A resolution of the inclusive flavor-breaking sum rule $\tau\, V_{us}$ puzzle

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A combination of continuum and lattice methods is used to investigate systematic issues in the finite-energy-sum-rule determination of V_{us} based on flavor-breaking combinations of hadronic τ decay data. Results for V_{us} obtained using assumptions for $D > 4$ OPE contributions employed in previous conventional implementations of this approach are shown to display significant unphysical dependences on the choice of sum rule weight, w, and upper limit, s_0, of the relevant experimental spectral integrals. Continuum and lattice results suggest the necessity of a new implementation of the flavor-breaking sum rule approach, in which not only $|V_{us}|$, but also $D > 4$ effective condensates are fit to data. Lattice results also provide a means of quantifying the truncation error for the slowly converging $D = 2$ OPE series. The new implementation is shown to produce $|V_{us}|$ results free of unphysical s_0- and w-dependences and typically ~ 0.0020 higher than the (unstable) results found using the conventional implementation. With preliminary new experimental results for the $K\pi$ branching fraction, the resulting $|V_{us}|$ is in excellent agreement with that obtained from $K_{\ell 3}$, and compatible within errors with expectations from three-family unitarity.

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1. Introduction

The conventional τ decay determination of $|V_{us}|$ is based on finite-energy sum rules (FESRs) involving flavor-breaking (FB) combinations of inclusive hadronic τ decay data [1]. With $\Pi_{V/A;ij}^{(J)}(s)$ the $J = 0, 1$ components of the flavor $ij = ud, us$, vector (V) or axial vector (A) current 2-point functions, $\rho_{V/A;ij}^{(J)}(s)$ their spectral functions, and $\Delta\Pi_\tau \equiv \left[\Pi_{V+A;ud}^{(0+1)} - \Pi_{V+A;us}^{(0+1)}\right]$, one has

$$\int_0^{s_0} w(s)\Delta\rho_\tau(s)\,ds = -\frac{1}{2\pi i}\oint_{|s|=s_0} w(s)\Delta\Pi_\tau(s)\,ds, \qquad (1.1)$$

valid for any s_0 and any analytic $w(s)$. The spectral function, $\Delta\rho_\tau$, of $\Delta\Pi_\tau$, is experimentally accessible in terms of the differential distribution, $dR_{V/A;ij}/ds$, of the normalized ratio $R_{V/A;ij} \equiv \Gamma[\tau^- \to \nu_\tau \text{hadrons}_{V/A;ij}(\gamma)]/\Gamma[\tau^- \to \nu_\tau e^- \bar{\nu}_e(\gamma)]$. Explicitly [2]

$$\frac{dR_{V/A;ij}}{ds} = c_\tau^{EW}|V_{ij}|^2 \left[w_\tau(s)\rho_{V/A;ij}^{(0+1)}(y_\tau) - w_L(y_\tau)\rho_{V/A;ij}^{(0)}(s)\right] \qquad (1.2)$$

with $y_\tau = s/m_\tau^2$, $w_\tau(y) = (1-y)^2(1+2y)$, $w_L(y) = 2y(1-y)^2$, c_τ^{EW} a known constant, and V_{ij} the flavor ij CKM matrix element. $\Delta\Pi_\tau$ on the RHS of Eq. (1.1) is to be treated using the OPE.

The reason for employing the $J = 0+1$ FESR Eq. (1.1), rather than the analogue involving the spectral function combination in Eq. (1.2), is the very bad behavior of the integrated $J = 0$, $D = 2$ OPE series [3]. $J = 0$ contributions to $dR_{V/A;ud;us}/ds$ are determined phenomenologically and subtracted, allowing $\rho_{V/A;ud,us}^{(0+1)}(s)$ to be obtained. The subtraction is dominated by the accurately known, non-chirally-suppressed π and K pole contributions. Continuum contributions to $\rho_{V/A;ud}^{(0)}$ are $\propto (m_d \mp m_u)^2$ and numerically negligible, while small, but not totally negligible, $(m_s \mp m_u)^2$-suppressed continuum $\rho_{V/A;us}^{(0)}$ contributions are fixed using highly constrained dispersive and sum rule methods [4, 5]. With $|V_{ud}|$ known [6], $\Delta\rho_\tau(s)$ is expressible in terms of experimental data and $|V_{us}|$. $|V_{us}|$ is then obtained by using the OPE for $\Delta\Pi_\tau$ on the RHS and data on the LHS of Eq. (1.1).

