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Gauge Invariant Currents in $SU(N)$ Gauge Theories

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The fermion and total currents generated by a gauge field are derived in the framework of a self-consistent consideration of the Dirac and Yang-Mills equations. Under the condition of gauge invariance, the obtained currents is found to be expressible in terms of the massive vector field generated by the initial gauge field. The total $SU(N)$ current satisfying the continuity equation is derived.

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1. Introduction

J.Schwinger pointed out the importance of the gauge invariance of a fermion current in the (1+1) quantum electrodynamics (QED_2) for the first time [1]. The calculated fermion current is found to be expressible in terms of some massive field. On the other hand, it was shown [2] that in the non-Abelian case a fermi theory in (1+1) dimensions is also equivalent to the local boson theory.

The two-dimensional physics has been further developed in the papers [3, 4] which are particularly devoted to the derivation of the spectra of observable mesons [3, 4] and baryons [3]. In terms of the QCD_2 action integral, it is shown [3, 4] that the meson mass is approximately proportional to the square root of the number of colors and flavors. The systematic presentation of the two-dimensional quantum field theory is given in Ref.[5].

Despite the obvious successes of the QCD_2 concept[6, 7] in treating experimental results[6, 8, 9], the problem how to describe the states of observable particles in the realistic QCD_4 dynamics has still been one of the most important problems in the strong interaction physics. There has also been concurrently a natural development of the QCD_2 models mentioned above. The dynamics of particles in QCD_4 is vastly more complicate as compared to the QCD_2 case because of increasing number of the dimensions of the studied problem. In this way, it is reasonable to think, that as well as in the case QCD_2, interaction of fermions with a gauge field can be expressed in terms of the massive gauge field. However, in calculating the total mass of a gauge field, we should note there is another source of the gauge field mass which is the self-interacting non-Abelian field.

The problem of a gluon mass was repeatedly raised[10, 11, 14, 12, 13, 15, 16, 17, 18, 19, 20, 21], mainly in the context of studying the low momentum properties of a gluon propagator in the pure Yang-Mills theory. The last investigations of the gluon propagator at low momentum[15, 16, 17, 18, 19, 20, 21], which also include the lattice studies[15, 20], show that the inverse gluon propagator always has non-vanishing asymptote at the small momentum, which is reasonable to be interpreted as a gluon mass.

In the present paper a self-consistent solution of the Dirac equation in a non-abelian gauge field is obtained in the absence of the additional restrictions[22, 23] to the field structure, which are the consideration of the Dirac equation either in the field of the non-abelian plane wave [22], or in the framework of the two-dimensional QCD[23]. On the basis of the derived formal solution of the Dirac equation, the fermion and total currents are obtained in an explicit form. The derived currents are found to be expressible in terms of the massive vector field, generated in the result of interaction between fermions and gauge fields. The obtained currents appear to be gauge invariant, and satisfies the continuity equation. In this way, the derived contribution to the mass is sufficiently dependent on the occupancy numbers of the fermion subsystem.

2. Solution of the Dirac equation

The Lagrangian governing the fermions interacting with an SU(N) gauge field is [24]

$$\mathscr{L} = \left\{ \frac{1}{2} \sum_f \left[\bar{\Psi}_f \gamma^\mu D_\mu \, \Psi_f - \bar{\Psi}_f \gamma^\mu \overleftarrow{D}_\mu \, \Psi_f - 2\bar{\Psi}_f \, m_f \, \Psi_f \right] - \frac{1}{4} F^a_{\mu\nu} \, F_a^{\mu\nu} \right\}, \tag{2.1}$$

where A^a_μ and Ψ_f are the gauge and fermion fields in the Minkowskii (3+1)-dimensional space-time with coordinates $x \equiv x^\mu = (x^0, \mathbf{x}) = (x^0, x^1, x^2, x^3)$, m_f is a fermion mass, g is the coupling constant, f means a quark flavor. In Eqs.(2.1) we introduce,

$$D_\mu = i\partial_\mu + gT_a A^a_\mu,\ \ \overleftarrow{D}_\mu = i\overleftarrow{\partial}_\mu - gT_a A^a_\mu;\ \ F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\, f^a_{bc}\, A^b_\mu\, A^c_\nu, \tag{2.2}$$

where γ^ν are the standard Dirac matrices, T_a are the infinitesimal operators satisfying the standard commutative relations and the normalization condition[24], $a,b,c = 1\ldots N^2-1$ are SU(N) group indices; $\partial_\mu = (\partial/\partial t, \nabla)$.

