

Study of the continuum limit of the Schwinger model using Wilson's lattice formulation

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We estimate the critical point of the Schwinger model in the θ vacuum by taking the continuum limit of Wilson's lattice formulation. The decorated tensor renormalization group which has been recently proposed by Dittrich *et al.* is employed for numerical investigation. We compare our numerical results with those of Byrnes *et al.*, which were derived from Kogut-Susskind's lattice formulation employing the density matrix renormalization group.

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1. Lattice Schwinger model in the Wilson formulation

We here employ the path-integral formulation of the lattice Schwinger model with one flavor of the Wilson fermion:

$$Z = \int \mathcal{D}U \det D[U] e^{-S_g[U]}. \quad (1.1)$$

The Wilson-Dirac matrix is given by

$$\begin{aligned} \bar{\psi} D[U] \psi &= \frac{1}{2\kappa} \sum_{n,\alpha} \bar{\psi}_{n,\alpha} \psi_{n,\alpha} \\ &\quad - \frac{1}{2} \sum_{n,\mu,\alpha,\beta} \bar{\psi}_{n,\alpha} \{ (1 - \gamma_\mu)_{\alpha,\beta} U_{n,\mu} \psi_{n+\hat{\mu},\beta} \\ &\quad + (1 + \gamma_\mu)_{\alpha,\beta} U_{n-\hat{\mu},\mu}^\dagger \psi_{n-\hat{\mu},\beta} \}, \end{aligned} \quad (1.2)$$

and the plaquette gauge action including the θ term is

$$S_g = -\beta \sum_p \cos \varphi_p - i\theta Q, \quad (1.3)$$

$$Q = \frac{1}{2\pi} \sum_p q_p, \quad (1.4)$$

$$q_p = \varphi_p \pmod{2\pi}, \quad (1.5)$$

where φ_p is a phase of an U(1) plaquette variable and θ is fixed to π throughout this paper. For the details of our notation, please refer to Ref. [1, 2]

2. Decorated tensor renormalization group

The decorated tensor renormalization group (DTRG) was recently proposed by Dittrich *et al.* [3] as a variation of the tensor renormalization group method which is suitable for lattice models having local gauge invariance. In order to get a decorated tensor network form of the partition function, the gauge part of the Boltzmann weight is decomposed by the character expansion [4, 5]:

$$\begin{aligned} &\exp \left\{ \beta \cos \varphi_p + i \frac{\theta}{2\pi} q_p \right\} \\ &= \sum_{m_p=-\infty}^{\infty} e^{im_p \varphi_p} \sum_{k=-\infty}^{\infty} I_k(\beta) \frac{2 \sin \left(\frac{\theta + 2\pi(m_p - k)}{2} \right)}{\theta + 2\pi(m_p - k)} \\ &\simeq \sum_{m_p=-N_{ce}}^{N_{ce}-1} e^{im_p \varphi_p} \sum_{k=-\infty}^{\infty} I_k(\beta) \frac{2 \sin \left(\frac{\theta + 2\pi(m_p - k)}{2} \right)}{\theta + 2\pi(m_p - k)}, \end{aligned} \quad (2.1)$$

where I_k is the modified Bessel function and we truncate the series up to N_{ce} -th terms. After the fermionic part is also decomposed by using the properties of Grassmann numbers, the partition function can be rewritten as

$$Z = \sum_{i_p, j_p, k_p, \dots} \int \sum_{i, j, k, \dots} T_{n; i, j, k, l}^{(i_p, j_p, k_p, l_p)} T_{n+1; m, o, i, p}^{(i_p, j_p, k_p, l_p)} T_{n+2; q, r, s, j}^{(i_p, j_p, k_p, l_p)} \dots, \quad (2.2)$$

