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Lifshitz-type SU(N) lattice gauge theory in five dimensions

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We present a lattice formulation of non-Abelian Lifshitz-type gauge theories. Due to anisotropic scaling of space and time, the theory is asymptotically free even in five dimensions. We show results of Monte Carlo simulations that suggest a smooth approach to the continuum limit.

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1. Introduction

Gauge theories exhibit a variety of phases depending on the dimensionality of spacetime. The critical dimensionality of Lorentz-invariant gauge theories is $3+1$, at which the coupling is classically marginal and quantum effects drive non-Abelian gauge theories with not-too-many fermion flavors to a strong-coupling confining phase with chiral symmetry breaking, whereas Abelian gauge theories hit a Landau pole at high energy and lacks a sensible definition at least within a perturbation theory. Gauge theories in lower dimensions are asymptotically free for any number of flavors and are relevant to strongly correlated lower-dimensional electronic materials. By contrast, at least within naive power counting, gauge theories living in higher than 4-dimensions lose renormalizability and seem to be ill-defined in the ultraviolet. However, some theoretical studies on physics beyond the Standard Model prompt considering gauge theories in higher dimensions, e.g. to solve the hierarchy problem in particle physics [1]. Given the lack of a continuum limit in the perturbative regime, one has to either conclude that a Yang-Mills theory in higher dimensions serves only as an infrared effective description of a more fundamental theory such as string theory, or, try to find their UV completion within quantum field theory. The latter calls for an intrinsically nonperturbative approach.

A powerful way to address nonperturbative problems in gauge theories is a lattice gauge theory pioneered by Wilson [2], in which one puts a theory on a discrete lattice keeping exact gauge symmetry. This proved to be a quite powerful numerical tool to elucidate the strong-coupling physics of Yang-Mills theory and QCD. Of course our ultimate interest is in physics in the continuum, so any lattice simulations of QCD have to be accompanied by a procedure of extrapolation to the continuum limit, which is quite well understood in QCD; it is the large-β limit that corresponds to the domain of infinitely long correlation length in lattice units. By contrast, the existence of a continuum limit is far from trivial in unconventional theories. In the case of five-dimensional Yang-Mills theories on a lattice, the phase diagram consists of a weakly coupled Coulomb phase and a strongly coupled confining phase, which are separated by a first-order phase transition [3]; so far the existence of a continuum limit is not confirmed yet. Although the so-called *layer phase* proposed by Fu and Nielsen [4] has been actively investigated with lattice simulations, the issue of continuum limit is still under a debate [5, 6, 7].

There is however a class of theories that enjoy Lifshitz-type scale invariance instead of Lorentz invariance. The original motivation for them came from anisotropic critical points in condensed matter systems, where the critical scaling exponents are spatially anisotropic [8]. In elementary particle physics, the idea of exploiting anisotropic scaling of space and time as a way of ameliorating UV divergences was explored by Hořava for quantum gravity [9] and by many authors for scalar and gauge theories, as reviewed in [10]. Although this deformed class of gauge theories seems to provide an intriguing UV completion, most of the analyses so far has been done in the continuum within a perturbative framework.

In this work, we report on a first nonperturbative lattice regularization of higher-dimensional non-Abelian Lifshitz-type gauge theories [11]. Besides a theoretical proposal of the formulation, we also performed a lattice Monte Carlo simulation based on our lattice action with the gauge group SU(3) and found a smooth crossover from strong to weak coupling. This opens up a novel arena to test ideas related to the beyond-Standard-Model physics and to seek for new dynamics of

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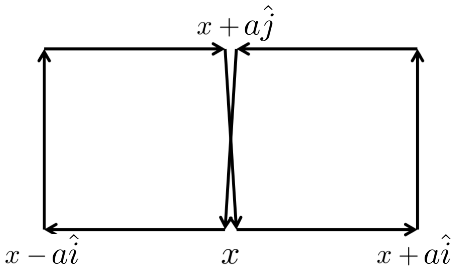

Figure 1: Twisted Wilson loop $T_{ij}(x)$ that starts and ends at a point x on the lattice. The variables $\hat{i}$ and $\hat{j}$ are unit vectors in x^i and x^j directions, respectively. Note that $T_{ij} \neq T_{ji}$. Figure taken from [11].

gauge fields unseen in the conventional setup in 4 dimensions.

