Precision tests via radiative μ and τ leptonic decays

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The branching fractions of radiative leptonic τ decays ($\tau \to l\nu\bar{\nu}\gamma$, $l = e, \mu$) were recently measured by the BABAR collaboration with a relative error of about 3%. The measurement of the branching ratio $\mathscr{B}(\tau \to e\bar{\nu}\nu\gamma)$, for a minimum photon energy of 10 MeV in the τ rest frame, differs from our recent SM prediction by 3.5 standard deviations, whereas our result agrees with BABAR's value for $\mathscr{B}(\tau \to \mu\bar{\nu}\nu\gamma)$. Our predictions also agree with the measurement of $\mathscr{B}(\mu \to e\bar{\nu}\nu\gamma)$ by the MEG collaboration. We also report on a recent proposal to test the τ dipole moments via precise measurements of radiative leptonic τ decays at high-luminosity B factories.

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1. Introduction

Muon and τ leptonic decays have been among the most powerful tools to study the Lorentz structure of weak interactions. Their precise theoretical formulation in terms of Bouchiat-Michel-Kinoshita-Sirlin parameters [1, 2, 3] places them in a unique position to investigate possible contributions beyond the *V−A* coupling of the Standard Model (SM). Radiative μ and τ leptonic decays, where an inner bremsstrahlung photon is emitted and detected, can be predicted with very high precision and provide independent and complementary tests [4, 5]. Radiative μ and τ leptonic decays also constitute an important source of background for experiments searching for charged lepton flavor violating decays, such as $\mu^{\pm} \to e^{\pm}\gamma$, $\tau^{\pm} \to l^{\pm}\gamma$ ($l = e, \mu$), and even $\mu^{\pm} \to e^{\pm}(e^+e^-)$ and $\tau^{\pm} \to l^{\pm}(e^+e^-)$, because of the internal conversion of photons to electron-positron pairs [6, 7, 8, 9].

Recently the BABAR collaboration measured the $\tau \to l\gamma\nu\bar{\nu}$ branching ratios for a minimum photon energy $\omega_0 = 10$ MeV in the τ rest frame [10, 11]. These measurements, with a relative error of about 3%, must be compared with the SM predictions of the branching fractions at next-to-leading order (NLO). Indeed these radiative corrections, recently computed in [12], are not protected from mass singularities by the Kinoshita-Lee-Nauenberg (KLN) theorem [13, 14, 15] and are of relative order $(\alpha/\pi)\ln(m_l/m_\tau)\ln(\omega_0/m_\tau)$, corresponding to a large 10% correction for $l = e$, and 3% for $l = \mu$. Radiative muon decays were measured long ago in [16], and new results were presented recently by the MEG [6] and PIBETA [17] collaborations.

Precise data on radiative τ leptonic decays also offer the opportunity to probe the τ anomalous magnetic moment $(g−2)$ and electric dipole moment (EDM). The short lifetime of the τ has so far prevented the direct measurement of its $g−2$ via the τ spin precession in a magnetic field (like in the electron and muon $g−2$ experiments) and the present bound is only of $O(10^{-2})$, more than an order of magnitude larger than the leading contribution $\alpha/(2\pi) \approx 0.001$. While experiments attempted the extraction of indirect bounds from τ pair production and decays by comparing sufficiently precise data with the SM predictions, in [18, 19, 20] we proposed the study of the τ electromagnetic dipole moments via its radiative leptonic τ decays by means of an effective Lagrangian approach.

In section 2 and 3 we present the SM predictions for differential decay rates and branching ratios of radiative μ and τ leptonic decays, and compare them with the experimental results. In section 4 we review the current status of the τ dipole moments and report on the sensitivities that can be expected at the upcoming Belle II experiment. Conclusions are drawn in section 5.

