

# Isospin splitting in Wilson chiral perturbation theory for lattice QCD with three non-degenerate quark flavours

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We have calculated the expansions of the masses of the pseudoscalar mesons in chiral perturbation theory at next-to-leading order for lattice QCD with three light quark flavours, taking the mass difference between the up and down quarks into account.

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In Quantum Chromodynamics the masses of quarks are free parameters. The knowledge of their numerical values is of great importance for phenomenological purposes [1]. Particularly delicate is the determination of the small masses of the up and down quarks. Their mass difference contributes significantly to the isospin splitting of hadron masses.

Lattice QCD offers the possibility to study quark masses by systematically investigating the dependence of hadronic observables on the quark masses, see e. g. [2]. An indispensable tool for such kind of studies is chiral perturbation theory, which describes analytically how hadron masses and other quantities depend on the quark masses. The systematic errors entailed by lattice discretisation can be taken into account in chiral perturbation theory, leading to the lattice spacing  $a$  being an additional expansion parameter. In the case of Wilson fermions this is called Wilson chiral perturbation theory. For reviews see [3, 4].

The isospin splitting produced by the small mass difference between up and down quarks has been investigated in lattice QCD in recent years [5, 6, 7]. We have calculated the isospin splittings of pseudoscalar meson masses in the framework of Wilson chiral perturbation theory with  $N_f = 1 + 1 + 1$  quark flavours. The purpose of these calculations is to reveal the effects of the lattice discretisation on the isospin splitting for pion and kaon meson masses. In addition to that, even the further limit of vanishing lattice spacing  $a$  is of interest, as it includes the effects of mixing in next-to-leading order, which are not completely included in the original work [8].

We have actually performed the calculations for twisted mass lattice QCD, which implements a chirally rotated mass term [9, 10]. Setting the twist angle to a value of  $\pi/2$  (“full twist”) implies automatic  $\mathcal{O}(a)$  Symanzik-improvement [11, 12]. In practice, simulations of twisted mass lattice QCD including strange quarks, are done differently, namely by adding a fourth heavy charm quark and implementing full twist in both the up-down and the strange-charm sector in order to ensure automatic  $\mathcal{O}(a)$  improvement in all observables [13, 14, 15]. The introduction of the fourth quark flavour is necessary for this purpose. Our calculations for three flavours explicitly show how the pion masses are influenced by the presence of strange quarks that are not accompanied by charm quarks, and in this way corroborate the inclusion of a fourth quark flavour in the simulations.

The calculations follow the techniques developed in [16, 17, 18, 19, 20, 21]. The masses of the members of the pseudoscalar meson octet have been obtained to next-to-leading order (NLO). The dependence on the lattice spacing  $a$  shows up in additional terms proportional to powers of  $a$ . The small isospin mass splitting  $\Delta m = m_d - m_u$  is included to order  $\Delta m^2$ , and lattice effects are included to order  $a^2$ . For details of the set-up and the method see [22].

In NLO the inverse propagator of pseudoscalar mesons is of the form

$$iG^{-1}(p^2) = p^2 \mathbf{1} + \mathfrak{M}_{\text{LO}} - A + p^2 B, \quad (1)$$

where  $\mathfrak{M}_{\text{LO}}$  is the LO mass squared matrix. The NLO matrices  $A$  and  $B$  are rather complicated and contain various off-diagonal terms that generate mixings in the  $\pi^0$ - $\eta$  sector and mixings among the kaon fields. Implementing field renormalisation via

$$iG^{-1}(p^2) = \left( \mathbf{1} + \frac{B}{2} \right) \left[ p^2 \mathbf{1} + \mathfrak{M}_{\text{LO}} - A - \frac{1}{2} \{B, \mathfrak{M}_{\text{LO}}\} \right] \left( \mathbf{1} + \frac{B}{2} \right) + \mathcal{O}(p^6), \quad (2)$$

the squared meson masses are obtained by diagonalisation of the matrix  $\mathfrak{M}_{\text{LO}} - A - \frac{1}{2} \{B, \mathfrak{M}_{\text{LO}}\}$ . With our conventions for the generators of the quark mass splitting the neutral pion is  $\pi_1 \equiv \pi^0$ , and

the charged pions  $\pi^\pm$  are linear combinations of the mass eigenstates  $\pi_2$  and  $\pi_3$ . For non-vanishing twist angle  $\omega$  the charged pions have different masses, which is a pure lattice artifact that vanishes in the continuum limit  $a \rightarrow 0$ .

The result for the mass splitting between the neutral pion and the charged pion  $\pi_2$ , presented to order  $a$  here, is given by

