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Calculation of Strange and Light Quark Condensate using Improved Staggered Fermions and Overlap Fermions

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We study quark condensates at zero temperature, which is the order parameter for chiral symmetry breaking as well as an important input parameter for the QCD sum rules and for heavy quark expansion. Here we review the continuum property of quark condensates and explain how to separate the contribution of the zero modes from the spectral decomposition. Then we review the staggered fermion formalism of quark condensate and explain how to separate the contribution of the zero modes and how to identify the quantum numbers of each zero mode. We also present preliminary numerical results of quark condensates calculated using HYP-smeared staggered fermions on the MILC asqtad lattice at $a \cong 0.12$ fm.

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PoS(LATTICE 2015)066

1. Quark condensate in the continuum

In continuum the quark condensate is given by

$$\langle\bar{\psi}\psi\rangle = \frac{1}{N_f}\langle 0|\bar{\psi}_f\psi_f|0\rangle = -\frac{1}{VN_f}\int d^4x \,\mathrm{Tr}\left(\frac{1}{D+m}\right), \tag{1.1}$$

where D is the Dirac operator, m is the quark mass, x is the space-time coordinate, V is the volume, and N_f is the number of flavors. The trace is a sum over spin and color. Let us think of the eigenvalues of the Dirac operator. D is anti-Hermitian, so its eigenvalues are purely imaginary or zero. Thus we can represent the eigenvalues of D as $i\lambda$ and their corresponding eigenvectors as $u_\lambda(x)$: $Du_\lambda(x) = i\lambda u_\lambda(x)$. By spectral decomposition,

$$S_f(x,y) = \langle\psi_f(x)\bar{\psi}_f(y)\rangle = \sum_\lambda \frac{1}{i\lambda+m}u_\lambda(x)u_\lambda^\dagger(y) \tag{1.2}$$

$$\langle\bar{\psi}\psi\rangle = -\frac{1}{V}\sum_\lambda \frac{1}{i\lambda+m}\int d^4x \,\mathrm{Tr}\left(u_\lambda(x)u_\lambda^\dagger(x)\right) \tag{1.3}$$

$$= -\frac{1}{V}\sum_\lambda \frac{1}{i\lambda+m}. \tag{1.4}$$

where we adopt a normalization convention of $\langle u_a|u_b\rangle = \int d^4x\, u_a^\dagger(x)u_b(x) = \delta_{ab}$. Let us define $u_{-\lambda} \equiv \gamma_5 u_\lambda$, and then $Du_{-\lambda} = -i\lambda u_{-\lambda}$. Hence, if there exists u_λ, then its partner eigenstate $u_{-\lambda}$ with negative eigenvalue $-i\lambda$ must exist accordingly as a pair except for zero modes with $\lambda = 0$. Now let us separate the zero mode contribution from the spectral decomposition.

$$\langle\bar{\psi}\psi\rangle = -\frac{1}{V}\sum_{\lambda>0}\frac{2m}{\lambda^2+m^2} - \frac{n_+ + n_-}{mV}. \tag{1.5}$$

Here, n_+ (n_-) is the number of right-handed (left-handed) zero modes per flavor.

We define the subtracted quark condensate $\langle\bar{\psi}\psi\rangle_{\mathrm{sub}}$:

$$\langle\bar{\psi}\psi\rangle_{\mathrm{sub}} = \langle\bar{\psi}\psi\rangle + \frac{n_+ + n_-}{mV} = -\frac{1}{V}\sum_{\lambda>0}\frac{2m}{\lambda^2+m^2} \tag{1.6}$$

The subtracted quark condensate $\langle\bar{\psi}\psi\rangle_{\mathrm{sub}}$ is expected to behave well in the chiral limit while the contribution from the zero modes are divergent as a simple pole in the chiral limit. Hence, in the numerical study on the lattice, it is important to identify the would-be zero modes which correspond to the zero modes in the continuum limit, and remove them in the calculation of quark condensate.

