} - 3F_{[\mu\nu}{}^{k}C_{\rho]kmn} \\
&\quad - \frac{3}{2}\,\varepsilon_{\alpha\beta}\,(\partial_{[m}C_{[\mu\nu}{}^{\alpha}C_{\rho]n]}{}^{\beta} + C_{[\mu|m|}{}^{\alpha}D_{\nu}C_{\rho]n}{}^{\beta})\;, \\
F_{\mu\nu\rho\sigma m} &\equiv \overline{F}_{\mu\nu\rho\sigma m} \\
&= 4D^{\rm KK}_{[\mu}C_{\nu\rho\sigma]m} + \partial_{m}C_{\mu\nu\rho\sigma} + 6F_{[\mu\nu}{}^{p}C_{\rho\sigma]pm} \\
&\quad + \frac{3}{2}\,\varepsilon_{\alpha\beta}F_{[\mu\nu}{}^{k}C_{\rho|m|}{}^{\alpha}C_{\sigma]k}{}^{\beta} - \frac{3}{4}\,\varepsilon_{\alpha\beta}C_{[\mu\nu}{}^{\alpha}\partial_{|m|}C_{\rho\sigma]}{}^{\beta} + \varepsilon_{\alpha\beta}\,C_{\mu\,m}{}^{\alpha}\mathscr{F}_{\nu\rho\sigma}{}^{\beta}\;, \\
F_{\mu\nu\rho\sigma\tau} &\equiv \overline{F}_{\mu\nu\rho\sigma\tau} = 5D^{\rm KK}_{[\mu}C_{\mu\nu\rho\sigma]} - 10F_{[\mu\nu}{}^{m}C_{\rho\sigma\tau]m} - \frac{15}{4}\,\varepsilon_{\alpha\beta}C_{[\mu\nu}{}^{\alpha}D^{\rm KK}_{\rho}C_{\sigma\tau]}{}^{\beta}\;.
\end{aligned}
\tag{4.24}
$$

4.3 External diffeomorphisms

In the previous subsection we have decomposed the IIB fields according to a 5+5 Kaluza-Klein split (without giving up the dependence on the 5 internal coordinates) and spelled out their transformations under internal diffeomorphisms and tensor gauge transformations after suitable redefinitions of the various components. Before fully establish the dictionary to the fields in the EFT basis, we will in this section compute the behaviour of the redefined IIB fields under external diffeomorphisms ξ^{μ}, whose parameter may in general also depend on all 10 coordinates.

Above, we have already discussed the transformation of the KK vector fields under external diffeomorphisms

$$
\delta^{\rm cov}_{\xi}A_{\mu}{}^{m} \;=\; \xi^{\nu}F_{\nu\mu}{}^{m} + \phi^{-\frac{2}{3}}G^{mn}g_{\mu\nu}\partial_{n}\xi^{\nu}\;, \tag{4.25}
$$

c.f. (3.9), which is in agreement with the EFT gauge vector transformations reduced to this component. Let us now test the remaining vector components from the IIB p-forms. For $C_{\mu m}{}^{\alpha}$, as redefined in (4.16), a straightforward calculation gives

$$
\begin{aligned}
\delta_{\xi}C_{\mu m}{}^{\alpha} \;&=\; \mathscr{L}_{\xi}C_{\mu m}{}^{\alpha} - \phi^{-\frac{2}{3}}G^{nk}C_{nm}{}^{\alpha}g_{\mu\nu}\partial_{k}\xi^{\nu} \\
&\quad + \partial_{m}\xi^{\nu}A_{\nu}{}^{n}C_{\mu n}{}^{\alpha} - A_{\mu}{}^{n}\partial_{n}\xi^{\nu}C_{\nu m}{}^{\alpha} + \partial_{m}\xi^{\nu}C_{\mu\nu}{}^{\alpha}\;,
\end{aligned}
\tag{4.26}
$$

under external diffeomorphisms. The origin of the second term is the corresponding variation of the Kaluza-Klein vector (4.25) which enters the redefined fields in (4.16). As for the Kaluza-Klein vector field, it follows that the last three terms are eliminated by field dependent gauge transformations with parameters (parameter redefinition)

$$\Lambda^m = -\xi^\nu A_\nu{}^m \,, \quad \overline{\lambda}_m{}^\alpha = -\xi^\nu C_{\nu m}{}^\alpha \,, \quad \overline{\lambda}_\mu{}^\alpha = -\xi^\nu C_{\nu\mu}{}^\alpha \,, \tag{4.27}$$

which render the action of the diffeomorphism manifestly gauge covariant. Together, the variation takes the form

$$\delta_\xi^{\rm cov} C_{\mu m}{}^\alpha = \xi^\nu F_{\nu\mu m}{}^\alpha - \phi^{-\frac{2}{3}} G^{nk} C_{nm}{}^\alpha g_{\mu\nu} \partial_k \xi^\nu \,. \tag{4.28}$$

Note in particular that the field strength entering this formula is the one defined in (4.22) which does not carry any scalar contributions. This is the form of variation that we will be able to match with the corresponding variation for the fields in the EFT basis.

Next let us consider the variation of the 4-form component $C_{\mu mnk}$. After standard Kaluza-Klein redefinition (4.16), some straightforward calculation yields

$$\begin{aligned}\delta_\xi \overline{C}_{\mu mnk} = \; & \xi^\nu \left(2D^{\rm KK}_{[\nu} \overline{C}_{\mu]mnk} + 3\partial_{[m} \overline{C}_{|\nu\mu|nk]} \right) + \mathcal{L}_{\xi^\nu A_\nu} \overline{C}_{\mu mnk} \\ & + D^{\rm KK}_\mu (\xi^\nu \overline{C}_{\nu mnk}) - 3\,\partial_{[m} (\xi^\nu \overline{C}_{|\nu\mu|nk]}) + \phi^{-\frac{2}{3}} G^{lp} C_{mnkl}\, g_{\mu\nu} \partial_p \xi^\nu \,, \end{aligned} \tag{4.29}$$

for the variation under external diffeomorphisms in terms of the redefined fields. In the first term we recognize the covariant field strength $F_{\nu\mu mnk}$ from (4.24) up to its bilinear contributions. These will be completed once we consider the variation of the redefined four form

$$\delta_\xi C_{\mu mnk} = \delta_\xi \overline{C}_{\mu mnk} - \frac{3}{8} \varepsilon_{\alpha\beta} \delta_\xi C_{\mu[m}{}^\alpha C_{nk]}{}^\beta - \frac{3}{8} \varepsilon_{\alpha\beta} C_{\mu[m}{}^\alpha \delta_\xi C_{nk]}{}^\beta \,, \tag{4.30}$$

with the second term obtained via (4.28), and the third term carrying

$$\delta_\xi C_{mn}{}^\alpha = \xi^\nu F_{\nu mn}{}^\alpha + 2\partial_{[m} (\xi^\nu C_{|\nu|n]}{}^\alpha) + \mathcal{L}_{\xi^\nu A_\nu} C_{mn}{}^\alpha \,. \tag{4.31}$$

Combining all these contributions and supplementing the variation by the gauge transformations with parameters (4.27), we arrive at the final form

$$\delta_\xi^{\rm cov} C_{\mu mnk} = \xi^\nu F_{\nu\mu mnk} + \phi^{-\frac{2}{3}} G^{lp} \left(C_{mnkl} + \frac{3}{8} \varepsilon_{\alpha\beta} C_{l[m}{}^\alpha C_{nk]}{}^\beta \right) g_{\mu\nu} \partial_p \xi^\nu \,. \tag{4.32}$$

In the next section, we will provide the complete dictionary between the Kaluza-Klein redefined fields of type IIB supergravity and the fundamental fields in the $E_{6(6)}$ EFT. In particular, matching the EFT equations against the IIB self-duality equations (4.7), we will explicitly reconstruct the remaining 4-form components $C_{\mu\nu\rho m}$, $C_{\mu\nu\rho\sigma}$.

5. Embedding of type IIB into $\mathrm{E}_{6(6)}$ Exceptional Field Theory

In this section, we provide an explicit dictionary between the Kaluza-Klein redefined fields of type IIB supergravity and those of the $E_{6(6)}$ exceptional field theory after picking solution (3.29) of the section constraint. We first show that the fundamental EFT fields can be identified among the redefined IIB fields on a pure kinematical level by comparing the transformation behaviour under diffeomorphisms and gauge transformations. We then show that the equivalence also holds on the dynamical level by reproducing the IIB self-duality equations (4.7) from the EFT field equations. In particular, this will allow us to obtain explicit expressions for the remaining 4-form components $C_{\mu\nu\rho m}$, $C_{\mu\nu\rho\sigma}$ which do not show up among the fundamental EFT fields, but whose existence follows from the EFT dynamics.