2. Lattice formulation

We consider a Lifshitz-type gauge theory in $(D+1)$-dimensional Euclidean spacetime studied by Hořava [12]. The action is given by

$$S = \frac{1}{2}\int \mathrm{d}x_0 \mathrm{d}^D x \left[\frac{1}{e^2}\,\mathrm{tr}\,F_{0i}^2 + \frac{1}{g^2}\,\mathrm{tr}(D_i F_{ik})(D_j F_{jk})\right], \tag{2.1}$$

where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$ and $D_i F_{jk} \equiv \partial_i F_{jk} + i[A_i, F_{jk}]$ $(i,j,k \in \{1,2,\dots,D\})$ with $A_\mu = A_\mu^a T^a$ the gauge field in the Lie algebra of SU(N). While the temporal part of the action is second order in derivatives, the spatial part is fourth order, hence this action has the dynamical critical exponent $z = 2$. The absence of the usual spatial kinetic term $\mathrm{tr}\,F_{ij}^2$ at first sight appears to be vulnerable to quantum corrections, but it has been shown in the stochastic quantization of Yang-Mills theory [13, 14] that $\mathrm{tr}\,F_{ij}^2$ is not engendered under renormalization. By the same token, other terms of the same scaling dimension (e.g., $\mathrm{tr}(F^3)$) are not generated either. This simplifying property makes (2.1) an attractive testbed for general Lifshitz-type gauge theories.

The two couplings e and g are independent at the classical level. Their β functions at one loop indicate that this theory is asymptotically free for $D \le 4$ [12]. For $D = 4$, a dynamical scale is expected to emerge in IR due to dimensional transmutation, much like in QCD. This attractive conjecture due to Hořava cannot be tested in a perturbative framework, however.

Now we proceed to a lattice regularization of (2.1). It is crucial to preserve gauge invariance at finite lattice spacing. We propose

$$S_{\mathrm{lat}} = \frac{\beta_e}{2N}\sum_x \sum_{i=1}^{D} \mathrm{Re}\,\mathrm{tr}\,[\mathbb{1} - P_{0i}(x)] + \frac{\beta_g}{2N}\sum_x \sum_{j=1}^{D} \mathrm{Re}\,\mathrm{tr}\left[\mathbb{1} - \prod_{\substack{i=1\\ i\neq j}}^{D} T_{ij}(x)\right]. \tag{2.2}$$

The temporal part of S_{lat} uses the standard Wilson's 1×1 plaquette P_{0i} lying in the (x^0, x^i)-plane, whereas T_{ij} in the spatial part is a twisted 2×1 Wilson loop in the (x^i, x^j)-plane, as shown in Fig. 1. The order of multiplication in $\prod T_{ij}$ is arbitrary because it only affects irrelevant terms in the continuum limit. The lattice couplings $(e_{\mathrm{lat}}, g_{\mathrm{lat}})$ are defined through

$$\beta_e = \frac{2N}{e_{\mathrm{lat}}^2} \quad \text{and} \quad \beta_g = \frac{2N}{g_{\mathrm{lat}}^2}. \tag{2.3}$$

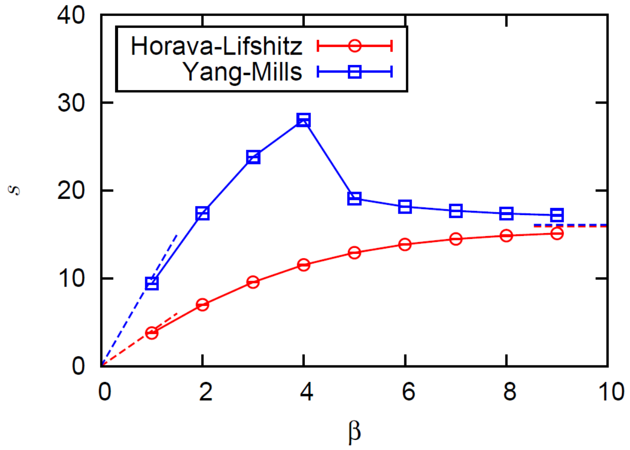

Figure 2: Action density $s \equiv \langle S_{\text{lat}} \rangle / N_{\text{lat}}$ with isotropic couplings $\beta_e = \beta_g \equiv \beta$ on a 6^5 lattice (red points). For comparison, simulation results for isotropic Yang-Mills theory on the same lattice is plotted (blue points). Dashed lines represent the leading order of weak and strong coupling expansions, respectively. Figure taken from [11].

It can be straightforwardly checked that S_{lat} recovers S in (2.1) at least in the *classical* continuum limit. On the lattice, we need to control the dimensionless couplings $\beta_{e,g}$ to find a second-order phase transition where the correlation length measured in lattice units tends to infinity. As shown in [12], the physical couplings (e, g) flow to zero in UV (asymptotic freedom), which motivates us to expect that the continuum limit of the lattice theory (2.2) may be defined by the weak-coupling limit $e_{\text{lat}}, g_{\text{lat}} \to 0$. We check this anticipation in full numerical simulations in the next section.