$$\begin{aligned}
M_{\pi_1}^2 - M_{\pi_2}^2 = & B_0 \Delta m^2 \left\{ \frac{1}{4(m_s - \hat{m})} \left[ 1 - \frac{aW_0(1 - \cos \omega)}{B_0(m_s - \hat{m})} \right] \right. \\
& + \frac{4B_0}{3F_0^2} (-3L_4^r + 2L_5^r + 6L_6^r - 78L_7^r - 18L_8^r) + \frac{4B_0}{F_0^2} \left( \frac{\hat{m}}{m_s - \hat{m}} \right) (-3L_4^r - 2L_5^r + 6L_6^r) \\
& + \frac{4aW_0}{F_0^2(m_s - \hat{m})} [(-L_5^r - 3W_4^r - W_5^r + 3W_6^r) \\
& \quad + (L_4^r + L_5^r + 20L_7^r + 4L_8^r + 2W_4^r + W_5^r - 3W_6^r - 10W_7^r - 2W_8^r)(1 - \cos \omega) \\
& \quad + \left( \frac{\hat{m}}{m_s - \hat{m}} \right) (3L_4^r + L_5^r + 4L_8^r + W_5^r - 3W_6^r - 2W_8^r)(1 - \cos \omega)] \\
& - \frac{B_0}{64\pi^2 F_0^2} \left[ \ln \left( \frac{\bar{m}_K^2}{\Lambda^2} \right) + 1 \right] \\
& - \frac{1}{192\pi^2 F_0^2(m_s - \hat{m})} \left[ 1 - \frac{aW_0(1 - \cos \omega)}{B_0(m_s - \hat{m})} \right] \\
& \quad \left. \left[ \bar{m}_\pi^2 \ln \left( \frac{\bar{m}_\pi^2}{\Lambda^2} \right) - 3\bar{m}_\eta^2 \ln \left( \frac{\bar{m}_\eta^2}{\Lambda^2} \right) - \frac{1}{2}(3\bar{m}_\eta^2 - \bar{m}_\pi^2) \ln \left( \frac{\bar{m}_K^2}{\Lambda^2} \right) + \frac{3}{2}(\bar{m}_\pi^2 + \bar{m}_\eta^2) \right] \right\} \\
& + \mathcal{O}(\Delta m^3, a^2). \tag{3}
\end{aligned}$$

We use the abbreviations  $\Delta m \doteq m_d - m_u$ ,  $\hat{m} \doteq (m_u + m_d)/2$ , and  $F_0$ ,  $B_0$ ,  $W_0$ ,  $L_i^r$  and  $W_i^r$  are the low-energy constants familiar in Wilson chiral perturbation theory. The quantities

$$\bar{m}_\pi^2 = 2B_0\hat{m} + 2aW_0 \cos(\omega) + \frac{a^2 W_0^2 \sin^2(\omega)}{B_0\hat{m}} + \mathcal{O}(a^3) \tag{4}$$

$$\bar{m}_K^2 = B_0(\hat{m} + m_s) + aW_0(1 + \cos(\omega)) + \frac{1}{2} \frac{a^2 W_0^2 \sin^2(\omega)}{B_0\hat{m}} + \mathcal{O}(a^3) \tag{5}$$

$$\bar{m}_\eta^2 = \frac{1}{3} \left[ 2B_0(\hat{m} + 2m_s) + 2aW_0(2 + \cos(\omega)) + \frac{a^2 W_0^2 \sin^2(\omega)}{B_0\hat{m}} \right] + \mathcal{O}(a^3) \tag{6}$$

are the LO masses squared.

The pion mass splitting is induced by the  $\pi^0$ - $\eta$  mixing and is of order  $\Delta m^2$ . The presence of order  $a$  terms even at full twist,  $\cos \omega = 0$ , illustrates the necessity of a fourth quark flavour for automatic  $\mathcal{O}(a)$  improvement.

Let us remark that in the pion masses themselves (which are not displayed in this conference contribution), terms proportional to the strange quark mass  $m_s$  show up. They are independent of  $\omega$  and persist in the continuum limit. As the pions are made from up and down quarks, these terms are counterintuitive. However, the prefactors  $L_4^r$  and  $L_6^r$  have very small empirical values [23].

In case of the kaons the mass splitting between neutral and charged kaons in NLO to order  $a$

turns out to be

$$\begin{aligned}
M_{K^0}^2 - M_{K^\pm}^2 = & B_0 \Delta m \left\{ 1 \right. \\
& + \frac{16B_0}{F_0^2} \left[ (-2L_4^r - L_5^r + 4L_6^r + 2L_8^r) \hat{m} + (-L_4^r - L_5^r + 2L_6^r + 2L_8^r) m_s \right] \\
& + \frac{8aW_0}{F_0^2} \left[ (-2L_5^r - 6W_4^r - 2W_5^r + 6W_6^r + 4W_8^r) \right. \\
& \quad \left. + (L_5^r + 4W_4^r + W_5^r - 4W_6^r - 2W_8^r) (1 - \cos \omega) \right] \\
& + \frac{1}{32\pi^2 F_0^2} \left[ \frac{2\bar{m}_\eta^2}{3} \ln \left( \frac{\bar{m}_\eta^2}{\Lambda^2} \right) \right. \\
& \quad \left. + \frac{1}{B_0(m_s - \hat{m})} \frac{\bar{m}_K^2}{2} \left[ \bar{m}_\eta^2 \ln \left( \frac{\bar{m}_\eta^2}{\Lambda^2} \right) - \bar{m}_\pi^2 \ln \left( \frac{\bar{m}_\pi^2}{\Lambda^2} \right) \right] \left( 1 - \frac{aW_0(1 - \cos \omega)}{B_0(m_s - \hat{m})} \right) \right] \left. \right\} \\
& + \mathcal{O}(\Delta m^2, a^2). \tag{7}
\end{aligned}$$

It has contributions linear in  $\Delta m$  in contrast to the pions. These partly stem from pion and eta loops. Thus, for studies of the quark mass difference  $\Delta m$ , the kaon masses are suited much better than the pion masses.

To conclude, we have obtained the masses of the pseudoscalar mesons in Wilson chiral perturbation theory, taking into account the mass difference between up and down quarks. The mass splitting between neutral and charged pions, which is of order  $\Delta m^2$ , and the splitting between the kaon masses, which is of order  $\Delta m$ , are presented to order  $a$  in the lattice spacing. Due to mixing with the  $\eta$  meson, automatic  $\mathcal{O}(a)$  improvement at maximal twist is lost.

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