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Supersymmetric Spacetimes from Curved Superspace

Sergei M. Kuzenko*
School of Physics M013, The University of Western Australia
35 Stirling Highway, Crawley W.A. 6009, Australia
E-mail: sergei.kuzenko@uwa.edu.au

We review the superspace technique to determine supersymmetric spacetimes in the framework of off-shell formulations for supergravity in diverse dimensions using the case of 3D $\mathcal{N}=2$ supergravity theories as an illustrative example. This geometric formalism has several advantages over other approaches advocated in the last four years. Firstly, the infinitesimal isometry transformations of a given curved superspace form, by construction, a finite-dimensional Lie superalgebra, with its odd part corresponding to the rigid supersymmetry transformations. Secondly, the generalised Killing spinor equation, which must be obeyed by the supersymmetry parameters, is a consequence of the more fundamental superfield Killing equation. Thirdly, general rigid supersymmetric theories on a curved spacetime are readily constructed in superspace by making use of the known off-shell supergravity-matter couplings and restricting them to the background chosen. It is the superspace techniques which make it possible to generate arbitrary off-shell supergravity-matter couplings. Fourthly, all maximally supersymmetric Lorentzian spaces correspond to those off-shell supergravity backgrounds for which the Grassmann-odd components of the superspace torsion and curvature tensors vanish, while the Grassmann-even components of these tensors are annihilated by the spinor derivatives.

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*Speaker.

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1. Introduction

Supersymmetric solutions of supergravity theories had already attracted much interest in the early 1980s in the context of Kaluza-Klein supergravity, see [1] for a review. At that period, the notion of Killing spinors[1] [2, 3] (see also [4]), which is crucial to the program of Kaluza-Klein supergravity, was introduced. The existence of such spinors guarantees some unbroken supersymmetry upon compactification. Since then, the Killing spinors on pseudo-Riemannian manifolds and their properties have been studied by mathematicians, see [5, 6] and references therein.

An additional impetus to study supersymmetric solutions of supergravity theories comes from superstring theory to which supergravity is a low-energy approximation. Due to certain non-renormalisation and stability properties they possess, such solutions are of special importance in the string-theoretic framework. A detailed discussion of the huge number of the supersymmetric solutions of supergravity constructed in diverse dimensions is beyond the scope of this conference paper. As an example of such constructions, it is pertinent to mention two papers [7] in which all supersymmetric solutions in minimal Poincaré and anti-de Sitter supergravity theories in five dimensions were constructed.

In off-shell supergravity, the superspace formalism to determine (super)symmetric backgrounds was elaborated twenty years ago [8] in the framework of the old minimal formulation [9, 10] for $\mathscr{N}=1$ supergravity in four dimensions (4D). The approach developed in [8] is universal, for it may be generalised to derive supersymmetric backgrounds associated with any supergravity theory formulated in superspace. In particular, it has already been used to construct rigid supersymmetric field theories in 5D $\mathscr{N}=1$ [11], 4D $\mathscr{N}=2$ [12, 13, 14] and 3D (p,q) [15, 16, 17] anti-de Sitter superspaces.

Recently, much progress has been made in deriving new exact results for observables (partition functions, Wilson loops) in rigid supersymmetric gauge theories on compact manifolds such as round spheres using localisation techniques [18, 19, 20, 21]. In order to apply these techniques, two technical prerequisites are required. Firstly, a curved space $\mathscr{M}$ has to admit some unbroken rigid supersymmetry. Secondly, the rigid supersymmetric theory on $\mathscr{M}$ should be off-shell. These conditions are met by those supersymmetric backgrounds that correspond to off-shell supergravity theories. This is why a number of publications have appeared which are devoted to the construction of supersymmetric backgrounds associated with off-shell supergravity theories in diverse dimensions, see [22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34] and references therein. Inspired by the work of Festuccia and Seiberg [22], these authors used component field considerations. In the case of 4D $\mathscr{N}=1$ supergravity, it was shown [35] how to derive the key component results of, e.g., [22, 27] from the more general superspace construction of [8]. Recently, the formalism of [8, 35] was extended to construct supersymmetric backgrounds [36] associated with all known off-shell formulations for 3D $\mathscr{N}=2$ supergravity [15, 37]. The results obtained are in agreement with the component considerations of [31, 32, 34]. The same formalism has also been used in [38] to derive supersymmetric backgrounds in off-shell formulations for 5D $\mathscr{N}=1$ supergravity.

In this paper, we give a pedagogical review of the formalism of [8, 35]. As an application of the formalism, we briefly describe the results of [36] devoted to the construction of supersymmetric backgrounds in all known off-shell formulations for 3D $\mathscr{N}=2$ supergravity [15, 37].

[1] F. Englert, one of the authors of [3], was awarded the 2013 Nobel Prize in Physics (shared with P. Higgs).

2. (Conformal) isometries of curved space

Before discussing supersymmetric backgrounds in off-shell supergravity, it is instructive to recall how the (conformal) isometries of a curved spacetime are defined within the Weyl invariant formulation for gravity [39, 40, 41]. Our presentation follows [38]. We start by recalling three known approaches to the description of gravity on a d-dimensional manifold $\mathscr{M}^d$: (i) the metric formulation; (ii) the vielbein formulation; and (iii) the Weyl invariant formulation.

In the standard metric approach, the gauge field is a symmetric metric tensor $g_{mn}(x)$ such that $g := \det(g_{mn}) \neq 0$. The infinitesimal gauge transformation of g_{mn} is

$$\delta g_{mn} = \nabla_m \xi_n + \nabla_n \xi_m \ , \tag{2.1}$$

with the gauge parameter $\xi = \xi^m(x)\partial_m$ being a vector field generating a one-parameter family of diffeomorphisms.

In the vielbein formulation, the gauge field is a vielbein $e^a := \mathrm{d}x^m e_m{}^a(x)$ that constitutes an orthonormal basis in the cotangent space $T^*_x\mathscr{M}^d$, for any spacetime point x, $e := \det(e_m{}^a) \neq 0$. The metric becomes a composite field defined by $g_{mn} = e_m{}^a e_n{}^b \eta_{ab}$, with η_{ab} the Minkowski metric. The gauge group is now larger than in the metric approach. It includes both general coordinate and local Lorentz transformations,

$$\delta \nabla_a = [\xi^b \nabla_b + \frac{1}{2} K^{bc} M_{bc}, \nabla_a] \ , \tag{2.2}$$

with the gauge parameters $\xi^a(x) = \xi^m(x) e_m{}^a(x)$ and $K^{ab}(x) = -K^{ba}(x)$ being completely arbitrary. The gauge transformation (2.2) makes use of the torsion-free covariant derivatives,

$$\nabla_a = e_a + \omega_a = e_a{}^m \partial_m + \frac{1}{2}\omega_a{}^{bc} M_{bc} \ , \qquad [\nabla_a, \nabla_b] = \frac{1}{2} R_{ab}{}^{cd} M_{cd} \ . \tag{2.3}$$

Here $M_{bc} = -M_{cb}$ denotes the Lorentz generators, $e_a{}^m(x)$ the inverse vielbein, $e_a{}^m e_m{}^b = \delta_a{}^b$, and $\omega_a{}^{bc}(x)$ the torsion-free Lorentz connection. Finally, $R_{ab}{}^{cd}(x)$ is the Riemann curvature tensor; its descendants are the Ricci tensor $R_{ab} := \eta^{cd} R_{cadb} = R_{ba}$ and the scalar curvature $R = \eta^{ab} R_{ab}$.

As is well-known, the torsion-free constraint

$$T_{ab}{}^c = 0 \quad \Longleftrightarrow \quad [\nabla_a, \nabla_b] \equiv T_{ab}{}^c \nabla_c + \frac{1}{2} R_{ab}{}^{cd} M_{cd} = \frac{1}{2} R_{ab}{}^{cd} M_{cd} \tag{2.4}$$

is invariant under Weyl (local scale) transformations of the form

$$\nabla_a \to \nabla'_a = \mathrm{e}^{\sigma} \Big(\nabla_a + (\nabla^b \sigma) M_{ba} \Big) \ , \tag{2.5}$$

with the parameter $\sigma(x)$ being completely arbitrary. Such a transformation is induced by that of the gravitational field

$$e_a{}^m \to \mathrm{e}^{\sigma} e_a{}^m \quad \Longrightarrow \quad g_{mn} \to \mathrm{e}^{-2\sigma} g_{mn} \ . \tag{2.6}$$

In general, Weyl invariant matter theories are curved-space extensions of ordinary conformally invariant theories. As an example, consider the model for a scalar field φ with action

$$S = -\frac{1}{2} \int \mathrm{d}^d x \, e \left\{ \nabla^a \varphi \nabla_a \varphi + \frac{1}{4} \frac{d-2}{d-1} R \varphi^2 + \lambda \varphi^{2d/(d-2)} \right\} \ , \qquad d \neq 2 \ , \tag{2.7}$$

where λ is a coupling constant. The action is Weyl invariant[2] provided φ transforms as

$$\varphi \to \varphi' = \mathrm{e}^{\frac{1}{2}(d-2)\sigma}\varphi \ . \tag{2.8}$$

The flat-space limit of (2.7) is a conformal field theory.

Most field theories in curved space do not possess Weyl invariance. In particular, the pure gravity action with a cosmological term

$$S_{\rm GR} = \frac{1}{2\kappa^2}\int \mathrm{d}^d x\, e\, R - \frac{\Lambda}{\kappa^2}\int \mathrm{d}^d x\, e \tag{2.9}$$

is not invariant under the Weyl transformations (2.5). However, any field theory in curved space can be made Weyl invariant by coupling it to a conformal compensator.