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5.1 Kinematics

Before identifying the details of the IIB embedding, let us first revisit the resulting field content of EFT after picking solution (3.29) of the section constraint. With the split (3.25), (3.26), the full p-form field content of the $\mathrm{E}_{6(6)}$ Lagrangian in this basis is given by (3.30)

$$\left\{\mathscr{A}_\mu{}^m, \mathscr{A}_{\mu\, m\alpha}, \mathscr{A}_{\mu\, kmn}, \mathscr{A}_{\mu\,\alpha}\right\} , \qquad \left\{\mathscr{B}_{\mu\nu}{}^\alpha, \mathscr{B}_{\mu\nu\, mn}, \mathscr{B}_{\mu\nu}{}^{m\alpha}\right\} , \tag{5.1}$$

where, more precisely, the Lagrangian depends on the 2-forms only under certain contractions with internal derivatives, c.f. (3.31). The EFT scalar sector is described by the fields parametrizing the $\mathrm{E}_{6(6)}$ generalized metric $\mathscr{M}_{MN}$ (3.44)

$$\{\Phi, m_{mn}, b_{mn}{}^\alpha, c_{klmn}\} . \tag{5.2}$$

Comparing the index structure of these fields to the field content of the Kaluza-Klein decomposition of IIB supergravity given in the previous section allows to give a first qualitative correspondence between the two formulations. With the discussion of section 3.1 in mind, it appears natural to relate the field $\mathscr{A}_\mu{}^m$ to the IIB Kaluza-Klein vector field $A_\mu{}^m$, and the scalars Φ, m_{mn}, to the remaining components of the internal IIB metric (4.13).

According to their index structure, the fields $\{b_{mn}{}^\alpha, \mathscr{A}_{\mu\, m\alpha}, \mathscr{B}_{\mu\nu}{}^\alpha\}$ from (5.1), (5.2) will relate to the different components of the SL(2) doublet of ten-dimensional two-forms. Similarly the fields $c_{klmn}, \mathscr{A}_{\mu\, kmn}, \mathscr{B}_{\mu\nu\, mn}$ will translate into the components of the (self-dual) IIB four-form. The remaining fields $\mathscr{A}_{\mu\,\alpha}, \mathscr{B}_{\mu\nu}{}^{m\alpha}$ descend from components of the doublet of dual six-forms. The two-form tensors $\mathscr{B}_{\mu\nu\, m}$ that complete the two-forms in (5.1) into the full **27** $\mathscr{B}_{\mu\nu\, M}$ of $\mathrm{E}_{6(6)}$ do not figure in the $\mathrm{E}_{6(6)}$ covariant Lagrangian. They represent the degrees of freedom on-shell dual to the Kaluza-Klein vector fields, i.e. descending from the ten-dimensional dual graviton.

Recall that in the EFT formulation, all vector fields in (5.1) appear with a Yang-Mills kinetic term whereas the two-forms couple via a topological term and are on-shell dual to the vector fields. In order to match the structure of IIB supergravity, we will thus have to trade the Yang-Mills vector fields $\mathscr{A}_{\mu\,\alpha}$ for a propagating two-form $\mathscr{B}_{\mu\nu}{}^\alpha$. Let us make this more explicit. The α-component of the EFT duality equations (2.26) yields

$$e\,\mathscr{M}^{\alpha\beta}\,\mathscr{F}^{\mu\nu}{}_\beta \;=\; -\frac{1}{6}\,\varepsilon^{\mu\nu\rho\sigma\tau}\,\mathscr{H}_{\rho\sigma\tau}{}^\alpha - e\,\mathscr{M}^\alpha{}_{\underline{M}}\,\mathscr{F}^{\mu\nu\underline{M}} , \tag{5.3}$$

where we have introduced the index split

$$\{X^M\} \longrightarrow \{X^{\underline{M}}, X_\alpha\} . \tag{5.4}$$

With the two-form fields $\tilde{\mathscr{B}}_{\mu\nu}{}^{k\beta}$ entering $\mathscr{F}_{\mu\nu\,\beta}$ on the l.h.s. of (5.3), this duality equation then allows to eliminate all $\tilde{\mathscr{B}}_{\mu\nu}{}^{k\beta}$ from the Lagrangian. The gauge symmetry (3.36) shows that in the process, the vector fields $\mathscr{A}_{\mu\,\alpha}$ also disappear from the Lagrangian.[8] We infer from (5.3) that the

[8]Strictly speaking, equation (5.3) only holds up to an x-dependent ‘integration constant’ $\mathscr{C}^{\mu\nu\,\alpha}(x)$, since it enters under y-derivative. To fix this freedom, we have to combine the equation with the vector field equations,

$$D_\nu\left(e\,\mathscr{M}^\alpha{}_M\,\mathscr{F}^{\nu\mu\, M}\right) \;=\; \frac{1}{4}\,\varepsilon^{\mu\nu\rho\sigma\tau}\,\varepsilon^{\alpha\beta}\,\mathscr{F}_{\nu\rho}{}^k\,\mathscr{F}_{\sigma\tau\, k\beta} , \tag{5.5}$$

and the Bianchi identity (3.40), leaving us with $D_\mu \mathscr{C}^{\mu\nu\,\alpha} = 0$. In the following we will directly set $\mathscr{C}^{\mu\nu\,\alpha} = 0$.

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kinetic term for the remaining vector fields changes into the form

$$e^{-1}\,\mathscr{L}_{\text{kin},1} \;=\; -\frac{1}{4}\,\mathscr{F}_{\mu\nu}{}^{M}\,\mathscr{F}^{\mu\nu N}\,\tilde{\mathscr{M}}_{MN}\;, \tag{5.6}$$

with $\tilde{\mathscr{M}}_{MN}$ from (3.47). At the same time, the two-forms $\tilde{B}_{\mu\nu}{}^{\alpha}$ are promoted into propagating fields with kinetic term

$$e^{-1}\,\mathscr{L}_{\text{kin},2} \;=\; -\frac{1}{12}\,e^{-5\Phi/3}\,m_{\alpha\beta}\,\tilde{\mathscr{H}}_{\mu\nu\rho}{}^{\alpha}\,\tilde{\mathscr{H}}^{\mu\nu\rho\,\beta}\;. \tag{5.7}$$

After this dualization, the remaining field content thus is given by

$$\left\{\Phi, m_{mn}, b_{mn}{}^{\alpha}, c_{klmn}, \mathscr{A}_{\mu}{}^{m}, \mathscr{A}_{\mu\,m\,\alpha}, \mathscr{A}_{\mu\,kmn}, \mathscr{B}_{\mu\nu}{}^{\alpha}, \mathscr{B}_{\mu\nu\,mn}\right\}\;, \tag{5.8}$$

with all except for the last field representing propagating degrees of freedom. In contrast, the two-form $\mathscr{B}_{\mu\nu\,mn}$ is related by a first order duality equation (2.26) to $\mathscr{A}_{\mu\,kmn}$, remnant of the IIB self-duality equations (4.7). In the following, we will make the dictionary fully explicit.

5.2 Dictionary and match of gauge symmetries

Having established the match of degrees of freedom between IIB supergravity and EFT upon choosing the IIB solution of the section condition, we can now make the map more precise by inspecting the gauge and diffeomorphism transformations on both sides. After Kaluza-Klein decomposition and redefinition of the IIB fields, as described in section 4.2, the resulting components turn out to be proportional to the EFT fields in their decomposition given in section 3.3 above. Specifically, comparing the variation of the EFT vector and two-form fields (3.34), (3.37), to the corresponding transformations in (4.17), (4.21), allows to establish the dictionary

$$\begin{aligned}
&A_{\mu}{}^{m} = \mathscr{A}_{\mu}{}^{m}\;, \qquad C_{\mu m}{}^{\alpha} = -\varepsilon^{\alpha\beta}\,\mathscr{A}_{\mu\,m\beta}\;, \qquad C_{\mu\nu}{}^{\alpha} = \tilde{\mathscr{B}}_{\mu\nu}{}^{\alpha}\;,\\
&C_{\mu\nu\,mn} = \frac{\sqrt{2}}{4}\,\tilde{\mathscr{B}}_{\mu\nu\,mn}\;, \qquad C_{\mu kmn} = \frac{\sqrt{2}}{4}\,\mathscr{A}_{\mu\,kmn} = \frac{\sqrt{2}}{8}\,\varepsilon_{mnkpq}\,\mathscr{A}_{\mu}{}^{pq}\;,
\end{aligned} \tag{5.9}$$

respectively. The corresponding gauge parameters translate with the same proportionality factors, and also the redefined IIB field strengths (4.22), (4.24) precisely translate into the EFT analogues

$$F_{\mu\nu}{}^{m} = \mathscr{F}_{\mu\nu}{}^{m}\;, \qquad F_{\mu\nu\,m}{}^{\alpha} = -\varepsilon^{\alpha\beta}\,\mathscr{F}_{\mu\nu\,m\beta}\;, \qquad F_{\mu\nu\,kmn} = \frac{\sqrt{2}}{4}\,\mathscr{F}_{\mu\nu\,kmn}\;. \tag{5.10}$$

