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Aspects of Supersymmetric Higher Spins

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Higher Super-Spins, are the irreducible representations of the Super-Poincaré group. We study the representation theory of this group in $4D$, $\mathscr{N}=1$ and demonstrate the off-shell Superspace realization of these theories. On the one hand, for the massless case, using gauge symmetry as a guide we can describe the arbitrary (integer or half-integer) superhelicity system. On the other hand for the massive case the general superspin case is still an open problem. However we would like to report on some recent progress towards that direction. We complete the picture by presenting various aspects of the Higher Super-Spin theories, such as the off-shell spectrum and degrees of freedom for the arbitrary superhelicity and how to construct $\mathscr{N}=2$ theories out of the $\mathscr{N}=1$ theories.

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1. Introduction

When, P. Dirac was trying to generalize his celebrated spin 1/2 equation said that '*the underlying theory is of considerable interest*'. This statement turned out to be absolutely correct, because for many years Higher Spins were one of the main driving forces behind many of the developments in physics. They introduced concepts and tools like gauge symmetry, dimensional reduction, Stüeckelberg and BRST formulations. It would be very interesting to examine if and how these structures appear in the Supersymmetric extension of spin theory.

Nowadays, most interest for higher spins comes from String theory. It is a well known fact that the spectrum of String theory includes an infinite tower of massive higher spin states. It turns out these states play a significant role to some of the most spectacular features of String theory such as planar duality, modular invariance and open-closed duality. Hence the study of the dynamics of higher spin states can help us understand some of the quantum properties of (Super) String theory.

Also String theory is conjectured to describe a spontaneously broken phase of an underlying Higher Spin gauge theory. The better understanding of this underlying theory can provide clues about String/M theory and (A)dS/CFT correspondence [1, 2, 3, 4, 5, 6].

The appropriate way to study the full structure of higher spins is through String Field Theory but this is not very practical, since the theory is far from being developed. Therefore we will follow the path of an Effective Supersymmetric Field Theoretic approach. In this way the goal is to build from the ground up a Supersymmetric invariant field theory that includes higher spins. A good starting point is the Superspace description of the irreducible representations of the $4D$, $\mathcal{N}=1$ of the Super-Poincaré group. In this way supersymmetry invariance will be manifest and the irreducible representations will provide the spin content.

This is where we hit our first major roadblock. After four decades of Supersymmetry we still do not have an off-shell description of the massive, free, irreducible representations. There has been made very little progress towards this direction and I would like to report on some recent developments.

In the first section, we briefly review the representation theory of the Super-Poincaré group for both massive and massless cases, introducing the concept of superspin and superhelicity respectively. In the second section we present the off-shell, superspace formulation of the massless, arbitrary superhelicity theory. The demonstration is done using the half-integer superhelicities as an example and includes a discussion of the component structure of the theory. Section three, is dedicated to the massive case. This problem is much harder to solve and we still do not know the general answer. We start with a warm up exercise for the massive vector multiplet, that illustrates the strategy of attack and then continue to a new formulation for massive superspin 3/2. This is the first non-trivial case and can provide clues regarding the more general case. The presentation of the material will be based on [7, 8] and references with in.

2. Irreducible Representations of $4D$, $\mathcal{N}=1$ Super-Poincaré group

The structure of the algebra is

$$\begin{aligned}
&[P_A, P_B\} = f_{AB}{}^{C} P_C \ , \quad P_A = \{Q_\alpha, \bar{Q}_{\dot{\alpha}}, P_m\} \ , \ A = \{\alpha, \ \dot{\alpha}, \ m\} \\
&[J, P_A] \sim P_A \ , \quad [J, J] \sim J
\end{aligned} \tag{2.1}$$

Because the Js form a subalgebra, the general group element can be factorized in the following way

$$g(\omega,x,\theta,\bar{\theta}) = \Omega(x,\theta,\bar{\theta})h(\omega)$$
$$\Omega(x,\theta,\bar{\theta}) = e^{-ix^m P_m + i\theta^\alpha Q_\alpha + i\bar{\theta}^{\dot{\alpha}}\bar{Q}_{\dot{\alpha}}}, \; h(\omega) = e^{\frac{i}{2}\omega^{mn}J_{mn}} \tag{2.2}$$

and therefore it can be parametrized by four bosonic and four fermionic variables. That defines Superspace ($\mathbb{R}^{4|4}$) and we can use it to define Superfields $\Phi(x,\theta,\bar{\theta})$, meaning mappings $\mathbb{R}^{4|4} \to \mathbb{R}^{4|4}$ that transform nicely under supersymmetry $\delta_S\Phi(x,\theta,\bar{\theta}) = i\left[\varepsilon^\alpha Q_\alpha, \Phi\right] + i\left[\bar{\varepsilon}^{\dot{\alpha}}\bar{Q}_{\dot{\alpha}}, \Phi\right]$. The goal is to find the unitary and finite, irreducible representations of the above algebra. Analogously to the Poincaré case, finiteness means we have to restrict to the super-Little group. That is the subset of transformations that respects both P_m and Q_α.

2.1 Massive case

For the massive case the algebra of the super-Little group is:

$$[Z_i, Z_j] = im\varepsilon_{ijk}Z^k \;,\; [Z_i, P_m] = 0 \;,\; [Z_i, Q_\alpha] = 0$$
$$Z_m = W_m - \frac{1}{4}(\bar{\sigma}_m)^{\dot{\alpha}\alpha}[Q_\alpha, \bar{Q}_{\dot{\alpha}}] \;,\; W^m = \frac{1}{2}\varepsilon^{mnrs}J_{nr}P_s \tag{2.3}$$

Keep in mind that J_{mn} are the generators of rotations over the entire superspace and not just the bosonic part of it. Therefore W^m is the supersymmetric extention of the Pauli-Lubanski vector. It is obvious that the three Z_i/m satisfy an $SU(2)$ algebra, hence one of the Casimir operators that will label the irreps is the Superspin operator $\vec{S}^2 = \frac{1}{m^2}\vec{Z}^2 = Y(Y+1)\mathbb{I}$. Its eigenvalues Y must take integer or half-integer values that are called superspin. The second Casimir operator is of course non-other than the mass $P^2 = -m^2$.

For our purpose, we want to find superfield realizations of the irreps. That means, to find appropriate types of superfields that satisfy some set of differential constraints in order to diagonalize the above Casimir operators. The answer is summerized in the following table:

$(Y{=}s)\ \Psi_{\alpha(s)\dot{\alpha}(s-1)}$	$(Y{=}s{+}1/2)\ H_{\alpha(s)\dot{\alpha}(s)}$
$\bar{\mathrm{D}}^2\Psi_{\alpha(s)\dot{\alpha}(s-1)}=0$	$H_{\alpha(s)\dot{\alpha}(s)}=\bar{H}_{\alpha(s)\dot{\alpha}(s)}$
$\mathrm{D}^{\alpha_s}\Psi_{\alpha(s)\dot{\alpha}(s-1)}=0$	$\mathrm{D}^{\alpha_s}H_{\alpha(s)\dot{\alpha}(s)}=0$
$\partial^{\gamma\dot{\gamma}}\Psi_{\gamma\alpha(s-1)\dot{\gamma}\dot{\alpha}(s-2)}=0$	$\Box H_{\alpha(s)\dot{\alpha}(s)}=m^2H_{\alpha(s)\dot{\alpha}(s)}$
$i\partial_{\alpha_s}{}^{\dot{\alpha}_s}\bar{\Psi}_{\alpha(s-1)\dot{\alpha}(s)}+m\Psi_{\alpha(s)\dot{\alpha}(s-1)}=0$	

Table 1: Constraints required in order to describe massive irreps of integer and half-integer superspin

For the integer case we have to consider a fermionic superfield $\Psi_{\alpha(s)\dot{\alpha}(s-1)}$ and for the half-integer case a real, bosonic superfield $H_{\alpha(s)\dot{\alpha}(s)}$. The notation $\alpha(n)$ means that there are n undotted symmetrized indices and similar for the dotted indices. Additionally, it is useful to know that within a supermultiplet Y there are four spin irreps. One with spin j=Y+1/2, two with spin j=Y and one with spin j=Y-1/2.