In the Weyl invariant formulation for gravity in $d \neq 2$ dimensions, the gravitational field is described in terms of two gauge fields. One of them is the vielbein $e_m{}^a(x)$ and the other is a conformal compensator $\varphi(x)$ with the Weyl transformation law (2.8). As compared with the matter model (2.7), the compensator is constrained to be nowhere vanishing, $\varphi \neq 0$. In this approach, the gravity gauge group is defined to consist of the general coordinate, local Lorentz and Weyl transformations

$$\delta\nabla_a = [\xi^b\nabla_b + \frac{1}{2}K^{bc}M_{bc}, \nabla_a] + \sigma\nabla_a + (\nabla^b\sigma)M_{ba} \equiv (\delta_{\mathscr{K}} + \delta_\sigma)\nabla_a \ , \tag{2.10a}$$

$$\delta\varphi = \xi^b\nabla_b\varphi + \frac{1}{2}(d-2)\sigma\varphi \equiv (\delta_{\mathscr{K}} + \delta_\sigma)\varphi \ , \tag{2.10b}$$

where we have denoted $\mathscr{K} := \xi^b\nabla_b + \frac{1}{2}K^{bc}M_{bc}$. Any dynamical system is required to be invariant under these transformations. In particular, the Weyl invariant gravity action is

$$S_{\rm GR} = \frac{1}{2}\int \mathrm{d}^d x\, e \left\{\nabla^a\varphi\nabla_a\varphi + \frac{1}{4}\frac{d-2}{d-1}R\varphi^2 + \lambda\,\varphi^{2d/(d-2)}\right\} . \tag{2.11}$$

Applying a finite Weyl transformation allows us to choose the gauge condition

$$\varphi = \frac{1}{2\kappa}\sqrt{\frac{d-1}{d-2}} \ , \tag{2.12}$$

in which the action (2.11) turns into (2.9).

Every gravity-matter system can be made Weyl invariant by replacing $e_a{}^m \to \varphi^{-2/(d-2)}e_a{}^m$ in the action. If the action of a Weyl invariant theory does not depend on φ, it describes conformal gravity coupled to matter. It is natural to use the notation $(\mathscr{M}^d, \nabla)$ in the case of conformal gravity and $(\mathscr{M}^d, \nabla, \varphi)$ for ordinary gravity. In both cases, the gravity gauge freedom is given by (2.10), but φ is not present in conformal gravity. One may understand conformal gravity as possessing an additional gauge freedom, $\nabla_a \to \nabla_a$ and $\varphi \to \mathrm{e}^\rho\varphi$, with the gauge parameter $\rho(x)$ being arbitrary.

Let us fix a background spacetime. A vector field $\xi = \xi^m\partial_m = \xi^a e_a$ on $(\mathscr{M}^d, \nabla)$ is said to be conformal Killing if there exist local Lorentz $K^{bc}[\xi]$ and Weyl $\sigma[\xi]$ parameters such that

$$\big(\delta_{\mathscr{K}[\xi]} + \delta_{\sigma[\xi]}\big)\nabla_a = 0 \ . \tag{2.13}$$

[2]The Weyl transformation of the scalar curvature is $R \to \mathrm{e}^{2\sigma}\Big\{R + 2(d-1)\nabla^a\nabla_a\sigma - (d-2)(d-1)(\nabla^a\sigma)\nabla_a\sigma\Big\}$.

A short calculation gives

$$K^{bc}[\xi] = \frac{1}{2}\big(\nabla^b\xi^c - \nabla^c\xi^b\big)~, \qquad \sigma[\xi] = \frac{1}{d}\nabla_b\xi^b~, \tag{2.14}$$

as well as the conformal Killing equation

$$\nabla^a\xi^b + \nabla^b\xi^a = 2\eta^{ab}\sigma[\xi]~. \tag{2.15}$$

The set of all conformal Killing vector fields on $(\mathscr{M}^d,\nabla)$ forms a finite-dimensional Lie algebra.[3] It is the conformal algebra of the spacetime, and its dimension cannot exceed that of $\mathfrak{so}(d,2)$, the conformal algebra of Minkowski space. The notion of a conformal Killing vector field does not make use of φ, and therefore two spacetimes $(\mathscr{M}^d,\nabla,\varphi)$ and $(\mathscr{M}^d,\nabla,\mathrm{e}^\rho\varphi)$ have the same conformal Killing vector fields, for an arbitrary scalar field $\rho(x)$.

Two spacetimes $(\mathscr{M}^d,\nabla,\varphi)$ and $(\mathscr{M}^d,\widetilde{\nabla},\widetilde{\varphi})$ are said to be conformally related if their gauge fields are obtained from each other by applying a finite Weyl transformation,

$$\widetilde{\nabla}_a = \mathrm{e}^\rho\Big(\nabla_a + (\nabla^b\rho)M_{ba}\Big)~, \qquad \widetilde{\varphi} = \mathrm{e}^{\frac12(d-2)\rho}\varphi~, \tag{2.16}$$

for some ρ. These spacetimes have the same conformal Killing vector fields,

$$\xi = \xi^a e_a = \tilde{\xi}^a\tilde{e}_a~. \tag{2.17}$$

The parameters $K^{cd}[\tilde{\xi}]$ and $\sigma[\tilde{\xi}]$ are related to $K^{cd}[\xi]$ and $\sigma[\xi]$ as follows:

$$\mathscr{K}[\tilde{\xi}] := \tilde{\xi}^b\widetilde{\nabla}_b + \frac{1}{2}K^{cd}[\tilde{\xi}]M_{cd} = \mathscr{K}[\xi]~, \tag{2.18}$$

$$\sigma[\tilde{\xi}] = \sigma[\xi] - \xi\rho~. \tag{2.19}$$

These relations are such that $(\delta_{\mathscr{K}[\tilde{\xi}]} + \delta_{\sigma[\tilde{\xi}]})\widetilde{\nabla}_a = 0$.

A conformal Killing vector field $\xi = \xi^a e_a$ on $(\mathscr{M}^d,\nabla,\varphi)$ is called Killing if the transformation $\delta_{\mathscr{K}[\xi]} + \delta_{\sigma[\xi]}$ does not change the compensator,

$$\xi\varphi + \frac{1}{2}(d-2)\sigma[\xi]\varphi = 0~. \tag{2.20}$$

The set of all Killing vector fields of the given spacetime $(\mathscr{M}^d,\nabla,\varphi)$ is a finite-dimensional Lie algebra. By construction, it is a subalgebra of the conformal algebra of $(\mathscr{M}^d,\nabla)$. The Killing equations (2.13) and (2.20) are Weyl invariant in the following sense. Given a conformally related spacetime $(\mathscr{M}^d,\widetilde{\nabla}_a,\widetilde{\varphi})$ defined by eq. (2.16), the Killing equations (2.13) and (2.20) have the same functional form when rewritten in terms of $\widetilde{\nabla}_a$ and $\widetilde{\varphi}$,

$$(\delta_{\mathscr{K}[\tilde{\xi}]} + \delta_{\sigma[\tilde{\xi}]})\widetilde{\nabla}_a = 0~, \qquad \xi\widetilde{\varphi} + \frac{1}{2}(d-2)\sigma[\tilde{\xi}]\widetilde{\varphi} = 0~. \tag{2.21}$$

The Weyl invariance allows us to choose the gauge condition

$$\varphi = 1~. \tag{2.22}$$

Then the Killing equations (2.21) turn into

$$\Big[\xi^b\nabla_b + \frac{1}{2}K^{bc}[\xi]M_{bc}, \nabla_a\Big] = 0~, \qquad \sigma[\xi] = 0~, \tag{2.23}$$

which is equivalent to the standard Killing equation

$$\nabla^a\xi^b + \nabla^b\xi^a = 0~. \tag{2.24}$$

[3] Introducing $\Upsilon := \{\xi^b, K^{bc}[\xi], \sigma[\xi], \nabla_b\sigma[\xi]\}$, one observes that $\nabla_a\Upsilon \subset \mathrm{span}(\Upsilon)$.

3. (Conformal) symmetries of curved superspace

The Weyl invariant approach to gravity and spacetime symmetries, which was reviewed in the previous section, has a natural superspace extension [8, 35, 36, 38] in all cases when Poincaré or anti-de Sitter supergravity is formulated as conformal supergravity coupled to certain conformal compensator(s). This is always possible for supergravity theories in $d \leq 6$ with up to eight supercharges, where off-shell conformal supergravity always exists.

Let $\mathscr{M}^{d|\delta}$ be a curved superspace, with d spacetime and δ fermionic dimensions, chosen to describe a given supergravity theory. We denote by $z^M = (x^m, \theta^{\hat{\mu}})$ the local coordinates for $\mathscr{M}^{d|\delta}$. Without loss of generality, we assume that the zero section of $\mathscr{M}^{d|\delta}$ defined by $\theta^{\hat{\mu}} = 0$ corresponds to the spacetime manifold $\mathscr{M}^d$.

The differential geometry of curved superspace $\mathscr{M}^{d|\delta}$ may be realised in terms of covariant derivatives of the form

$$\mathscr{D}_A = (\mathscr{D}_a, \mathscr{D}_{\hat{\alpha}}) = E_A + \Omega_A + \Phi_A \ . \tag{3.1}$$

Here $E_A = E_A{}^M(z)\partial/\partial z^M$ denotes the inverse superspace vielbein, $\Omega_A = \frac{1}{2}\Omega_A{}^{bc}(z)M_{bc}$ is the Lorentz connection, and $\Phi = \Phi_A{}^I(z)J_I$ the R-symmetry connection.[4] The index $\hat{\alpha}$ of the fermionic operator $\mathscr{D}_{\hat{\alpha}}$ is, in general, composite; it is comprised of a spinor index α and an R-symmetry index.