Given the $J = 0$-subtracted $dR_{V+A;ud,us}^{(0+1)}/ds$, it is straightforward to define re-weighted $J = 0+1$ versions of $R_{V+A;ud,us}$, $R_{V+A;ij}^w(s_0) \equiv \int_0^{s_0} ds\,\frac{w(s)}{w_\tau(s)}\frac{dR_{V+A;ij}^{(0+1)}(s)}{ds}$, for any w and $s_0 \leq m_\tau^2$. With $\delta R_{V+A}^{w,OPE}(s_0)$ the OPE representation of $\delta R_{V+A}^w(s_0) = \frac{R_{V+A;ud}^w(s_0)}{|V_{ud}|^2} - \frac{R_{V+A;us}^w(s_0)}{|V_{us}|^2}$, one then has

$$|V_{us}| = \sqrt{R_{V+A;us}^w(s_0)/\left[\frac{R_{V+A;ud}^w(s_0)}{|V_{ud}|^2} - \delta R_{V+A}^{w,OPE}(s_0)\right]}. \qquad (1.3)$$

The resulting $|V_{us}|$ should be independent of $w(s)$ and s_0, provided external experimental and theoretical inputs, and any assumptions employed in evaluating $\delta R_{V+A}^{w,OPE}(s_0)$, are reliable. Since integrated $D = 2k+2$ OPE contributions scale as $1/s_0^k$, problems with assumptions about higher D non-perturbative contributions will show up as instabilities in $|V_{us}|$ as a function of s_0.

The conventional implementation of Eq. (1.3) [1] employs $w = w_\tau$ and $s_0 = m_\tau^2$. This has the advantage that the spectral integrals $R_{V+A;ud,us}^{w_\tau}(m_\tau^2)$ can be determined using only the inclusive non-strange and strange hadronic τ branching fractions, but the disadvantage that assumptions have to be made about the higher dimension $D = 6, 8$ OPE contributions in priniciple present for a degree 3 weight like w_τ. The restriction to a single w and single s_0 precludes subjecting these assumptions to w- and s_0-independence self-consistency tests. It is this conventional implementation

which leads to the long-standing puzzle of inclusive τ $|V_{us}|$ determinations $> 3\sigma$ low relative the 3-family-unitarity expectations ($|V_{us}| = 0.2258(9)$ for the $|V_{ud}|$ of Ref. [6]), the most recent such determination, $|V_{us}| = 0.2176(21)$ [7], e.g., lying 3.6σ low. Tests of the conventional implementation performed using variable s_0 and alternate weight choices [8], however, show sizeable s_0- and w-dependence [8] (see, e.g., the left panel, and solid lines in the right panel, of Fig. 1), indicating systematic problems with at least some aspects of the conventional implementation. The dashed lines in the right panel show the results of an alternate implementation to be discussed below.
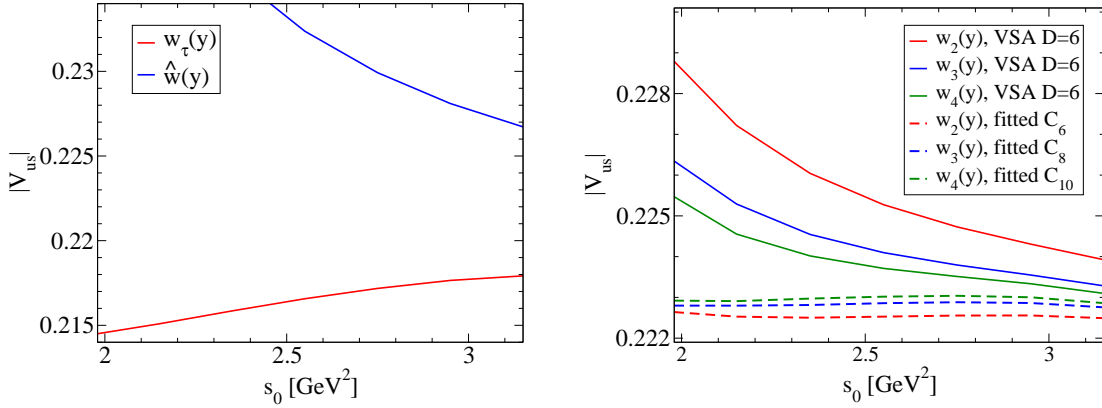