2.2 Massless case

Similar discussion can be made for the massless case. The algebra of the super-Little group for the massless case is:

$$[Z^1,Z^2]=0 \ , \ [Z^1,Z^3]=-iEZ_2 \ , \ [Z^2,Z^3]=iEZ_1$$
$$[Z_m,P_n]=0 \ , \ [Z_m,Q_\alpha]=0 \ , \ Z^m=W^m-\frac{1}{8}(\bar{\sigma}^m)^{\dot{\alpha}\alpha}[Q_\alpha,\bar{Q}_{\dot{\alpha}}] \tag{2.4}$$

In this case we do not get the $SU(2)$ algebra as before, but the $E2$ algebra (Euclidean 2 group) which does not have finite dimensional irreps. We can see that from the fact that Z^1 and Z^2 commute and can be interpreted as translation generators. Fineteness forces us to further constraint ourselves to the case of Z^1=0=Z^2. As a consequence the vector Z_m becomes proportional to momentum P_m. The constant of proportionality defines the superhelicity Y

$$Z_m=(Y+\frac{1}{4})P_m \ . \tag{2.5}$$

The superfield realization for the integer and half-integer superhelicities is:

(Y=s) $F_{\alpha(2s)}$	(Y=s+1/2) $F_{\alpha(2s+1)}$
$\bar{\mathrm{D}}_{\dot{\gamma}}F_{\alpha(2s)}$=0	$\bar{\mathrm{D}}_{\dot{\gamma}}F_{\alpha(2s+1)}$=0
$\mathrm{D}^{\alpha_{2s}}F_{\alpha(2s)}$=0	$\mathrm{D}^{\alpha_{2s+1}}F_{\alpha(2s+1)}$=0

Table 2: Constraints required in order to describe massless irreps of integer and half-integer superhelicity

Finally, the helicity content of a supermultiplet with superhelicity Y is j=Y+1/2 and j=Y. This concludes the tour of the representation theory of the Super-Poincaré group. In the next section, we focus in the massless case and construct Superspace actions such that, the equations of motion they generate, reproduce the above constraints. Therefore the on-shell system they describe is that of an irreducible representation. We will demonstrate this construction for the case of half-integer superhelicity.

3. Off-Shell Formulation for the Massless case

3.1 Building Blocks

For the half-integer superhelicity (Y=s+1/2), we must consider a chiral superfield $F_{\alpha(2s+1)}$. This can be considered as the higher spin supersymmetric analog of F_{mn}, the field strength of Maxwell's theory. Not only that, but F_{mn} satisfies similar type of constraints ($\partial_{[k}F_{mn]}$=0 , $\partial^m F_{mn}$=0). In that case what we do, is to solve $\partial_{[k}F_{mn]}$=0 by expressing F_{mn} in terms of a vector field A_m. A useful observation is that a vector field is the object that describes the corresponding massive theory. In a practical level, this is convenient because by taking the massless limit of the massive theory we transit smoothly to the F_{mn} theory. So we apply the same logic to the superfield $F_{\alpha(2s+1)}$. We solve the chirality condition by expressing $F_{\alpha(2s+1)}$ in terms of a real bosonic superfield $H_{\alpha(s)\dot{\alpha}(s)}$ (same type of superfield as the corresponding massive theory)

$$\bar{\mathrm{D}}_{\dot{\gamma}}F_{\alpha(2s+1)}=0 \ \rightarrow \ F_{\alpha(2s+1)}=\bar{\mathrm{D}}^2\mathrm{D}_{(\alpha_{2s+1}}\partial_{\alpha_{2s}}{}^{\dot{\alpha}_s}\dots\partial_{\alpha_{s+1}}{}^{\dot{\alpha}_1}H_{\alpha(s))\dot{\alpha}(s)} \ . \tag{3.1}$$

On the other hand, this can be interpreted as a statement that $F_{\alpha(2s+1)}$ is not the fundamental object, but $H_{\alpha(s)\dot{\alpha}(s)}$ is. However we know that $F_{\alpha(2s+1)}$ carries the physical degrees of freedom. In the case of Maxwell's theory, by doing an experiment we measure F_{mn} (electric and magnetic fields). Furthermore the on-shell degrees of freedom of $H_{\alpha(s)\dot{\alpha}(s)}$ do not match the physical degrees of freedom of the system. The same problem of matching the degrees of freedom appears in Maxwell's theory. The solution out of this, following in the steps of Maxwell's theory, is to introduce a redundancy $R_{\alpha(s)\dot{\alpha}(s)}$ and declare that the two configurations $H_{\alpha(s)\dot{\alpha}(s)}$ and $H_{\alpha(s)\dot{\alpha}(s)}$+$R_{\alpha(s)\dot{\alpha}(s)}$ are to be identified.

$$H_{\alpha(s)\dot{\alpha}(s)} \sim H_{\alpha(s)\dot{\alpha}(s)} + R_{\alpha(s)\dot{\alpha}(s)} \tag{3.2}$$

In this consideration the real bosonic superfield $H_{\alpha(s)\dot{\alpha}(s)}$ is promoted to an equivalence class $[H_{\alpha(s)\dot{\alpha}(s)}]$ with the above equivalence relation. This promotion will make obvious the non-physical (non-observable) status of $H_{\alpha(s)\dot{\alpha}(s)}$ and has potential to fix the mismatch of the degrees of freedom. The redundancy $R_{\alpha(s)\dot{\alpha}(s)}$ whatever it is, has to respect the physical degrees of freedom of $F_{\alpha(2s+1)}$. Therefore it must satisfy the following:

$$\bar{\mathrm{D}}^2\mathrm{D}_{(\alpha_{2s+1}}\partial_{\alpha_{2s}}{}^{\dot{\alpha}_s}\ldots\partial_{\alpha_{s+1}}{}^{\dot{\alpha}_1}R_{\alpha(s))\dot{\alpha}(s)} = 0 \rightarrow R_{\alpha(s)\dot{\alpha}(s)} = \frac{1}{s!}\mathrm{D}_{(\alpha_s}\bar{L}_{\alpha(s-1))\dot{\alpha}(s)} - \frac{1}{s!}\bar{\mathrm{D}}_{(\dot{\alpha}_s}L_{\alpha(s)\dot{\alpha}(s-1))} \tag{3.3}$$

As in Maxwell's theory, this is badly called a gauge symmetry. There is no symmetry, just the fact that the variable we have choosen to describe the system is an equivalence class. This reflects on the fact about physics that there is a discontinuity in the degrees of freedom as we go from $m \neq 0$ to $m = 0$. However because this is the established nomenclature we will keep on using this terminology.

3.2 The Superspace Action

Thus far, we know the Superspace space action must be quadratic to the superfield $H_{\alpha(s)\dot{\alpha}(s)}$. Its mass dimension is zero (0) because its highest rank component, the symmetric part of the $\theta\bar{\theta}$ term, must be a propagating boson. That also means that the action must include exactly four covariant derivatives in order to be dimensionless. Finally it must be invariant under the gauge transformation $\delta_G H_{\alpha(s)\dot{\alpha}(s)}$=$\frac{1}{s!}\mathrm{D}_{(\alpha_s}\bar{L}_{\alpha(s-1))\dot{\alpha}(s)}$-$\frac{1}{s!}\bar{\mathrm{D}}_{(\dot{\alpha}_s}L_{\alpha(s)\dot{\alpha}(s-1))}$. The most general action we can write is:

$$\begin{aligned} S = \int d^8z \Big\{ & a_1 H^{\alpha(s)\dot{\alpha}(s)}\mathrm{D}^{\gamma}\bar{\mathrm{D}}^2\mathrm{D}_{\gamma}H_{\alpha(s)\dot{\alpha}(s)} \\ & + a_2 H^{\alpha(s)\dot{\alpha}(s)}\left\{\mathrm{D}^2,\bar{\mathrm{D}}^2\right\}H_{\alpha(s)\dot{\alpha}(s)} \\ & + a_3 H^{\alpha(s)\dot{\alpha}(s)}\mathrm{D}_{\alpha_s}\bar{\mathrm{D}}^2\mathrm{D}^{\gamma}H_{\gamma\alpha(s-1)\dot{\alpha}(s)} + c.c. \\ & + a_4\, H^{\alpha(s)\dot{\alpha}(s)}\mathrm{D}_{\alpha_s}\bar{\mathrm{D}}_{\dot{\alpha}_s}\mathrm{D}^{\gamma}\bar{\mathrm{D}}^{\dot{\gamma}}H_{\gamma\alpha(s-1)\dot{\gamma}\dot{\alpha}(s-1)} + c.c. \Big\} \end{aligned} \tag{3.4}$$

The strategy to fix the coefficients is gauge invariance. For this purpose we calculate the deformation of the action under the above transformation.