The covariant derivatives obey the (anti-)commutation relations of the form

$$[\mathscr{D}_A, \mathscr{D}_B\} = \mathscr{T}_{AB}{}^C\mathscr{D}_C + \frac{1}{2}\mathscr{R}_{AB}{}^{cd}M_{cd} + \mathscr{R}_{AB}{}^I J_I \ , \tag{3.2}$$

where $\mathscr{T}_{AB}{}^C(z)$ is the torsion tensor, $\mathscr{R}_{AB}{}^{cd}(z)$ and $\mathscr{R}_{AB}{}^I(z)$ are the Lorentz and R-symmetry curvature tensors, respectively. In order to describe conformal supergravity, the superspace torsion $\mathscr{T}_{AB}{}^C$ has to obey certain algebraic constraints, which may be thought of as generalisations of the torsion-free constraint in gravity, eq. (2.4), and which have to be Lorentz and R-symmetry invariant. Unlike the gravity case, there is no universal expression for such constraints, since their explicit form depends on the dimension of spacetime d as well as on the supersymmetry type chosen. However, certain guiding principles leading to proper torsion constraints are described in important papers by Gates et al. [42] and Howe [43], and are also reviewed in textbooks [8, 44].

The supergravity gauge group includes a subgroup generated by local transformations

$$\delta_{\mathscr{K}}\mathscr{D}_A = [\mathscr{K}, \mathscr{D}_A] \ , \qquad \mathscr{K} := \xi^B(z)\mathscr{D}_B + \frac{1}{2}K^{bc}(z)M_{bc} + K^I(z)J_I \ , \tag{3.3a}$$

where the gauge parameters ξ^A, $K^{bc} = -K^{cb}$ and K^I obey standard reality conditions but are otherwise arbitrary. Given a tensor superfield $\Phi(z)$ (with suppressed Lorentz and R-symmetry indices), its transformation law under (3.3a) is

$$\delta_{\mathscr{K}}\Phi = \mathscr{K}\Phi \ . \tag{3.3b}$$

In order to describe conformal supergravity, the constraints imposed on the superspace torsion should be invariant under super-Weyl transformations of the form

$$\delta_\sigma\mathscr{D}_a = \sigma\mathscr{D}_a + \cdots \ , \qquad \delta_\sigma\mathscr{D}_{\hat{\alpha}} = \frac{1}{2}\sigma\mathscr{D}_{\hat{\alpha}} + \cdots \ , \tag{3.4a}$$

[4] The superspace structure group, $\mathrm{Spin}(d-1,1)\times G_R$, is a subgroup of the isometry group of Minkowski superspace $\mathbb{R}^{d|\delta}$. This subgroup is the isotropy group of the origin in $\mathbb{R}^{d|\delta}$.

PoS(CORFU2014)140

where the scale parameter σ is an arbitrary real superfield. The ellipsis in the expression for $\delta_\sigma \mathscr{D}_a$ includes, in general, a linear combination of the spinor covariant derivatives $\mathscr{D}_{\hat\beta}$ and the structure group generators M_{cd} and J_K. The ellipsis in $\delta_\sigma \mathscr{D}_{\hat\alpha}$ stands for a linear combination of the generators of the structure group. The super-Weyl transformation (3.4a) is a natural generalisation of the Weyl transformation (2.5) in gravity. In most cases of interest, matter superfields may be chosen to be primary under the super-Weyl group,

$$\delta_\sigma \Phi = w_\Phi \sigma \Phi \ , \tag{3.4b}$$

with w_Φ a super-Weyl weight. The transformations (3.3a) and (3.4a) generate the gauge group of conformal supergravity.

An important difference between the superspace covariant derivatives (3.1) and the spacetime ones, eq. (2.3), is that the superspace structure group includes not only the Lorentz group, but also the R-symmetry group G_R. In principle, it is always possible to deal with an alternative superspace geometry such that its structure group coincides with the Lorentz group, similar to the Wess-Zumino formulation [9] of 4D $\mathscr{N}=1$ supergravity. The local G_R group will then appear as an additional invariance of the superspace constraints (similar to the (super-)Weyl invariance in (super)gravity). In many cases, however, such a formulation is technically less useful due to the presence of dimension-1/2 constraints, as explained by Howe [43] in the four-dimensional case.

It should be mentioned that there exist alternative approaches to conformal gravity and conformal supergravity. Conformal gravity in d dimensions can be formulated as a gauge theory of the conformal group, see, e.g., [45] for a review. In such a formulation, the local special conformal transformations may be used to gauge away the dilatation connection. This will lead to the realisation for conformal gravity described in the previous section. Analogously, conformal supergravity in diverse dimensions $d \leq 6$ can be obtained by gauging the relevant superconformal group in superspace [46, 45, 47]. The resulting formulation, known as conformal superspace, may be viewed as a superspace version of the superconformal tensor calculus, see, e.g., [48] for a review. The formulation for conformal supergravity described above is obtained from conformal superspace by gauge fixing certain local symmetries. It is completely adequate to study (conformal) isometries of curved superspace backgrounds; this is why we will not discuss conformal superspace here.

To describe Poincaré or anti-de Sitter supergravity theories, the conformal supergravity multiplet has to be coupled to some off-shell conformal compensators that will be symbolically denoted Ξ. In general, the compensators are Lorentz scalars, and at least one of them has to have a non-zero super-Weyl weight $w_\Xi \neq 0$,

$$\delta_\sigma \Xi = w_\Xi \sigma \Xi \ . \tag{3.5}$$

They may also transform in some representations of the R-symmetry group. The compensators are required to be nowhere vanishing in the sense that the R-symmetry singlets $|\Xi|^2$ should be strictly positive. Different off-shell supergravity theories correspond to different choices of Ξ. The notion of conformally related superspaces can be defined in complete analogy with the non-supersymmetric case considered in the previous section.

Let us now fix a background superspace. A real vector field $\xi = \xi^B E_B$ on $(\mathscr{M}^{d|\delta}, \mathscr{D})$ is called conformal Killing if

$$(\delta_{\mathscr{K}} + \delta_\sigma)\mathscr{D}_A = 0 \ , \tag{3.6}$$

for some Lorentz K^{bc}, R-symmetry K^I and super-Weyl σ parameters. For any dimension $d \leq 6$ and any conformal supergravity with up to eight supercharges, the following properties hold:

- All parameters K^{bc}, K^I and σ are uniquely determined in terms of ξ^B, which allows us to write $K^{bc} = K^{bc}[\xi]$, $K^I = K^I[\xi]$ and $\sigma = \sigma[\xi]$;
- The spinor component $\xi^{\hat{\beta}}$ is uniquely determined in terms of ξ^b;
- The vector component ξ^b obeys an equation that contains all information about the conformal Killing vector field.

For example, in the case of $\mathscr{N} = 1$ supergravity in four dimensions the equation on ξ^b reads [8]

$$\mathscr{D}_{(\alpha}\xi_{\beta)\dot{\beta}} = \bar{\mathscr{D}}_{(\dot{\alpha}}\xi_{\beta\dot{\beta})} = 0 , \tag{3.7}$$

where the vector index of ξ^b is replaced by a pair of spinor ones, undotted and dotted. In the case of 3D $\mathscr{N} = 2$ supergravity studied in [36], the equation on ξ^b is given by (4.22) in the next section.

By construction, the set of conformal Killing vectors on $(\mathscr{M}^{d|\delta}, \mathscr{D})$ is a Lie superalgebra with respect to the standard Lie bracket. This is the superconformal algebra of $(\mathscr{M}^{d|\delta}, \mathscr{D})$. One can show that it is finite-dimensional (the argument one uses is similar to that described in the next section in the three-dimensional case).

Let $\xi = \xi^B E_B$ be a conformal Killing vector field on $(\mathscr{M}^{d|\delta}, \mathscr{D})$,

$$(\delta_{\mathscr{K}[\xi]} + \delta_{\sigma[\xi]})\mathscr{D}_A = 0 , \tag{3.8a}$$

for uniquely determined parameters $K^{bc}[\xi]$, $K^I[\xi]$ and $\sigma[\xi]$. It is called a Killing vector field on $(\mathscr{M}^{d|\delta}, \mathscr{D}, \Xi)$ if the compensators are invariant,

$$(\delta_{\mathscr{K}[\xi]} + w_{\Xi}\sigma[\xi])\Xi = 0 . \tag{3.8b}$$

The set of Killing vectors on $(\mathscr{M}^{d|\delta}, \mathscr{D}, \Xi)$ is a Lie superalgebra. The Killing equations (3.8a) and (3.8b) are super-Weyl invariant in the sense that they hold for all conformally related superspace geometries.

Using the compensators Ξ we can always construct a superfield $\phi = \phi(\Xi)$ that is a singlet under the structure group and has the properties: (i) it is an algebraic function of Ξ; (ii) it is nowhere vanishing; and (iii) it has a non-zero super-Weyl weight w_ϕ, $\delta_\sigma\phi = w_\phi\sigma\phi$. It follows from (3.8b) that

$$(\delta_{\mathscr{K}[\xi]} + w_\phi\sigma[\xi])\phi = 0 . \tag{3.9}$$

The super-Weyl invariance may be used to impose the gauge condition $\phi = 1$. Then eq. (3.9) reduces to $\sigma[\xi] = 0$, and the Killing equations (3.8a) and (3.8b) take the form

$$\big[\mathscr{K}[\xi], \mathscr{D}_A\big] = 0 , \tag{3.10a}$$

$$\mathscr{K}[\xi]\Xi = 0 . \tag{3.10b}$$

Of special interest are those backgrounds $(\mathscr{M}^{d|\delta}, \mathscr{D}, \Xi)$ which admit at least one (conformal) supersymmetry. Such a superspace possesses a conformal Killing vector field ξ^A of the type

$$\xi^a| = 0 , \qquad \xi^{\hat\alpha}| \neq 0 . \tag{3.11}$$

Here, as always, the bar-projection of a superfield $U(z) = U(x,\theta)$ is defined by $U| := U(x,\theta)|_{\theta=0}$. We are usually interested in purely bosonic backgrounds with the property that all fermionic components of the superspace torsion and curvature tensors, eq. (3.2), have vanishing bar-projections,

$$\varepsilon(\mathscr{T}_{\dots}{}^{\dots}) = 1 \to \mathscr{T}_{\dots}{}^{\dots}| = 0 , \qquad \varepsilon(\mathscr{R}_{\dots}{}^{\dots}) = 1 \to \mathscr{R}_{\dots}{}^{\dots}| = 0 , \tag{3.12}$$

where ε denotes the Grassmann parity, $\varepsilon = 0$ for bosons and $\varepsilon = 1$ for fermions. If ξ^A is a Killing vector field with $\sigma[\xi] = 0$, then the bosonic requirements (3.12) naturally arise as consistency conditions. Indeed, let us suppose that $\mathscr{B}$ is a bosonic part, $\varepsilon(\mathscr{B}) = 0$, of the superspace torsion or curvature. For $\sigma[\xi] = 0$, the transformation of $\mathscr{B}|$ is $\delta\mathscr{B}| = \mathscr{K}[\xi]\mathscr{B}| = \xi^{\hat\alpha}|\mathscr{D}_{\hat\alpha}\mathscr{B}|$, assuming all other bosonic parameters, $K^{bc}[\xi]|$ and $K^I[\xi]|$, vanish. On the other hand, it must hold that $\delta\mathscr{B}| = 0$, since the geometry does not change under the transformation associated with ξ^A. This is consistent provided $\mathscr{D}_{\hat\alpha}\mathscr{B}| = 0$, which indicates that all fermionic components of the superspace torsion and curvature tensors should vanish.