The Lagrangian (2.1) leads to the Dirac equation:

$$\left\{ i\gamma^\mu \left(\partial_\mu - ig \cdot A^a_\mu(x)\, T_a \right) - m_f \right\} \Psi_f(x) = 0, \tag{2.3}$$

The solution of Eq. (2.3) can be formally written in the operator form as follows:

$$\Psi_f(x) = \{T_{l(x_0;x)} \exp\} \left\{ igT_a \int dx^\mu A^a_\mu \right\} \psi_f(x), \tag{2.4}$$

where the symbol $\{T_{l(x_0;x)} \exp\}$ means that the integration is to be carried out along the line from the point x_0 to the point x such that the factors in exponent expansion are chronologically ordered from x_0 to x. In this way, $\psi_f(x)$ obeys the free Dirac equation:

$$\left\{ i\gamma^\mu \partial_\mu - m_f \right\} \psi_f(x) = 0. \tag{2.5}$$

The general solution of Eq.(2.5) can be presented as a superposition of the Dirac plane waves

$$\psi_{\sigma c f}(x) = u_\sigma(P)\, \frac{\exp(-iP_\mu x^\mu)}{\sqrt{2\varepsilon(\vec p)}}\, v_{c,f}(x_0), \qquad \varepsilon^2(\vec p) = \vec p^{\ 2} + m_f^2,\ P \equiv P^\mu = (\varepsilon(\vec p),\ \vec p) \tag{2.6}$$

where P^μ is the 4-momentum, $u_\sigma(P)$ are the free Dirac bispinors, normalized by the doubled mass ($\bar u u = 2m_f$). The symbol $v_{c,f}(x_0)$ is a vector in the charged space, which also depends on a point in the Minkowskii space-time. We take $v_{c,f}(x_0)$ to be normalized by the standard condition:

$$(v^\dagger)_{c'f'}(x_0) v^{cf}(x_0) = \delta^c_{c'} \delta^f_{f'}. \tag{2.7}$$

Here, σ and c denote the spin and color variables.

The chronological exponent in Eq.(2.4) acts on the vector $v_{c,f}(x_0)$, and carries out the parallel shift of this vector along the geodesic line from the point x_0 to the point x in the Minkowskii space-time. Taking into account Eqs.(2.4) and (2.6) we can write the general solution of Eq. (2.3) as follows:

$$\Psi_{c,f}(x) = \int \frac{d^3\vec p}{(2\pi)^3} \sum_{\sigma\lambda} \left[u_\sigma(P)\, a_f(P,\sigma,\lambda,c)\, \theta(P^0)\ +\ u_\sigma(-P)\, a_f(-P,\sigma,\lambda,c)\, \theta(-P^0) \right]$$
$$\times \frac{\exp(-iP_\mu x^\mu)}{\sqrt{2\varepsilon(\vec p)}} \{T_{l(x_0;x)} \exp\} \left\{ igT_a \int dx^\mu A^a_\mu \right\}\ v_{c,f}(x_0), \tag{2.8}$$

where summation with respect to λ means summing over all the possible trajectories of a fermion which connect the points x_0 and x in the Minkowskii space-time. The subscribes c and f enumerate the colors and flavor states, respectively; $\theta(z)$ is the unit step function. The coefficients

$a_f(P,\sigma,\lambda,c)$ are related to particles or anti-particles, and satisfy the standard commutative relations for the Fermi operators under the field quantization:

$$[a_f(P,\sigma,\lambda,c)\ ,\ a_f^{\dagger}(P',\sigma',\lambda',c')]_+ = (2\pi)^3\delta(\vec{p}\ -\ \vec{p}\ ')\ \delta_{\sigma\sigma'}\ \delta_{cc'}\ \delta_{ff'}\ \delta_{\lambda\lambda'}. \tag{2.9}$$

The δ-symbol with respect to the "variable" λ eliminates interception of the particle trajectories what is the direct consequence of the superposition principle.