2. Radiative μ and τ leptonic decays: differential rates

The SM prediction, at NLO, for the differential rate of the radiative leptonic decays

$$\mu^{\pm} \to e^{\pm}\nu\bar{\nu}\gamma, \tag{2.1}$$

$$\tau^{\pm} \to l^{\pm}\nu\bar{\nu}\gamma, \tag{2.2}$$

with $l = e$ or μ, of a polarized $\mu^{\pm}$ or $\tau^{\pm}$ in their rest frame is

$$\frac{d^6\Gamma^{\pm}(y_0)}{dx\,dy\,d\Omega_l\,d\Omega_\gamma} = \frac{\alpha\,G_F^2 M^5}{(4\pi)^6}\frac{x\beta_l}{1+\delta_{\rm w}}\left[G \mp x\beta_l\hat{n}\cdot\hat{p}_l J \mp y\hat{n}\cdot\hat{p}_\gamma K + xy\beta_l\hat{n}\cdot\left(\hat{p}_l\times\hat{p}_\gamma\right)L\right], \tag{2.3}$$

where $G_F = 1.1663787(6) \times 10^{-5}$ GeV^{-2} [21] is the Fermi constant determined by the muon lifetime, and $\alpha = 1/137.035999157(33)$ is the fine-structure constant [22]. Calling m and M the masses of the final and initial charged leptons (neutrinos and antineutrinos are considered massless) we define $r = m/M$ and $r_{\rm w} = M/M_{\rm w}$, where $M_{\rm w}$ is the W-boson mass; p and $n = (0, \hat{n})$ are the four-momentum and polarization vector of the initial τ or muon, with $n^2 = -1$ and $n \cdot p = 0$. Also, $x = 2E_l/M$, $y = 2\omega/M$ and $\beta_l \equiv |\vec{p}_l|/E_l = \sqrt{1 - 4r^2/x^2}$, where $p_l = (E_l, \vec{p}_l)$ and $p_\gamma = (\omega, \vec{p}_\gamma)$ are the four-momenta of the final charged lepton and photon, respectively. The final charged lepton and photon are emitted at solid angles Ω_l and Ω_γ, with normalized three-momenta $\hat{p}_l$ and $\hat{p}_\gamma$, and c is the cosine of the angle between $\hat{p}_l$ and $\hat{p}_\gamma$. The term $\delta_{\rm w} = 1.04 \times 10^{-6}$ is the tree-level correction to muon decay induced by the W-boson propagator [23, 24]. Equation (2.3) includes the possible emission of an additional soft photon with normalized energy y' lower than the detection threshold y_0 (with $y_0 \ll 1$): $y' < y_0 < y$. The function $G(x, y, c, y_0)$ and, analogously, J and K, are given by

$$G(x,y,c,y_0) = = \frac{4}{3yz^2}\left[g_0(x,y,z) + r_{\rm w}^2\, g_{\rm w}(x,y,z) + \frac{\alpha}{\pi}\, g_{\rm NLO}(x,y,z,y_0)\right], \qquad (2.4)$$

where $z = xy(1 - c\beta_l)/2$; the LO function $g_0(x,y,z)$, computed in [25, 26, 27], arises from the pure Fermi V–A interaction, whereas $g_{\rm w}(x,y,z)$ is the LO contribution of the W-boson propagator derived in [24]. The NLO term $g_{\rm NLO}(x,y,z,y_0)$ is the sum of the virtual and soft bremsstrahlung contributions calculated in [12] (see also [28, 29]). The function $L(x,y,z)$, appearing in front of the product $\hat{n} \cdot (\hat{p}_l \times \hat{p}_\gamma)$, does not depend on y_0; it is only induced by the loop corrections and is therefore of $\mathcal{O}(\alpha/\pi)$. The (lengthy) explicit expressions of G, J, K and L are provided in [12]. If the initial $\mu^\pm$ or $\tau^\pm$ are not polarized, Eq. (2.3) simplifies to

$$\frac{d^3\Gamma^\pm(y_0)}{dx\,dc\,dy} = \frac{\alpha G_F^2 M^5}{(4\pi)^6} \frac{x\beta_l}{1 + \delta_{\rm w}}\, 8\pi^2\, G(x,y,c,y_0). \qquad (2.5)$$

3. Radiative μ and τ leptonic decays: branching ratios

The analytic integration of the LO part of the differential rate (2.5) over the allowed kinematic ranges for a minimum photon energy $y_0 = 2\omega_0/M$ gives [26, 30]

$$\Gamma_0(y_0) = \frac{G_F^2 M^5}{192\pi^3} \frac{\alpha}{3\pi}\left[3\,\mathrm{Li}_2(y_0) - \frac{\pi^2}{2} + \left(\ln r + \frac{17}{12}\right)\left(6\ln y_0 + 6\bar{y}_0 + \bar{y}_0^4\right) + \qquad (3.1)\right.$$