The conditions (3.12) imply that at the component level all fermionic fields may be gauged away. In particular, the background gravitini are purely gauge degrees of freedom.

4. Backgrounds with (conformal) isometries in 3D $\mathscr{N} = 2$ supergravity

As an application of the formalism described in the previous section, we review the results of [36] devoted to the construction of supersymmetric backgrounds in all known off-shell formulations for 3D $\mathscr{N} = 2$ supergravity [15, 37]. We consider a curved superspace in three spacetime dimensions, $\mathscr{M}^{3|4}$, parametrised by local bosonic (x^m) and fermionic ($\theta^\mu, \bar\theta_\mu$) coordinates $z^M = (x^m, \theta^\mu, \bar\theta_\mu)$, where $m = 0,1,2$ and $\mu = 1,2$. The Grassmann variables θ^μ and $\bar\theta_\mu$ are related to each other by complex conjugation: $\overline{\theta^\mu} = \bar\theta^\mu$.

4.1 $\mathscr{N} = 2$ conformal supergravity in three dimensions

As discussed in section 2, conformal gravity can be described in terms of the frame field $e_a = e_a{}^m(x)\partial_m$ defined modulo the gauge transformations (2.10a). Here we review the generalisation of that formulation to the case of 3D $\mathscr{N} = 2$ conformal supergravity, following [49, 37, 15].

The superspace structure group is chosen to be $\mathsf{SL}(2,\mathbb{R}) \times \mathsf{U}(1)_R$, and the covariant derivatives $\mathscr{D}_A = (\mathscr{D}_a, \mathscr{D}_\alpha, \bar{\mathscr{D}}^\alpha)$ have the form

$$\mathscr{D}_A = E_A + \Omega_A + \mathrm{i}\,\Phi_A J , \tag{4.1}$$

with J the R-symmetry generator. The Lorentz connection can be written in three different forms,

$$\Omega_A = \frac{1}{2}\Omega_A{}^{bc}M_{bc} = \frac{1}{2}\Omega_A{}^{\beta\gamma}M_{\beta\gamma} = -\Omega_A{}^c M_c , \tag{4.2}$$

depending on whether we use the Lorentz generators with two vector indices ($M_{ab}=-M_{ba}$), one vector index (M_a) and two spinor indices ($M_{\alpha\beta}=M_{\beta\alpha}$).[5] The R-symmetry and Lorentz generators act on the covariant derivatives as follows:

$$[J,\mathscr{D}_\alpha]=\mathscr{D}_\alpha\ , \qquad [J,\bar{\mathscr{D}}^\alpha]=-\bar{\mathscr{D}}^\alpha\ , \qquad [J,\mathscr{D}_a]=0\ , \tag{4.3a}$$

$$[M_{\alpha\beta},\mathscr{D}_\gamma]=\varepsilon_{\gamma(\alpha}\mathscr{D}_{\beta)}\ , \qquad [M_{\alpha\beta},\bar{\mathscr{D}}_\gamma]=\varepsilon_{\gamma(\alpha}\bar{\mathscr{D}}_{\beta)}\ , \quad [M_{ab},\mathscr{D}_c]=2\eta_{c[a}\mathscr{D}_{b]}\ . \tag{4.3b}$$

The supergravity gauge group includes local $\mathscr{K}$-transformations of the form

$$\delta_{\mathscr{K}}\mathscr{D}_A=[\mathscr{K},\mathscr{D}_A]\ , \qquad \mathscr{K}=\xi^C\mathscr{D}_C+\frac{1}{2}K^{cd}M_{cd}+\mathrm{i}\,\tau J\ , \tag{4.4}$$

with the gauge parameters obeying natural reality conditions, but otherwise arbitrary.

In order to describe $\mathscr{N}=2$ conformal supergravity, the torsion has to obey the covariant constraints proposed in [49]. The resulting algebra of covariant derivatives is [37, 15]

$$\{\mathscr{D}_\alpha,\mathscr{D}_\beta\} = -4\bar{\mathscr{R}}M_{\alpha\beta}\ , \qquad \{\bar{\mathscr{D}}_\alpha,\bar{\mathscr{D}}_\beta\}=4\mathscr{R}M_{\alpha\beta}\ , \tag{4.5a}$$

$$\{\mathscr{D}_\alpha,\bar{\mathscr{D}}_\beta\} = -2\mathrm{i}(\gamma^c)_{\alpha\beta}\mathscr{D}_c-2\mathscr{C}_{\alpha\beta}J-4\mathrm{i}\varepsilon_{\alpha\beta}\mathscr{S}J+4\mathrm{i}\mathscr{S}M_{\alpha\beta}-2\varepsilon_{\alpha\beta}\mathscr{C}^{\gamma\delta}M_{\gamma\delta}\ . \tag{4.5b}$$

The explicit expressions for commutators $[\mathscr{D}_a,\mathscr{D}_\beta]$, $[\mathscr{D}_a,\bar{\mathscr{D}}_\beta]$ and $[\mathscr{D}_a,\mathscr{D}_b]$ are given in [15] and [36]. The algebra involves three dimension-1 torsion superfields: a real scalar $\mathscr{S}$, a complex scalar $\mathscr{R}$ and its conjugate $\bar{\mathscr{R}}$, and a real vector $\mathscr{C}_a$. The $\mathrm{U}(1)_R$ charge of $\mathscr{R}$ is -2. The torsion superfields obey certain constraints implied by the Bianchi identities. Some of these constraints are

$$\bar{\mathscr{D}}_\alpha\mathscr{R} = 0\ , \tag{4.6a}$$

$$(\bar{\mathscr{D}}^2-4\mathscr{R})\mathscr{S} = 0\ , \qquad \bar{\mathscr{S}}=\mathscr{S}\ . \tag{4.6b}$$

Thus R is covariantly chiral, and $\mathscr{S}$ covariantly linear.

The algebra of covariant derivatives given by (4.5) does not change under the super-Weyl transformation [15, 37]

$$\mathscr{D}'_\alpha = \mathrm{e}^{\frac{1}{2}\sigma}\Big(\mathscr{D}_\alpha+(\mathscr{D}^\gamma\sigma)M_{\gamma\alpha}-(\mathscr{D}_\alpha\sigma)J\Big)\ , \tag{4.7a}$$

$$\begin{aligned}\mathscr{D}'_a = \mathrm{e}^{\sigma}\Big(&\mathscr{D}_a-\frac{\mathrm{i}}{2}(\gamma_a)^{\gamma\delta}(\mathscr{D}_\gamma\sigma)\bar{\mathscr{D}}_\delta-\frac{\mathrm{i}}{2}(\gamma_a)^{\gamma\delta}(\bar{\mathscr{D}}_\gamma\sigma)\mathscr{D}_\delta\\ &+\varepsilon_{abc}(\mathscr{D}^b\sigma)M^c-\frac{\mathrm{i}}{2}(\mathscr{D}^\gamma\sigma)(\bar{\mathscr{D}}_\gamma\sigma)M_a\\ &-\frac{\mathrm{i}}{8}(\gamma_a)^{\gamma\delta}([\mathscr{D}_\gamma,\bar{\mathscr{D}}_\delta]\sigma)J-\frac{3\mathrm{i}}{4}(\gamma_a)^{\gamma\delta}(\mathscr{D}_\gamma\sigma)(\bar{\mathscr{D}}_\delta\sigma)J\Big)\ ,\end{aligned} \tag{4.7b}$$

which induces the following transformation of the torsion tensors:

$$\mathscr{S}' = \mathrm{e}^{\sigma}\Big(\mathscr{S}+\frac{\mathrm{i}}{4}\mathscr{D}^\gamma\bar{\mathscr{D}}_\gamma\sigma\Big)\ , \tag{4.7c}$$

$$\mathscr{C}'_a = \mathrm{e}^{\sigma}\Big(\mathscr{C}_a+\frac{1}{8}(\gamma_a)^{\gamma\delta}[\mathscr{D}_\gamma,\bar{\mathscr{D}}_\delta]\sigma+\frac{1}{4}(\gamma_a)^{\gamma\delta}(\mathscr{D}_\gamma\sigma)\bar{\mathscr{D}}_\delta\sigma\Big)\ , \tag{4.7d}$$

$$\mathscr{R}' = \mathrm{e}^{\sigma}\Big(\mathscr{R}+\frac{1}{4}\bar{\mathscr{D}}^2\sigma-\frac{1}{4}(\bar{\mathscr{D}}_\gamma\sigma)\bar{\mathscr{D}}^\gamma\sigma\Big)\ . \tag{4.7e}$$

[5]These generators are related to each other as follows: $M_a=\frac{1}{2}\varepsilon_{abc}M^{bc}$, $M_{ab}=-\varepsilon_{abc}M^c$, $M_{\alpha\beta}=(\gamma^a)_{\alpha\beta}M_a$ and $M_a=-\frac{1}{2}(\gamma_a)^{\alpha\beta}M_{\alpha\beta}$.