This dictionary may be further confirmed upon comparing the action of external diffeomorphisms on both sides. Indeed, the variations calculated in (4.25), (4.28), (4.32) above, precisely reproduce the EFT transformation law (2.18) for the vectors $\mathscr{A}_{\mu}{}^{M}$, provided we identify the components of the scalar matrix $\mathscr{M}^{MN}$ (3.56) with the IIB fields according to

$$\phi^{-\frac{2}{3}}G^{mn} \;=\; e^{4\Phi/3}m^{mn}\;, \qquad C_{mn}{}^{\alpha} \;=\; -2b_{mn}{}^{\alpha}\;, \qquad C_{mnkl} = -4\,c_{mnkl}\;. \tag{5.11}$$

This last identification is precisely compatible with the gauge transformation behaviour (3.50) as compared to the scalar components of (4.17), (4.21). Let us also note, that with this dictionary the

EFT covariant derivatives (3.49) for the scalar fields precisely translate into the components of the IIB field strengths

$$\begin{aligned} \mathscr{D}_\mu b_{mn}{}^\alpha &= -\frac{1}{2}\overline{F}_{\mu mn}{}^\alpha \,, \\ \widehat{\mathscr{D}}_\mu c_{klmn} &= -4\overline{F}_{\mu klmn} \,, \end{aligned} \tag{5.12}$$

with $\widehat{\mathscr{D}}_\mu c_{klmn}$ from (3.52). Similarly, we have the identification

$$\partial_{[k} c_{lmnp]} + 12\,\varepsilon_{\alpha\beta}\, b_{[kl}{}^\alpha \partial_m b_{np]}{}^\beta X_{klmnp} = -\frac{4}{5}\overline{F}_{klmnp} \,, \tag{5.13}$$

with X_{klmnp} from (3.54).

We have thus identified the elementary EFT fields among the Kaluza-Klein components of the IIB fields. So far, the identification has been solely based on the matching of gauge symmetries on both sides. We will in the following show that the embedding of IIB into EFT also holds dynamically on the level of the equations of motion.

5.3 Dynamics and reconstruction of 3- and 4-forms

In this section, we will show how the full IIB self-duality equations (4.7) follow from the EFT dynamics. Along the way, we will establish explicit expressions for the remaining components of the ten-dimensional 4-form, thereby completing the explicit embedding of the IIB theory. To begin with, it is useful to first rewrite the various components of the IIB self-duality equations in terms of the Kaluza-Klein decomposed fields introduced in section 4.2 above. With the IIB metric (4.13) given in term of the EFT fields as

$$G_{\hat\mu\hat\nu} = \begin{pmatrix} e^{5\Phi/6}\, g_{\mu\nu} + \mathscr{A}_\mu{}^m \mathscr{A}_\nu{}^n \,\phi_{mn} & e^{-\Phi/2}\, m_{kn}\, \mathscr{A}_\mu{}^k \\ e^{-\Phi/2}\, m_{mk}\, \mathscr{A}_\nu{}^k & e^{-\Phi/2}\, m_{mn} \end{pmatrix} , \tag{5.14}$$

the IIB self-duality equations (4.7) split into the following three components

$$\overline{F}_{\mu\nu\rho\, mn} = \frac{1}{12}\, e^{2\Phi/3}\sqrt{-g}\,\varepsilon_{\mu\nu\rho\sigma\tau}\varepsilon_{mnklp}\,\overline{F}^{\sigma\tau}{}_{qrs}\, m^{kq} m^{lr} m^{pq} \,, \tag{5.15}$$

$$\overline{F}_{\mu\nu\rho\sigma m} = -\frac{1}{24}\, e^{2\Phi}\sqrt{-g}\,\varepsilon_{\mu\nu\rho\sigma\tau} m_{mn}\varepsilon^{nklpq}\,\overline{F}^{\tau}{}_{klpq} \,, \tag{5.16}$$

$$\overline{F}_{\mu\nu\rho\sigma\tau} = \frac{1}{120}\, e^{10\Phi/3}\sqrt{-g}\,\varepsilon_{\mu\nu\rho\sigma\tau}\varepsilon^{mnklp}\,\overline{F}_{mnklp} \,. \tag{5.17}$$

On the r.h.s. all external indices are raised and lowered with the metric $g_{\mu\nu}$, and both ε-symbols denote the numerical tensor densities. All explicit appearance of Kaluza-Klein vectors $\mathscr{A}_\mu{}^m$ from (5.14) is absorbed in the redefined $\overline{F}$'s. We will now reproduce these equations one by one from the EFT dynamics.

Let us start from the $[mn]$ component of the EFT duality equations (2.26) which can be integrated to

$$\widetilde{\mathscr{H}}_{\mu\nu\rho\, mn} + \mathscr{O}_{mn\,\mu\nu\rho} = \frac{1}{2}\, e\varepsilon_{\mu\nu\rho\sigma\tau}\, \mathscr{M}_{mn,M}\, \mathscr{F}^{\sigma\tau M} \,, \tag{5.18}$$

where the $\mathscr{O}_{mn\,\mu\nu\rho}$ keeps track of the integration ambiguity and satisfies

$$\partial_{[k}\mathscr{O}_{mn]\,\mu\nu\rho} = 0 \qquad \Longrightarrow \qquad \mathscr{O}_{mn\,\mu\nu\rho} \equiv \partial_{[m}\xi_{n]\,\mu\nu\rho} \quad \text{(locally)}\,. \tag{5.19}$$

Eliminating $\mathscr{F}_{\mu\nu\,\alpha}$ on the r.h.s. of (5.18) by means of (5.3), turns $\mathscr{M}_{MN}$ into $\tilde{\mathscr{M}}_{MN}$, such that upon using the explicit expressions (3.48), we obtain

$$\begin{aligned}\partial_{[m}\xi_{n]\,\mu\nu\rho} \;=\; & \frac{1}{12}\,e^{2\Phi/3}\,e\,\varepsilon_{\mu\nu\rho\sigma\tau}\varepsilon_{mnklp}m^{kq}m^{lr}m^{ps}\,\widehat{\mathscr{F}}^{\sigma\tau}{}_{qrs}\\ & -\tilde{\mathscr{H}}_{\mu\nu\rho\,mn} - \sqrt{2}\,\varepsilon_{\alpha\beta}\,b_{mn}{}^{\alpha}\,\tilde{\mathscr{H}}_{\mu\nu\rho}{}^{\beta}\,,\end{aligned} \tag{5.20}$$

with

$$\begin{aligned}\widehat{\mathscr{F}}_{\mu\nu\,klm} \equiv\; & \mathscr{F}_{\mu\nu\,klm} + 3\sqrt{2}\,b_{[kl}{}^{\alpha}\mathscr{F}_{|\mu\nu|m]\alpha} + 3\sqrt{2}\,\varepsilon_{\alpha\beta}\,b_{n[k}{}^{\alpha}b_{lm]}{}^{\beta}\,\mathscr{F}_{\mu\nu}{}^{n} + \frac{1}{2}\sqrt{2}\,c_{klmn}\,\mathscr{F}_{\mu\nu}{}^{n}\,.\\ =\; & 2\sqrt{2}\,\overline{F}_{\mu\nu\,klm}\,,\end{aligned} \tag{5.21}$$

where the last identity is easily confirmed upon using the dictionary of field strengths (4.24), (5.10) and scalars (5.11). Together, the relation (5.20) then gives rise to

$$F_{\mu\nu\rho\,mn} - \frac{1}{4}\,\varepsilon_{\alpha\beta}\,C_{mn}{}^{\alpha}\,\overline{F}_{\mu\nu\rho}{}^{\beta} \;=\; \frac{1}{12}\,e^{2\Phi/3}\,e\,\varepsilon_{\mu\nu\rho\sigma\tau}\varepsilon_{mnklp}m^{kq}m^{lr}m^{ps}\,\overline{F}^{\sigma\tau}{}_{qrs}\,, \tag{5.22}$$

and thus precisely reproduces (5.15) if we identify the 3-form component $C_{\mu\nu\rho\,m}$ from (4.19) as

$$C_{\mu\nu\rho\,m} \;=\; -\frac{1}{8}\sqrt{2}\,\xi_{\mu\nu\rho\,m}\,. \tag{5.23}$$

We have thus reproduced the first of the components of the IIB self-duality equations and along the way identified one of the missing components (5.23) of the IIB four-form, that is not among the fundamental EFT fields. It is defined by the first order differential equations (5.22) in terms of the EFT fields up to a gradient

$$C_{\mu\nu\rho\,m} \longrightarrow C_{\mu\nu\rho\,m} + \partial_m\lambda_{\mu\nu\rho}\,, \tag{5.24}$$

corresponding to a gauge transformation in the IIB theory.