$$\left. + \frac{1}{48}\left(125 + 45y_0 - 33y_0^2 + 7y_0^3\right)\bar{y}_0 - \frac{1}{2}\left(6 + \bar{y}_0^3\right)\bar{y}_0\ln\bar{y}_0\right], \qquad (3.2)$$

where $\bar{y}_0 = 1 - y_0$ and the dilogarithm is defined by $\mathrm{Li}_2(z) = -\int_0^z dt\, \frac{\ln(1-t)}{t}$. Terms depending on the mass ratio r have been neglected in the expression for $\Gamma_0(y_0)$, with the obvious exception of the logarithmic contribution which diverges in the limit $r \to 0$. However, terms in the integrand $g_0(x,y,c)$ (see (2.4)) proportional to r^2 were not neglected when performing the integral to obtain (3.2), as they lead to terms of $\mathcal{O}(1)$ in the integrated result $\Gamma_0(y_0)$. This feature was first pointed out in [15]. We also note that the presence of the mass singularity $\ln r$ in the integrated decay rate $\Gamma_0(y_0)$ does not contradict the KLN theorem, which applies only to total decay rates [13, 14, 15]. The tiny corrections induced by the W-boson propagator were neglected in eq. (3.2).

	$\tau \to e\bar{\nu}\nu\gamma$ [10]	$\tau \to \mu\bar{\nu}\nu\gamma$ [10]	$\mu \to e\nu\bar{\nu}\gamma$ [16]	$\mu \to e\nu\bar{\nu}\gamma$ [6]
$\mathscr{B}_{\mathrm{LO}}$	$1.834 \cdot 10^{-2}$	$3.663 \cdot 10^{-3}$	$1.308 \cdot 10^{-2}$	$6.204 \cdot 10^{-8}$
$\mathscr{B}_{\mathrm{NLO}}^{\mathrm{Inc}}$	$-1.06(1)_n(10)_N \cdot 10^{-3}$	$-5.8(1)_n(2)_N \cdot 10^{-5}$	$-1.91(5)_n(6)_N \cdot 10^{-4}$	$-3.61(8)_n(21)_N \cdot 10^{-9}$
$\mathscr{B}_{\mathrm{NLO}}^{\mathrm{Exc}}$	$-1.89(1)_n(19)_N \cdot 10^{-3}$	$-9.1(1)_n(3)_N \cdot 10^{-5}$	$-2.25(5)_n(7)_N \cdot 10^{-4}$	$-3.61(8)_n(21)_N \cdot 10^{-9}$
$\mathscr{B}^{\mathrm{Inc}}$	$1.728(10)_{\mathrm{th}}(3)_\tau \cdot 10^{-2}$	$3.605(2)_{\mathrm{th}}(6)_\tau \cdot 10^{-3}$	$1.289(1)_{\mathrm{th}} \cdot 10^{-2}$	$5.84(2)_{\mathrm{th}} \cdot 10^{-8}$
$\mathscr{B}^{\mathrm{Exc}}$	$1.645(19)_{\mathrm{th}}(3)_\tau \cdot 10^{-2}$	$3.572(3)_{\mathrm{th}}(6)_\tau \cdot 10^{-3}$	$1.286(1)_{\mathrm{th}} \cdot 10^{-2}$	$5.84(2)_{\mathrm{th}} \cdot 10^{-8}$
$\mathscr{B}_{\mathrm{EXP}}$	$1.847(15)_{\mathrm{st}}(52)_{\mathrm{sy}} \cdot 10^{-2}$	$3.69(3)_{\mathrm{st}}(10)_{\mathrm{sy}} \cdot 10^{-3}$	$1.4(4) \cdot 10^{-2}$	$6.03(14)_{\mathrm{st}}(53)_{\mathrm{sy}} \cdot 10^{-8}$

Table 1: Branching ratios of radiative μ and τ leptonic decays. The minimum photon energy ω_0 is 10 MeV, except for the last column, where $\omega_0 = 40$ MeV and $E_e^{\mathrm{min}} = 45$ MeV. Inclusive and exclusive ($\mathscr{B}^{\mathrm{Inc/Exc}}$) predictions are separated into LO contributions ($\mathscr{B}_{\mathrm{LO}}$) and NLO corrections ($\mathscr{B}_{\mathrm{NLO}}^{\mathrm{Inc/Exc}}$). Uncertainties were estimated for uncomputed NNLO corrections (N), numerical errors (n), and the experimental errors of the lifetimes (τ). The first two types of errors were combined to provide the total theoretical uncertainty (th). The last line reports the experimental measurements of Refs. [10, 16, 6].