$$
\begin{aligned}
\delta_G S = \int d^8z \Big\{ & \left[A \mathrm{D}^2 \bar{\mathrm{D}}_{\dot{\alpha}_s} H^{\alpha(s)\dot{\alpha}(s)} - B \mathrm{D}^{\alpha_s} \bar{\mathrm{D}}_{\dot{\gamma}} \mathrm{D}_{\gamma} H^{\gamma\alpha(s-1)\dot{\gamma}\dot{\alpha}(s-1)}\right] \left(\bar{\mathrm{D}}^2 L_{\alpha(s)\dot{\alpha}(s-1)} + \mathrm{D}^{\alpha_{s+1}} \Lambda_{\alpha(s+1)\dot{\alpha}(s-1)}\right) \\
& + \Gamma H^{\alpha(s)\dot{\alpha}(s)} \mathrm{D}^2 \bar{\mathrm{D}}^2 \mathrm{D}_{\alpha_s} \bar{L}_{\alpha(s-1)\dot{\alpha}(s)} \\
& - \Delta \bar{\mathrm{D}}_{\dot{\beta}} \mathrm{D}_{\gamma} \bar{\mathrm{D}}_{\dot{\gamma}} H^{\gamma\alpha(s-1)\dot{\beta}\dot{\gamma}\dot{\alpha}(s-2)} \left(\bar{\mathrm{D}}^{\dot{\alpha}_{s-1}} \mathrm{D}^{\alpha_s} L_{\alpha(s)\dot{\alpha}(s-1)} + \frac{s-1}{s} \mathrm{D}^{\alpha_s} \bar{\mathrm{D}}^{\dot{\alpha}_{s-1}} L_{a(s)\dot{\alpha}(s-1)} + \bar{\mathrm{D}}_{\dot{\alpha}_{s-2}} J_{\alpha(s-1)\dot{\alpha}(s-3)}\right) \\
& + c.c. \Big\}
\end{aligned}
$$

where $A = -2a_1 + 2\frac{s+1}{s}a_3 + 2a_4$, $B = 2a_3 + \frac{s+1}{s}a_4$, $\Gamma = 2a_2$ and $\Delta = 2a_4$. A few commends are in order at this point. First of all, there are two new terms appearing $\mathrm{D}^{\alpha_{s+1}}\Lambda_{\alpha(s+1)\dot{\alpha}(s-1)}$ and $\bar{\mathrm{D}}_{\dot{\alpha}_{s-2}} J_{\alpha(s-1)\dot{\alpha}(s-3)}$. Because of the D algebra these terms identically vanish. It is like adding zero. The reason they are introduced, is to illustrate that the action has a bit more symmetry than it was originally thought. Secondly, it should be also obvious that there is no non-trivial choice of coefficients in order to make the right hand part of the equation to vanish. So insisting with invariance, force us to introduce a compensator. In fact there are two ways of doing that.

- We can pick $\Gamma = \Delta = 0$ and introduce a compensator $\chi_{\alpha(s)\dot{\alpha}(s-1)}$ with a transformation $\delta_G \chi_{\alpha(s)\dot{\alpha}(s-1)} = \bar{\mathrm{D}}^2 L_{\alpha(s)\dot{\alpha}(s-1)} + \mathrm{D}^{\alpha_{s+1}} \Lambda_{\alpha(s+1)\dot{\alpha}(s-1)}$
- Or pick $A = B = 0$ and introduce a compensator $\chi_{\alpha(s-1)\dot{\alpha}(s-2)}$ with transformation $\delta_G \chi_{\alpha(s-1)\dot{\alpha}(s-2)} = \bar{\mathrm{D}}^{\dot{\alpha}_{s-1}} \mathrm{D}^{\alpha_s} L_{\alpha(s)\dot{\alpha}(s-1)} + \frac{s-1}{s} \mathrm{D}^{\alpha_s} \bar{\mathrm{D}}^{\dot{\alpha}_{s-1}} L_{a(s)\dot{\alpha}(s-1)} + \bar{\mathrm{D}}_{\dot{\alpha}_{s-2}} J_{\alpha(s-1)\dot{\alpha}(s-3)}$

From now on, we will focus on the first case. By introducing the compensator, updating the action with its interaction term with $H_{\alpha(s)\dot{\alpha}(s)}$ and its kinetic energy terms and demanding invariance of the action we get the final expression for the Superspace action:

$$
\begin{aligned}
S = \int d^8z \Big\{ & c\, H^{\alpha(s)\dot{\alpha}(s)} \mathrm{D}^{\gamma} \bar{\mathrm{D}}^2 \mathrm{D}_{\gamma} H_{\alpha(s)\dot{\alpha}(s)} \\
& - 2c\, H^{\alpha(s)\dot{\alpha}(s)} \bar{\mathrm{D}}_{\dot{\alpha}_s} \mathrm{D}^2 \chi_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \\
& - \frac{s+1}{s} c\, \chi^{\alpha(s)\dot{\alpha}(s-1)} \mathrm{D}^2 \chi_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \\
& + 2c\, \chi^{\alpha(s)\dot{\alpha}(s-1)} \mathrm{D}_{\alpha_s} \bar{\mathrm{D}}^{\dot{\alpha}_s} \bar{\chi}_{\alpha(s-1)\dot{\alpha}(s)} \Big\}
\end{aligned}
\tag{3.5}
$$

The gauge symmetry of this action, is revealed by the following two Bianchi identities:

$$\bar{\mathrm{D}}^{\dot{\alpha}_s} T_{\alpha(s)\dot{\alpha}(s)} - \bar{\mathrm{D}}^2 G_{\alpha(s))\dot{\alpha}(s-1)} = 0 \tag{3.6a}$$

$$\frac{1}{(s+1)!} \mathrm{D}_{(\alpha_{s+1}} G_{a(s))\dot{\alpha}(s-1)} = 0 \tag{3.6b}$$

where superfields $T_{\alpha(s)\dot{\alpha}(s)}$ and $G_{\alpha(s)\dot{\alpha}(s-1)}$ are the two equations of motion (variations of the action with respect to $H_{\alpha(s)\dot{\alpha}(s)}$ and $\chi_{\alpha(s)\dot{\alpha}(s-1)}$). However there is one more Bianchi type of identity that relates $T_{\alpha(s)\dot{\alpha}(s)}$, $G_{\alpha(s)\dot{\alpha}(s-1)}$ and $F_{\alpha(2s+1)}$. One can prove that the following holds identically:

$$
\begin{aligned}
\frac{1}{(2s+1)!} \mathrm{D}^{\alpha_{2s+1}} F_{\alpha(2s+1)} = & \frac{1}{2c} \frac{1}{(2s)!} \partial_{(\alpha_{2s}}{}^{\dot{\alpha}_s} \dots \partial_{\alpha_{s+1}}{}^{\dot{\alpha}_1} T_{\alpha(s))\dot{\alpha}(s)} \\
& + \frac{i}{2c} \frac{s}{2s+1} \frac{1}{(2s)!} \mathrm{D}_{(\alpha_{2s}} \bar{\mathrm{D}}^2 \partial_{\alpha_{2s-1}}{}^{\dot{\alpha}_{s-1}} \dots \partial_{\alpha_{s+1}}{}^{\dot{\alpha}_1} G_{\alpha(s))\dot{\alpha}(s-1)} \\
& + \frac{1}{2c} \frac{s}{2s+1} \frac{1}{(2s)!} \mathrm{D}_{(\alpha_{2s}} \partial_{\alpha_{2s-1}}{}^{\dot{\alpha}_s} \dots \partial_{\alpha_s}{}^{\dot{\alpha}_1} \bar{G}_{\alpha(s-1))\dot{\alpha}(s)}
\end{aligned}
\tag{3.7}
$$

That goes to prove that on-shell, when superfields $T_{\alpha(s)\dot{\alpha}(s)}$ and $G_{\alpha(s)\dot{\alpha}(s-1)}$ vanish, we get the constraints of Table 2, required for the description of half-integer superhelicity.

3.3 The Component Action

Having an exact expression of the Superspace action, we can use it to extract all kind of pieces of information about the system. Particularly we would like to obtain the off-shell component description of it. Of course we know that at the component level, the action must reduce to the (Fang-)Fronsdal[9] action for the dynamics of a helicity (j=s+$1/2$) j=s+1, plus a set of auxiliary fields. Consequently, the most important piece of information we would like to get is, how many and what type of auxiliary components the theory includes. The second most important piece of information we would like to know is with what sign each auxiliary component appears in the action. It turns out, that the set of auxiliary components can be extracted from superspace without doing any calculations. But in order to figure out their signs, one has to do the entire projection process. The importance of the sign of the coefficient for each auxiliary field is not actually important for the free theory, but it will play a significant role when we introduce higher derivative terms. Typical examples for such behavior are the higher curvature theories of supergravity.

In order to get the component action out of superspace, we have to integrate (which is equivalent to differentiation) over the θ coordinates. That includes doing a lot of D-algebra, defining components for the various superfields through the covariant derivatives and then doing field redefinitions in order to eliminate spurious components. For simple theories, such as the vector multiplet (Y=$1/2$) this can be easily done. However for more complicated theories, such as the arbitrary half-integer superhelicity, this process is not very practical.