3. Fermion and total currents

The initial Lagrangian (2.1) implies the self-consistency with respect to the interacting fermions and gauge fields, that leads to a very complicate structure of the currents generated by this Lagrangian. However, it is reasonable to expect some simplification in calculations of fermion currents in external fields owing to the derived solution (2.8) of the Dirac equation, since any current is the convolution of Ψ-functions.

Let us calculate the fermion current generated by the external non-Abelian field. In deriving such a current, we follow Schwinger's idea consisting of the consideration of J_a^μ as a limit:

$$J_a^\mu(x) = g\sum_f \left\{\bar{\Psi}_f(x)\gamma^\mu T_a\Psi_f(x')\right\}, \quad x' \to x. \tag{3.1}$$

We note, that because of the trace operation with respect to the color variable in Eq.(3.1), the current $J_a^\mu(x)$ contains the factor

$$\{T_{l(x_0;x)}\exp\}[ig\ T_a\int_x^{x'} A_\mu^a dx^\mu]. \tag{3.2}$$

Then, expanding the operator exponent in the last equation as a series with respect to $(x'-x)\to 0$, we get

$$\begin{aligned}\{T_{l(x_0;x)}\exp\}\left\{ig\ T_a\int_x^{x'} A_\mu^a dx^\mu\right\} &= 1 + ig\ T_a(x'-x)^\mu A_\mu^a(\xi) + \frac{i}{2}g\ T_a(x'-x)^\mu(x'-x)^\nu\partial_\nu A_\mu^a(\xi)\\ &- g^2\ (T_aT_b)(\tilde{x}'-\tilde{x})^\mu(x'-x)^\nu A_\mu^a(\tilde{\xi})A_\nu^b(\xi)\theta(\tilde{\xi}-\xi) + .. \end{aligned} \tag{3.3}$$

where $\tilde{\xi}\in[\tilde{x},\tilde{x}']$, and $\xi\in[x,x']$; $x'\to x$.

Let us take the limits $(\tilde{x}'-\tilde{x})\to 0$ and $(x'-x)\to 0$, such that $(\tilde{x}'-\tilde{x})/(x'-x)\to 0$. Then, the last term in the expansion in Eq. (3.3) is equal to zero. Substituting the expansion of the chronological exponent which is given by Eq. (3.3) into Eq.(3.1), we obtain at $(x'-x)\to 0$

$$\begin{aligned}J_a^\mu &= g^2\int\frac{d^4P}{(2\pi)^3}\sum_{f,c,\lambda}\sum_\sigma\Bigg\{n_f(P,\sigma,\lambda,c)\ P^\mu\left(-\frac{\partial}{\partial P_\nu}\exp\left(-iP(x'-x)\right)\right)\\ &\times\frac{\left[\delta(P^0+\varepsilon(\vec{p}))\ +\ \delta(P^0-\varepsilon(\vec{p}))\right]}{\varepsilon(\vec{p})}\left(A_\nu^a(\xi)+\frac{1}{2}(x'-x)^\beta\partial_\nu A_\beta^a(\xi)\right)\Bigg\},\end{aligned} \tag{3.4}$$

where $n_f(P,\sigma,\lambda,c)$ denotes the occupancy numbers of fermion states:

$$n_f(P,\sigma,\lambda,c) = \langle a_f^{\dagger}(P,\sigma,\lambda,c)a_f(P,\sigma,\lambda,c)\rangle, \tag{3.5}$$

where the angle brackets mean averaging over all the possible sets of the quantum numbers determining the fermion states.