Here the parameter σ is an arbitrary real scalar superfield. The infinitesimal version of super-Weyl transformation (4.7) provides a concrete realisation of (3.4a).

The gauge group of conformal supergravity is defined to be spanned by the $\mathscr{K}$-transformations (4.4) and the super-Weyl transformations. The super-Weyl invariance is the reason why the superspace geometry introduced describes the conformal supergravity multiplet.

Using the super-Weyl transformation laws (4.7), one may check that the real symmetric spinor superfield [50]

$$\mathscr{W}_{\alpha\beta} := \frac{\mathrm{i}}{2}[\mathscr{D}^\gamma, \bar{\mathscr{D}}_\gamma]\mathscr{C}_{\alpha\beta} - [\mathscr{D}_{(\alpha}, \bar{\mathscr{D}}_{\beta)}]\mathscr{S} - 4\mathscr{S}\mathscr{C}_{\alpha\beta} \tag{4.8}$$

transforms homogeneously,

$$\mathscr{W}'_{\alpha\beta} = \mathrm{e}^{2\sigma}\,\mathscr{W}_{\alpha\beta}\ . \tag{4.9}$$

This superfield is the $\mathscr{N}=2$ supersymmetric generalisation of the Cotton tensor

$$W_{ab} := \frac{1}{2}\varepsilon_{acd}W^{cd}{}_b = W_{ba}\ , \qquad W_{abc} = 2\nabla_{[a}R_{b]c} + \frac{1}{2}\eta_{c[a}\nabla_{b]}R \tag{4.10}$$

in 3D pseudo-Riemannian geometry. A curved superspace background $(\mathscr{M}^{3|4}, \mathscr{D})$ is conformally flat iff the super-Cotton tensor $\mathscr{W}_{\alpha\beta}$ vanishes [45].

4.2 Compensators

In order to describe 3D $\mathscr{N}=2$ Poincaré or anti-de Sitter supergravity theories, the conformal supergravity multiplet has to be coupled to a certain conformal compensator Ξ and its conjugate. In general, Ξ is a scalar superfield of super-Weyl weight $w \neq 0$ and $\mathrm{U}(1)_R$ charge q,

$$\delta_\sigma \Xi = w\sigma\Xi\ , \qquad J\Xi = q\Xi\ , \tag{4.11}$$

chosen to be nowhere vanishing, $\Xi \neq 0$. It is assumed that $q=0$ if and only if Ξ is real, which is the case for $\mathscr{N}=2$ supergravity with a real linear compensator (see below). Different off-shell supergravity theories correspond to different superfield types of Ξ.

Type I minimal supergravity [15, 37] is a 3D analogue of the old minimal formulation for 4D $\mathscr{N}=1$ supergravity [9, 10]. It makes use of two compensators, a covariantly chiral scalar Ψ and its conjugate $\bar{\Psi}$ with the properties

$$\bar{\mathscr{D}}_\alpha \Psi = 0\ , \qquad \delta_\sigma \Psi = \frac{1}{2}\sigma\Psi\ , \qquad J\Psi = -\frac{1}{2}\Psi\ . \tag{4.12}$$

The freedom to perform the super-Weyl and local $\mathrm{U}(1)_R$ transformations allows us to choose a gauge $\Psi = 1$, which implies the consistency conditions

$$\mathscr{S} = 0\ , \qquad \Phi_\alpha = 0\ , \qquad \Phi_a = \mathscr{C}_a\ . \tag{4.13}$$

This reduces the superspace structure group from $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{U}(1)_R$ to its subgroup $\mathrm{SL}(2,\mathbb{R})$.

Type II minimal supergravity [15, 37] is a 3D analogue of the new minimal formulation for 4D $\mathscr{N}=1$ supergravity [51]. It makes use of a real covariantly linear compensator $\mathbb{G}$ with the properties

$$(\bar{\mathscr{D}}^2 - 4\mathscr{R})\mathbb{G} = (\mathscr{D}^2 - 4\bar{\mathscr{R}})\mathbb{G} = 0\ , \qquad \delta_\sigma \mathbb{G} = \sigma\mathbb{G}\ . \tag{4.14}$$

The super-Weyl invariance allows us to choose the gauge $\mathbb{G} = 1$, which implies

$$\mathscr{R} = \bar{\mathscr{R}} = 0 . \tag{4.15}$$

Unlike the 4D case, this formulation is suitable to describe anti-de Sitter supergravity [15].

Non-minimal $\mathscr{N} = 2$ Poincaré supergravity [15, 37] is a 3D analogue of the non-minimal 4D $\mathscr{N} = 1$ supergravity (see [8, 44] for reviews). It makes use of a complex covariantly linear superfield Σ and its conjugate $\bar{\Sigma}$. The superfield Σ is characterised by the properties [37]

$$(\bar{\mathscr{D}}^2 - 4\mathscr{R})\Sigma = 0 , \qquad \delta_\sigma \Sigma = w\sigma\Sigma , \qquad J\Sigma = (1-w)\Sigma , \tag{4.16}$$

for some real parameter w. No reality condition is imposed on Σ. The only way to describe anti-de Sitter supergravity using a non-minimal formulation [15] (in complete analogy with the four-dimensional $\mathscr{N} = 1$ case [52]) consists in choosing $w = -1$ in (4.16) and replacing the constraint $(\bar{\mathscr{D}}^2 - 4\mathscr{R})\Sigma = 0$ with a deformed one,

$$-\frac{1}{4}(\bar{\mathscr{D}}^2 - 4\mathscr{R})\Gamma = \mu = \text{const} . \tag{4.17}$$

The freedom to perform the super-Weyl and local $\mathrm{U}(1)_R$ transformations allows us to choose a gauge $\Sigma = 1$.

Supersymmetric spacetimes in non-minimal supergravity are analogous to (but more restrictive than) those in Type I supergravity [36]. This is why we will not consider non-minimal supergravity in what follows.

4.3 Conformal Killing vector fields on $(\mathscr{M}^{3|4}, \mathscr{D})$

Let $\xi = \xi^A E_A$ be a real vector field on $(\mathscr{M}^{3|4}, \mathscr{D})$, with $\xi^A \equiv (\xi^a, \xi^\alpha, \bar{\xi}_\alpha)$. It is conformal Killing provided eq. (3.6) holds. Since the vector covariant derivative $\mathscr{D}_a$ is given in terms of an anti-commutator of two spinor ones, eq. (4.5b), it suffices to analyse the implications of

$$(\delta_{\mathscr{K}} + \delta_\sigma)\mathscr{D}_\alpha = 0 . \tag{4.18}$$

We should stress that the other requirement contained in (3.6),

$$(\delta_{\mathscr{K}} + \delta_\sigma)\mathscr{D}_a = 0 , \tag{4.19}$$

is automatically satisfied provided (4.18) holds.

The left-hand side of (4.18) is a linear combination of the five linearly independent operators $\mathscr{D}^\beta$, $\bar{\mathscr{D}}^\beta$, $\mathscr{D}^{\beta\gamma}$, $M^{\beta\gamma}$ and J. Therefore, eq. (4.18) gives five different equations. Let us consider in some detail the equations associated with the operators $\mathscr{D}^\beta$ and $\mathscr{D}^{\beta\gamma}$, which are

$$\mathscr{D}_\alpha \xi_\beta = -\frac{1}{2}\varepsilon_{\alpha\beta}(\sigma + 2\mathrm{i}\tau) - \mathrm{i}\xi_{(\alpha}{}^\gamma \mathscr{C}_{\beta)\gamma} + \xi_{\alpha\beta}\mathscr{S} + \frac{1}{2}K_{\alpha\beta} , \tag{4.20a}$$

$$\mathscr{D}_\alpha \xi_{\beta\gamma} = 4\mathrm{i}\varepsilon_{\alpha(\beta}\bar{\xi}_{\gamma)} , \tag{4.20b}$$

and their complex conjugate equations. These relations imply that the parameters ξ^α, $\bar\xi_\alpha$, $K_{\alpha\beta}$, σ and τ are uniquely expressed in terms of ξ^a and its covariant derivatives as follows:

$$\xi^\alpha = -\frac{\rm i}{6}\bar{\mathscr{D}}_\beta \xi^{\beta\alpha}~, \qquad \bar\xi_\alpha = -\frac{\rm i}{6}\mathscr{D}^\beta \xi_{\beta\alpha}~, \tag{4.21a}$$

$$\sigma[\xi] = \frac{1}{2}\big(\mathscr{D}_\alpha\xi^\alpha + \bar{\mathscr{D}}^\alpha\bar\xi_\alpha\big)~, \tag{4.21b}$$

$$\tau[\xi] = -\frac{\rm i}{4}\big(\mathscr{D}_\alpha\xi^\alpha - \bar{\mathscr{D}}^\alpha\bar\xi_\alpha\big)~, \tag{4.21c}$$

$$K_{\alpha\beta}[\xi] = \mathscr{D}_{(\alpha}\xi_{\beta)} - \bar{\mathscr{D}}_{(\alpha}\bar\xi_{\beta)} - 2\xi_{\alpha\beta}\mathscr{S}~. \tag{4.21d}$$

In accordance with (4.20b), the remaining vector parameter ξ^a satisfies the equation[6]

$$\mathscr{D}_{(\alpha}\xi_{\beta\gamma)} = 0 \tag{4.22}$$

and its complex conjugate. From (4.22) one may deduce the conformal Killing equation

$$\mathscr{D}_a\xi_b + \mathscr{D}_b\xi_a = \frac{2}{3}\eta_{ab}\mathscr{D}^c\xi_c~. \tag{4.23}$$

Eq. (4.22) is fundamental in the sense that it implies $(\delta_{\mathscr{K}}+\delta_\sigma)\mathscr{D}_A \equiv 0$ provided the parameters ξ^α, $K_{\alpha\beta}$, σ and τ are defined as in (4.21). Therefore, every conformal Killing vector field on $(\mathscr{M}^{3|4},\mathscr{D})$ is a real vector field

$$\xi = \xi^A E_A~, \qquad \xi^A = (\xi^a,\xi^\alpha,\bar\xi_\alpha) := \Big(\xi^a, -\frac{\rm i}{6}\bar{\mathscr{D}}_\beta\xi^{\beta\alpha}, -\frac{\rm i}{6}\mathscr{D}^\beta\xi_{\beta\alpha}\Big)~, \tag{4.24}$$

which obeys the master equation (4.22). If ξ_1 and ξ_2 are two conformal Killing vector fields, their Lie bracket $[\xi_1,\xi_2]$ is a conformal Killing vector field.