In search for a more efficient method, we do a few observations. By doing enough redefinitions we should be able to bring the component Lagrangian in a diagonal form, meaning the Lagrangian describing the s+1 helicity plus the Lagrangian describing the s+$1/2$ helicity plus algebraic terms, that involve only the auxiliary components in a way that each auxiliary component appears in exactly one term. In this configuration the spin content of the on-shell theory is obvious and the auxiliary status of the auxiliary components is evident too. That is because each auxiliary component appears in exactly one term in an algebraic manner (A^2 or AB). Hence their equations of motion will make them vanish on-shell (A=0 or B=0). For the same reason, the auxiliary components must by gauge invariant objects $\delta_G A$=0. The entire action is gauge invariant, the actions describing the spin dynamics are invariant too and each auxiliary appears in exactly one algebraic term.

The conclusion is that if we want to be efficient, we can define the auxiliary components to vanish on-shell and to be gauge invariant. The only question is, what are the superspace objects that will allow us to define the auxiliary fields in such a way. However the Superspace consideration of the system naturally provides us with two superfields $T_{\alpha(s)\dot{\alpha}(s)}$ and $G_{\alpha(s)\dot{\alpha}(s-1)}$ that have these exact properties. They vanish identically on-shell because they are the equations of motion and they are gauge invariant because they are generated by a gauge invariant action. So, the message is that if we want to find the auxiliary structure of the theory then we must look at the components of superfields $T_{\alpha(s)\dot{\alpha}(s)}$ and $G_{\alpha(s)\dot{\alpha}(s-1)}$.

What is more, $T_{\alpha(s)\dot{\alpha}(s)}$ and $G_{\alpha(s)\dot{\alpha}(s-1)}$ satisfy the Bianchi identities (3.6), which at the component level further reduce the free components that can play the role of auxiliary fields.

$$\begin{aligned}
&\bar{\mathrm{D}}^{\dot{\alpha}_s}T_{\alpha(s)\dot{\alpha}(s)}-\bar{\mathrm{D}}^2G_{\alpha(s))\dot{\alpha}(s-1)}=0\ \to\ \mathrm{D}^2T_{\alpha(s)\dot{\alpha}(s)}=0,\ \bar{\mathrm{D}}^2T_{\alpha(s)\dot{\alpha}(s)}=0,\\
&\frac{1}{(s+1)!}\mathrm{D}_{(\alpha_{s+1}}G_{a(s))\dot{\alpha}(s-1)}=0\quad\to\quad \mathrm{D}^2G_{a(s))\dot{\alpha}(s-1)}=0
\end{aligned}\tag{3.8}$$

So just by looking the above constraints we immediately realize that the only set of auxiliary fields this theory can have is:

- Bosons: $\bar{\mathrm{D}}^{\dot{\alpha}_{s-1}}G_{\alpha(s)\dot{\alpha}(s-1)}|$, $\bar{\mathrm{D}}_{(\dot{\alpha}_s}G_{\alpha(s)\dot{\alpha}(s-1))}|$, $T_{\alpha(s)\dot{\alpha}(s)}|$, $\mathrm{D}^{\alpha_s}G_{\alpha(s)\dot{\alpha}(s-1)}|$,
- Fermions: $G_{\alpha(s)\dot{\alpha}(s-1)}|$, $\mathrm{D}_{(\alpha_s}\bar{\mathrm{D}}^{\dot{\alpha}_s}\bar{G}_{\alpha(s-1))\dot{\alpha}(s)}|$

The lesson here is, in order to get the list of auxiliary fields one has to look for the components of the superfields, that play the role of equations of motion, that are left unaffected by the Bianchi identities, emerging from the gauge invariance. On the other hand if we want to get the full details then we have to proceed with the projection procedure but in a way that introduces the superfields $T_{\alpha(s)\dot{\alpha}(s)}$ and $G_{\alpha(s)\dot{\alpha}(s-1)}$ as a whole. The answer is to re-express the action (3.5) in the following way:

$$S=\int d^8z\left\{\frac{1}{4}H^{\alpha(s)\dot{\alpha}(s)}T_{\alpha(s)\dot{\alpha}(s)}+\frac{1}{2}\chi^{\alpha(s)\dot{\alpha}(s-1)}G_{\alpha(s)\dot{\alpha}(s-1)}+c.c.\right\}\tag{3.9}$$

After projection, one finds for the Fermionic Component Lagrangian

$$\begin{aligned}
\mathscr{L}_F=&i\ \bar{\psi}^{\alpha(s)\dot{\alpha}(s+1)}\partial^{\alpha_{s+1}}{}_{\dot{\alpha}_{s+1}}\psi_{\alpha(s+1)\dot{\alpha}(s)}\\
&+i\left[\frac{s}{s+1}\right]\psi^{\alpha(s+1)\dot{\alpha}(s)}\partial_{\alpha_{s+1}\dot{\alpha}_s}\psi_{\alpha(s)\dot{\alpha}(s-1)}+c.c.\\
&-i\left[\frac{2s+1}{(s+1)^2}\right]\bar{\psi}^{\alpha(s-1)\dot{\alpha}(s)}\partial^{\alpha_s}{}_{\dot{\alpha}_s}\psi_{\alpha(s)\dot{\alpha}(s-1)}\\
&+i\ \psi^{\alpha(s)\dot{\alpha}(s-1)}\partial_{\alpha_s\dot{\alpha}_{s-1}}\psi_{\alpha(s-1)\dot{\alpha}(s-2)}+c.c.\\
&-i\ \bar{\psi}^{\alpha(s-2)\dot{\alpha}(s-1)}\partial^{\alpha_{s-1}}{}_{\dot{\alpha}_{s-1}}\psi_{\alpha(s-1)\dot{\alpha}(s-2)}\\
&+\rho^{\alpha(s)\dot{\alpha}(s-1)}\beta_{\alpha(s)\dot{\alpha}(s-1)}+c.c.
\end{aligned}\tag{3.10}$$

The first 5 lines of it, is the part of the Lagrangian that describes the helicity $s+1/2$ dynamics and the last line, in an expected algebraic fashion, provides the two auxiliary fields. The detailed definition of all the fermionic components is:

$$\begin{aligned}
\rho_{\alpha(s)\dot{\alpha}(s-1)}&\equiv G_{\alpha(s)\dot{\alpha}(s-1)}|\\
\beta_{\alpha(s)\dot{\alpha}(s-1)}&\equiv-\frac{1}{2s!}\left\{\frac{s}{s+1}\mathrm{D}_{(\alpha_s}\bar{\mathrm{D}}^{\dot{\alpha}_s}\bar{G}_{\alpha(s-1))\dot{\alpha}(s)}-\frac{i}{2}\partial_{(\alpha_s}{}^{\dot{\alpha}_s}\bar{G}_{\alpha(s-1))\dot{\alpha}(s)}\right\}|\\
\psi_{a(s+1)\dot{\alpha}(s)}&\equiv\frac{\sqrt{2}}{(s+1)!}\bar{\mathrm{D}}^2\mathrm{D}_{(\alpha_{s+1}}H_{\alpha(s))\dot{\alpha}(s)}|\\
\psi_{\alpha(s)\dot{\alpha}(s-1)}&\equiv-\sqrt{2}\left\{\mathrm{D}^2\bar{\mathrm{D}}^{\dot{\alpha}_s}H_{\alpha(s)\dot{\alpha}(s)}+\frac{s+1}{s}\mathrm{D}^2\chi_{\alpha(s)\dot{\alpha}(s-1)}\right\}|\\
\psi_{\alpha(s-1)\dot{\alpha}(s-2)}&\equiv-\sqrt{2}\frac{(s-1)}{s}\bar{\mathrm{D}}^{\dot{\alpha}_{s-1}}\mathrm{D}^{\alpha_s}\chi_{\alpha(s)\dot{\alpha}(s-1)}|
\end{aligned}\tag{3.11}$$