3. Monte Carlo simulation

We simulated the lattice theory (2.2) with standard Monte Carlo techniques. We studied the $N = 3$ theory on lattices of size $N_{\text{lat}} = 6^5$ and 10^5. In Fig. 2 the action density for isotropic couplings is plotted. The transition from strong to weak coupling appears to be a smooth crossover. At $\beta \ll 1$ and $\gg 1$ one can estimate the behavior of the action density analytically and our numerical results are in full accord with those limits. For the purpose of comparison we have also simulated the conventional Yang-Mills theory with Wilson plaquette action on the same lattice, as plotted in blue in Fig. 2. There is a sharp drop at $\beta \sim 4.5$, which is a known bulk transition from a confining phase at small β to a deconfined Coulomb phase at large β [15]. This is consistent with perturbative non-renormalizability of the Yang-Mills theory in five dimensions. (A similar result was obtained for the gauge group SU(2) long time ago [3]; for a more recent study, see [16].) Confirmation of this dramatic disparity between the two theories is the main result of this work.

We also performed simulations with anisotropic couplings ($\beta_e \neq \beta_g$). The action density, plotted in Fig. 3, varies smoothly with the couplings and shows no indication of a phase transition.

Next we measured temporal Wilson loops W_{0i} of edge lengths t and x to extract the color-singlet potential between infinitely heavy quarks

$$V(x) \equiv -\lim_{t\to\infty} \frac{1}{t} \log \langle \operatorname{tr} W_{0i}(t,x) \rangle . \tag{3.1}$$

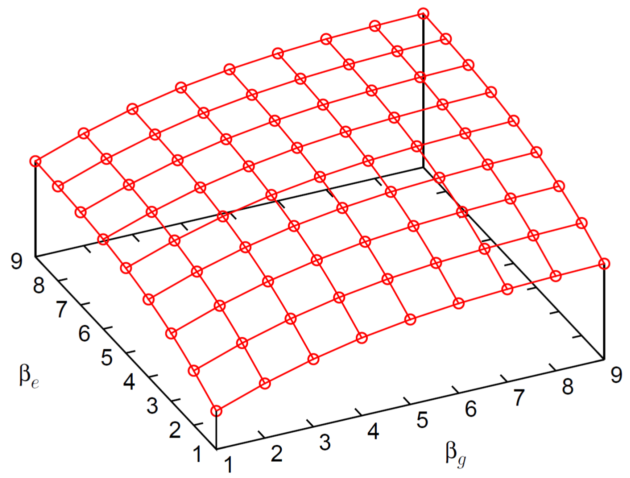

Figure 3: Action density on a 6^5 lattice for various β_e and β_g. Figure taken from [11].

In actual simulations, the extrapolation to $t = \infty$ is replaced with a large but finite t. The numerical result for the potential is presented in Fig. 4. The data points indicate a linear confining potential between heavy quarks in this theory. Due to limited volume, we could not extract the potential at very short and long distances. In perturbation theory, one expects a one-gluon-exchange potential of the form $V(x) \sim \int \mathrm{d}^D p \exp(-ipx)/p^2 \sim 1/x^2$ at small x. Confirmation of this behavior is postponed to future work.

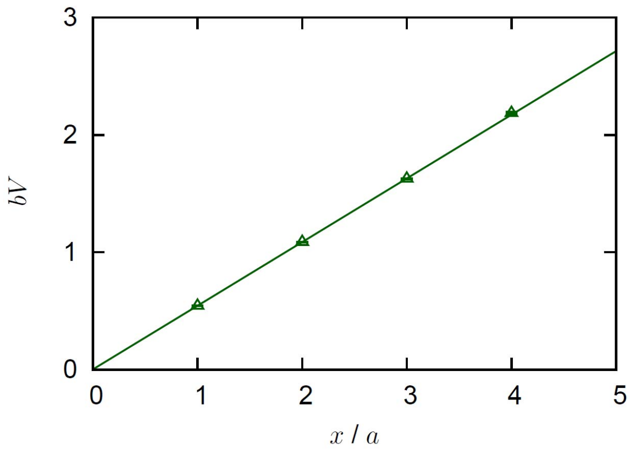

Figure 4: Color-singlet potential $V(x)$ measured on a 10^5 lattice with $\beta_e = \beta_g = 9$. The axes are in lattice units. Figure taken from [11].

We have also measured spatial Wilson loops W_{ij} and spatial plaquettes P_{ij}. Their statistical averages were found to be zero within errors. This is natural, considering that the spatial kinetic term for gluons $\mathrm{tr} F_{ij}^2$ should not be generated by quantum corrections.