The equation (4.18) implies some additional results that have not been discussed above. Defining $\Upsilon := \big\{\xi^B, K^{\beta\gamma}[\xi], \tau[\xi], \sigma[\xi], \mathscr{D}_B\sigma[\xi]\big\}$, it turns out that the descendants $\mathscr{D}_A\Upsilon$ are linear combinations of the elements of Υ. This means that the Lie superalgebra of conformal Killing vector fields on $(\mathscr{M}^{3|4},\mathscr{D})$ is finite dimensional. The number of its even and odd generators cannot exceed those in the $\mathscr{N}=2$ superconformal algebra $\mathfrak{osp}(2|4)$.

4.4 Killing vector fields on $(\mathscr{M}^{3|4},\mathscr{D},\Xi)$

A conformal Killing vector field $\xi=\xi^A E_A$ on $(\mathscr{M}^{3|4},\mathscr{D})$ is said to be a Killing vector field on $(\mathscr{M}^{3|4},\mathscr{D},\Xi)$ if the following conditions hold:

$$\Big[\xi^B\mathscr{D}_B + \frac{1}{2}K^{bc}[\xi]M_{bc} + {\rm i}\tau[\xi]J, \mathscr{D}_A\Big] + \delta_{\sigma[\xi]}\mathscr{D}_A = 0~, \tag{4.25a}$$

$$\Big(\xi^B\mathscr{D}_B + {\rm i}q\tau[\xi] + w\sigma[\xi]\Big)\Xi = 0~, \tag{4.25b}$$

with the parameters $K^{bc}[\xi]$, $\tau[\xi]$ and $\sigma[\xi]$ defined as in (4.21). The set of all Killing vector fields on $(\mathscr{M}^{3|4},\mathscr{D},\Xi)$ is a Lie superalgebra. The Killing vector fields generate the symmetries of rigid supersymmetric field theories defined on this superspace.

[6]The equation (4.22) is analogous to the conformal Killing equation, $\nabla_{(\alpha\beta}V_{\gamma\delta)}=0$, on a pseudo-Riemannian three-dimensional manifold.

The Killing equations (4.25) are super-Weyl invariant in the sense that they have the same form for conformally related superspaces. The super-Weyl and local $\mathrm{U}(1)_R$ symmetries allow us to choose the useful gauge

$$\Xi = 1 \,, \tag{4.26}$$

which characterises the off-shell supergravity formulation chosen. If $q \neq 0$, there remain no residual super-Weyl and local $\mathrm{U}(1)_R$ symmetries in this gauge. If $q = 0$, the local $\mathrm{U}(1)_R$ symmetry remains unbroken while the super-Weyl freedom is completely fixed.

In the gauge (4.26), the Killing equation (4.25b) becomes

$$\mathrm{i}q\Big(\xi^B\Phi_B + \tau[\xi]\Big) + w\sigma[\xi] = 0 \,, \tag{4.27}$$

where Φ_B is the $\mathrm{U}(1)_R$ connection, eq. (4.1). Hence, the isometry transformations are generated by those conformal Killing supervector fields which respect the conditions

$$\sigma[\xi] \,=\, 0 \,, \tag{4.28a}$$

$$q \neq 0 \quad \Longrightarrow \quad \tau[\xi] \,=\, -\xi^B\Phi_B \,. \tag{4.28b}$$

These properties provide the main rationale for choosing the gauge condition (4.26) which is: for any off-shell supergravity formulation, the isometry transformations are characterised by the condition $\sigma[\xi] = 0$, which eliminates super-Weyl transformations.

5. Supersymmetric three-dimensional spacetimes

Let us look for curved superspace backgrounds $(\mathcal{M}^{3|4}, \mathcal{D})$ which admit at least one conformal supersymmetry. Such a superspace must possess a conformal Killing vector field with the property

$$\xi^a| = 0 \,, \qquad \varepsilon^\alpha(x) := \xi^\alpha| \neq 0 \,. \tag{5.1}$$

All other bosonic parameters are assumed to vanish, $\sigma| = \tau| = K_{\alpha\beta}| = 0$. Then any parameter of the type $(\mathcal{D}_{B_1}\cdots\mathcal{D}_{B_n}\xi^A)|$ is expressed in terms of the two spinor parameters: Q-supersymmetry $\varepsilon^\alpha(x)$ and S-supersymmetry $\eta_\alpha(x) := \mathcal{D}_\alpha\sigma|$. This follows from the general properties of the conformal Killing vector fields on $(\mathcal{M}^{3|4}, \mathcal{D})$ discussed above.

In the 3D $\mathcal{N} = 2$ case, all bosonic superspace backgrounds, which possess no covariant fermionic fields, are characterised by the conditions:

$$\mathcal{D}_\alpha\mathcal{S}| = 0 \,, \qquad \mathcal{D}_\alpha\mathcal{R}| = 0 \,, \qquad \mathcal{D}_\alpha\mathcal{C}_{\beta\gamma}| = 0 \,. \tag{5.2}$$

These conditions mean that the gravitini can be gauged away such that

$$\mathcal{D}_a| = \mathbf{D}_a := e_a{}^m(x)\partial_m + \frac{1}{2}\omega_a{}^{bc}(x)M_{bc} + \mathrm{i}b_a(x)J = \nabla_a + \mathrm{i}b_a(x)J \,, \tag{5.3}$$

where ∇_a stands for the torsion-free covariant derivative (2.3). Introduce scalar and vector fields associated with the superspace torsion:

$$s(x) := \mathcal{S}| \,, \qquad r(x) := \mathcal{R}| \,, \qquad c_a(x) := \mathcal{C}_a| \,. \tag{5.4}$$

The spinor parameter $\varepsilon = (\varepsilon_\alpha)$ proves to obey the equation

$$\mathbf{D}_a \varepsilon + \frac{\mathrm{i}}{2}\gamma_a \bar\eta + \mathrm{i}\varepsilon_{abc}\, c^b \gamma^c \varepsilon - s\gamma_a \varepsilon - \mathrm{i} r \gamma_a \bar\varepsilon = 0 \,. \tag{5.5}$$

This equation is obtained by bar-projecting the relation

$$\begin{aligned} 0 = \mathscr{D}_a \xi_\alpha &+ \frac{\mathrm{i}}{2}(\gamma_a)_\alpha{}^\beta \bar{\mathscr{D}}_\beta \sigma - \mathrm{i}\varepsilon_{abc}(\gamma^b)_\alpha{}^\beta \mathscr{C}^c \xi_\beta - (\gamma_a)_\alpha{}^\beta (\xi_\beta \mathscr{S} + \bar\xi_\beta R) \\ &+ \frac{1}{2}\varepsilon_{abc}\xi^b (\gamma^c)^{\beta\gamma}\Big(\bar{\mathscr{D}}_{(\alpha}\mathscr{C}_{\beta\gamma)} + \frac{4\mathrm{i}}{3}\varepsilon_{\alpha(\beta}\bar{\mathscr{D}}_{\gamma)}\mathscr{S} + \frac{2}{3}\varepsilon_{\alpha(\beta}\mathscr{D}_{\gamma)} R\Big) \,, \end{aligned} \tag{5.6}$$

which is one of the implications of (4.19). We recall that (4.19) is automatically satisfied if the equation (4.18) holds.

Eq. (5.5) contains two pieces of information. Firstly, it allows one to express the spinor parameter $\bar\eta = (\bar\eta_\alpha)$ via ε, its conjugate $\bar\varepsilon$ and covariant derivative $\mathbf{D}_a \varepsilon$:

$$\bar\eta_\alpha = -\frac{2\mathrm{i}}{3}\Big((\gamma^a \mathbf{D}_a \varepsilon)_\alpha + 2\mathrm{i}(\gamma^a \varepsilon)_\alpha c_a + 3s\varepsilon_\alpha + 3\mathrm{i} r \bar\varepsilon_\alpha\Big) \,. \tag{5.7a}$$

Secondly, it gives a closed-form equation on ε:

$$\Big(\mathbf{D}_{(\alpha\beta} - \mathrm{i} c_{(\alpha\beta}\Big)\varepsilon_{\gamma)} = \Big(\nabla_{(\alpha\beta} - \mathrm{i}(b+c)_{(\alpha\beta}\Big)\varepsilon_{\gamma)} = 0 \,. \tag{5.7b}$$

Equation (5.7b) tells us that ε is a *charged conformal Killing spinor*, since (5.7b) can be rewritten in the form [32]

$$\widetilde\nabla_{(\alpha\beta}\varepsilon_{\gamma)} = 0 \,, \qquad \widetilde\nabla_a \varepsilon := (\nabla_a - \mathrm{i} A_a)\varepsilon \,, \tag{5.8}$$

where $A_a = b_a + c_a$. Switching off the U(1) connection A in (5.8) gives the equation for conformal Killing spinors. We point out that the more conventional form of writing (5.8) is

$$(\widetilde\nabla_a - \frac{1}{3}\gamma_a \gamma^b \widetilde\nabla_b)\varepsilon = 0 \,. \tag{5.9}$$

Choose ε_α to be a bosonic (commuting) spinor. Then, by analogy with, e.g., the 5D analysis in [7], we deduce from (5.8) that the real vector field $V_a := (\gamma_a)^{\alpha\beta}\bar\varepsilon_\alpha \varepsilon_\beta$ has the following properties: (i) V_a is a conformal Killing vector field, $\nabla_{(\alpha\beta}V_{\gamma\delta)} = 0$; and (ii) V_a is null or time-like, since $V^a V_a = (\bar\varepsilon^\alpha \varepsilon_\alpha)^2 \leq 0$. This vector field is null if and only if $\bar\varepsilon_\alpha \propto \varepsilon_\alpha$. These properties were first observed in [32].