Let us continue towards the other components (5.16), (5.17), of the self-duality relations. Consider the external curl of (5.18), which reads

$$4D_{[\mu}\tilde{\mathscr{H}}_{\nu\rho\sigma]\,mn} + 4D^{\rm KK}_{[\mu}\,\mathscr{O}_{\nu\rho\sigma]\,mn} \;=\; 2\,e\varepsilon_{\tau\lambda[\nu\rho\sigma}\,D^{\rm KK}_{\mu]}\left(\mathscr{M}_{mn,N}\,\mathscr{F}^{\tau\lambda\,N}\right)\,, \tag{5.25}$$

and use the Bianchi identity (3.41) to find

$$\begin{aligned}4\,\partial_m\left(D^{\rm KK}_{[\mu}\,\xi_{\nu\rho\sigma]\,n}\right) \;=\; & 6\,\mathscr{F}_{[\mu\nu}{}^{k}\mathscr{F}_{\rho\sigma]\,kmn} + 3\sqrt{2}\,\varepsilon^{\alpha\beta}\,\mathscr{F}_{[\mu\nu\,|m\alpha|}\mathscr{F}_{\rho\sigma]\,n\beta}\\ & + 4\sqrt{2}\,\partial_m\mathscr{H}_{[\mu\nu\rho}{}^{\alpha}\,\mathscr{A}_{\sigma]\,n\alpha} - e\varepsilon_{\mu\nu\rho\sigma\lambda}\,D^{\rm KK}_{\tau}\left(\mathscr{M}_{mn,N}\,\mathscr{F}^{\tau\lambda\,N}\right)\\ & - 3\sqrt{2}\,\partial_m\left(\varepsilon_{\alpha\beta}\,\tilde{\mathscr{B}}_{[\mu\nu}{}^{\alpha}\partial_{|n|}\tilde{\mathscr{B}}_{\rho\sigma]}{}^{\beta}\right) + 12\,\partial_m\left(\mathscr{F}_{[\mu\nu}{}^{k}\tilde{\mathscr{B}}_{\rho\sigma]\,kn}\right)\\ & + 6\sqrt{2}\,\partial_m\left(\varepsilon^{\alpha\beta}\,\mathscr{A}_{[\mu\,|n\alpha|}\mathscr{F}_{\nu\rho}{}^{k}\mathscr{A}_{\sigma]\,k\beta}\right)\,,\end{aligned} \tag{5.26}$$

where both, left and right hand side are supposed to be explicitly projected onto their part antisymmetric in $[mn]$.

In order to simplify the second line, we make use of the equations of motion obtained by varying the Lagrangian (2.19) w.r.t. the vector fields $\mathscr{A}_\mu{}^{mn}$ and using the duality equation (5.3) in order to eliminate $\mathscr{F}_{\mu\nu\alpha}$

$$\begin{aligned} 0 \;=\; & -\frac{1}{24}\sqrt{2}\,\partial_{[m}\left(e^{2\Phi}m_{n]k}\,\widehat{\mathscr{D}}^{\mu}c_{pqrs}\varepsilon^{kpqrs}\right)+D^{\rm KK}_{\nu}\left(\mathscr{M}_{mn,M}\mathscr{F}^{\nu\mu M}\right) \\ & +\frac{1}{6}\sqrt{2}\,\varepsilon^{\mu\nu\rho\sigma\tau}\,\partial_{[m}\mathscr{A}_{|\nu|n]\alpha}\tilde{\mathscr{H}}_{\rho\sigma\tau}{}^{\alpha}-\frac{1}{12}\sqrt{2}\,\varepsilon^{\mu\nu\rho\sigma\tau}\,\mathscr{A}_{\nu\,[m|\alpha|}\partial_{n]}\tilde{\mathscr{H}}_{\rho\sigma\tau}{}^{\alpha} \\ & +\frac{3}{4}\,\varepsilon^{\mu\nu\rho\sigma\tau}\left(\frac{\sqrt{2}}{6}\,\varepsilon^{\alpha\beta}\,\mathscr{F}_{\nu\rho\,m\alpha}\mathscr{F}_{\sigma\tau n\beta}+\frac{1}{3}\,\mathscr{F}_{\nu\rho\,mnp}\mathscr{F}_{\sigma\tau}{}^{p}+\frac{\sqrt{2}}{9}\,\mathscr{A}_{\nu[m|\alpha|}\,\partial_{n]}\tilde{\mathscr{H}}_{\rho\sigma\tau}{}^{\alpha}\right) \end{aligned} \tag{5.27}$$

Together we find for (5.26)

$$\begin{aligned} 4\,\partial_m\left(D^{\rm KK}_{\mu}\,\xi_{\nu\rho\sigma n}\right) \;=\; & -\frac{1}{24}\sqrt{2}\,e\varepsilon_{\mu\nu\rho\sigma\lambda}\,\partial_m\left(e^{2\Phi}m_{nk}\,\widehat{\mathscr{D}}^{\lambda}c_{pqrs}\varepsilon^{kpqrs}\right) \\ & -3\sqrt{2}\,\partial_m\left(\varepsilon_{\alpha\beta}\,\tilde{\mathscr{B}}_{\mu\nu}{}^{\alpha}\partial_n\tilde{\mathscr{B}}_{\rho\sigma}{}^{\beta}\right)+12\,\partial_m\left(\mathscr{F}_{\mu\nu}{}^{k}\tilde{\mathscr{B}}_{\rho\sigma kn}\right) \\ & +6\sqrt{2}\,\partial_m\left(\mathscr{A}_{\mu n\alpha}\varepsilon^{\alpha\beta}\mathscr{F}_{\nu\rho}{}^{k}\mathscr{A}_{\sigma k\beta}\right)-4\sqrt{2}\,\partial_m\left(\mathscr{A}_{\mu n\alpha}\tilde{\mathscr{H}}_{\nu\rho\sigma}{}^{\alpha}\right)\;, \end{aligned} \tag{5.28}$$

again, projected onto the antisymmetric part $[mn]$. The entire equation thus takes the form of an internal curl and can be integrated to

$$\begin{aligned} -\frac{1}{24}\sqrt{2}\,e\varepsilon_{\mu\nu\rho\sigma\lambda}\,e^{2\Phi}m_{nk}\,\widehat{\mathscr{D}}^{\lambda}c_{pqrs}\varepsilon^{kpqrs} \;=\; & 4\,D^{\rm KK}_{[\mu}\,\xi_{\nu\rho\sigma]\,n}+3\sqrt{2}\,\varepsilon_{\alpha\beta}\,\tilde{\mathscr{B}}_{[\mu\nu}{}^{\alpha}\partial_{|n|}\tilde{\mathscr{B}}_{\rho\sigma]}{}^{\beta} \\ & -12\,F_{[\mu\nu}{}^{k}\tilde{\mathscr{B}}_{\rho\sigma]\,kn}-6\sqrt{2}\,\varepsilon^{\alpha\beta}\,\mathscr{F}_{[\mu\nu}{}^{k}\,\mathscr{A}_{\rho\,|n\alpha|}\mathscr{A}_{\sigma]\,k\beta} \\ & +4\sqrt{2}\,\mathscr{A}_{\mu n\alpha}\tilde{\mathscr{H}}_{\nu\rho\sigma}{}^{\alpha}+\partial_n\xi_{\mu\nu\rho\sigma}\;, \end{aligned} \tag{5.29}$$

up to an internal gradient $\partial_n\xi_{\mu\nu\rho\sigma}$. Applying the dictionary (5.9), (5.10) to translate all fields into the IIB components, this equation becomes

$$-\frac{1}{24}\,e\varepsilon_{\mu\nu\rho\sigma\lambda}\,\varepsilon^{kpqrs}\,e^{2\Phi}m_{nk}\,\overline{F}^{\lambda}{}_{pqrs} \;=\; \overline{F}_{\mu\nu\rho\sigma n}-\partial_n\left(C_{\mu\nu\rho\sigma}+\frac{1}{8}\sqrt{2}\,\xi_{\mu\nu\rho\sigma}\right)\;, \tag{5.30}$$

i.e. reproduces equation (5.16), provided we identify the last missing component of the 4-form as

$$C_{\mu\nu\rho\sigma} \;=\; -\frac{1}{8}\sqrt{2}\,\xi_{\mu\nu\rho\sigma}\;. \tag{5.31}$$

We have thus also reproduced the second component of the IIB self-duality equations and along the way identified the last missing components (5.31) of the IIB four-form, that is not among the fundamental EFT fields. It is defined by the first order differential equations (5.29) in terms of the EFT fields up to an additive function

$$C_{\mu\nu\rho\sigma} \;\longrightarrow\; C_{\mu\nu\rho\sigma}+\Lambda_{\mu\nu\rho\sigma}(x)\;, \tag{5.32}$$