If we multiply the analytic result for $\Gamma_0(y_0)$ in eq. (3.2) by the lifetimes $\tau_{\mu,\tau}$ with a threshold $\omega_0 = 10$ MeV we obtain the following LO predictions for the branching ratios: 1.83×10^{-2} ($\tau \to e\bar{\nu}\nu\gamma$), 3.58×10^{-3} ($\tau \to \mu\bar{\nu}\nu\gamma$), and 1.31×10^{-2} ($\mu \to e\nu\bar{\nu}\gamma$). These values are in good agreement with the results $\mathscr{B}_{\mathrm{LO}}$ reported in table 1, obtained integrating numerically the LO part of the differential rate (2.5), with the exception of the $\tau \to \mu\bar{\nu}\nu\gamma$ value; this difference is due to the terms neglected in the analytic result (3.2).

At NLO, which allows for double photon emission, the branching ratios of the radiative decays (2.1,2.2) can be distinguished in two types:

- "Inclusive" measurements of the branching ratios, $\mathscr{B}^{\mathrm{Inc}}(y_0)$, where there is *at least one* photon in the final state with energy higher than y_0;

- "Exclusive" measurements of the branching ratios, $\mathscr{B}^{\mathrm{Exc}}(y_0)$, where there is *one, and only one*, photon in the final state with energy larger than the detection threshold y_0.

Exclusive and inclusive branching ratios for the radiative decays (2.1,2.2) were computed in [12] for a threshold $\omega_0 = y_0(M/2) = 10$ MeV, and are reported in table 1. Uncertainties were estimated for uncomputed NNLO corrections, numerical errors, and the experimental errors of the lifetimes. For $\omega_0 = 10$ MeV, the former were estimated to be $\delta\mathscr{B}_{\mathrm{NLO}}^{\mathrm{Exc/Inc}} \sim (\alpha/\pi) \ln r \ln(\omega_0/M) \mathscr{B}_{\mathrm{NLO}}^{\mathrm{Exc/Inc}}$; they are about 10%, 3% and 3% for $\tau \to e\bar{\nu}\nu\gamma$, $\tau \to \mu\bar{\nu}\nu\gamma$ and $\mu \to e\nu\bar{\nu}\gamma$, respectively (they appear with the subscript "N" in table 1). Numerical errors, labeled by the subscript "n", are smaller than those induced by missing radiative corrections. These two kinds of uncertainties were combined to provide the total theoretical error of $\mathscr{B}^{\mathrm{Exc/Inc}}$, indicated by the subscript "th". The uncertainty due to the experimental error of the lifetimes is labeled by the subscript "τ".

BABAR's recent measurements of the branching ratios of the radiative decays $\tau \to l\bar{\nu}\nu\gamma$, with $l = e$ and μ, for a minimum photon energy $\omega_0 = 10$ MeV in the τ rest frame, are [10, 11]:

$$\mathscr{B}_{\mathrm{EXP}}(\tau \to e\bar{\nu}\nu\gamma) = 1.847(15)_{\mathrm{st}}(52)_{\mathrm{sy}} \times 10^{-2}, \tag{3.3}$$

$$\mathscr{B}_{\mathrm{EXP}}(\tau \to \mu\bar{\nu}\nu\gamma) = 3.69(3)_{\mathrm{st}}(10)_{\mathrm{sy}} \times 10^{-3}, \tag{3.4}$$

where the first error is statistical and the second is systematic. These results are substantially more precise than the previous measurements of the CLEO collaboration [31]. The experimental values in Eqs. (3.3,3.4) were obtained requiring a signal with either a muon or an electron, plus a single photon; they must therefore be compared with our predictions for the exclusive branching ratios in table 1. For $\tau \to \mu \bar{\nu} \nu \gamma$ decays, the branching ratio measurement and prediction agree within 1.1 standard deviations (1.1σ). On the contrary, the experimental and theoretical values for $\tau \to e \bar{\nu} \nu \gamma$ decays differ by $2.02\,(57) \times 10^{-3}$, i.e. by 3.5$\sigma$. If BABAR's measurement (3.3) were inclusive, this discrepancy would decrease to 2.2σ. This puzzling discrepancy deserves further researches.