For the Bosonic part of the story, we get:

$$
\begin{aligned}
\mathscr{L}_B = & \frac{1}{4}\left[\frac{s-1}{s+1}\right]\ U^{\alpha(s)\dot{\alpha}(s-2)}U_{\alpha(s)\dot{\alpha}(s-2)} + c.c. \\
& +\frac{1}{2}\left[\frac{s}{2s+1}\right]\ u^{\alpha(s)\dot{\alpha}(s)}u_{\alpha(s)\dot{\alpha}(s)} - \left[\frac{s}{2}\right]\ v^{\alpha(s)\dot{\alpha}(s)}v_{\alpha(s)\dot{\alpha}(s)} + \frac{1}{8}\left[\frac{2s+1}{s+1}\right]\ A^{\alpha(s)\dot{\alpha}(s)}A_{\alpha(s)\dot{\alpha}(s)} \\
& -\frac{1}{2}\left[\frac{s^2}{(s+1)^2}\right]\ S^{\alpha(s-1)\dot{\alpha}(s-1)}S_{\alpha(s-1)\dot{\alpha}(s-1)} - \frac{1}{2}\left[\frac{s^2}{(s+1)^2}\right]\ P^{\alpha(s-1)\dot{\alpha}(s-1)}P_{\alpha(s-1)\dot{\alpha}(s-1)} \\
& +h^{\alpha(s+1)\dot{\alpha}(s+1)}\Box h_{\alpha(s+1)\dot{\alpha}(s+1)} - \left[\frac{s+1}{2}\right]\ h^{\alpha(s+1)\dot{\alpha}(s+1)}\partial_{\alpha_{s+1}\dot{\alpha}_{s+1}}\partial^{\gamma\dot{\gamma}}h_{\gamma\alpha(s)\dot{\gamma}\dot{\alpha}(s)} \\
& +[s(s+1)]\ h^{\alpha(s+1)\dot{\alpha}(s+1)}\partial_{\alpha_{s+1}\dot{\alpha}_{s+1}}\partial_{\alpha_s\dot{\alpha}_s}h_{\alpha(s-1)\dot{\alpha}(s-1)} \\
& -[(s+1)(2s+1)]\ h^{\alpha(s-1)\dot{\alpha}(s-1)}\Box h_{\alpha(s-1)\dot{\alpha}(s-1)} \\
& -\left[\frac{(s+1)(s-1)^2}{2}\right]\ h^{\alpha(s-1)\dot{\alpha}(s-1)}\partial_{\alpha_{s-1}\dot{\alpha}_{s-1}}\partial^{\gamma\dot{\gamma}}h_{\gamma\alpha(s-2)\dot{\gamma}\dot{\alpha}(s-2)}
\end{aligned}
\tag{3.12}
$$

The last four lines describe the dynamics of helicity s+1 and the rest of the terms are the expected auxiliary fields. Their precise definition is:

$$
\begin{aligned}
U_{\alpha(s)\dot{\alpha}(s-2)} &\equiv \bar{\mathrm{D}}^{\dot{\alpha}_{s-1}}G_{\alpha(s))\dot{\alpha}(s-1)}| \\
u_{\alpha(s)\dot{\alpha}(s)} &\equiv \frac{1}{2s!}\left\{\mathrm{D}_{(\alpha_s}\bar{G}_{\alpha(s-1))\dot{\alpha}(s)} - \bar{\mathrm{D}}_{(\dot{\alpha}_s}G_{\alpha(s)\dot{\alpha}(s-1))}\right\}| \\
v_{\alpha(s)\dot{\alpha}(s)} &\equiv -\frac{i}{2s!}\left\{\mathrm{D}_{(\alpha_s}\bar{G}_{\alpha(s-1))\dot{\alpha}(s)} + \bar{\mathrm{D}}_{(\dot{\alpha}_s}G_{\alpha(s)\dot{\alpha}(s-1))}\right\}| \\
A_{\alpha(s)\dot{\alpha}(s)} &\equiv T_{\alpha(s)\dot{\alpha}(s)}| + \frac{s}{2s+1}\frac{1}{s!}\left(\mathrm{D}_{(\alpha_s}\bar{G}_{\alpha(s-1))\dot{\alpha}(s)} - \bar{\mathrm{D}}_{(\dot{\alpha}_s}G_{\alpha(s)\dot{\alpha}(s-1))}\right)| \\
S_{\alpha(s-1)\dot{\alpha}(s-1)} &\equiv \frac{1}{2}\left\{\mathrm{D}^{\alpha_s}G_{\alpha(s)\dot{\alpha}(s-1)} + \bar{\mathrm{D}}^{\dot{\alpha}_s}\bar{G}_{\alpha(s)\dot{\alpha}(s-1)}\right\}| \\
P_{\alpha(s-1)\dot{\alpha}(s-1)} &\equiv -\frac{i}{2}\left\{\mathrm{D}^{\alpha_s}G_{\alpha(s)\dot{\alpha}(s-1)} - \bar{\mathrm{D}}^{\dot{\alpha}_s}\bar{G}_{\alpha(s)\dot{\alpha}(s-1)}\right\}| \\
h_{\alpha(s+1)\dot{\alpha}(s+1)} &\equiv \frac{1}{2}\frac{1}{(s+1)!^2}\left[\mathrm{D}_{(\alpha_{s+1}},\mathrm{D}_{(\dot{\alpha}_{s+1}}\right]H_{\alpha(s))\dot{\alpha}(s))}| \\
h_{\alpha(s-1)\dot{\alpha}(s-1)} &\equiv \frac{1}{2}\frac{s}{(s+1)^2}\left[\mathrm{D}^{\alpha_s},\bar{\mathrm{D}}^{\dot{\alpha}_s}\right]H_{\alpha(s)\dot{\alpha}(s)}| + \frac{1}{s+1}\left(\mathrm{D}^{\alpha_s}\chi_{\alpha(s)\dot{\alpha}(s-1)} + \bar{\mathrm{D}}^{\dot{\alpha}_s}\bar{\chi}_{\alpha(s-1)\dot{\alpha}(s)}\right)|
\end{aligned}
\tag{3.13}
$$

To complete the component discussion we will do a counting of the off-shell degrees of freedom of the theory. The fact that the auxiliary fields are gauge invariant will make the counting extremely easy. Obviously we are expecting the same number of off-shell degrees of freedom for bosons and fermions. After all the theory is supersymmetric. The answer is, this theory has $8s^2+8s+4$ / $8s^2+8s+4$ degrees of freedom:

fields	*d.o.f*	*redundancy*	*net*
$h_{\alpha(s+1)\dot{\alpha}(s+1)}$	$(s+2)^2$	$(s+1)^2$	s^2+2s+3
$h_{\alpha(s-1)\dot{\alpha}(s-1)}$	s^2		
$u_{\alpha(s)\dot{\alpha}(s)}$	$(s+1)^2$	0	$(s+1)^2$
$v_{\alpha(s)\dot{\alpha}(s)}$	$(s+1)^2$	0	$(s+1)^2$
$A_{\alpha(s)\dot{\alpha}(s)}$	$(s+1)^2$	0	$(s+1)^2$
$U_{\alpha(s)\dot{\alpha}(s-2)}$	$2(s+1)(s-1)$	0	$2(s+1)(s-1)$
$S_{\alpha(s-1)\dot{\alpha}(s-1)}$	s^2	0	s^2
$P_{\alpha(s-1)\dot{\alpha}(s-1)}$	s^2	0	s^2
		Total	$8s^2+8s+4$

fields	*d.o.f*	*redundancy*	*net*
$\psi_{\alpha(s+1)\dot{\alpha}(s)}$	$2(s+2)(s+1)$	$2(s+1)s$	$4s^2+4s+4$
$\psi_{\alpha(s)\dot{\alpha}(s-1)}$	$2(s+1)s$		
$\psi_{\alpha(s-1)\dot{\alpha}(s-2)}$	$2s(s-1)$		
$\rho_{\alpha(s)\dot{\alpha}(s-1)}$	$2(s+1)s$	0	$2(s+1)s$
$\beta_{\alpha(s)\dot{\alpha}(s-1)}$	$2(s+1)s$	0	$2(s+1)s$
		Total	$8s^2+8s+4$

3.4 The General Picture

So far, we have demonstrated the development of the arbitrary half-integer (Y=s+$1/2$) superhelicity theory. However as it was explained earlier there is another, not-equivalent, off-shell formulation of this irreducible representation. Finally there is also the case of integer superhelicities. The general picture can be visualized in Figure 1