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which we will fix in the following. In order to find the last component (5.17) of the self-duality equations, we take the external curl of (5.29)

$$
\begin{aligned}
-\partial_n D^{\rm KK}_{[\mu}\,\xi_{\nu\rho\sigma\tau]} \;=\; & -\frac{1}{120}\sqrt{2}\,e\varepsilon_{\mu\nu\rho\sigma\tau}D^{\rm KK}_{\lambda}\left(e^{2\Phi}m_{nk}\,\widehat{\mathscr{D}}^{\lambda}c_{pqrs}\varepsilon^{kpqrs}\right)+2\sqrt{2}\,\mathscr{F}_{[\mu\nu\,|n\alpha|}\mathscr{H}_{\rho\sigma\tau]}{}^{\alpha}\\
&+4\,\mathscr{F}_{[\mu\nu}{}^{k}\left(\mathscr{H}_{\rho\sigma\tau]\,kn}+\partial_{[k}\xi_{\rho\sigma\tau]\,n]}\right)+2\sqrt{2}\,\varepsilon_{\alpha\beta}\partial_n\tilde{\mathscr{B}}_{[\mu\nu}{}^{\beta}\mathscr{H}_{\rho\sigma\tau]}{}^{\alpha}\\
&-2\sqrt{2}\,\varepsilon_{\alpha\beta}\,\mathscr{H}_{[\mu\nu\rho}{}^{\alpha}\partial_{|n|}\tilde{\mathscr{B}}_{\sigma\tau]}{}^{\beta}-6\sqrt{2}\varepsilon^{\alpha\beta}\mathscr{F}_{[\mu\nu}{}^{k}\mathscr{A}_{\rho\,|n\alpha|}\mathscr{F}_{\sigma\tau]\,k\beta}\\
&+6\sqrt{2}\,\varepsilon^{\alpha\beta}\mathscr{A}_{[\mu\,|n\alpha|}\mathscr{F}_{\nu\rho}{}^{k}\mathscr{F}_{\sigma\tau]k\beta}-3\sqrt{2}\,\partial_n\left(\varepsilon_{\alpha\beta}\,\tilde{\mathscr{B}}_{[\mu\nu}{}^{\alpha}D_{\rho}\tilde{\mathscr{B}}_{\sigma\tau]}{}^{\beta}\right)\\
&+2\partial_n\left(\mathscr{F}_{[\mu\nu}{}^{k}\xi_{\rho\sigma\tau]\,k}\right)\,,
\end{aligned}
\tag{5.33}
$$

which after using the equations of motion for c_{klmn} turns into a full internal gradient and can be integrated to the equation

$$
D^{\rm KK}_{[\mu}\,\xi_{\nu\rho\sigma\tau]}+3\sqrt{2}\,\varepsilon_{\alpha\beta}\,\tilde{\mathscr{B}}_{[\nu\rho}{}^{\alpha}D_{\mu}\tilde{\mathscr{B}}_{\sigma\tau]}{}^{\beta}-2\mathscr{F}_{[\mu\nu}{}^{k}\xi_{\rho\sigma\tau]\,k}\;=\;\frac{\sqrt{2}}{120}\,e\varepsilon_{\mu\nu\rho\sigma\tau}\varepsilon^{klmnp}\,e^{10\Phi/3}X_{klmnp}\,,
\tag{5.34}
$$

with X from (3.54), up to some y-independent function. The latter can be set to zero by properly fixing the freedom (5.32). After translating (5.34) into the IIB fields, we thus find

$$
5D^{\rm KK}_{[\mu}C_{\nu\rho\sigma\tau]}-\frac{15}{4}\,\varepsilon_{\alpha\beta}\,\overline{C}_{[\nu\rho}{}^{\alpha}D^{\rm KK}_{\mu}\overline{C}_{\sigma\tau]}{}^{\beta}-10\mathscr{F}_{[\mu\nu}{}^{k}C_{\rho\sigma\tau]\,k}\;=\;\frac{1}{120}\,e\varepsilon_{\mu\nu\rho\sigma\tau}\varepsilon^{klmnp}\,e^{10\Phi/3}\,\overline{F}_{klmnp}\,.
\tag{5.35}
$$

Thereby we find the last missing component (5.17) of the IIB self-duality equation. We have thus shown that the full IIB self-duality equations (4.7) follow from the EFT dynamics, provided we identify by (5.23), (5.31) the remaining components of the IIB 4-form. Together with the dictionary established in section (5.2), this defines all the IIB fields in terms of the fundamental fields from EFT.

5.4 Complementary checks

We have in the preceding sections established the full dictionary between the IIB theory and the EFT fields upon choosing the explicit solution (3.29) of the section constraint. In particular, we have defined all the components of the IIB fields (4.1) in terms of the fundamental EFT fields and shown that the EFT dynamics implies the full IIB self-duality equations (4.7). Via integrability this also implies the IIB second order field equations for the 4-form. The remaining equations of motion of the IIB theory can be verified in a more straightforward manner, similar to the analogous discussion for the embedding of $D=11$ supergravity [13], by using the explicit dictionary.

As an example, let us collect the contributions to the kinetic terms for the IIB two-form doublet $\hat{C}_{\hat{\mu}\hat{\nu}}{}^{\alpha}$. According to their Kaluza-Klein decomposition, these contributions descend from different terms of the EFT Lagrangian: the kinetic terms (3.51), (5.6), (5.7), and the scalar potential (3.53), giving rise to

$$
\begin{aligned}
e^{-1}\,\mathscr{L}_{2\text{-form}} \;=\; & -e^{\Phi}\,\mathscr{D}_{\mu}b_{mn}{}^{\alpha}\mathscr{D}^{\mu}b_{kl}{}^{\beta}\,m^{km}m^{ln}\,m_{\alpha\beta}-\frac{1}{4}\,e^{-\Phi/3}\,m^{mn}\,m_{\alpha\beta}\,\mathscr{F}_{\mu\nu\,m}{}^{\alpha}\mathscr{F}^{\mu\nu}{}_{n}{}^{\beta}\\
&-\frac{1}{12}\,e^{-5\Phi/3}\,m_{\alpha\beta}\,\mathscr{H}_{\mu\nu\rho}{}^{\alpha}\,\mathscr{H}^{\mu\nu\rho\,\beta}-3\,e^{7\Phi/3}\,\partial_{[k}b_{mn]}{}^{\alpha}\partial_{l}b_{pq}{}^{\beta}\,m_{\alpha\beta}\,m^{kl}m^{mp}m^{nq}\,.
\end{aligned}
\tag{5.36}
$$

Upon translating these fields into the IIB components via (5.10)–(5.12), the Lagrangian takes the form

$$\mathscr{L}_{2-\text{form}} = -\frac{1}{12}\sqrt{|G|}\Big(3\,F_{\mu mn}{}^{\alpha}F^{\mu mn\,\beta}+3\,F_{\mu\nu m}{}^{\alpha}F^{\mu\nu m\,\beta} \\ +F_{\mu\nu\rho}{}^{\alpha}F^{\mu\nu\rho\,\beta}+F_{kmn}{}^{\alpha}F^{kmn\,\beta}\Big)m_{\alpha\beta}\;,$$

where now all indices on the r.h.s. are raised and lowered with the full IIB metric (5.14). The result thus precisely agrees with the corresponding kinetic term of the IIB (pseudo-)action (4.6). Similarly, we find from collecting all the EFT contributions to the 5-form kinetic term

$$\mathscr{L}_{5-\text{form}} = -\frac{1}{15}\sqrt{|G|}\left(F_{klmnp}F^{klmnp}+5\,F_{\mu klmn}F^{\mu klmn}+10\,F_{\mu\nu klm}F^{\mu\nu klm}\right)\;, \tag{5.37}$$

which reproduces half of the components of the corresponding term in the pseudo-action (4.6), with the other half doubling the contribution due to the self-duality equations (4.7).[9]

6. Generalized Scherk-Schwarz compactification

The manifestly covariant formulation of EFT described in the previous sections has proven a rather powerful tool in order to describe consistent truncations by means of a generalization of the Scherk-Schwarz ansatz [63] to the exceptional space-time [39]. This relates to gauged supergravity theories in lower dimensions (in this case to $D=5$ supergravities), formulated in the embedding tensor formalism. Via the explicit dictionary of EFT to $D=11$ and type IIB supergravity, this ansatz then provides the full Kaluza-Klein embedding of various consistent truncations.