Figure 1: Left panel: $|V_{us}|$ obtained from the w_τ and $\hat{w}$ FESRs using the conventional implementation [1] OPE treatment, including use of the CIPT prescription for the $D = 2$ series. Right panel: Comparison of the conventional implementation results for $|V_{us}|$ from the $w_{2,3,4}$ FESRs with those obtained using the central fitted values of $C_{6,8,10}$, now using the FOPT $D = 2$ prescription favored by lattice results.

Two obvious potential sources exist for these instabilities. The first lies in the treatment of $D = 6$, 8 OPE contributions. In both the conventional implementation and generalized versions just mentioned [8], $D = 6$ contributions are estimated using the vacuum saturation approximation (VSA), and $D = 8$ contributions neglected. The resulting $D = 6$ estimate is very small due to significant cancellations, first in the individual ud and us V+A sums and, second, in the FB difference of these sums. Such strong cancellations make use of the VSA estimate potentially dangerous, given the sizeable, channel-dependent VSA breaking observed in the flavor ud V and A channels [9]. The second possibility concerns the slow convergence, at the correlator level, of the $D = 2$ OPE series. With $\bar{a} = \alpha_s(Q^2)/\pi$, and $\alpha_s(Q^2)$, $m_s(Q^2)$ the running coupling and strange quark mass in the $\overline{MS}$ scheme, one has, to four loops [10] (neglecting $O(m_{u,d}^2/m_s^2)$ corrections)

$$\left[\Delta\Pi_\tau(Q^2)\right]_{D=2}^{OPE} = \frac{3}{2\pi^2}\frac{m_s(Q^2)}{Q^2}\left[1 + \frac{7}{3}\bar{a} + 19.93\bar{a}^2 + 208.75\bar{a}^3 + \cdots\right]. \qquad (1.4)$$

With $\bar{a}(m_\tau^2) \simeq 0.1$, convergence at the spacelike point on the contour $|s| = s_0$ is marginal at best. This raises the question of truncation order and truncation error estimates for the corresponding integrated series. The $D = 2$ convergence issue also shows up in the significant difference (increasing from ~ 0.0010 to ~ 0.0020 between 3- and 5-loop truncation order) in $|V_{us}|$ results obtained using alternate (fixed-order (FOPT) and contour-improved (CIPT)) prescriptions for the truncated integrated $D = 2$ series which differ only by terms beyond the common truncation order [8].

In what follows, we first investigate the treatment of the $D = 2$ OPE series using lattice data for $\Delta\Pi_\tau$, then test the $D = 6, 8$ assumptions of the conventional implementation by comparing FESR results for a judiciously chosen pair of weights, $w_\tau(y)$ and $\hat{w}(y) = (1-y)^3$, $y = s/s_0$. We then present results obtained employing an alternate implementation of the FB FESR approach suggested by these investigations.