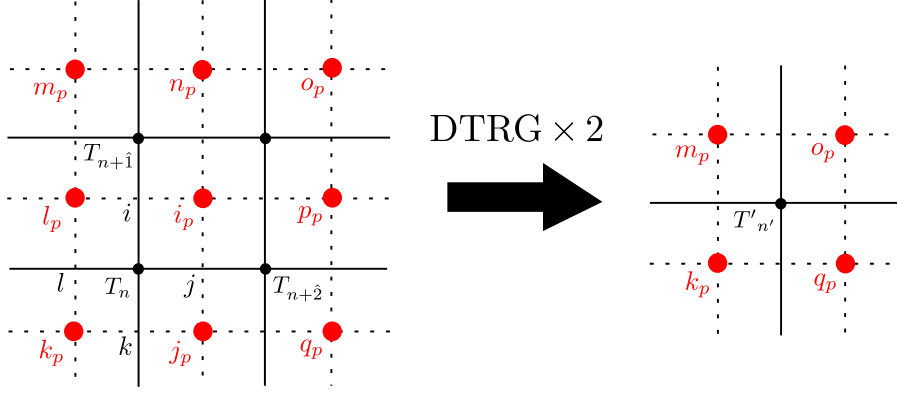


Figure 1: Decorated tensor network and its coarse-graining. Each tensor which sits on a lattice site is decorated by labels of the character expansion which surround the site. After iterations of the DTRG procedure, a part of labels (i_p, j_p, l_p, n_p and p_p in the figure) are integrated out, but the others (k_p, m_p, o_p and q_p in the figure) remain.

where $T^{(i_p, j_p, k_p, l_p)}$ is a Grassmann-valued tensor decorated by labels of the character expansion $\{i_p, j_p, k_p, l_p\}$, whose detail is given in Ref. [6]. Using the algorithm for two-dimensional systems described in Ref. [3], the decorated tensor network is coarse-grained and we get an approximate representation of the partition function. Then, each coarse-grained tensor has new indices which emerge from the singular value decomposition in the DTRG procedure, while keeping the decoration labels the same as their originals (See Figure 1.). Thanks to this property, the Gauss constraints are preserved under coarse-graining and that allows us to reduce computational cost remarkably. The details in case including fermions are discussed in Ref. [6].

3. Numerical results

To estimate the critical point at $\theta = \pi$, we calculate the following quantity:

$$X \equiv \frac{\sum_{i_p} \sum_{i,j} (-1)^{i_f j_f} T_{n;i,j,i,j}^{(i_p, i_p, i_p, i_p)}}{\sum_{i_p, j_p} \sum_{i,j,k,l} (-1)^{i_f + i_f j_f + k_f l_f} T_{n;i,j,k,j}^{(i_p, j_p, j_p, i_p)} T_{n+1;k,l,i,l}^{(j_p, i_p, i_p, j_p)}}, \quad (3.1)$$

where $i_f = 0$ ($i_f = 1$) if the index i corresponds to a bosonic (fermionic) state. This quantity was originally proposed by Gu and Wen [7] for bosonic cases. In the infinite volume (infrared) limit, it represents the degeneracy of the ground state. Figure 2 shows DTRG flows of X for $\beta = 10.0$. While the flow for $\kappa = 0.24135$ reaches $X = 1$, namely, the symmetric phase, the one for $\kappa = 0.24130$ reaches $X = 2$, the Z(2)-broken phase. Convergence values of X for various κ are given in Fig. 3. It clearly shows a phase transition in $0.24130 < \kappa < 0.24135$. It should be noted that there is a truncation error of the singular value decomposition in the DTRG procedure and we have to take care of its effect on our evaluation of κ_c . In this paper, we refer to the truncation number in each sector divided by decoration labels and Grassmann parity as D . Figure 4 shows the D dependence of κ_c evaluated from X . As you can see, κ_c is converged well where $D \geq 15$ in this case. We also have a truncation error of the character expansion in Eq.(2.1). In Figs. 5, DTRG flows of X for various N_{ce} are presented. Fortunately, the higher-order terms seem not to contribute to the infinite