The generalized Scherk-Schwarz ansatz in EFT is governed by a group-valued twist matrix $U\in\mathrm{E}_{6(6)}$, depending on the internal coordinates, which rotates each fundamental group index. For instance, for the generalized metric the ansatz reads

$$\mathscr{M}_{MN}(x,Y) = U_M{}^{\underline{K}}(Y)\,U_N{}^{\underline{L}}(Y)\,M_{\underline{KL}}(x)\;, \tag{6.1}$$

where $M_{\underline{MN}}$ becomes the $\mathrm{E}_{6(6)}$-valued scalar matrix of five-dimensional gauged supergravity. This ansatz is invariant under a global $\mathrm{E}_{6(6)}$ symmetry acting on the underlined indices. Indeed, gauged supergravity in the embedding tensor formalism is covariant w.r.t. a global duality group ($\mathrm{E}_{6(6)}$ in the present case), although this is not a physical symmetry but rather relates different gauged supergravities to each other. In addition to the group valued twist matrix, consistency requires that we also introduce a scale factor ρ, depending only on the internal coordinates, for fields carrying a non-zero density weight λ, for which the ansatz contains $\rho^{-3\lambda}$. We thus write the general reduction ansatz for all bosonic fields of the $\mathrm{E}_{6(6)}$ EFT (1.2) as [39]

$$\begin{aligned}
\mathscr{M}_{MN}(x,Y) &= U_M{}^{\underline{K}}(Y)\,U_N{}^{\underline{L}}(Y)\,M_{\underline{KL}}(x)\;,\\
g_{\mu\nu}(x,Y) &= \rho^{-2}(Y)\,\mathbf{g}_{\mu\nu}(x)\;,\\
\mathscr{A}_\mu{}^M(x,Y) &= \rho^{-1}(Y)A_\mu{}^{\underline{N}}(x)(U^{-1})_{\underline{N}}{}^M(Y)\;,\\
\mathscr{B}_{\mu\nu\,M}(x,Y) &= \rho^{-2}(Y)U_M{}^{\underline{N}}(Y)\,B_{\mu\nu\,\underline{N}}(x)\;.
\end{aligned} \tag{6.2}$$

[9]Again, it is important that the self-duality equation (4.7) is to be used in the pseudo-action (4.6) only after deriving the field equations by variation. Strictly speaking, our proof of equivalence holds on the level of the field equations.

We will call the above ansatz consistent if the twist matrix U and the function ρ factor out of all covariant expressions in the action, the gauge transformations or the equations of motion. If this is established, it follows that the reduction is consistent in the strong Kaluza-Klein sense that any solution of the lower-dimensional theory can be uplifted to a solution of the full theory, with the uplift formulas being (6.2). Let us explain the required consistency conditions for the gauge transformations under internal generalized diffeomorphisms, for which the gauge parameter is subject to the same ansatz as the one-form gauge field,

$$\Lambda^M(x,Y) \;=\; \rho^{-1}(Y)(U^{-1})_{\underline{N}}{}^M(Y)\blacksquare^{\underline{N}}(x)\,. \tag{6.3}$$

We start with the field $g_{\mu\nu}$ that transforms as a scalar density of weight $\lambda = \frac{2}{3}$. Consistency of the ansatz (6.2) requires that under gauge transformations we have

$$\delta_\Lambda g_{\mu\nu}(x,Y) \;=\; \rho^{-2}(Y)\delta_\Lambda \mathbf{g}_{\mu\nu}(x)\,, \tag{6.4}$$

where the expression for $\delta_\Lambda \mathbf{g}_{\mu\nu}$ is Y-independent and can hence consistently be interpreted as the gauge transformation for the lower-dimensional metric. The variation on the left-hand side yields, upon insertion of (6.3),

$$\begin{aligned}\delta_\Lambda g_{\mu\nu} \;&=\; \Lambda^N\partial_N g_{\mu\nu} + \tfrac{2}{3}\,\partial_N\Lambda^N g_{\mu\nu}\\ &=\; \rho^{-1}(U^{-1})_{\underline{K}}{}^N\blacksquare^{\underline{K}}\partial_N(\rho^{-2}\mathbf{g}_{\mu\nu}) + \tfrac{2}{3}\,\partial_N(\rho^{-1}(U^{-1})_{\underline{K}}{}^N)\blacksquare^{\underline{K}}\rho^{-2}\mathbf{g}_{\mu\nu}\\ &=\; \tfrac{2}{3}\rho^{-3}\Big[\partial_N(U^{-1})_{\underline{K}}{}^N - 4\,(U^{-1})_{\underline{K}}{}^N\rho^{-1}\partial_N\rho\Big]\blacksquare^{\underline{K}}\mathbf{g}_{\mu\nu}\,.\end{aligned} \tag{6.5}$$

If we now demand that

$$\partial_N(U^{-1})_{\underline{K}}{}^N - 4\,(U^{-1})_{\underline{K}}{}^N\rho^{-1}\partial_N\rho \;=\; 3\,\rho\,\vartheta_{\underline{K}}\,, \tag{6.6}$$

where $\vartheta_{\underline{K}}$ is *constant*, then the ansatz (6.4) is established with

$$\delta_\Lambda \mathbf{g}_{\mu\nu} \;=\; 2\blacksquare^{\underline{M}}\,\vartheta_{\underline{M}}\,\mathbf{g}_{\mu\nu}\,. \tag{6.7}$$

This corresponds to a gauging of the so-called trombone symmetry that rescales the metric and the other tensor fields of the theory with specific weights. Here, $\vartheta_{\underline{K}}$ is the embedding tensor component for the trombone gauging, as introduced in [64]. An important consistency condition is that (6.6) is a covariant equation under internal generalized diffeomorphisms. Treating the (inverse) twist matrix as a vector of weight zero, its divergence $\partial_N(U^{-1})_{\underline{M}}{}^N$ (recalling that the underlined index is inert) is not a scalar. Indeed, a quick computation with (2.6) using the section constraint shows that it transforms as a scalar density of weight $\lambda = -\frac{1}{3}$, except for the following anomalous term in the transformation

$$\Delta^{\rm nc}_\Lambda\big(\partial_N(U^{-1})_{\underline{M}}{}^N\big) \;=\; -\tfrac{4}{3}\,\partial_N(\partial\cdot\Lambda)(U^{-1})_{\underline{M}}{}^N\,. \tag{6.8}$$

This contribution is precisely cancelled by the anomalous variation of the second term in (6.6), provided ρ is a scalar density of weight $\lambda(\rho) = -\frac{1}{3}$. Then both sides of (6.6) are scalar densities of weight $\lambda = -\frac{1}{3}$ and the equation is gauge covariant.

Let us now turn to the consistency conditions required for fields with a non-trivial tensor structure under internal generalized diffeomorphisms, as the generalized metric. In parallel to the above discussion we require that the twist matrices consistently factor out, i.e.

$$\delta_\Lambda \mathcal{M}_{MN}(x,Y) \;=\; U_M{}^{\underline{K}}(Y)U_N{}^{\underline{L}}(Y)\delta_\Lambda M_{\underline{KL}}(x)\,. \tag{6.9}$$

Using the explicit form of the gauge transformations given by generalized Lie derivatives (2.6) one may verify by direct computation that this leads to consistent gauge transformations

$$\delta_\Lambda M_{\underline{MN}}(x) \;=\; 2\blacksquare^{\underline{L}}(x)\left(\Theta_{\underline{L}}{}^{\alpha}+\tfrac{9}{2}\,\vartheta_{\underline{R}}\,(t^{\alpha})_{\underline{L}}{}^{\underline{R}}\right)(t_\alpha)_{(\underline{M}}{}^{\underline{P}}M_{\underline{N})\underline{P}}(x)\,, \tag{6.10}$$

provided we assume the consistency conditions

$$\left[(U^{-1})_{\underline{M}}{}^{K}(U^{-1})_{\underline{N}}{}^{L}\partial_K U_L{}^{\underline{P}}\right]_{\mathbf{351}} \;=\; \tfrac{1}{5}\,\rho\,\Theta_{\underline{M}}{}^{\alpha}(t_\alpha)_{\underline{N}}{}^{\underline{P}}\,, \tag{6.11}$$

where the *constant* $\Theta_{\underline{M}}{}^{\alpha}$ is the embedding tensor encoding conventional (i.e. non-trombone) gaugings, and the left-hand side is projected onto the **351** sub-representation. Specifically, writing the derivatives of U in terms of

$$\mathscr{X}_{\underline{MN}}{}^{\underline{K}} \;\equiv\; (U^{-1})_{\underline{M}}{}^{K}(U^{-1})_{\underline{N}}{}^{L}\partial_K U_L{}^{\underline{K}} \;\equiv\; \mathscr{X}_{\underline{M}}{}^{\alpha}(t_\alpha)_{\underline{N}}{}^{\underline{K}}\,, \tag{6.12}$$

where we used that since U is group valued, $U^{-1}\partial U$ is Lie algebra valued (in the indices $\underline{N}$, $\underline{K}$), so that we can expand it in terms of generators as done in the second equality, the projector acts as (c.f. eq. (4.13) in [65]),

$$\begin{aligned}\left[\,\mathscr{X}_{\underline{M}}{}^{\alpha}\,\right]_{\mathbf{351}} \;&\equiv\; (\mathbb{P}_{\mathbf{351}})_{\underline{M}}{}^{\alpha\underline{N}}{}_{\beta}\,\mathscr{X}_{\underline{N}}{}^{\beta}\\ &=\; \tfrac{1}{5}\Big(\,\mathscr{X}_{\underline{M}}{}^{\alpha}-6\,(t^{\alpha})_{\underline{P}}{}^{\underline{N}}(t_\beta)_{\underline{M}}{}^{\underline{P}}\,\mathscr{X}_{\underline{N}}{}^{\beta}+\tfrac{3}{2}\,(t^{\alpha})_{\underline{M}}{}^{\underline{P}}(t_\beta)_{\underline{P}}{}^{\underline{N}}\,\mathscr{X}_{\underline{N}}{}^{\beta}\Big)\,.\end{aligned} \tag{6.13}$$