The occupancy numbers $n_f(P,\sigma,\lambda,c)$ should be physically treated as the average number of the fermions with the quantum numbers (P,σ,c,f) , which propagate along the fixed trajectory identified by the number λ. Thereat, the trajectories appear to be not interferant owing to the commutative relations (2.9), that relates to the superposition principle for fermion fields.

In obtaining the last equation, we have successively calculated a trace, and introduced the additional integration with respect to the zeroth component of the vector P^{μ}. We note that on taking the partial derivative $\partial^2 \equiv \partial^2_{(x)}$ on $(x'-x)^{\beta}\partial_{\nu(x)}A^b_{\beta}(\xi)$, we get [23]

$$\lim_{x'\to x}\left((x'-x)^{\lambda}\partial_{\nu}A^b_{\lambda}(\xi)\right) = -2\frac{\partial^{\beta}\partial_{\nu}}{\partial^2}A^b_{\beta}(x). \tag{3.6}$$

Integrating partially with respect to P^{μ} in Eq.(3.4), we derive:

$$J^{\mu}_a(x) = M^2(N_c,N_f)\left(A^{\mu}_a(x) - \frac{\partial_{\lambda}\partial^{\mu}}{\partial^2}A^{\lambda}_a(x)\right), \tag{3.7}$$

where N_c and N_f are the number of colors and flavors, respectively. The operator of the inverse squared derivative in Eq.(3.7) means division by the offshell squared 4-momentum in the momentum representation.

The factor $M^2(N_c,N_f)$ in Eq.(3.7) which has the dimension of the squared mass, is equal to

$$M^2(N_c,N_f) = \frac{g^2}{8}\int\frac{d^4P}{(2\pi)^3}\sum_{f,c,\lambda}\sum_{\sigma}\frac{\partial}{\partial P_{\nu}}\left\{n_f(P,\sigma,\lambda,c)\frac{P^{\nu}\left[\delta(P^0+\varepsilon(\vec{p}))+\delta(P^0-\varepsilon(\vec{p}))\right]}{\varepsilon(\vec{p})}\right\}. \tag{3.8}$$

The obtained mass $M(N_c,N_f)$ can be physically interpreted as the mass of a fermion field which is carried by a gauge field due to interaction between the fermion and gluon fields. Provided that the occupancy numbers $n_f(P,\sigma,\lambda,c)$ are approximately independent on N_c and N_f we obtain that $M^2(N_c,N_f) \propto N_f N_c$, which is in a good agreement with the results obtained earlier in Refs.[3, 4].

We add the current, induced by the self-interaction of a gauge field[24], to $J^{\mu}_a(x)$ given by Eq.(3.7). Then, the total fermion current $I^{\mu}_a(x)$, generated by the non-Abelian external field is:

$$I^{\mu}_a(x) = M^2(N_c,N_f)\left(A^{\mu}_a(x) - \frac{\partial_{\lambda}\partial^{\mu}}{\partial^2}A^{\lambda}_a(x)\right) - gf_{ab}{}^{c}A^b_{\nu}F^{\mu\nu}_c. \tag{3.9}$$

The obtained current (3.9) is gauge invariant, and is obviously satisfies the continuity equation:

$$\partial_{\mu}I^{\mu}_a(x) = 0. \tag{3.10}$$

We should note that the currents, which are given by Eq. (3.7), and derived in the paper [1], appear to be very similar. However such similarity is only formal, since the current Eq. (3.7) depends strongly on the mass of a vector field which is governed by the occupancy numbers of the fermion subsystem.

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4. Conclusion

The self-consistent dynamics of fermion and boson fields in the gauge $SU(N)$ model is considered beyond the perturbative approximation. The formal solution of the Dirac equation is derived. On the basis of the obtained solution, the fermion and total currents in the $SU(N)$ gauge model in the (3+1) Minkowskii space-time are derived. The obtained currents are gauge invariant, and satisfy the continuity equation.

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