5.1 Supersymmetric backgrounds

As discussed in section 3, using the compensators Ξ one can construct a nowhere vanishing real scalar ϕ with the super-Weyl transformation $\delta_\sigma \phi = w_\phi \sigma \phi$, where the super-Weyl weight w_ϕ is non-zero. The super-Weyl gauge freedom can be fixed by choosing the gauge $\phi = 1$ in which $\sigma[\xi] = 0$. One may choose ϕ to be (i) $\bar\Psi\Psi$ in Type I supergravity; (ii) $\mathbb{G}$ in Type II supergravity; and (iii) $\bar\Sigma\Sigma$ in non-minimal supergravity.

In the super-Weyl gauge $\phi = 1$, every rigid supersymmetry transformation is characterised by

$$\sigma[\xi] = 0 \quad \Longrightarrow \quad \eta_\alpha = 0 \,. \tag{5.10}$$

Then the conformal Killing spinor equation (5.5) turns into

$$\mathbf{D}_a \varepsilon = -\mathrm{i}\varepsilon_{abc}c^b\gamma^c\varepsilon + s\gamma_a\varepsilon + \mathrm{i}r\gamma_a\bar{\varepsilon} \ . \tag{5.11}$$

We recall that the covariant derivative $\mathbf{D}_a$ is defined by (5.3). It contains a $\mathrm{U}(1)_R$ connection, and the algebra of covariant derivatives is

$$[\mathbf{D}_a, \mathbf{D}_b] = \frac{1}{2}R_{ab}{}^{cd}M_{cd} + \mathrm{i}F_{ab}J = [\nabla_a, \nabla_b] + \mathrm{i}F_{ab}J \ . \tag{5.12}$$

Eq. (5.11) is a generalised Killing spinor equation. Along with the frame field $e_a = e_a{}^m(x)\partial_m$, it involves four other background fields, which are: the $\mathrm{U}(1)_R$ gauge connection $b_a(x)$, the vector field $c_a(x)$, the real scalar field $s(x)$ and the complex scalar one $r(x)$.

5.2 Maximally supersymmetric backgrounds

The existence of rigid supersymmetries, i.e. solutions of the equation (5.11), imposes non-trivial restrictions on the background fields. In the case of four supercharges, these restrictions have been analysed in [36]. They are:

$$\nabla_a s = 0 \ , \qquad \mathbf{D}_a r = (\nabla_a - 2\mathrm{i}b_a)r = 0 \ , \qquad \nabla_a c_b = 2\varepsilon_{abc}c^c s \ , \tag{5.13a}$$

$$rs = 0 \ , \qquad rc_a = 0 \ . \tag{5.13b}$$

It follows that c_a is a Killing vector field,

$$\nabla_a c_b + \nabla_b c_a = 0 \ , \tag{5.14}$$

such that $c^2 := \eta_{ab}c^a c^b = \mathrm{const}$. The $\mathrm{U}(1)_R$ field strength proves to vanish,

$$F_{ab} = 0 \ . \tag{5.15}$$

For the Ricci tensor we obtain

$$R_{ab} = 4\Big[c_a c_b - \eta_{ab}\big\{c^2 + 2\big(s^2 + \bar{r}r\big)\big\}\Big] \ . \tag{5.16}$$

Using this result, for the Cotton tensor defined by (4.10) we read off the following expression:

$$W_{ab} = -24s\big[c_a c_b - \frac{1}{3}\eta_{ab}c^2\big] \ . \tag{5.17}$$

It is clear that the spacetime is conformally flat if $sc_a = 0$.

The above restrictions are given in terms of component fields. They may be recast in the language of superspace and superfields using a 3D analogue of the 5D observation in [38]. For any 3D $\mathscr{N} = 2$ supergravity background admitting four supercharges, if there exists a tensor superfield T such that its bar-projection vanishes, $T| = 0$, and this condition is supersymmetric, then the entire superfield is zero, $T = 0$. In particular, the supersymmetric conditions (5.2) imply

$$\mathscr{D}_\alpha \mathscr{S} = 0 \ , \qquad \mathscr{D}_\alpha \mathscr{R} = 0 \ , \qquad \mathscr{D}_\alpha \mathscr{C}_{\beta\gamma} = 0 \ . \tag{5.18}$$

Further superfield conditions follow from (5.13). As follows from (4.8), the super-Cotton tensor takes the form

$$\mathscr{W}_{\alpha\beta} = -4\mathscr{S}\mathscr{C}_{\alpha\beta} \ . \tag{5.19}$$

Up to this point, no specific compensator has been chosen, and all the results so far obtained are applicable to every off-shell formulation for 3D $\mathscr{N} = 2$ supergravity. We now turn to making a specific choice of compensators.

5.3 Maximally supersymmetric backgrounds in Type I supergravity

As discussed in subsection 4.2, in Type I supergravity the super-Weyl and local $\mathrm{U}(1)_R$ transformations can be used to impose the gauge $\Psi = 1$, which leads to the consistency conditions (4.13). The corresponding Killing spinor equation is obtained from (5.11) by setting $s = 0$ and $b_a = c_a$, which gives

$$\nabla_a \varepsilon = \mathrm{i} c_a \varepsilon - \mathrm{i}\varepsilon_{abc} c^b \gamma^c \varepsilon + \mathrm{i} r \gamma_a \bar{\varepsilon} \ . \tag{5.20}$$

In the case of maximally supersymmetric backgrounds, the dimension-1 torsion superfields obey the constraints:

$$\mathscr{S} = 0 \ , \qquad \mathscr{R}\mathscr{C}_a = 0 \ , \qquad \mathscr{D}_A \mathscr{R} = 0 \ , \qquad \mathscr{D}_A \mathscr{C}_b = 0 \ . \tag{5.21}$$

The complete algebra of covariant derivatives is

$$\{\mathscr{D}_\alpha, \mathscr{D}_\beta\} = -4\bar{\mathscr{R}} M_{\alpha\beta} \ , \qquad \{\bar{\mathscr{D}}_\alpha, \bar{\mathscr{D}}_\beta\} = 4\mathscr{R} M_{\alpha\beta} \ , \tag{5.22a}$$

$$\{\mathscr{D}_\alpha, \bar{\mathscr{D}}_\beta\} = -2\mathrm{i}(\gamma^c)_{\alpha\beta}\Big(\mathscr{D}_c - \mathrm{i}\mathscr{C}_c J\Big) + 4\varepsilon_{\alpha\beta}\mathscr{C}^c M_c \ , \tag{5.22b}$$

$$[\mathscr{D}_a, \mathscr{D}_\beta] = \mathrm{i}\varepsilon_{abc}(\gamma^b)_\beta{}^\gamma \mathscr{C}^c \mathscr{D}_\gamma - \mathrm{i}(\gamma_a)_{\beta\gamma}\bar{\mathscr{R}}\bar{\mathscr{D}}^\gamma \ , \tag{5.22c}$$

$$[\mathscr{D}_a, \bar{\mathscr{D}}_\beta] = -\mathrm{i}\varepsilon_{abc}(\gamma^b)_\beta{}^\gamma \mathscr{C}^c \bar{\mathscr{D}}_\gamma - \mathrm{i}(\gamma_a)_\beta{}^\gamma \mathscr{R}\mathscr{D}_\gamma \ , \tag{5.22d}$$

$$[\mathscr{D}_a, \mathscr{D}_b] = 4\varepsilon_{abc}\Big(\mathscr{C}^c \mathscr{C}_d + \delta^c{}_d \bar{\mathscr{R}}\mathscr{R}\Big) M^d \ . \tag{5.22e}$$

Re-defining the covariant derivatives $\mathscr{D}_A = (\mathscr{D}_a, \mathscr{D}_\alpha, \bar{\mathscr{D}}^\alpha) \to \widetilde{\mathscr{D}}_A = (\mathscr{D}_a - \mathrm{i}\mathscr{C}_a J, \mathscr{D}_\alpha, \bar{\mathscr{D}}^\alpha)$ results in a supergeometry without $\mathrm{U}(1)_R$ curvature, which means that the $\mathrm{U}(1)_R$ connection can be gauged away. As follows from (5.19) and (5.21), the super-Cotton tensor is equal to zero, and thus the superspace (and spacetime) geometry is conformally flat.

There are four different maximally supersymmetric backgrounds described by the superalgebra (5.22), with $\mathscr{R}$ and $\mathscr{C}_a$ constrained by (5.21). The case $\mathscr{R} \neq 0$ and $\mathscr{C}_a = 0$ corresponds to (1,1) AdS superspace [15]. The other three cases are characterised by $\mathscr{R} = 0$ and correspond to different choices for a covariantly constant vector field $c_a(x) = \mathscr{C}_a|$, which are timelike, spacelike or null.