2. Lattice and continuum investigations of the OPE representation of $\Delta\Pi_\tau$

Data for $\Delta\Pi_\tau(Q^2)$ over a wide range of Euclidean Q^2 can be generated using the lattice, with an appropriate cylinder cut applied to avoid lattice artifacts at high Q^2. This issue has been studied in detail for the ensemble employed here in a recent analysis focused on determining α_s from lattice current-current two-point function data [11]. Here we first consider data at Q^2 high enough that $[\Delta\Pi_\tau]_{OPE}$ will be safely dominated by its leading $D = 2$ and 4 contributions. $D = 4$ contributions are determined by light and strange quark masses and condensates and hence known. We take FLAG results for the physical quark masses [12], the light condensate from GMOR, and the strange condensate from $\langle\bar{s}s\rangle/\langle\bar{u}u\rangle$. The latter is determined using the HPQCD physical-m_q version of this ratio [13], translated to the m_q of the ensemble employed using NLO ChPT [14]. We then consider various combinations of truncation order and schemes for resumming logs for the $D = 2$ OPE series, investigating whether any of these choices produce a good match between the resulting $D = 2 + 4$ OPE sum and lattice data in the high-Q^2 region.
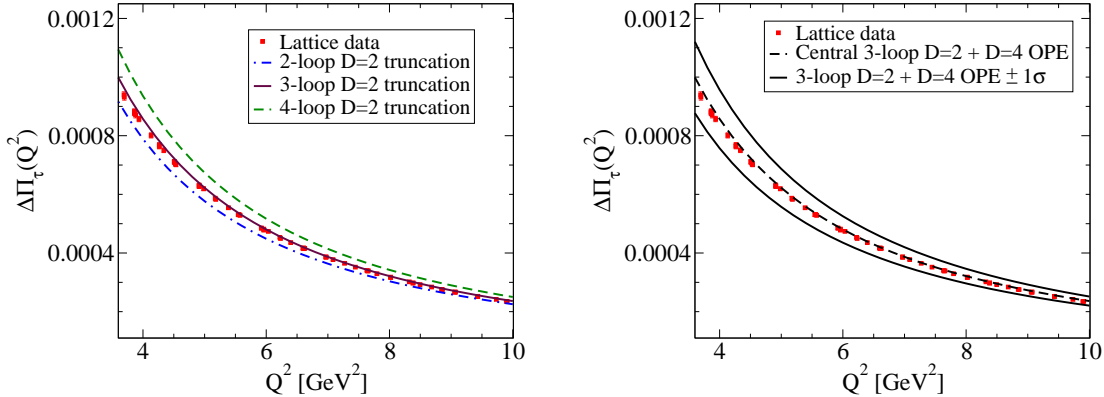

Figure 2: Left panel: comparison of lattice data and OPE $D = 2 + 4$ expectations for the various truncation orders and the fixed-scale treatment of the $D = 2$ series. Right panel: lattice data and the $D = 2 + 4$ OPE sum, with conventional OPE error estimates, for the 3-loop-truncated, fixed-scale $D = 2$ treatment

For this high-Q^2 study, we employ the RBC/UKQCD $n_f = 2 + 1$, $32^3 \times 64$, $1/a = 2.38$ GeV, $m_\pi \sim 300$ MeV domain wall fermion ensemble [15]. We find (see the left panel of Fig. 2) that 3-loop $D = 2$ truncation with fixed-scale (the analogue of the FOPT FESR prescription) provides an excellent OPE-lattice match over a wide range of Q^2, extending from ~ 10 GeV^2 down to ~ 4 GeV^2. The fixed-scale choice is, moreover, superior to the alternate local-scale ($\mu^2 = Q^2$) choice (analogous to the CIPT FESR prescription). The right panel of Fig.2 shows the $D = 2 + 4$ OPE error band obtained using the 3-loop-truncated, fixed-scale $D = 2$ OPE treatment and conventional OPE error estimate methods. The resulting, nominally naive error turns out to be, in fact, extremely

conservative, despite the very slow convergence of the $D = 2$ series. Clear evidence (to be detailed elsewhere) also exists for the onset, below $Q^2 \sim 4\,\mathrm{GeV}^2$, of significantly larger $D > 4$ contributions than expected based on the VSA $D = 6$ condensate employed in the conventional implementation. With no means of selectively isolating contributions of different $D > 4$ in the Euclidean lattice data, further investigation of the higher D question requires continuum FESR methods.