Before proceeding, let us briefly go through the index theorem. In the continuum, the axial Ward identity is

$$\partial_\mu A_\mu(x) = 2mP(x) + 2N_f q(x). \tag{1.7}$$

Here $A_\mu \equiv \bar{\psi}\gamma_\mu\gamma_5\psi$ is the axial vector current in the flavor singlet representation, $P \equiv \bar{\psi}\gamma_5\psi$ is the corresponding pseudo-scalar operator, and $q \equiv \frac{1}{32\pi^2}F^a_{\mu\nu}\widetilde{F}^a_{\mu\nu}$ is the topological charge density (= winding number density). Now the topological charge Q is

$$Q \equiv \int d^4x\,\langle q(x)\rangle = \frac{1}{2}\int d^4x\,\langle\partial_\mu A_\mu(x) - 2mP(x)\rangle = -\frac{m}{N_f}\int d^4x\,\langle\bar{\psi}\gamma_5\psi\rangle \tag{1.8}$$

Using the spectral decomposition, we can rewrite Q as follows,

$$Q = m \sum_{\lambda} \frac{1}{i\lambda + m} \int d^4x \left[u_\lambda^\dagger(x) \gamma_5 u_\lambda(x) \right] . \tag{1.9}$$

By the way, $\gamma_5 u_\lambda(x) = u_{-\lambda}(x)$, and so

$$\int d^4x \left[u_\lambda^\dagger(x) \gamma_5 u_\lambda(x) \right] = \langle u_\lambda | u_{-\lambda} \rangle = 0 \qquad \text{unless} \quad \lambda = 0 . \tag{1.10}$$

Hence, only zero modes with $\lambda = 0$ contribute to Q. For the zero modes, it is convenient to choose the helicity eigenstates as the basis vectors so that $\langle u_0^L | \gamma_5 | u_0^L \rangle = -1$ and $\langle u_0^R | \gamma_5 | u_0^R \rangle = +1$, where the superscripts L, R represent left-handed and right-handed helicity, respectively. Then, it is straightforward to derive the index theorem [1]: $Q = n_+ - n_-$, where n_+ (n_-) is the number of right-handed (left-handed) zero modes.

2. Quark condensate using improved staggered fermions

In the staggered fermion formalism, there are a number of improved versions such as HYP-smeared staggered fermions, asqtad fermions, and HISQ fermions. Here, we call all of them "staggered fermions" collectively. Staggered fermions have four tastes per flavor by construction. Hence, quark condensate for staggered fermions is defined as

$$\langle \bar{\chi} \chi \rangle = -\frac{1}{V N_t} \left\langle \mathrm{Tr} \, \frac{1}{D_s + m} \right\rangle_U , \tag{2.1}$$

where D_s is the staggered Dirac operator for a single valence flavor, V is the lattice volume, and N_t is the number of tastes. We measure the quark condensate using the stochastic method.

$$(D_s + m)_{x,y} \psi_s(y) = \xi(x) \tag{2.2}$$

$$\psi_s(x) = \left[\frac{1}{D_s + m} \right]_{x,y} \xi(y) \tag{2.3}$$

$$\mathrm{Tr} \, \frac{1}{D_s + m} = \lim_{N_\xi \to \infty} \frac{1}{N_\xi} \sum_{\xi} \sum_{y} \xi^\dagger(y) \psi_s(y) , \tag{2.4}$$

where x, y are representative indices which represent the space-time coordinate and taste, color indices collectively. Here, $\xi(x)$ represents either Gaussian random numbers or U(1) noise random numbers which satisfy a simple identity:

$$\lim_{N_\xi \to \infty} \frac{1}{N_\xi} \sum_{\xi} \xi^\dagger(x) \xi(y) = \delta_{xy} ,$$

where N_ξ is the number of the random vector samples.

Staggered fermions have a taste symmetry of $\mathrm{SU}(4)_L \otimes \mathrm{SU}(4)_R \otimes \mathrm{U}(1)_V$ in the continuum limit at $a = 0$. However, this symmetry breaks down to a subgroup of $\mathrm{U}(1)_V \otimes \mathrm{U}(1)_A$ on the lattice with $a \neq 0$ [2,3]. This remaining axial symmetry of $\mathrm{U}(1)_A$ plays an important role in protecting the quark mass from receiving an additive renormalization. In addition, it does not have any anomaly.

PoS(LATTICE 2015)066

Let us turn to the spectral decomposition of the staggered Dirac operator. As in the continuum, the staggered Dirac operator D_s is anti-Hermitian and so its eigenvalues are purely imaginary or zero.

$$D_s f_\lambda(x) = +i\lambda f_\lambda(x) \tag{2.5}$$

$$\langle \chi(x)\bar{\chi}(y)\rangle = S_{\rm st}(x,y) = \sum_\lambda \frac{1}{i\lambda + m} f_\lambda(x) f_\lambda^\dagger(y) \tag{2.6}$$

where $f_\lambda(x)$ is an eigenstate for an eigenvalue $i\lambda$. Hence, using Eq. (2.6), we can rewrite the quark condensate as follows,

$$\langle \bar{\chi}\chi\rangle = -\frac{1}{VN_t}\sum_\lambda \frac{1}{i\lambda + m}\langle f_\lambda | f_\lambda\rangle = -\frac{1}{VN_t}\sum_\lambda \frac{1}{i\lambda + m}\,, \tag{2.7}$$

where $\langle f_\lambda | f_{\lambda'}\rangle = \int d^4x f_\lambda^\dagger(x) f_{\lambda'}(x) = \delta_{\lambda\lambda'}$.