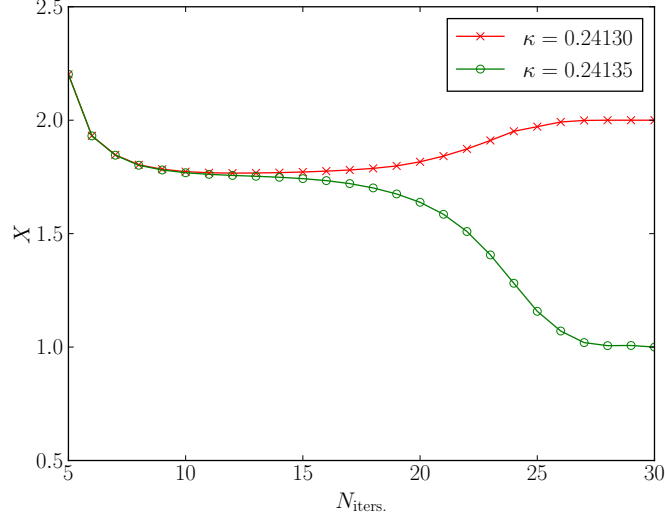


Figure 2: DTRG flows of X for $\beta = 10.0$, $D = 16$ and $N_{\text{ce}} = 20$. The red symbols which converge to $X = 2$ are the results for $\kappa = 0.24130$ and the green ones which converge to $X = 1$ are for $\kappa = 0.24135$.

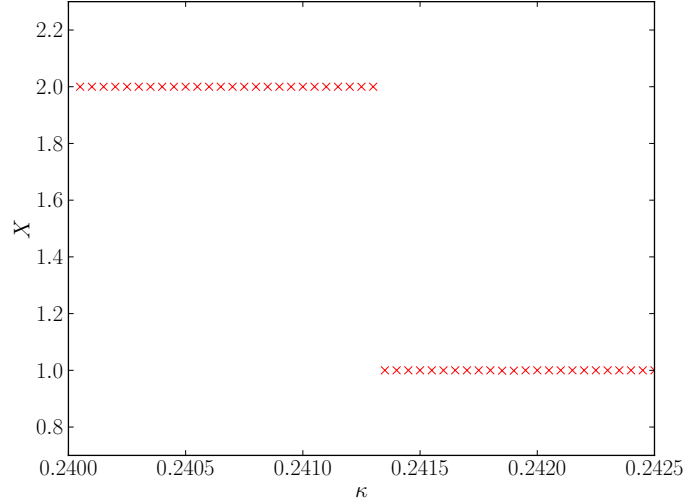


Figure 3: Convergence values of X after 30 iterations of the DTRG procedure as a function of κ for $\beta = 10.0$, $D = 16$ and $N_{\text{ce}} = 20$. A phase transition is observed in $0.24130 < \kappa < 0.24135$.

volume limit. It means that the character expansion is quite powerful to investigate infrared physics. To estimate the continuum limit of the critical point, we perform the same analysis as above up to $\beta = 700.0$. The continuum limit of the critical value of $\frac{m}{g}$ with $m = \frac{1}{2\kappa} - 2$ and $g = \frac{1}{\sqrt{\beta}}$ has been already evaluated by Byrnes *et al.* [8] using the Hamiltonian lattice formulation and the density matrix renormalization group technique. Although the lattice formulation we employ is different from that of Byrnes *et al.*, our estimate should converge to the same value in the continuum limit because of the universality of the continuum limit. Figure 6 presents our results of $\frac{m}{g}$ from $\beta = 10.0$

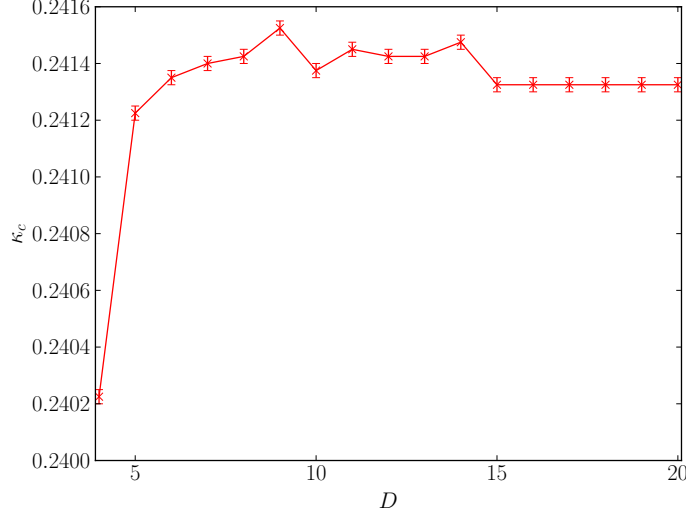


Figure 4: Convergence of κ_c as a function of the truncation number D for $\beta = 10.0$ and $N_{cc} = 20$.