For our continuum FESR studies, we employ the $D = 2$ and 4 OPE treatment favored by lattice data, detailed above. As input on the spectral integral sides, we employ $\pi_{\mu 2}$, $K_{\mu 2}$ and Standard Model expectations for the π and K pole contributions, recent ALEPH data for the continuum ud V+A distribution [16], BaBar [17] and Belle [18] results for the $K^- \pi^0$ and $\bar{K}^0 \pi^-$ distributions, BaBar results [19] for the $K^- \pi^+ \pi^-$ distribution, Belle results [20] for the $\bar{K}^0 \pi^- \pi^0$ distribution and 1999 ALEPH results [21] for the "residual" distribution involving strange modes not remeasured by the B-factory experiments. The unit-normalized BaBar and Belle exclusive mode distributions must be normalized using experimental branching fractions. In the results quoted below we employ HFAG strange exclusive mode branching fractions, with the exception of $K^- \pi^0$, for which we employ the update contained in the recent BaBar Adametz thesis [22], performing an accompanying very small rescaling of the continuum ud V+A distribution to restore unitarity.

Neglecting α_s-suppressed logarithmic corrections, the $D > 4$ OPE contributions to $\Delta \Pi_\tau(Q^2)$ can be written $\sum_{D>4} C_D/Q^D$ with C_D effective condensates of dimension D. The degree 3 weights $w_\tau(y) = 1 - 3y^2 + 2y^3$ and $\hat{w}(y) = 1 - 3y + 3y^2 - y^3$ generate integrated OPE contributions up to $D = 8$. The associated $D > 4$ contributions,

$$-\frac{3C_6}{s_0^2} - \frac{2C_8}{s_0^3} \;\; \text{for } w_\tau \qquad \text{and} \qquad \frac{3C_6}{s_0^2} + \frac{C_8}{s_0^3} \;\; \text{for } \hat{w}, \qquad (2.1)$$

differ in sign, with the two $D = 6$ contributions identical in magnitude and the magnitude of the $\hat{w}$ $D = 8$ contribution half that of w_τ. It follows that, if the assumptions of the conventional implementation are correct and $D = 6, 8$ contributions are basically negligible in the w_τ FESR, this will necessarily also be the case for the $\hat{w}$ FESR. The $|V_{us}|$ results obtained from the two FESRs should then agree and, moreover, be s_0-independent. In contrast, if the $D = 6$ and/or 8 contributions to the w_τ FESR are not, in fact, negligible, one should see s_0-instabilities of opposite sign, decreasing in magnitude with increasing s_0, for the results of $|V_{us}|$ obtained from the two FESRs. The left panel of Fig. 1 shows that it is the latter scenario which is realized. The sizeable s_0- and weight-choice dependences demonstrate unambiguously that the assumptions underlying the conventional implementation are untenable, and that the 3σ low $|V_{us}|$ results obtained employing them are afflicted with significant previously unquantified systematic uncertainties.

3. An alternate implementation of the FB FESR approach

With previously employed methods for estimating $D > 4$ effective OPE condensates shown to be unreliable, one has no option but to fit these condensates to data. This requires working with FESRs involving variable s_0 and hence precludes determining the required spectral integrals solely in terms of inclusive hadronic branching fractions. To suppress possible duality violating contributions, we restrict our attention to FESRs with weights having at least a double zero at $s = s_0$. The weights $w_N(y) = 1 - \frac{N}{N-1}y + \frac{1}{N-1}y^N$, $N \geq 2$ [23] are particularly convenient since they