The branching ratio of radiative muon decays was measured long ago for a minimum photon energy $\omega_0 = 10$ MeV in the μ rest frame [16], and more recently by the MEG collaboration for $\omega_0 = 40$ MeV and minimum electron energy $E_e^{\min} = 45$ MeV (in this case, $\mathscr{B}^{\text{Inc}}$ and $\mathscr{B}^{\text{Exc}}$ coincide):

$$\mathscr{B}_{\text{EXP}}\left(\mu \to e \bar{\nu} \nu \gamma, \omega_0 = 10 \text{ MeV}\right) = 1.4\,(4) \times 10^{-2} \text{ [16]}, \tag{3.5}$$

$$\mathscr{B}_{\text{EXP}}\left(\mu \to e \bar{\nu} \nu \gamma, \omega_0 = 40 \text{ MeV}, E_e^{\min} = 45 \text{ MeV}\right) = 6.03\,(14)_{\text{st}}(53)_{\text{sy}} \times 10^{-8} \text{ [6]}. \tag{3.6}$$

Both measurements agree with our theoretical predictions (see table 1). New precise results are expected from the MEG [6] and PIBETA [17] collaborations.

The relative magnitude of radiative corrections were also studied in the specific final-state configuration of the decays (2.1,2.2) when the neutrino energies (E_ν and $E_{\bar{\nu}}$) are very small, i.e. when the photon and the final charged lepton are almost back-to-back. As already mentioned in the introduction, this phase-space region is of particular interest for experiments searching for the charged lepton flavor violating decays $\mu \to e\gamma$ or $\tau \to l\gamma$. Indeed the SM decays (2.1,2.2) are indistinguishable from the signal ($\mu \to e\gamma$ or $\tau \to l\gamma$), except for the energy carried away by the neutrinos. This SM background can be suppressed via a precise determination of the final state momenta: the total energy of the $e\gamma$ final state (or $l\gamma$) must be as close as possible to m_μ (or m_τ).

The upper panel of figure 1 shows, for radiative muon decays, the SM prediction at NLO of the branching fraction $\mathscr{B}^{\text{SM}}(\not{E}_{\max})$, defined as the integral of (2.5) over the phase space region satisfying $\not{E} = E_\nu + E_{\bar{\nu}} = m_\mu - E_e - \omega \le \not{E}_{\max}$. The maximum missing energy $\not{E}_{\max}$ is assumed to be lower than the detection threshold ω_0 and much lower than the muon mass: $\not{E}_{\max} < \omega_0 \ll m_\mu$. For this reason, $\not{E}_{\max}$ plays the role of infrared cut-off. Indeed if $\not{E}_{\max} \ll m_\mu$, the photon energy must be of the order of $(m_\mu - \not{E}_{\max})/2$, a value much larger than the threshold ω_0. Moreover, at NLO also a second soft photon can be emitted, but its energy is always below the threshold ω_0 – and therefore invisible – since it cannot exceed $\not{E}_{\max}$. We calculated and included these second-soft-photon effects in $\mathscr{B}^{\text{SM}}(\not{E}_{\max})$ adopting the same $y_0' \to 0$ limit described in [12] for the numerical evaluation of the exclusive and inclusive branching fractions in table 1. The lower panel of figure 1 shows the ratio of NLO corrections with respect to the LO branching ratio. The relative magnitude of these corrections can be as large as $8 - 12\%$ for an invisible energy cut $\not{E}_{\max}$ ranging from 1 to 6 MeV. The purple band represents the theoretical error assigned to this ratio for uncomputed NNLO corrections; it is estimated to be $\delta \mathscr{B}_{\text{NLO}}^{\text{SM}}(\not{E}_{\max}) \sim (\alpha/\pi)(\ln r)(\ln \frac{\not{E}_{\max}}{m_\mu}) \mathscr{B}_{\text{NLO}}^{\text{SM}}(\not{E}_{\max})$.