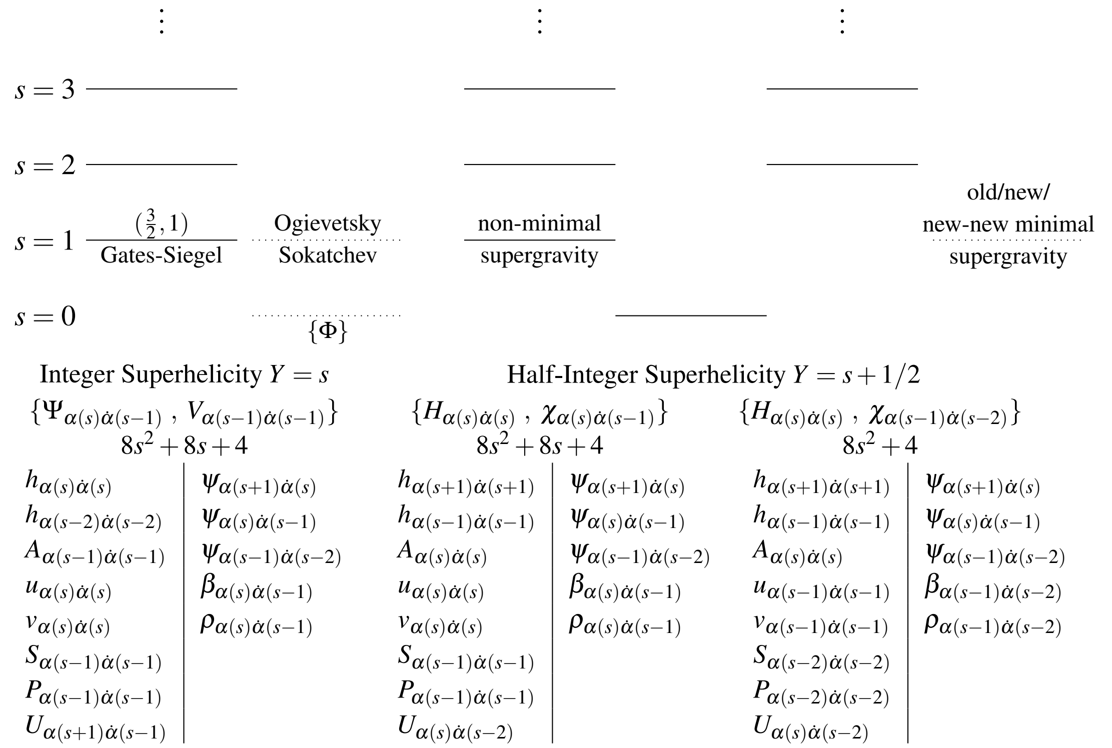

Figure 1: Landscape of highest superhelicities

For the description of the highest possible superhelicity given a specific index strucuture, the results are three infinite towers, one for the integer case and two for the half-integer case. Each solid line denotes an irreducible representation. Under each tower, it is given the set of superfields participating in the superspace action, the off-shell number of degrees of freedom and the list of components. There are also a few dotted lines. These represent "low spin accidents", meaning alternative formulations of the corresponding representations that can not be generalized.

A very intriguing observation is that, the degrees of freedom of Y=s exactly match the degrees of freedom of one of the two Y=s+$1/2$. This suggest that for every boson in one theory there is a fermion in the other one. Therefore if we put the two theories together there is another direction of supersymmetry. So this very simple counting, reveales how we can construct an $\mathcal{N}$=2 theory out of $\mathcal{N}$=1 building blocks. This was done by trial an error in [10], but now we understand why this can work.

4. Off-Shell Formulation for the Massive case

Now we attempt to repeat the process for the off-shell description of the massive irreducible representations. Of course this is a much harder problem because there are no guidelines such as gauge symmetry and the progress that has been made is not big [11, 12, 13]. If we assume that such a construction can be made, then we have to be able to take its massless limit. A reasonable request is that this limit results to the massless theory of corresponding superhelicity with the possibility of a few extra stuff that decouple. After all, it is a well known fact that massive spins can be expressed as the direct sum of a tower of massless spins. This property has been proven for the supersymmetric case only on-shell and in component formulation[14].

Nevertheless, the strategy will be to work in a case by case manner starting from the low superspins in order to build intuition and understanding. Then we attempt to generalize to arbitrary superspins. The construction of the superspace action that describes on-shell the massive irreps will be done by starting with the corresponding massless action and adding all possible mass deformations.

On top of that we may be required to introduce new, auxiliary superfields but in a way that they decouple in the massless limit. This is also motivated from the structure of the action that describes a massive spin. An integer massive spin s requires the presence of s real tensors of increasing rank $0, 1, \ldots, s-3, s-2, s$. For the massive half integer spin $s+1/2$ we have a not-identical but similar behavior. This is the Singh(-Hagen) [15] descriptions and are expected to be recovered at the component projection of the massive superspace action. It is obvious that such a big number of components (without even taking into account the supersymmetric auxiliary fields) can not be generated by only the superfields that participate into the massless action. So the solution of this hard problem resides in the understanding of the set of the auxiliary superfields.

Hopefully the low superspin cases will not be extremely hard to solve and they will reveal enough of this structure in order to allow us to induce the solution of the general, arbitrary superspin, theory. In this process, the latest contribution, is the case of superspin Y=3/2, which corresponds to the massive extension of linearized non-minimal supergravity. It is the first non-trivial case that shade some light to the first member of this list of auxiliary superfields.

4.1 Warming up: Massive vector multiplet

The above suggested algorithm starts with the massless theory. All highest superhelicity, massless theories are included in Figure 1. In principle we can pick any one of them and start working on the corresponding massive theory. However, our goal is to generalize to arbitrary superhelicities. This suggest that we should start with a massless theory that is a member of one of the towers and therefore can be generalized. This argument excludes immediately the "low spin accidents". To be clear, it does not mean that these cases do not have a massive extension because they do. It is just the fact that whatever insight we gain by studying these theories, it can not be extrapolated to the general case.

Continuing with the half-integer case, the easiest possible starting point is the massive vector multiplet (Y=1/2). This will be a warm up exercise and then we move on to the harder problem of Y=3/2. The massless vector multiplet superspace action is a very simple one,

$$S=\int d^8z\left\{H\mathrm{D}^\gamma\bar{\mathrm{D}}^2\mathrm{D}_\gamma H\right\} \tag{4.1}$$

Then we add to it all possible mass deformations, which are only two terms

$$S=\int d^8z\left\{H\mathrm{D}^\gamma\bar{\mathrm{D}}^2\mathrm{D}_\gamma H+a_1mH(\mathrm{D}^2H+\bar{\mathrm{D}}^2H)+a_2m^2H^2\right\} \tag{4.2}$$

The free coefficients will be determined by the requirement to generate the constraints of Table 1, which in this case become D^2H=0 and $\Box H$=m^2H. In order to generate the first of them we act with D^2 on the equation of motion $\mathscr{E}^{(H)}$:

$$\begin{aligned}\mathscr{E}^{(H)}&=2\mathrm{D}^\gamma\bar{\mathrm{D}}^2\mathrm{D}_\gamma H+2a_1m\left(\mathrm{D}^2H+\bar{\mathrm{D}}^2H\right)+2a_2m^2H\\ \mathrm{D}^2\mathscr{E}^{(H)}&=2a_1m\mathrm{D}^2\bar{\mathrm{D}}^2H+2a_2m^2\mathrm{D}^2H\xrightarrow[a_1=0]{a_2\neq0}\mathrm{D}^2H=0\end{aligned} \tag{4.3}$$

Under this constraint and due to the D algebra, the Klein-Gordon equation follows from the equation of motion:

$$\mathscr{E}^{(H)}=-2\Box H+2a_2m^2H\xrightarrow{a_2=1}\Box H=m^2H \tag{4.4}$$

Therefore the superspace action that on-shell describes superspin Y=1/2 is:

$$S=\int d^8z\left\{H\mathrm{D}^\gamma\bar{\mathrm{D}}^2\mathrm{D}_\gamma H+m^2H^2\right\} \tag{4.5}$$

4.2 Superspin Y=3/2 - massive, linear, non-minimal supergravity

We repeat the procedure for superspin Y=3/2. The corresponding massless theory is:

$$\begin{aligned}S=\int d^8z\Big\{&H^{\alpha\dot\alpha}\mathrm{D}^\gamma\bar{\mathrm{D}}^2\mathrm{D}_\gamma H_{\alpha\dot\alpha}\\ &-2\,H^{\alpha\dot\alpha}\bar{\mathrm{D}}_{\dot\alpha}\mathrm{D}^2\chi_\alpha+c.c.\\ &-2\,\chi^\alpha\mathrm{D}^2\chi_\alpha+c.c.\\ &+2\,\chi^\alpha\mathrm{D}_\alpha\bar{\mathrm{D}}^{\dot\alpha}\bar\chi_{\dot\alpha}\Big\}\end{aligned} \tag{4.6}$$