yield a single $D > 4$ OPE contribution (with $D = 2N + 2$). With $D = 2 + 4$ OPE contributions under control, as discussed above, this leaves $|V_{us}|$ and C_{2N+2} as the only parameters to be fit to the w_N-weighted, s_0-dependent spectral integrals. Further tests of the analysis are provided by checking that (i) the $|V_{us}|$ obtained from the different w_N FESRs are in agreement and (ii) the fitted C_D are physically sensible (i.e., show FB cancellation relative to the results of Ref. [9] for the corresponding flavor *ud* condensates). We have analyzed the w_N FESRs for $N = 2, 3, 4$ and verified that the results pass these self-consistency tests. In the right panel of Fig. 1 we display the results obtained by taking the central values for the C_6, C_8 and C_{10} obtained in this analysis as input and solving Eq. (1.3) for $|V_{us}|$, as a function of s_0, for each of the w_2, w_3 and w_4 FESRs. The figure illustrates (i) the underlying excellent match between the fitted OPE and spectral integral sets, (ii) the excellent agreement between the results of the different FESR analyses and (iii) the dramatic decrease in s_0- and weight-dependence produced by using $D > 4$ OPE contributions fit to data in place of those based on the assumptions of the conventional implementation. In addition, one sees that, as expected, the fitted $|V_{us}|$ lie between the s_0-unstable results produced by the conventional implementation of the w_τ and $\hat{w}$ FESRs, and are ~ 0.0020 higher than the results of the conventional w_τ implementation.

With the results from the different w_N FESRs showing good compatibility and s_0-stability, our final result for $|V_{us}|$ is obtained by performing a combined fit to the w_2, w_3 and w_4 FESRs. We find[1]

$$|V_{us}| = 0.2228(23)_{exp}(5)_{th} , \qquad (3.1)$$

in excellent agreement with the results, $0.2235(4)_{exp}(9)_{th}$ and $0.2231(4)_{exp}(7)_{th}$, obtained using the 2014 FlaviaNet experimental $K_{\ell3}$ update [25] and most recent $n_f = 2 + 1$ [26] and $n_f = 2 + 1 + 1$ [27] lattice results for $f_+(0)$. It is also compatible within errors with (i) the results, $0.2251(3)_{exp}(9)_{th}$ and $0.02250(3)_{exp}(7)_{th}$ obtained using the 2014 experimental $\Gamma[K_{\mu2}]/\Gamma[\pi_{\mu2}]$ update [25] and most recent $n_f = 2 + 1$ [28] and $n_f = 2 + 1 + 1$ [29] lattice determinations of f_K/f_π and (ii) the expectations of 3-family unitarity. It is worth noting that, among these methods, the one having the smallest theory error is the FB FESR determination, which error, as we have seen, is very conservative. At present the experimental error on the FB FESR determination (resulting almost entirely from uncertainties in the *us* exclusive mode distributions) is larger than those of the competing methods, but this error is currently dominated by the uncertainty on the branching fraction normalizations for the exclusive strange modes, and systematically improvable in the near future.

References

[1] E. Gamiz *et al.*, *JHEP* **0301** (2003) 060; *Phys. Rev. Lett.* **94** (2005) 011803; *PoS KAON 2007* (2008) 008.

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[1]An analogous analysis with $K\pi$ normalization not employing the $B[K^-\pi^0\nu_\tau]$ update of Ref. [22] yields $|V_{us}| = 0.2200(23)_{exp}(5)_{th}$. Of the 0.0024 difference between this result and the conventional implementation result [7] noted above, 0.0005 results from the use of $K_{\mu2}$ for the K pole contribution; the remainder is due to presence of the $D = 6, 8$ contributions not correctly accounted for by the assumptions of the conventional implementation. Note that the normalization of the two-mode $K\pi$ sum produced by the $B[K^-\pi^0\nu_\tau]$ update is in good agreement with the results of the dispersive study of $K\pi$ detailed in Ref. [24].

[2] Y.-S. Tsai, *Phys. Rev. D* **4** (1971) 2821.

[3] K. Maltman, *Phys. Rev. D* **58** (1998) 093015; K. Maltman and J. Kambor, *ibid.* **64** (2001) 093014.

[4] M. Jamin, J.A. Oller and A. Pich, *Nucl. Phys. B* **587** (2000) 331; *ibid.* **622** (2002) 279; *Phys. Rev. D* **74** (2006) 074009.