In the continuum, the chiral symmetry insures that there must be an partner eigenstate with $-i\lambda$ for each eigenstate with its eigenvalue $+i\lambda$. Similarly, for staggered fermions, the $\mathrm{U}(1)_A$ symmetry guarantees the same kind of duality in the eigenstates. The generator for the $\mathrm{U}(1)_A$ symmetry is $\varepsilon = [\gamma_5 \otimes \xi_5]$. It anti-commutes with the staggered Dirac operator: $\varepsilon D_s = -D_s \varepsilon$. Hence, $f_{-\lambda}(x) = \varepsilon f_\lambda(x)$, if $\lambda \neq 0$. Since the non-zero eigenvalues of D_s exist as $\pm$ pairs, we can rewrite the quark condensate as follows,

$$\langle \bar{\chi}\chi\rangle = -\frac{1}{VN_t}\sum_{\lambda>0}\frac{2m_v}{\lambda^2 + m_v^2} - \frac{\tilde{n}_+ + \tilde{n}_-}{m_v V N_t} = \frac{1}{VN_t}\sum_{\lambda>0}\frac{2m_v}{\lambda^2 + m_v^2} - \frac{n_+ + n_-}{m_v V N_t}\,, \tag{2.8}$$

where m_v is the valence quark mass, and $\tilde{n}_\pm$ are the numbers of zero modes with $P_\pm = (1\pm\varepsilon)/2$ projection, respectively. Here we can define the subtracted quark condensate as

$$\langle \bar{\chi}\chi\rangle_{\rm sub} = \langle \bar{\chi}\chi\rangle + \frac{n_+ + n_-}{m_v V N_t} = \frac{1}{VN_t}\sum_{\lambda>0}\frac{2m_v}{\lambda^2 + m_v^2} \tag{2.9}$$

Once more, it is important to identify the zero modes in the staggered fermion formulation.

Now let us turn to the zero modes and index theorem in the staggered fermion formalism. The generator for the anomalous $\mathrm{U}(1)_A^{\rm anom}$ symmetry is $A_5 = [\gamma_5 \otimes \mathbb{1}]$ in the staggered fermion formalism. This $\mathrm{U}(1)_A^{\rm anom}$ symmetry is broken on the lattice with $a > 0$, and restored in the continuum limit of $a = 0$ which, however, is broken by the well-known anomaly. Hence, the Ward identity of Eq. (1.7) does not have its correspondence in the staggered fermion formalism. Then, we may raise a question: how does the index theorem work in the staggered fermion formalism? A simple answer is to use the spectral flow method [4, 5].

Let us consider a Hermitian operator H_s:

$$H_s \equiv -iD_s + \mu[\gamma_5 \otimes \mathbb{1}]\,, \tag{2.10}$$

Where μ is just a tuning parameter for the eigenvalue $\lambda_s(\mu)$. Since H_s is Hermitian, its eigenvalue must be real.

$$H_s f_s(x,\mu) = \lambda_s(\mu) f_s(x,\mu)\,, \tag{2.11}$$

where f_s is the corresponding eigenstate. In the limit of $\mu = 0$, $\lambda_s(\mu = 0) = \lambda$ and $f_s(x,\mu = 0) = f_\lambda(x)$. We can express $\lambda_s(\mu)$ as

$$\lambda_s(\mu) = \langle f_s(\mu)|H_s|f_s(\mu)\rangle\,. \tag{2.12}$$

Now let us take a derivative with respect to μ and use the normalization convention of $\langle f_s(\mu)|f_t(\mu)\rangle = \delta_{st}$. Then, we obtain the following expression for the chirality.

$$\lambda_s'(\mu) = \langle f_s(\mu)|[\gamma_5 \otimes \mathbb{1}]|f_s(\mu)\rangle\,. \tag{2.13}$$