The existence of a covariantly constant vector field c^a means that spacetime is decomposable in the non-null case (see, e.g., [53]). For $c^2 \neq 0$ the spacetime is the product of a two- and a one-dimensional manifold. We can choose a coordinate frame $x^m = (x^{\hat{m}}, \zeta)$, where $\hat{m} = 1,2$, such that the vector field $c^a e_a$ is proportional to $\partial/\partial\zeta$ and the metric reads

$$\mathrm{d}s_3^2 = g_{\hat{m}\hat{n}}(x^{\hat{r}})\mathrm{d}x^{\hat{m}}\mathrm{d}x^{\hat{n}} + \kappa(\mathrm{d}\zeta)^2 = \eta_{\hat{a}\hat{b}} e^{\hat{a}} e^{\hat{b}} + \kappa(\mathrm{d}\zeta)^2 \ , \qquad e^{\hat{a}} := \mathrm{d}x^{\hat{m}} e_{\hat{m}}{}^{\hat{a}}(x^{\hat{n}}) \ , \tag{5.23}$$

where $\kappa = -1$ when c^a is timelike, and $\kappa = +1$ when c^a is spacelike. The two-dimensional metric $\mathrm{d}s_2^2 = g_{\hat{m}\hat{n}}(x^{\hat{r}})\mathrm{d}x^{\hat{m}}\mathrm{d}x^{\hat{n}}$ corresponds to a two-dimensional submanifold $\mathscr{N}^2$ of $\mathscr{M}^3$ orthogonal to $c^a e_a$. We denote by $\mathfrak{R}_{\hat{a}\hat{b}}$ the Ricci tensor for $\mathscr{N}^2$. Since c^a is covariantly constant, $R_{abc}{}^{b} = 0$, which means $R_{a\zeta} = 0$. From (5.16) we then read off $R_{\hat{a}\hat{b}} = -4c^2\eta_{\hat{a}\hat{b}}$. This means that the submanifold $\mathscr{N}^2$ is (i) S^2 if c^a is timelike; and (ii) AdS_2 if c^a is spacelike. Finally, in the case that c^a is null, the corresponding spacetime is a special example of pp-waves.

5.4 Maximally supersymmetric backgrounds in Type II supergravity

As discussed in subsection 4.2, in Type II supergravity the super-Weyl invariance can be used to impose the gauge $\mathbb{G} = 1$, which leads to the consistency conditions (4.15). The corresponding Killing spinor equation is obtained from (5.11) by setting $r = 0$,

$$\mathbf{D}_a \varepsilon = -\mathrm{i}\varepsilon_{abc} c^b \gamma^c \varepsilon + s\gamma_a \varepsilon . \tag{5.24}$$

In the case of maximally supersymmetric backgrounds, the dimension-1 torsion superfields obey the constraints:

$$\mathscr{R} = 0 , \qquad \mathscr{D}_A \mathscr{S} = 0 , \qquad \mathscr{D}_\alpha \mathscr{C}_b = 0 \quad \Longrightarrow \quad \mathscr{D}_a \mathscr{C}_b = 2\varepsilon_{abc} \mathscr{C}^c \mathscr{S} , \tag{5.25}$$

and hence $\mathscr{C}^b \mathscr{C}_b = \text{const}$. The corresponding algebra of covariant derivatives is

$$\{\mathscr{D}_\alpha, \mathscr{D}_\beta\} = 0 , \qquad \{\bar{\mathscr{D}}_\alpha, \bar{\mathscr{D}}_\beta\} = 0 , \tag{5.26a}$$

$$\{\mathscr{D}_\alpha, \bar{\mathscr{D}}_\beta\} = -2\mathrm{i}(\gamma^c)_{\alpha\beta}\Big(\mathscr{D}_c - 2\mathscr{S} M_c - \mathrm{i}\mathscr{C}_c J\Big) + 4\varepsilon_{\alpha\beta}\Big(\mathscr{C}^c M_c - \mathrm{i}\mathscr{S} J\Big) , \tag{5.26b}$$

$$[\mathscr{D}_a, \mathscr{D}_\beta] = \mathrm{i}\varepsilon_{abc}(\gamma^b)_\beta{}^\gamma \mathscr{C}^c \mathscr{D}_\gamma + (\gamma_a)_\beta{}^\gamma \mathscr{S} \mathscr{D}_\gamma , \tag{5.26c}$$

$$[\mathscr{D}_a, \bar{\mathscr{D}}_\beta] = -\mathrm{i}\varepsilon_{abc}(\gamma^b)_\beta{}^\gamma \mathscr{C}^c \bar{\mathscr{D}}_\gamma + (\gamma_a)_\beta{}^\gamma \mathscr{S} \bar{\mathscr{D}}_\gamma , \tag{5.26d}$$

$$[\mathscr{D}_a, \mathscr{D}_b] = 4\varepsilon_{abc}\Big(\mathscr{C}^c \mathscr{C}_d + \delta^c{}_d \mathscr{S}^2\Big) M^d . \tag{5.26e}$$

The solution with $\mathscr{C}_a = 0$ corresponds to (2,0) AdS superspace [15]. The algebras (5.22) and (5.26) coincide under the conditions $\mathscr{R} = \mathscr{S} = 0$.

Curved backgrounds of the type (5.26) are solutions to the equations of motion for topologically massive Type II supergravity with a cosmological term. These equations are [36]

$$\mathrm{i}\mathscr{D}^\alpha \bar{\mathscr{D}}_\alpha \ln \mathbb{G} - 4\mathscr{S} - 2\lambda \mathbb{G} = 0 , \tag{5.27a}$$

$$\frac{1}{g}\mathscr{W}_{\alpha\beta} - \frac{1}{\mathbb{G}}\mathscr{D}_{(\alpha}\mathbb{G}\bar{\mathscr{D}}_{\beta)}\mathbb{G} + \frac{1}{4}[\mathscr{D}_{(\alpha}, \bar{\mathscr{D}}_{\beta)}]\mathbb{G} + \mathscr{C}_{\alpha\beta}\mathbb{G} = 0 . \tag{5.27b}$$

Here λ is the cosmological constant, and g the coupling constant appearing in the conformal supergravity action (Newton's constant is set equal to one). In the super-Weyl gauge $\mathbb{G} = 1$ these equations turn into

$$\mathscr{S} + \frac{1}{2}\lambda = 0 , \tag{5.28a}$$

$$\frac{\mathrm{i}}{2}[\mathscr{D}^\gamma, \bar{\mathscr{D}}_\gamma]\mathscr{C}_{\alpha\beta} + (g + 2\lambda)\mathscr{C}_{\alpha\beta} = 0 , \tag{5.28b}$$

where we have used the explicit expression for the super-Cotton tensor (4.8). For a solution with a non-vanishing $\mathscr{C}_{\alpha\beta}$ constrained by $\mathscr{D}_\gamma \mathscr{C}_{\alpha\beta} = 0$, one can satisfy eq. (5.28b) if the coupling constants g and λ are related to each other as

$$g + 2\lambda = 0 . \tag{5.29}$$

The bosonic solutions of topologically massive $\mathscr{N} = 2$ supergravity with a cosmological term were classified in [54]. Supersymmetric spacetime (5.26) is of type N (for C_a null), type $\mathrm{D_s}$ (for C_a spacelike) or $\mathrm{D_t}$ (for C_a timelike) in the Petrov-Segre classification, see [54] for more details.

PoS(CORFU2014)140

6. Concluding comments

In this note we reviewed the superspace formalism to determine supersymmetric spacetimes from off-shell supergravity in diverse dimensions. For a given supergravity theory, we showed that a purely bosonic background admits rigid supersymmetry transformations provided the corresponding curved superspace possesses a Killing vector field of the type (3.11). Thus the superspace must possess nontrivial isometries that, by construction, form a finite-dimensional supergroup.

Within the component approaches to supersymmetric backgrounds in off-shell supergravity theories [22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34], the analysis amounts to classifying all solutions of generalised Killing spinor equations (such as eqs. (5.20) and (5.24) in the case of $\mathscr{N} = 2$ supergravity theories in three dimensions) obtained as the condition for the gravitino variation to vanish. Given such a solution, special analysis is required to understand whether there exists a superalgebra to which the generators of rigid supersymmetry transformations belong. In the superspace setting, this issue does not occur since the rigid supersymmetry transformations belong to the isometry group of the background superspace.

The superspace formalism provides a simple geometric realisation for maximally supersymmetric spacetimes. They correspond to those off-shell supergravity backgrounds for which the Grassmann-odd components of the superspace torsion and curvature tensors vanish, while the Grassmann-even components of these tensors are annihilated by the spinor derivatives. This follows from the observation that, for every maximally supersymmetric background, if there exists a tensor superfield T such that its bar-projection vanishes, $T| = 0$, and this condition is supersymmetric, then the entire superfield is zero, $T = 0$. As a simple corollary of this result, one can readily deduce that all maximally supersymmetric spacetimes are conformally flat for certain supergravity theories. For instance, in the case of 4D $\mathscr{N} = 1$ supergravity, the super-Weyl tensor is a completely symmetric spinor superfield $W_{\alpha\beta\gamma}$ [9]. Since it must vanish for every maximally supersymmetric spacetime, the corresponding Weyl tensor is equal to zero. This vanishing of the Weyl tensor was observed in [22], but no explanation of this result was given. Another example is provided by 3D $\mathscr{N} = 1$ supergravity in which the super-Cotton tensor is again a symmetric spinor superfield $W_{\alpha\beta\gamma}$ [55, 45]. The Cotton tensor is one of the components fields contained in $W_{\alpha\beta\gamma}$. Since $W_{\alpha\beta\gamma}$ must vanish for every maximally supersymmetric 3D spacetime, the corresponding Cotton tensor is equal to zero. Our last example is provided by 3D $\mathscr{N} = 3$ supergravity in which the super-Cotton tensor is a spinor superfield W_α [45]. Since W_α must vanish for every maximally supersymmetric background of $\mathscr{N} = 3$ supergravity, the corresponding 3D spacetime is conformally flat.

A striking feature of superspace techniques is that they make it possible to generate arbitrary off-shell supergravity-matter couplings (such as the off-shell locally supersymmetric sigma models in 5D $\mathscr{N} = 1$ [56], 4D $\mathscr{N} = 2$ [57] and 3D $\mathscr{N} = 3$ and $\mathscr{N} = 4$ [37] supergravity theories). Restricting these couplings to a given background allows one to construct general rigid supersymmetric theories on such a spacetime.

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