Now we add all possible mass deformations. The massless theory, besides the main superfield $H_{\alpha\dot\alpha}$, provides another superfield χ_α that played the role of the compensator but in the massive theory will play the role of an auxiliary superfield. Therefore there are more terms proportional to m and m^2 in comparison to the previous simple example

$$\begin{aligned}S=\int d^8z\Big\{&H^{\alpha\dot\alpha}\mathrm{D}^\gamma\bar{\mathrm{D}}^2\mathrm{D}_\gamma H_{\alpha\dot\alpha} &&+a_1mH^{\alpha\dot\alpha}(\bar{\mathrm{D}}_{\dot\alpha}\chi_\alpha-\mathrm{D}_\alpha\bar\chi_{\dot\alpha})\\ &-2\,H^{\alpha\dot\alpha}\bar{\mathrm{D}}_{\dot\alpha}\mathrm{D}^2\chi_\alpha+c.c. &&+a_2mH^{\alpha\dot\alpha}(\mathrm{D}^2H_{\alpha\dot\alpha}+\bar{\mathrm{D}}^2H_{\alpha\dot\alpha})\\ &-2\,\chi^\alpha\mathrm{D}^2\chi_\alpha+c.c. &&+a_3m\chi^\alpha\chi_\alpha+c.c.\\ &+2\,\chi^\alpha\mathrm{D}_\alpha\bar{\mathrm{D}}^{\dot\alpha}\bar\chi_{\dot\alpha} &&+a_4m^2H^{\alpha\dot\alpha}H_{\alpha\dot\alpha}\Big\}\end{aligned} \tag{4.7}$$

This time, the goal is to prove that on-shell we get the constraints χ_α=0, $\mathrm{D}^\alpha H_{\alpha\dot\alpha}$=0 and $\Box H_{\alpha\dot\alpha}=m^2H_{\alpha\dot\alpha}$. The equations of motion as derived from the above action are:

$$\mathscr{E}^{(H)}_{\alpha\dot\alpha} = 2\mathrm{D}^{\gamma}\bar{\mathrm{D}}^2\mathrm{D}_{\gamma}H_{\alpha\dot\alpha} + 2(\mathrm{D}_{\alpha}\bar{\mathrm{D}}^2\bar\chi_{\dot\alpha} - \bar{\mathrm{D}}_{\dot\alpha}\mathrm{D}^2\chi_{\alpha}) + a_1 m(\bar{\mathrm{D}}_{\dot\alpha}\chi_{\alpha} - \mathrm{D}_{\alpha}\bar\chi_{\dot\alpha}) \\ +2a_2 m(\mathrm{D}^2 H_{\alpha\dot\alpha} + \bar{\mathrm{D}}^2 H_{\alpha\dot\alpha}) + 2a_4 m^2 H_{\alpha\dot\alpha} \tag{4.8}$$

$$\mathscr{E}^{(\chi)}_{\alpha} = -4\mathrm{D}^2\chi_{\alpha} + 2\mathrm{D}_{\alpha}\bar{\mathrm{D}}^{\dot\alpha}\bar\chi_{\dot\alpha} - 2\mathrm{D}^2\bar{\mathrm{D}}^{\dot\alpha}H_{\alpha\dot\alpha} + a_1 m\bar{\mathrm{D}}^{\dot\alpha}H_{\alpha\dot\alpha} + 2a_3 m\chi_{\alpha} \tag{4.9}$$

It will be desirable to start with the first constraint (χ_{α}=0), since it will restrict $H_{\alpha\dot\alpha}$ too and hence help us prove the other two ($\mathrm{D}^{\alpha}H_{\alpha\dot\alpha}$=0, $\Box H_{\alpha\dot\alpha} = m^2 H_{\alpha\dot\alpha}$) . However the only place where χ_{α} appears algebraically, is in the last term of $\mathscr{E}^{(\chi)}$. In order to isolate this term, we attempt to generate an equation that depends only in χ_{α} (no $H_{\alpha\dot\alpha}$ dependence) and afterwards check if we can select coefficients to isolate the specific term. For this purpose consider the following combination:

$$\begin{aligned} I_{\alpha} &= A\mathrm{D}^2\bar{\mathrm{D}}^{\dot\alpha}\mathscr{E}^{(H)}_{\alpha\dot\alpha} + B\mathrm{D}^2\bar{\mathrm{D}}^2\mathscr{E}^{(\chi)}_{\alpha} + m^2\mathscr{E}^{(\chi)}_{\alpha} \\ &= -2(A+B)\Box\mathrm{D}^2\bar{\mathrm{D}}^{\dot\alpha}H_{\alpha\dot\alpha} \quad +2(A+B)\mathrm{D}^2\bar{\mathrm{D}}^2\mathrm{D}_{\alpha}\bar{\mathrm{D}}^{\dot\alpha}\bar\chi_{\dot\alpha} - Aa_1 m\mathrm{D}^2\bar{\mathrm{D}}^{\dot\alpha}\mathrm{D}_{\alpha}\bar\chi_{\dot\alpha} \\ &\quad +2(Aa_4-1)m^2\mathrm{D}^2\bar{\mathrm{D}}^{\dot\alpha}H_{\alpha\dot\alpha} - 4(A+B)\Box\mathrm{D}^2\chi_{\alpha} \quad -4m^2\mathrm{D}^2\chi_{\alpha} \\ &\quad +a_1 m^3\bar{\mathrm{D}}^{\dot\alpha}H_{\alpha\dot\alpha} \quad +2(Aa_1+Ba_3)m\mathrm{D}^2\bar{\mathrm{D}}^2\chi_{\alpha} + 2m^2\mathrm{D}_{\alpha}\bar{\mathrm{D}}^{\dot\alpha}\bar\chi_{\dot\alpha} \\ &\quad +2a_3 m^3\chi_{\alpha} \end{aligned} \tag{4.10}$$

To remove any $H_{\alpha\dot\alpha}$ dependence we make the following choice:

$$A+B=0,\ Aa_4-1=0,\ a_1=0 \tag{$\Sigma 1$}$$

therfore we get:

$$I_{\alpha} = -4m^2\mathrm{D}^2\chi_{\alpha} + 2Ba_3 m\mathrm{D}^2\bar{\mathrm{D}}^2\chi_{\alpha} + 2m^2\mathrm{D}_{\alpha}\bar{\mathrm{D}}^{\dot\alpha}\bar\chi_{\dot\alpha} + 2a_3 m^3\chi_{\alpha}$$

From the above expression it is obvious that there is no more freedom left in order to isolate the mass term. This is a hint that, the superfields provided by the massless theory are not enough for the description of the massive system and more degrees of freedom must be introduced. Following this argument, we introduce a new, fermionic, auxiliary, superfield u_{α} that couples only with χ_{α} and only through a mass term $m\chi^{\alpha}u_{\alpha}$, so in the massless limit u_{α} completely decouples from the massless theory.

The superspace action has to be updated with the presence of u_{α}, meaning we have to add the interaction term with χ_{α} and all possible kinetic energy terms of u_{α}, so it will not be just a Lagrange multiplier (enforcing constraints by hand). The new action has the form:

$$\begin{aligned} S=\int d^8z\Big\{ & H^{\alpha\dot\alpha}\mathrm{D}^{\gamma}\bar{\mathrm{D}}^2\mathrm{D}_{\gamma}H_{\alpha\dot\alpha} \quad +a_2 mH^{\alpha\dot\alpha}\mathrm{D}^2H_{\alpha\dot\alpha} + c.c. \quad +b_1 u^{\alpha}\mathrm{D}^2u_{\alpha} + c.c. \\ & -2\,H^{\alpha\dot\alpha}\bar{\mathrm{D}}_{\dot\alpha}\mathrm{D}^2\chi_{\alpha} + c.c. \quad +a_3 m\chi^{\alpha}\chi_{\alpha} + c.c. \quad +b_2 u^{\alpha}\bar{\mathrm{D}}^2u_{\alpha} + c.c. \\ & -2\,\chi^{\alpha}\mathrm{D}^2\chi_{\alpha} + c.c. \quad +a_4 m^2 H^{\alpha\dot\alpha}H_{\alpha\dot\alpha} \quad +b_3 u^{\alpha}\bar{\mathrm{D}}^{\dot\alpha}\mathrm{D}_{\alpha}\bar u_{\dot\alpha} \\ & +2\,\chi^{\alpha}\mathrm{D}_{\alpha}\bar{\mathrm{D}}^{\dot\alpha}\bar\chi_{\dot\alpha} \quad +\gamma mu^{\alpha}\chi_{\alpha} + c.c. \quad +b_4 u^{\alpha}\mathrm{D}_{\alpha}\bar{\mathrm{D}}^{\dot\alpha}\bar u_{\dot\alpha} \\ & +b_5 mu^{\alpha}u_{\alpha}\Big\} \end{aligned} \tag{4.11}$$