Here, we define the chirality operator as follows,

$$\langle f_s(\mu)|[\gamma_5 \otimes \mathbb{1}]|f_s(\mu)\rangle \equiv \int d^4x\, f_s^\dagger(x_A,\mu)\overline{(\gamma_5 \otimes \mathbb{1})}_{AB} U(x_A,x_B) f_s(x_B,\mu) \tag{2.14}$$

$$\overline{(\gamma_S \otimes \xi_T)}_{AB} = \frac{1}{4}\mathrm{Tr}\,(\gamma_A^\dagger \gamma_S \gamma_B \gamma_T^\dagger) \tag{2.15}$$

$$U(x_A,x_B) = \mathbb{P}_{\mathrm{SU(3)}}\left[\sum_{p\in\mathscr{C}} V(x_A,x_{p_1})V(x_{p_1},x_{p_2})V(x_{p_2},x_{p_3})V(x_{p_3},x_B)\right] \tag{2.16}$$

where $x_A = 2x + A$, and A,B are the hypercubic vectors with $A_\mu, B_\mu \in \{0,1\}$, $V(x,y)$ is a HYP-smeared gluon link, $\mathbb{P}_{\mathrm{SU(3)}}$ represents the SU(3) projection, and $\mathscr{C}$ represents a complete set of the shortest paths from A to B. The advantage of the chirality operator in Eq. (2.14) is that it satisfies the same recursion relationship as the continuum chirality operator: for $n > 0$ and $n \in Z$,

$$[\gamma_5 \otimes \mathbb{1}]^{2n+1} = [\gamma_5 \otimes \mathbb{1}]\,, \tag{2.17}$$

$$[\gamma_5 \otimes \mathbb{1}]^{2n} = [\mathbb{1} \otimes \mathbb{1}]\,, \tag{2.18}$$

$$[\frac{1}{2}(1 \pm \gamma_5) \otimes \mathbb{1}]^n = [\frac{1}{2}(1 \pm \gamma_5) \otimes \mathbb{1}]\,, \tag{2.19}$$

$$[\frac{1}{2}(1 + \gamma_5) \otimes \mathbb{1}][\frac{1}{2}(1 - \gamma_5) \otimes \mathbb{1}] = 0\,. \tag{2.20}$$

From Eq. (2.13), we can define the chirality (X) of the zero modes in the staggered fermion formalism as

$$X = \pm 1 = \mathrm{sign}\left(\lim_{\lambda\to 0}\left[\lim_{\mu\to 0} \lambda_s'(\mu)\right]\right) \tag{2.21}$$

Here, note that $\lambda_s(\mu = 0) = \lambda$. This is called the spectral flow method. Therefore, it is rigorously possible to determine the chirality index of the zero modes with staggered fermions using the spectral flow method. Hence, even though we can**not** derive the index theorem directly from the chiral Ward identity as in the continuum, we can still get around the difficulty using the spectral flow method, and determine the chirality index correctly, and identify the index theorem and the topological contribution to the quark condensate rigorously in the staggered fermion formalism.

We have not proved yet that $\tilde{n}_+ + \tilde{n}_- = n_+ + n_-$. Let us consider the following relationship:

$$\varepsilon[\gamma_5 \otimes \mathbb{1}] = [\gamma_5 \otimes \mathbb{1}]\varepsilon\,, \tag{2.22}$$

$$\varepsilon[\mathbb{1} \otimes \xi_5] = [\mathbb{1} \otimes \xi_5]\varepsilon\,, \tag{2.23}$$

where $[\mathbb{1}\otimes\xi_5]$ is defined as

$$\langle f_s(\mu)|[\mathbb{1}\otimes\xi_5]|f_s(\mu)\rangle \equiv \int d^4x\, f_s^\dagger(x_A,\mu)\overline{(\mathbb{1}\otimes\xi_5)}_{AB}U(x_A,x_B)f_s(x_B,\mu)\,, \tag{2.24}$$

and satisfies the recursion relationship: for $n>0$ and $n\in Z$,

$$[\mathbb{1}\otimes\xi_5]^{2n+1} = [\mathbb{1}\otimes\xi_5]\,, \tag{2.25}$$

$$[\mathbb{1}\otimes\xi_5]^{2n} = [\mathbb{1}\otimes\mathbb{1}]\,, \tag{2.26}$$

$$\varepsilon = [\gamma_5\otimes\xi_5] = [\gamma_5\otimes\mathbb{1}][\mathbb{1}\otimes\xi_5] = [\mathbb{1}\otimes\xi_5][\gamma_5\otimes\mathbb{1}] \tag{2.27}$$