Also the condition (6.11) is covariant under internal diffeomorphisms. This can be explicitly verified in the same way as the covariance of the torsion tensor (2.36), which lives in the same representation. Let us emphasize that solving the consistency equations (6.6) and (6.11) for U and ρ in general is a rather non-trivial problem. It would be important to develop a general theory for doing this, which plausibly may require a better understanding of large generalized diffeomorphisms, as in [66–69].

The consistency conditions (6.6) and (6.11) can equivalently be encoded in the structure of a 'generalized parallelization', see [70]. To this end, the twist matrix U and the scale factor ρ are combined into a vector of non-zero weight,

$$(\widehat{U}^{-1})_{\underline{M}}{}^{N} \;\equiv\; \rho^{-1}\,(U^{-1})_{\underline{M}}{}^{N}\,. \tag{6.14}$$

Since ρ carries weight $-\frac{1}{3}$ this is a generalized vector of weight $\frac{1}{3}$, the same as for the gauge parameter, so that the generalized Lie derivative w.r.t. $\widehat{U}^{-1}$ is well-defined. Both consistency conditions (6.6) and (6.11) can then be encoded in the single manifestly covariant equation

$$\mathbb{L}_{\widehat{U}^{-1}_{\underline{M}}}\,\widehat{U}^{-1}_{\underline{N}} \;\equiv\; -X_{\underline{MN}}{}^{\underline{K}}\,\widehat{U}^{-1}_{\underline{K}}\,, \tag{6.15}$$

with $X_{\underline{MN}}{}^{\underline{K}}$ constant and related to the $D=5$ embedding tensor as

$$X_{\underline{MN}}{}^{\underline{K}} \;=\; \left(\Theta_{\underline{M}}{}^{\alpha}+\tfrac{9}{2}\,\vartheta_{\underline{L}}(t^{\alpha})_{\underline{M}}{}^{\underline{L}}\right)(t_\alpha)_{\underline{N}}{}^{\underline{K}}-\delta_{\underline{N}}{}^{\underline{K}}\,\vartheta_{\underline{M}}\,, \tag{6.16}$$

as we briefly verify in the following. In particular, equation (6.15) implies that

$$\mathbb{L}_{\widehat{U}^{-1}_{\underline{M}}}\,\rho \;=\; -\vartheta_{\underline{M}}\,\rho\,. \tag{6.17}$$

The left-hand side of (6.15) reads

$$\left(\mathbb{L}_{\widehat{U}_{\underline{M}}^{-1}}\widehat{U}_{\underline{N}}^{-1}\right)^K = (\widehat{U}^{-1})_{\underline{M}}{}^N\partial_N(\widehat{U}^{-1})_{\underline{N}}{}^K - 6\,(t^\alpha)_L{}^K(t_\alpha)_Q{}^P\,\partial_P(\widehat{U}^{-1})_{\underline{M}}{}^Q\,(\widehat{U}^{-1})_{\underline{N}}{}^L + \tfrac{1}{3}\,\partial_P(\widehat{U}^{-1})_{\underline{M}}{}^P\,(\widehat{U}^{-1})_{\underline{N}}{}^K\,. \tag{6.18}$$

Expressing this in terms of U and ρ, writing the derivatives of U in terms of (6.12), and multiplying both sides by $\widehat{U}_K{}^{\underline{K}}$, a quick computation yields

$$\widehat{U}_K{}^{\underline{K}}\left(\mathbb{L}_{\widehat{U}_{\underline{M}}^{-1}}\widehat{U}_{\underline{N}}^{-1}\right)^K = -\rho^{-1}(t_\alpha)_{\underline{N}}{}^{\underline{K}}\left(\mathscr{X}_{\underline{M}}{}^\alpha - 6\,(t^\alpha)_{\underline{P}}{}^{\underline{Q}}(t_\beta)_{\underline{M}}{}^{\underline{P}}\,\mathscr{X}_{\underline{Q}}{}^\beta\right) - \tfrac{1}{3}\,\rho^{-1}\,\mathscr{X}_{\underline{PM}}{}^{\underline{P}}\,\delta_{\underline{N}}{}^{\underline{K}} + \left(6\,(t^\alpha)_{\underline{N}}{}^{\underline{K}}(t_\alpha)_{\underline{M}}{}^{\underline{Q}}(U^{-1})_{\underline{Q}}{}^P - \tfrac{4}{3}\,(U^{-1})_{\underline{M}}{}^P\,\delta_{\underline{N}}{}^{\underline{K}}\right)\rho^{-2}\partial_P\rho\,. \tag{6.19}$$

Next, the form of the projector (6.13) onto the **351** allows to rewrite the terms in parenthesis in the first line of (6.19). One finds

$$\widehat{U}_K{}^{\underline{K}}\left(\mathbb{L}_{\widehat{U}_{\underline{M}}^{-1}}\widehat{U}_{\underline{N}}^{-1}\right)^K = -5\,\rho^{-1}\left[\mathscr{X}_{\underline{MN}}{}^{\underline{K}}\right]_{\mathbf{351}} + \tfrac{1}{3}\,\rho^{-1}\,\delta_{\underline{N}}{}^{\underline{K}}\left(\partial_P(U^{-1})_{\underline{M}}{}^P - 4\,(U^{-1})_{\underline{M}}{}^P\,\rho^{-1}\partial_P\rho\right) - \tfrac{3}{2}\,\rho^{-1}(t_\alpha)_{\underline{N}}{}^{\underline{K}}(t^\alpha)_{\underline{M}}{}^{\underline{Q}}\left(\partial_P(U^{-1})_{\underline{Q}}{}^P - 4\,(U^{-1})_{\underline{Q}}{}^P\,\rho^{-1}\partial_P\rho\right)\,. \tag{6.20}$$

Finally inserting (6.6) and (6.11), we obtain

$$\widehat{U}_K{}^{\underline{K}}\left(\mathbb{L}_{\widehat{U}_{\underline{M}}^{-1}}\widehat{U}_{\underline{N}}^{-1}\right)^K = -\Theta_{\underline{M}}{}^\alpha(t_\alpha)_{\underline{N}}{}^{\underline{K}} + \delta_{\underline{N}}{}^{\underline{K}}\,\vartheta_{\underline{M}} - \tfrac{9}{2}\,(t_\alpha)_{\underline{N}}{}^{\underline{K}}(t^\alpha)_{\underline{M}}{}^{\underline{Q}}\,\vartheta_{\underline{Q}}\,, \tag{6.21}$$

which implies (6.16) for the structure constants defined in (6.15), thereby verifying the equivalence with (6.6), (6.11).