[5] K. Maltman and J. Kambor, *Phys. Rev. D* **65** (2002) 074013.

[6] J. C. Hardy and I. S. Towner, *Phys. Rev. C* **91** (2015) 015501

[7] A. Lusiani, arXiv: 1411.4526 [hep-ex]; E. Passemar, presentation at CKM 2014

[8] K. Maltman and C.E. Wolfe, *Phys. Lett.* **B650** (2007) 27; K. Maltman *et al. Int. J. Mod. Phys. A* **23** (2008) 3191; K. Maltman, C.E. Wolfe, S. Banerjee, J.M. Roney and I. Nugent, *Nucl. Phys. Proc. Suppl.* **189** (2009) 175; K. Maltman, *Nucl. Phys. Proc. Suppl.* **218** (2011) 146.

[9] D. Boito, *et al.*, *Phys. Rev. D* **85** (2012) 093015; *ibid.* **D91** (2015) 034003.

[10] K.G. Chetyrkin and A. Kwiatkowski, *Z. Phys. C* **59** (1993) 525 and hep-ph/9805232; P.A. Baikov, K.G. Chetyrkin and J.H. Kuhn, *Phys. Rev. Lett.* **95** (2005) 012003.

[11] R. J. Hudspith, R. Lewis, K. Maltman and E. Shintani, arXiv:1510.04890 [hep-lat].

[12] S. Aoki *et al.*, *Eur. Phys. J. C* **74** (2014) 2890.

[13] C. McNeile, *et al.*, *Phys. Rev. D* (2013) 034503

[14] J. Gasser and H. Leutwyler, *Nucl. Phys. B* **250** (1985) 465.

[15] Y. Aoki *et al.* [RBC and UKQCD Collaborations], *Phys. Rev. D* **83** (2011) 074508.

[16] M. Davier, A. Hoecker, B. Malaescu, C. Z. Yuan and Z. Zhang, *Eur. Phys. J. C* **74** (2014) 2803.

[17] B. Aubert, *et al.* [BaBar Collaboration], *Phys. Rev. D* **76** (2007) 051104.

[18] D. Epifanov, *et al.* [Belle Collaboration], *Phys. Lett. B* **654** (2007) 65. The $K_s\pi^-$ invariant mass spectrum is available at belle.kek.jp/belle/preprint/2007-28/tau_kspinu.dat. Thanks to Denis Epifanov for providing access.

[19] I.M. Nugent, *et al.* [BaBar Collaboration], *Nucl. Phys. Proc. Suppl.* **253-255** (2014) 38. Thanks to Ian Nugent for providing the unfolded $K^-\pi^-\pi^+$ distribution and covariances employed here.

[20] S. Ryu, *et al.* [Belle Collaboration], *Nucl. Phys. Proc. Suppl.* **253-255** (2014) 33.

[21] R. Barate, *et al.* [ALEPH Collaboration], *Eur. Phys. J. C* **11** (1999) 599.

[22] A. Adametz, "Measurement of τ decays into charged hadron accompanied by neutral π mesons and determination of the CKM matrix element $|V_{us}|$", University of Heidelberg PhD thesis, July 2011 and the BaBar Collaboration, in progress.

[23] K. Maltman and T. Yavin, *Phys. Rev. D* **78** (2008) 094020.

[24] M. Antonelli, V. Cirigliano, A. Lusiani and E. Passemar, *JHEP* **1310** (2013) 070.

[25] For a recent assessment and references to earlier work, see M. Moulson, arXiv:1411.5252 [hep-ex].

[26] P. A. Boyle, *et al.*, *JHEP* **1506** (2015) 164.

[27] T. Primer, *et al.* (FNAL and MILC Collaborations), *Pos* **LATTICE2014** (2014) 374.

[28] T. Blum, *et al.* (RBC/UKQCD Collaboration), arXiv:1411.7017 [hep-lat]

[29] A. Bazavov, *et al.* (FNAL and MILC Collaborations), *Phys. Rev. D* **90** (2014) 074509.