4. τ dipole moments via radiative leptonic τ decays

In this section we report on a recent proposal to determine the τ dipole moments via radiative leptonic τ decays [18, 19, 20]. The most general vertex function describing the interaction between
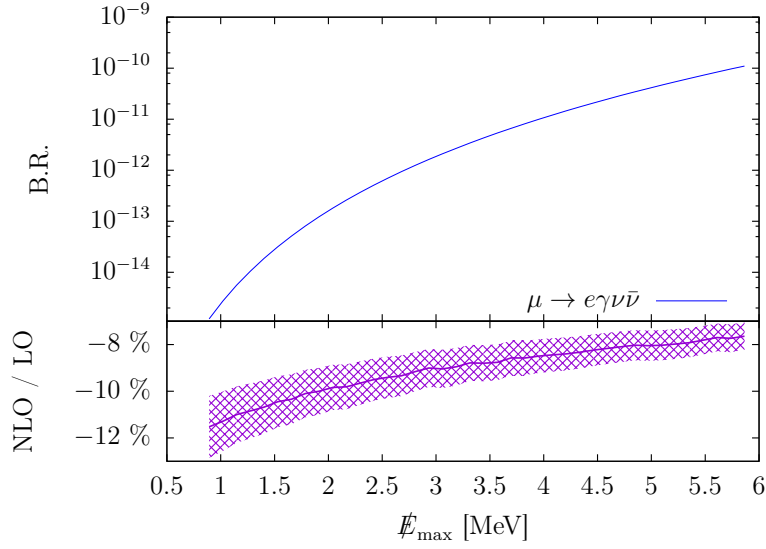
Figure 1: Top panel: branching ratio of the radiative muon decay (2.1) as a function of the invisible energy cut $\not{E}_{\max}$. Lower panel: the ratio of NLO corrections with respect to the LO branching ratio. The purple band represents the assigned theoretical error due to uncomputed NNLO corrections.

a photon and the initial and final states of an on-shell τ lepton can be written in the form

$$\Gamma^{\mu}(q^2) = ie \left\{ \gamma^{\mu} F_1(q^2) + \frac{\sigma^{\mu\nu} q_{\nu}}{2m_{\tau}} \left[iF_2(q^2) + F_3(q^2)\gamma_5 \right] + \left(\gamma^{\mu} - \frac{2m_{\tau} q^{\mu}}{q^2} \right) \gamma_5 F_4(q^2) \right\}, \quad (4.1)$$

where $e > 0$ is the positron charge, $\sigma_{\mu\nu} = i[\gamma_{\mu}, \gamma_{\nu}]/2$, and q is the ingoing four-momentum of the off-shell photon. In the limit $q^2 \to 0$, $F_2(0) = a_{\tau}$ and $F_3(0) = -d_{\tau}(2m_{\tau}/e)$, where $a_{\tau} = (g_{\tau} - 2)/2$ and d_{τ} are the anomalous magnetic moment and EDM of the τ, respectively.

Deviations of the τ dipole moments from the SM values can be analyzed in the framework of dimension-six gauge-invariant operators. Out of the complete set of 59 independent gauge invariant operators in [32, 33], only two of them can directly contribute to the τ $g-2$ and EDM at tree level (i.e., not through loop effects), $Q_{lW}^{33} = (\bar{l}_{\tau}\sigma^{\mu\nu}\tau_R) \sigma^I \varphi W_{\mu\nu}^I$ and $Q_{lB}^{33} = (\bar{l}_{\tau}\sigma^{\mu\nu}\tau_R) \varphi B_{\mu\nu}$, where φ and $l_{\tau} = (\nu_{\tau}, \tau_L)$ are the Higgs and the left-handed SU(2) doublets, σ^I the Pauli matrices, and $W_{\mu\nu}^I$ and $B_{\mu\nu}$ are the gauge field strength tensors. The leading non-standard effective Lagrangian relevant for our study is therefore given by

$$\mathscr{L}_{\text{eff}} = \frac{1}{\Lambda^2} \left[C_{lW}^{33} Q_{lW}^{33} + C_{lB}^{33} Q_{lB}^{33} + \text{h.c.} \right]. \quad (4.2)$$

After the electroweak symmetry breaking, the two operators mix and give additional, beyond the SM, contributions to the τ anomalous magnetic moment and EDM:

$$\tilde{a}_{\tau} = \frac{2m_{\tau}}{e} \frac{\sqrt{2}v}{\Lambda^2} \text{Re} \left[\cos\theta_{\text{w}} C_{lB}^{33} - \sin\theta_{\text{w}} C_{lW}^{33} \right], \quad \tilde{d}_{\tau} = \frac{\sqrt{2}v}{\Lambda^2} \text{Im} \left[\cos\theta_{\text{w}} C_{lB}^{33} - \sin\theta_{\text{w}} C_{lW}^{33} \right], \quad (4.3)$$

where $v = 246$ GeV and $\sin\theta_{\text{w}}$ is the weak mixing angle.