4. Conclusions

In this work we reported a nonperturbative lattice regularization of Lifshitz-type gauge theories with anisotropic scale invariance. The results of first numerical simulation of our lattice theory in 5 dimensions reported here suggests that the continuum limit can be defined in the weak-coupling limit, in a way quite analogous to the standard lattice Yang-Mills theory in 4 dimensions. We measured the heavy-quark potential and found that quark confinement takes place in our lattice theory even at weak coupling. All these findings are theoretically interesting as a new pathway to extend the realm of cutoff-free gauge theories into higher dimensions. While our first simulation was limited to a relatively small volume and a restricted coupling parameter region, there seems to be no fundamental difficulty in performing more thorough simulations based on existing techniques in lattice QCD.

There are numerous questions left unanswered in this pilot study. Can we extend the lattice formulation to the case of dynamical exponents $z > 2$? What is the mechanism of quark confinement in 5 dimensions? Can we introduce quarks into the theory without spoiling good renormalization properties? If possible, then could there be a spontaneous flavor symmetry breaking of quarks? Is there any applications to the theories of quantum criticality with anisotropic scale invariance in condensed matter systems [17, 18]? What can be learned about the Standard Model from this theory? These issues should be investigated elsewhere.

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References

[1] L. Randall and R. Sundrum, *A Large mass hierarchy from a small extra dimension*, *Phys. Rev. Lett.* **83** (1999) 3370–3373, [hep-ph/9905221].

[2] K. G. Wilson, *Confinement of Quarks*, *Phys.Rev.* **D10** (1974) 2445–2459.

[3] M. Creutz, *Confinement and the Critical Dimensionality of Space-Time*, *Phys.Rev.Lett.* **43** (1979) 553–556.

[4] Y. K. Fu and H. B. Nielsen, *A Layer Phase in a Nonisotropic U(1) Lattice Gauge Theory: Dimensional Reduction a New Way*, *Nucl. Phys.* **B236** (1984) 167.

[5] K. Farakos and S. Vrentzos, *Exploration of the phase diagram of 5d anisotropic SU(2) gauge theory*, *Nucl.Phys.* **B862** (2012) 633–649, [arXiv:1007.4442].

[6] F. Knechtli, M. Luz, and A. Rago, *On the phase structure of five-dimensional SU(2) gauge theories with anisotropic couplings*, *Nucl. Phys.* **B856** (2012) 74–94, [arXiv:1110.4210].

[7] L. Del Debbio, R. D. Kenway, E. Lambrou, and E. Rinaldi, *The transition to a layered phase in the anisotropic five-dimensional SU(2) Yang-Mills theory*, *Phys.Lett.* **B724** (2013), no. 1-3 133–137, [arXiv:1305.0752].

[8] P. M. Chaikin and T. C. Lubensky, *Principles of condensed matter physics*. Cambridge Univ Press, Cambridge, 2000.

[9] P. Horava, *Quantum Gravity at a Lifshitz Point*, *Phys.Rev.* **D79** (2009) 084008, [arXiv:0901.3775].

[10] J. Alexandre, *Lifshitz-type Quantum Field Theories in Particle Physics*, *Int.J.Mod.Phys.* **A26** (2011) 4523–4541, [arXiv:1109.5629].

[11] T. Kanazawa and A. Yamamoto, *Asymptotically free lattice gauge theory in five dimensions*, *Phys. Rev.* **D91** (2015), no. 7 074508, [arXiv:1411.4667].

[12] P. Horava, *Quantum Criticality and Yang-Mills Gauge Theory*, *Phys.Lett.* **B694** (2010) 172–176, [arXiv:0811.2217].

[13] J. Zinn-Justin, *Renormalization and Stochastic Quantization*, *Nucl.Phys.* **B275** (1986) 135.

[14] K. Okano, *Background Field Method in Stochastic Quantization*, *Nucl.Phys.* **B289** (1987) 109.

[15] E. Itou, K. Kashiwa, and N. Nakamoto, *Phase structure of pure SU(3) lattice gauge theory in 5 dimensions*, arXiv:1403.6277.

[16] P. de Forcrand, A. Kurkela, and M. Panero, *The phase diagram of Yang-Mills theory with a compact extra dimension*, *JHEP* **1006** (2010) 050, [arXiv:1003.4643].

[17] E. Ardonne, P. Fendley, and E. Fradkin, *Topological order and conformal quantum critical points*, *Annals Phys.* **310** (2004) 493–551, [cond-mat/0311466].

[18] M. Freedman, C. Nayak, and K. Shtengel, *A line of critical points in 2+1 dimensions: quantum critical loop gases and non-Abelian gauge theory*, *Phys.Rev.Lett.* **94** (2005) 147205, [cond-mat/0408257].