It is straightforward to verify that subject to (6.15), the gauge transformations of all bosonic fields in (6.2) reduce to the correct gauge transformations in gauged supergravity. Let us illustrate this for a vector of generic weight λ, for which the Scherk-Schwarz ansatz reads

$$V^M(x,Y) = \rho^{-3\lambda}(U^{-1})_{\underline{N}}{}^M(Y)\,V^{\underline{N}}(x) = \rho^{-3\lambda+1}(\widehat{U}^{-1})_{\underline{N}}{}^M(Y)\,V^{\underline{N}}(x)\,. \tag{6.22}$$

Using (6.15) and (6.17), its gauge transformation then takes the form

$$\begin{aligned}\delta_\Lambda V^M &= \mathbb{L}_{\blacksquare^{\underline{K}}\widehat{U}_{\underline{K}}^{-1}}\left(\rho^{-3\lambda+1}(\widehat{U}^{-1})_{\underline{N}}{}^M\right)V^{\underline{N}} \\ &= \blacksquare^{\underline{K}}\left((-3\lambda+1)\left(\mathbb{L}_{\widehat{U}_{\underline{K}}^{-1}}\rho\right)\rho^{-3\lambda}\,(\widehat{U}^{-1})_{\underline{N}}{}^M + \rho^{-3\lambda+1}\mathbb{L}_{\widehat{U}_{\underline{K}}^{-1}}(\widehat{U}^{-1})_{\underline{N}}{}^M\right)V^{\underline{N}} \\ &= \rho^{-3\lambda+1}(\widehat{U}^{-1})_{\underline{N}}{}^M\left((3\lambda-1)\blacksquare^{\underline{K}}\,\vartheta_{\underline{K}}\,V^{\underline{N}} - \blacksquare^{\underline{K}}\,X_{\underline{KL}}{}^{\underline{N}}\,V^{\underline{L}}\right)\,,\end{aligned} \tag{6.23}$$

from which we read off, inserting (6.16),

$$\delta_\blacksquare V^{\underline{N}} = -\blacksquare^{\underline{K}}\left(\Theta_{\underline{K}}{}^\alpha + \tfrac{9}{2}\,\vartheta_{\underline{P}}(t^\alpha)_{\underline{K}}{}^{\underline{P}}\right)(t_\alpha)_{\underline{L}}{}^{\underline{N}}\,V^{\underline{L}} + 3\,\lambda\,\blacksquare^{\underline{K}}\,\vartheta_{\underline{K}}\,V^{\underline{N}}\,. \tag{6.24}$$

This is the expected transformation in gauged supergravity with general trombone gauging and in particular is compatible with (6.10) and (6.7) for $\lambda=0$ and $\lambda=\frac{2}{3}$, respectively. As the covariant derivatives and field strengths are defined in terms of generalized Lie derivatives (or its anti-symmetrization, the E-bracket), it follows immediately that also these objects reduce 'covariantly' under Scherk-Schwarz, e.g.,

$$\begin{aligned}\mathscr{D}_\mu g_{\nu\rho}(x,Y) &= \rho^{-2}\left(\partial_\mu - A_\mu{}^N\vartheta_N\right)\mathrm{g}_{\nu\rho}\,, \\ \mathscr{D}_\mu\mathscr{M}_{MN}(x,Y) &= U_M{}^{\underline{P}}U_N{}^{\underline{Q}}\left(\partial_\mu M_{\underline{PQ}} - 2A_\mu{}^{\underline{L}}\left(\Theta_{\underline{L}}{}^\alpha + \tfrac{9}{2}\,\vartheta_{\underline{R}}\,(t^\alpha)_{\underline{L}}{}^{\underline{R}}\right)(t_\alpha)_{(\underline{M}}{}^{\underline{P}}M_{\underline{N})\underline{P}}\right)\,.\end{aligned} \tag{6.25}$$

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In addition, the covariant two-form field strength reduces consistently,

$$\mathscr{F}_{\mu\nu}{}^{M}(x,Y) \;=\; \rho^{-1}\,(U^{-1})_{\underline{N}}{}^{M}\,F_{\mu\nu}{}^{\underline{N}}(x)\;, \tag{6.26}$$

with the $D=5$ covariant field strength $F_{\mu\nu}{}^{N}$ given by

$$F_{\mu\nu}{}^{\underline{M}} \equiv 2\partial_{[\mu}A_{\nu]}{}^{\underline{M}} + X_{\underline{KL}}{}^{\underline{M}}A_{[\mu}{}^{\underline{K}}A_{\nu]}{}^{\underline{L}} + d^{\underline{MKL}}X_{\underline{KL}}{}^{\underline{N}}B_{\mu\nu\,\underline{N}}\;, \tag{6.27}$$

and similarly for the three-form curvature. Finally, one can verify that *internal* covariant derivatives ∇_M, whose connection components are only partially determined in terms of the physical fields, reduce covariantly under Scherk-Schwarz reduction for those contractions/projections that are fully determined. To this end one may start from the vielbein postulate that relates the Christoffel-type connections to the USp(8) valued 'spin-connections' and use the covariant constraints that determine projections of the Christoffel connection, e.g., the generalized torsion constraint (2.36). The latter then determines, via (6.11), the corresponding projections of the spin connection in terms of the embedding tensor. The general analysis proceedes in complete parallel to the discussion in [39]. In particular, with the geometric definition (2.49) of the curvature scalar, which is independent of undetermined connections, it follows that the potential reduces consistently and thus yields the scalar potential of five-dimensional gauged supergravity, whose form is uniquely determined by supersymmetry.

Let us finally discuss the fermions $\psi_\mu{}^i$ and χ^{ijk}, which transform under the local Lorentz group $\mathrm{USp}(8)$ and are scalar densities of weight $\frac{1}{6}$ and $-\frac{1}{6}$, respectively. Accordingly, the Scherk-Schwarz ansatz simply reads

$$\psi_\mu{}^i(x,Y) \;=\; \rho^{-\frac{1}{2}}(Y)\,\psi_\mu{}^i(x)\;, \qquad \chi^{ijk}(x,Y) \;=\; \rho^{\frac{1}{2}}(Y)\,\chi^{ijk}(x)\;. \tag{6.28}$$

Note in particular that the ansatz does not involve a 'rotation' of the $\mathrm{USp}(8)$ indices by Killing spinors, in contrast to conventional Kaluza-Klein compactifications. This is in accord with the fact that such a rotation is a $\mathrm{USp}(8)$ transformation, which in the context of EFT is a gauge symmetry, and so would correspond to a deformation that is pure gauge and hence irrelevant. By the above discussion, the supersymmetry variations (2.53), (2.54) reduce consistently under Scherk-Schwarz. In particular, the terms in the fermion variations of (2.53) depending on the internal covariant derivatives ∇_M, whose connection components are fully determined, reduce to the projections of the embedding tensor (more precisely, the 'flattened' embedding tensor often referred to as the 'T-tensor') that determine the tensors A_1 and A_2 defining the fermion shifts in gauged supergravity.

To summarize, the reduction ansatz (6.2), (6.28) describes a consistent truncation of $\mathrm{E}_{6(6)}$ EFT to a $D=5$ maximal gauged supergravity, provided the twist matrices satisfy the consistency conditions (6.6) and (6.11). It is intriguing, that the match with lower-dimensional gauged supergravity, does in fact not explicitly use the section constraint (provided the initial scalar potential is written in an appropriate form) [25, 39, 45]. Formally this allows to reproduce all $D=5$ maximal gauged supergravities, and it is intriguing to speculate about their possible higher-dimensional embedding upon a possible relaxation of the section constraints that would define a genuine extension of the original supergravity theories. For the moment it is probably fair to say that our understanding of a consistent extension of the framework is still limited. If on the other hand the twist matrices U do obey the section constraint (2.1), the reduction ansatz (6.2), (6.28) translates into a consistent truncation of the original $D = 11$ or type IIB supergravity, respectively, depending on to which solution of the section constraint the twist matrices U belong. With the explicit dictionary between EFT and the original supergravities, given above for type IIB and in [13] for $D = 11$, the simple factorization ansatz (6.2), (6.28) then translates into a highly non-linear ansatz for the consistent embedding of the lower-dimensional theory.

7. Summary and Outlook

We have reviewed the $E_{6(6)}$ exceptional field theory and established the precise embedding of ten-dimensional type IIB supergravity upon picking the corresponding solution of the section constraint. Given that as shown here, the resulting theory admits the full ten-dimensional diffeomorphism invariance, maximal supersymmetry and the global $SL(2,\mathbb{R})$ S-duality invariance, its equivalence to type IIB supergravity is guaranteed on general grounds. It is nevertheless useful to work out the explicit embedding. We have done so in this review by first matching the gauge symmetries on both sides. On the type IIB supergravity side, this requires a number of field redefinitions, which are largely analogous to those needed in conventional Kaluza-Klein compactifications. On the exceptional field theory side, this requires a suitable parametrization of the $E_{6(6)}$ valued '27-bein'. We have then given the explicit dictionary from the various components of the IIB fields to the EFT fields after solution of the section constraint. We also established the on-shell equivalence of both theories and in particular showed how the three- and four-forms of type IIB, originating from components of the self-dual four-form in ten dimensions, are reconstructed on-shell in exceptional field theory in which these fields are not present from the start.

Having determined the precise embedding of type IIB into $E_{6(6)}$ exceptional field theory, we can use the results of [39] on generalized Scherk-Schwarz compactifications in exceptional field theory to give the explicit embedding of various consistent Kaluza-Klein truncations of type IIB. The details will appear in [52]. In particular, this establishes the Kaluza-Klein consistency of $AdS_5 \times S^5$ in type IIB and, more importantly, gives the precise embedding formulas. This requires the precise interplay between various identities whose validity appears somewhat miraculous from the point of view of conventional geometry but which find a natural interpretation within the extended geometry of exceptional field theory.

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