The present resolution on the τ anomalous magnetic moment is only of $\mathscr{O}(10^{-2})$, more than an order of magnitude larger than its precise SM prediction $a_{\tau}^{\text{SM}} = 117721(5) \times 10^{-8}$ [34]. In fact, the

short lifetime of the τ (2.9×10^{-13} s) has so far prevented the determination of a_τ by measuring the τ spin precession in a magnetic field, like in the electron and muon $g-2$ experiments. The present PDG limit on the τ $g-2$ was derived by the DELPHI collaboration from $e^+e^- \to e^+e^-\tau^+\tau^-$ total cross section measurements at LEP2 (the study of a_τ via this channel was proposed in [35]). Limits were derived comparing these measurements with the SM values, assuming that possible deviations were due to non-SM contributions to a_τ. The obtained limit at 95% CL is [36]

$$-0.052 < \tilde{a}_\tau < 0.013. \tag{4.4}$$

The reanalysis of Ref. [37] of various LEP and SLD measurements – mainly of the $e^+e^- \to \tau^+\tau^-$ cross section – allowed the authors to set the indirect 2σ confidence interval

$$-0.007 < \tilde{a}_\tau < 0.005, \tag{4.5}$$

a bound stronger than that in (4.4). This analysis assumed $\tilde{d}_\tau = 0$. The bound (4.5) has been confirmed by a recent update in Ref. [20].

Lepton EDMs are predicted to be extremely small in the SM, of the $\mathcal{O}(10^{-38} - 10^{-35})\, e \cdot \mathrm{cm}$ [38], far below the current experimental reach. The present PDG limit on the τ EDM at 95% CL is

$$-2.2 < \mathrm{Re}(d_\tau) < 4.5\ (10^{-17}\, e \cdot \mathrm{cm}), \quad -2.5 < \mathrm{Im}(d_\tau) < 0.8\ (10^{-17}\, e \cdot \mathrm{cm}); \tag{4.6}$$

it was obtained by the Belle collaboration [39] following the analysis of Ref. [40] for the impact of an effective operator for the τ EDM in the process $e^+e^- \to \tau^+\tau^-$.

At the LHC, bounds on the τ dipole moments are expected to be set in τ pair production via Drell-Yan [41] or double photon scattering processes [42]. The best limits achievable with the former are estimated to be comparable with the current ones if one assumes that the total cross section for τ pair production will be measured at the 14% level. Earlier proposals to set bounds on the τ dipole moments can be found in [43, 44, 45].

The Belle II experiment at the upcoming high-luminosity B factory SuperKEKB [46] will offer new opportunities to improve the determination of the τ electromagnetic properties. The authors of Refs. [47, 48] proposed to determine the Pauli form factor $F_2(q^2)$ of the τ via $\tau^+\tau^-$ production in e^+e^- collisions at the Υ resonances ($\Upsilon(1S)$, $\Upsilon(2S)$ and $\Upsilon(3S)$) with a sensitivity of $O(10^{-5})$ or even better (of course, the center-of-mass energy at super B factories is $\sqrt{s} \sim M_{\Upsilon(4S)} \approx 10\,\mathrm{GeV}$, so that the form factor $F_2(q^2)$ is not the anomalous magnetic moment). The contributions to the $e^+e^- \to \tau^+\tau^-$ cross section arise not only from the usual s-channel one-loop vertex corrections, but also from box diagrams, which should be somehow subtracted out. The strategy proposed in [47, 48] to eliminate their contamination is to measure the observables on top of the Υ resonances, where the non-resonant box diagrams should be numerically negligible. However, it is very difficult to resolve the narrow peaks of the $\Upsilon(1S, 2S, 3S)$ ($\Gamma_\Upsilon \sim 20$–50 keV) in the $\tau^+\tau^-$ decay channel (the $\Upsilon(4S)$ decays almost entirely in $B\bar{B}$) because of the natural irreducible beam energy spread associated to any e^+e^- synchrotron (5.45 MeV at SuperKEKB). In Ref. [20] it was shown that, at the Belle II experiment, the $\tau^+\tau^-$ events produced with beams at a center-of-mass energy $\sqrt{s} \sim M_\Upsilon$ will be mostly due to non-resonant contributions. The situation at Belle was similar (the energy spread at KEKB was 5.24 MeV). Therefore, the measurement of the $e^+e^- \to \tau^+\tau^-$ cross section on top of the Υ resonances will not eliminate the contamination of the non-resonant contributions.