Since ε, $[\gamma_5\otimes\mathbb{1}]$, and $[\mathbb{1}\otimes\xi_5]$ commutes with one another, it is possible to use the quantum numbers of these operators to identify the zero modes. Let us define them as $Z=\pm1$ for ε, $X=\pm1$ for $[\gamma_5\otimes\mathbb{1}]$, and $Y=\pm1$ for $[\mathbb{1}\otimes\xi_5]$. Here, note that $Z=XY$. Hence, we can sort out zero modes as in Table 1. Therefore, it is obvious that

n_{XY}	X	Y	$Z=XY$
n_{++}	$+1$	$+1$	$+1$
n_{+-}	$+1$	-1	-1
n_{-+}	-1	$+1$	-1
n_{--}	-1	-1	$+1$

Table 1: n_{XY} represents the number of zero modes with X and Y quantum numbers.

$$n_+ = n_{++}+n_{+-} \tag{2.28}$$

$$n_- = n_{-+}+n_{--} \tag{2.29}$$

$$\tilde{n}_+ = n_{++}+n_{--} \tag{2.30}$$

$$\tilde{n}_- = n_{+-}+n_{-+} \tag{2.31}$$

At this stage, it becomes trivial to prove that $\tilde{n}_+ + \tilde{n}_- = n_+ + n_-$.

One may raise a question of how to determine the quantum number Y. A simple answer is that since X and Z are good quantum numbers, $Y=XZ$ must be so. However, we need a more elaborate and rigorous answer to this question, since, if Y is a good quantum number, there should be a direct method to determine it definitely.

Let us consider a Hermitian operator, $\tilde{H}_s \equiv -iD_s + \tilde{\mu}[\mathbb{1}\otimes\xi_5]$ Since it is Hermitian, its eigenvalues must be real and so $\tilde{H}_s\tilde{f}_s(x,\tilde{\mu}) = \tilde{\lambda}_s(\tilde{\mu})\tilde{f}_s(x,\tilde{\mu})$. We can apply the same logic of the spectral flow method to the eigenvalue, and then, in the end of the day, we obtain the following relation.

$$\tilde{\lambda}_s'(\tilde{\mu}) = \frac{d\tilde{\lambda}_s(\tilde{\mu})}{d\tilde{\mu}} = \langle \tilde{f}_s(\tilde{\mu})|[\mathbb{1}\otimes\xi_5]|\tilde{f}_s(\tilde{\mu})\rangle\,. \tag{2.32}$$

From this, we can determine Y in the staggered fermion formalism as follows,

$$Y = \pm1 = \mathsf{sign}\left(\lim_{\lambda\to0}\left[\lim_{\tilde{\mu}\to0}\tilde{\lambda}_s'(\tilde{\mu})\right]\right) \tag{2.33}$$

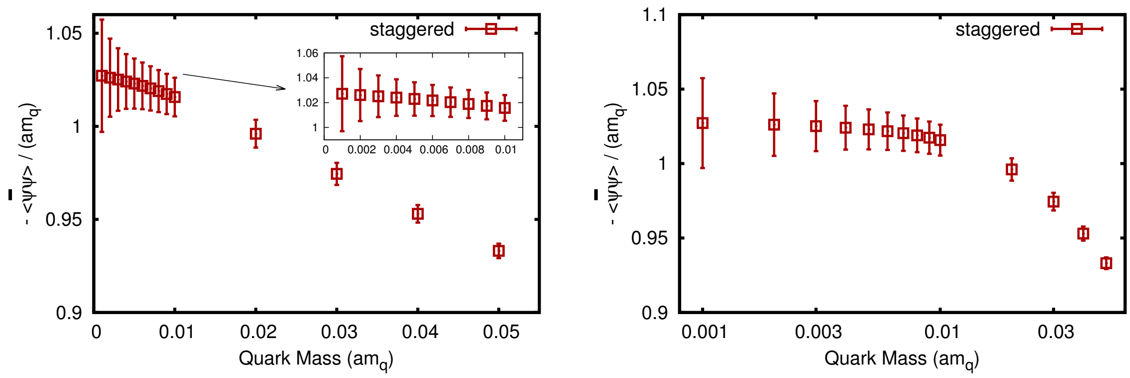

Figure 1: Quark condensates. Results are preliminary and unrenormalized bare.

Here, note that $\tilde{\lambda}_s(\tilde{\mu} = 0) = \lambda$. Using this method similar to the spectral flow method, it is possible to determine the quantum number Y rigorously.

In Fig. 1, we show a preliminary result of the quark condensate calculated using HYP-smeared staggered fermions as valence quarks on the MILC asqtad lattices at $a \cong 0.12$ fm with $m_\ell/m_s = 0.01/0.05$ [6]. Here, we have not yet nail down all the zero modes with their quantum numbers X, Y, Z identified. We plan to do it in near future.

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