Now we repeat the previous calculations aiming towards the on-shell vanishing of u_{α} and χ_{α}. The equation of motion of $H_{\alpha\dot\alpha}$ has not changed because u_{α} does not interact with it, so with the same choice of coefficients as (Σ1) we can eliminate all $H_{\alpha\dot\alpha}$ dependence. The next step is to remove all χ_{α} dependence and get an equation that involves only u_{α}. For that reason, we consider the following combination:

$$
\begin{aligned}
J_{\alpha} = & I_{\alpha} + mK\mathrm{D}^2\mathscr{E}^{(u)}_{\alpha} + m\Lambda \mathrm{D}_{\alpha}\bar{\mathrm{D}}^{\dot{\alpha}}\bar{\mathscr{E}}^{(u)}_{\dot{\alpha}} \\
= & [2Ba_3]\mathrm{D}^2\bar{\mathrm{D}}^2\chi_{\alpha} && +[B\gamma+2Kb_2+\Lambda b_3]m\mathrm{D}^2\bar{\mathrm{D}}^{\dot{\alpha}}u_{\alpha} \\
& +[-4+K\gamma]m^2\mathrm{D}^2\chi_{\alpha} && +[Kb_3+2\Lambda b_2]m\mathrm{D}^2\bar{\mathrm{D}}^{\dot{\alpha}}\mathrm{D}_{\alpha}\bar{u}_{\dot{\alpha}} \\
& +[2+\Lambda\gamma]m^2\mathrm{D}_{\alpha}\bar{\mathrm{D}}^{\dot{\alpha}}\bar{\chi}_{\dot{\alpha}} && +[\Lambda(2b_4-b_3)]\mathrm{D}_{\alpha}\bar{\mathrm{D}}^2\mathrm{D}^{\beta}u_{\beta} \\
& +[2a_3]m^3\chi_{\alpha} && +\gamma m^3 u_{\alpha} \\
& +[Kb_5]m^2\mathrm{D}^2u_{\alpha} && +[\Lambda b_5]m^2\mathrm{D}_{\alpha}\bar{\mathrm{D}}^{\dot{\alpha}}\bar{u}_{\dot{\alpha}}
\end{aligned}
\tag{4.12}
$$

This can be achieved with the choice

$$a_3=0,\ -4+K\gamma=0,\ 2+\Lambda\gamma=0,\ b_5=0 \tag{$\Sigma 2$}$$

and the equation for u_{α} is:

$$
\begin{aligned}
J_{\alpha} = & [B\gamma+2Kb_2+\Lambda b_3]m\mathrm{D}^2\bar{\mathrm{D}}^{\dot{\alpha}}u_{\alpha}+[Kb_3+2\Lambda b_2]m\mathrm{D}^2\bar{\mathrm{D}}^{\dot{\alpha}}\mathrm{D}_{\alpha}\bar{u}_{\dot{\alpha}} \\
& +[\Lambda(2b_4-b_3)]\mathrm{D}_{\alpha}\bar{\mathrm{D}}^2\mathrm{D}^{\beta}u_{\beta}+\gamma m^3u_{\alpha}
\end{aligned}
\tag{4.13}
$$

It is obvious that we have more freedom left and we can pick coefficients to isolate the last term. The following choice of coefficients

$$B\gamma+2Kb_2+\Lambda b_3=0,\ Kb_3+2\Lambda b_2=0,\ 2b_4-b_3=0,\gamma\neq 0 \tag{$\Sigma 3$}$$

will force u_{α} to vanish on-shell. This will start a cascading event that will lead to the vanishing of χ_{α} and the constraints on $H_{\alpha\dot{\alpha}}$. Let us see how that works. The equation of motion of u_{α} will make χ_{α} vanish on-shell

$$\mathscr{E}^{(u)}_{\alpha}=2b_1\mathrm{D}^2u_{\alpha}+2b_2\bar{\mathrm{D}}^2u_{\alpha}+b_3\bar{\mathrm{D}}^{\dot{\alpha}}\mathrm{D}_{\alpha}\bar{u}_{\dot{\alpha}}+b_4\mathrm{D}_{\alpha}\bar{\mathrm{D}}^{\dot{\alpha}}\bar{u}_{\dot{\alpha}}+\gamma m\chi_{\alpha}\xrightarrow{u_{\alpha}=0}\chi_{\alpha}=0 \tag{4.14}$$

and the vanishing of χ_{α} will lead to:

$$
\begin{aligned}
&\mathscr{E}^{(\chi)}_{\alpha}=-2\mathrm{D}^2\bar{\mathrm{D}}^{\dot{\alpha}}H_{\alpha\dot{\alpha}}\rightarrow \mathrm{D}^2\bar{\mathrm{D}}^{\dot{\alpha}}H_{\alpha\dot{\alpha}}=0, \\
&\mathrm{D}^{\alpha}\mathscr{E}^{(H)}_{\alpha\dot{\alpha}}=-2\mathrm{D}^2\bar{\mathrm{D}}^2\mathrm{D}^{\alpha}H_{\alpha\dot{\alpha}}+2a_2m\mathrm{D}^{\alpha}\bar{\mathrm{D}}^2H_{\alpha\dot{\alpha}}+2a_4m^2\mathrm{D}^{\alpha}H_{\alpha\dot{\alpha}}\xrightarrow{a_2=0\ (\Sigma 4)}\mathrm{D}^{\alpha}H_{\alpha\dot{\alpha}}=0, \\
&\mathscr{E}^{(H)}_{\alpha\dot{\alpha}}=-2\Box H_{\alpha\dot{\alpha}}+2a_4m^2H_{\alpha\dot{\alpha}}\xrightarrow{a_4=1\ (\Sigma 5)}\Box H_{\alpha\dot{\alpha}}=m^2H_{\alpha\dot{\alpha}}
\end{aligned}
\tag{4.15}
$$

There is one more thing that needs to be checked and that is, if all these choices of coefficients (Σ1-Σ5) are compatible to each other and give a non-trivial solution. The answer is that, they are compatible and they give a unique one parameter family of solutions:

a_1=0	a_2=0	a_3=0	a_4=1	
b_1=free	b_2=1/6	b_3=1/6	b_4=1/12	b_5=0
γ=1	A=1	B=−1	K=4	Λ=−2

Table 3: Solution for the coefficients

The final expression of the superspace action that describes superspin Y=3/2 is (up to an overall coefficient):

$$
\begin{aligned}
S=\int d^8z\Big\{ & H^{\alpha\dot{\alpha}}\mathrm{D}^{\gamma}\bar{\mathrm{D}}^2\mathrm{D}_{\gamma}H_{\alpha\dot{\alpha}} && +mu^{\alpha}\chi_{\alpha}+c.c. \\
& -2\,H^{\alpha\dot{\alpha}}\bar{\mathrm{D}}_{\dot{\alpha}}\mathrm{D}^2\chi_{\alpha}+c.c. && +\frac{1}{6}u^{\alpha}\bar{\mathrm{D}}^2u_{\alpha}+c.c. \\
& -2\,\chi^{\alpha}\mathrm{D}^2\chi_{\alpha}+c.c. && +\frac{1}{6}u^{\alpha}\bar{\mathrm{D}}^{\dot{\alpha}}\mathrm{D}_{\alpha}\bar{u}_{\dot{\alpha}} \\
& +2\,\chi^{\alpha}\mathrm{D}_{\alpha}\bar{\mathrm{D}}^{\dot{\alpha}}\bar{\chi}_{\dot{\alpha}} && +\frac{1}{12}u^{\alpha}\mathrm{D}_{\alpha}\bar{\mathrm{D}}^{\dot{\alpha}}\bar{u}_{\dot{\alpha}} \\
& && +m^2H^{\alpha\dot{\alpha}}H_{\alpha\dot{\alpha}}\Big\}
\end{aligned}
\tag{4.16}
$$

This process has to be generalized for the higher superspin theories, but it should be obvious by now that it is not an easy task. As it was mentioned before, the major difficulty is to determine the number and the type of auxiliary superfields that are required beyond the massless theory.

5. Conclusion

To conclude, studying the representation theory of the $4D$, $\mathcal{N}=1$ Super-Poincaré group we learn all about the building blocks and the constraints the have to satisfy, in order to describe the various irreps. Then we managed to realize these descriptions for the massless, arbitrary superhelicity case by constructing a superspace action that can generate on-shell all the required constraints. Furthermore, we demonstrated how we can extract all the auxiliary component structure through the superfields that play the role of equations of motion and their properties (Bianchi identities). The counting of the off-shell degrees of freedom will reveal patterns that can be explored towards the construction of $\mathcal{N}=2$ theories.

For the general massive case, the problem is still open but we managed to get some insight by studying low superspins. The latest contribution in this programm was the case of massive superspin $3/2$ for the 20/20 supermultiplet. This is the linearized non-minimal supergravity.

Nowadays, there is a lot of interest for modified supergravities and the massive extension of non-minimal supergravity falls in this category. Another type of modification would be to consider higher curvature theories, with $R+R^2$ to be the simplest one. This extension for linearized non-minimal supergravity has been recently done in [16].

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