The effective Lagrangian (4.2) generates additional non-standard contributions to the differential decay rate in Eq. (2.3). For a $\tau^\pm$ decay they can be summarised in the shifts [20]

$$G \to G + \tilde{a}_\tau G_a, \quad J \to J + \tilde{a}_\tau J_a, \quad K \to K + \tilde{a}_\tau K_a, \quad L \to L \mp (m_\tau/e)\, \tilde{d}_\tau L_d. \qquad (4.7)$$

Tiny terms of $O(\tilde{a}_\tau^2)$, $O(\tilde{d}_\tau^2)$ and $O(\tilde{a}_\tau \tilde{d}_\tau)$ were neglected. Deviations of the τ dipole moments from the SM values can be determined comparing the SM prediction for the differential rate in Eq. (2.3), modified by the terms G_a, J_a, K_a and L_d, with sufficiently precise data.

The possibility to set bounds on $\tilde{a}_\tau$ via radiative leptonic τ decays was suggested long ago in Ref. [49]. In that article the authors proposed to take advantage of a radiation zero of the differential decay rate in (2.3). This zero occurs when, in the τ rest frame, the final lepton l and the photon are back-to-back, and l has maximal energy. Since a non-standard contribution to a_τ spoils this radiation zero, precise measurements of this phase-space region could be used to set bounds on its value. However, a Monte Carlo simulation in the conditions of the Belle experiment shows no significant improvement of the existing limits for a_τ [20].

A more powerful method to extract $\tilde{a}_\tau$ and $\tilde{d}_\tau$ consists in the use of an unbinned maximum likelihood fit of events in the full phase space [20]. In this approach, we considered $e^+ e^- \to \tau^+ \tau^-$ events where both τ leptons decay subsequently into a particular final state: $\tau^\mp$ (signal side) decays to the radiative leptonic mode, and the other $\tau^\pm$ (tag side) decays to some well known mode with a large branching fraction. As a tag decay mode we chose $\tau^\pm \to \rho^\pm \nu \to \pi^\pm \pi^0 \nu$, which also serves as spin analyser and allows us to be sensitive to the spin-dependent part of the differential decay width of the signal decay using effects of spin-spin correlation of the τ leptons [50]. With this technique we analyzed a data sample of $(\ell^\mp \nu \nu \gamma,\ \pi^\pm \pi^0 \nu)$ events corresponding to the total amount of data available at Belle and the one planned at the Belle II experiment.

The feasibility study of Ref. [20] shows that the experimental sensitivity on $\tilde{a}_\tau$ that can be reached at the Belle II experiment can improve the DELPHI bound (4.4). On the other hand, the expected sensitivity on the τ EDM is still worse than the most precise measurement of $\tilde{d}_\tau$ performed at Belle in τ pair production [39].

5. Conclusions

We discussed the SM predictions of the differential rates and branching ratios of the decays $\mu \to e \gamma \nu \bar{\nu}$ and $\tau \to l \gamma \nu \bar{\nu}$ ($l = \mu, e$) at NLO recently derived in Ref. [12]. Our predictions agree with the measurements of the branching ratio $\mathscr{B}(\mu \to e \bar{\nu} \nu \gamma)$ obtained by MEG and in Ref. [16]. Also the recent precise measurement by BABAR of the branching ratio $\mathscr{B}(\tau \to \mu \bar{\nu} \nu \gamma)$, for $\omega_0 = 10$ MeV, agrees with our prediction within 1.1 standard deviations (1.1σ). On the contrary, BABAR's recent measurement of the branching ratio $\mathscr{B}(\tau \to e \bar{\nu} \nu \gamma)$, for the same threshold ω_0, differs from our prediction by 3.5σ. This puzzling discrepancy deserves further researches.

We proposed to determine the τ dipole moments via precise measurements of radiative leptonic τ decays at high-luminosity B factories. Deviations of the τ $g-2$ and EDM from the SM predictions can be determined via an effective Lagrangian approach. Our dedicated feasibility study in Ref. [20] showed that the measurement of the τ anomalous magnetic moment at the upcoming Belle II experiment can improve the current bound of the DELPHI experiment, while the foreseen sensitivity is not expected to lower the current limit on the τ EDM.

Acknowledgments

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