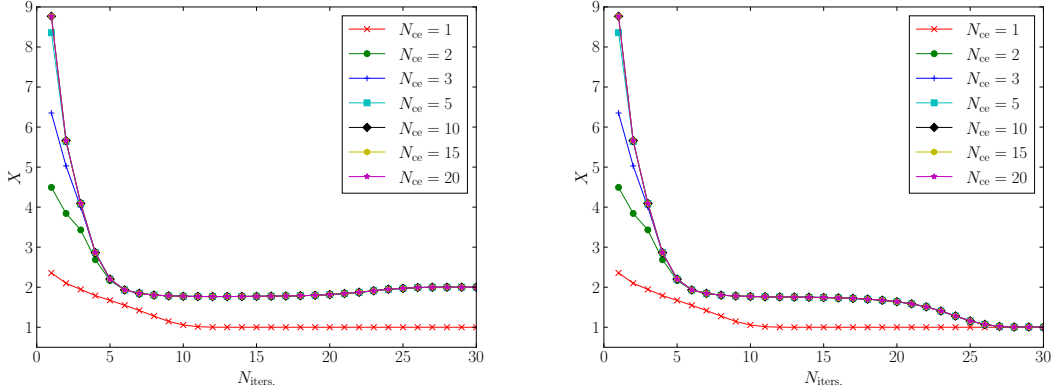


Figure 5: DTRG flows of X for various values of N_{cc} at $\beta = 10.0$ with $D = 16$. The left (right) panel shows the results for $\kappa = 0.24130$ ($\kappa = 0.24135$). All the flows for $N_{cc} \geq 2$ converge to the same value.

up to $\beta = 700.0$. A quadratic fit in g with a data set $\beta = 50.0, 100.0, 200.0, 300.0, 500.0, 700.0$ is performed and our estimate of the continuum extrapolation is thus

$$\left(\frac{m}{g}\right)_c = 0.330(9), \quad (3.2)$$

which agrees with the estimate by Byrnes *et al.* of $\left(\frac{m}{g}\right)_c = 0.3335(2)$. The reason why ours is less accurate is thought to be that we employ the unimproved Wilson fermion which has $O(a)$ discretization error while Byrnes *et al.* employ the Kogut-Susskind fermion whose discretization error is $O(a^2)$.

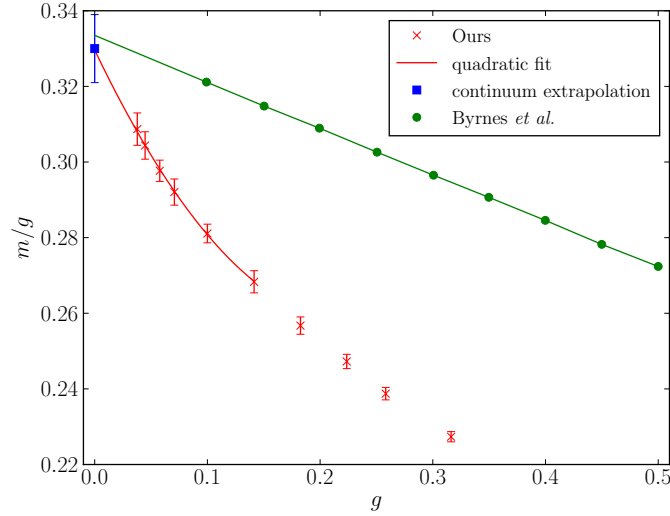


Figure 6: Critical value of $\frac{m}{g}$ as a function of g . The estimates by Byrnes *et al.* reported in Ref. [8] are also plotted. The red line indicates a quadratic fit in g with a data set $\beta = 50.0, 100.0, 200.0, 300.0, 500.0, 700.0$ and the blue symbol is the continuum extrapolation derived from the fit.

Acknowledgments

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