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Renormalizability of the Schrödinger Functional

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Symanzik showed that quantum field theory can be formulated on a space with boundaries by including suitable surface interactions in the action to implement boundary conditions. We show that to all orders in perturbation theory all the divergences induced by these surface interactions can be absorbed by a renormalization of their coefficients.

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1. Introduction

It is of interest to study quantum field theories with boundaries for a variety of reasons: the Casimir effect [1] describes a quantized electromagnetic field with mirrors as spatial boundaries; and the Schrödinger functional [2] describes a quantum field theory with specified initial and final field configurations, which has interesting applications to renormalization problems in lattice gauge theories and lattice QCD [3, 4]. Symanzik showed that such boundary conditions may be imposed by including suitable surface interactions in the action [2]. Our goal is to establish that quantum field theories with boundary conditions imposed in this way are renormalizable to all orders in perturbation theory.

Our proof is independent of the choice of regulator: we shall use continuum notation for simplicity, but results apply equally well with a lattice regulator. The proof is in Euclidean space; extension to Minkowski space, where the Green's functions are distributions rather than functions presumably follows by partial integration onto test functions just as for theories without boundaries [5]. We shall only present a proof for a scalar field here.

2. Boundary Conditions

For simplicity in this summary we describe the proof for a scalar field ϕ with the Lagrangian density $\mathscr{L} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{4!}\lambda\phi^4$ to which we add the surface term $K = \frac{1}{2}c\phi_-\delta'(\sigma)\phi_+$, with $c = \pm 1$ so as to decouple the two sides of the wall. This is similar but not identical to Symanzik's surface interaction. The function σ vanishes on the wall, and specifies the location of the boundary. For a planar wall that is orthogonal to a unit vector w and a distance ℓ from the origin we could take $\sigma(x) = x\cdot w - \ell$. In general we can take σ to be a smooth function corresponding to a wall that is topologically equivalent to a plane.

2.1 Quadratic Interactions

Observe that the boundary conditions are imposed by a local interaction that is quadratic in the field ϕ, and moreover there is no small parameter associated with this wall interaction. This is analogous to the mass term $\frac{1}{2}m^2\phi^2$: we can either treat this as part of the propagator, $(k^2+m^2)^{-1}$, or treat it "perturbatively" as a two-point vertex $-m^2$ with the massless propagator $\Delta = 1/k^2$. In the latter case we can sum the two-point interactions to all orders in m

$$\begin{aligned}\Delta_M &= \Delta + \Delta(-m^2)\Delta + \Delta(-m^2)\Delta(-m^2)\Delta + \cdots = \Delta \sum_{n=0}^{\infty} \left[(-m^2)\Delta\right]^n \\ &= \Delta + \Delta(-m^2)\Delta_M = \frac{\Delta}{1+m^2\Delta} = \frac{1}{k^2+m^2}.\end{aligned} \tag{2.1}$$

Of course, this series only converges for $k^2 \le m^2$, but it has a unique analytic continuation $\forall k^2 \neq -m^2$, even though there is no small parameter.

This equivalence should be familiar: the mass renormalization is $m^2 \to m^2 + \delta m^2$, where δm^2 is treated as a countervertex order by order in the loop expansion.

2.2 Integral Equation

The Green's function $H(x,y)$ for the quadratic kernel without walls

$$L(x,y) = \delta(x-y)(-\partial^2 + m^2) \tag{2.2}$$

satisfies $\int dz L(x,z)H(z,y) = \delta(x,y)$, which we shall abbreviate as $LH = 1$. Of course, we must also specify suitable boundary conditions to uniquely specify the Green's function, so We require that $\lim_{|x-y|\to\infty} H(x-y) = 0$. Because there are no walls H is translationally invariant and is only a function of $x-y$. The Green's function $G(x,y)$ for the full quadratic kernel $L+K$ where the wall interaction is

$$K(x,y) = \int dz\, \delta(x-z_-)\delta(y-z_+)\delta'(\sigma(z)) \tag{2.3}$$

which satisfies $(L+K)G = 1$, where $z_\pm = z \pm \varepsilon\partial\sigma$ with $\varepsilon \to 0$. Naturally we also chose the boundary conditions that $G(x,y) = 0$ as $|x| \to \infty$ or $|y| \to \infty$.

Following Symanzik we may thus find G "non-perturbatively" by solving the integral equation $(L+K)G = 1$; upon multiplying on the left by H this gives $H(L+K)G = H \Rightarrow (1+HK)G = H$. $G(x,y)$ is not translationally invariant, so it is not just a function of $x-y$. We require $G(x_-,x_+) = 0$, so the two sides of the wall are decoupled; moreover all the propagator's derivatives must also vanish across the wall. Since the propagator G vanishes across the wall so does any connected Green's function that couples points on opposite sides of the wall, as it is a convolution of propagators.

For simplicity we now assume the wall is the plane orthogonal to the x_1 axis and passing through the origin. We may then reduce the problem of finding the Green's function G to a one-dimensional one by considering a single Fourier mode in the $D-1$ directions parallel to the wall with frequency ω.

Inserting the explicit solution $H(x_1,y_1) = e^{-\omega|x_1-y_1|}/2\omega$ this leads to the linear system

$$\begin{pmatrix} 1-\frac{c}{2} & -\frac{c}{2} & -\frac{c}{2\omega} & -\frac{c}{2\omega} \\ \frac{c}{2} & 1+\frac{c}{2} & -\frac{c}{2\omega} & -\frac{c}{2\omega} \\ \frac{c\omega}{2} & \frac{c\omega}{2} & 1+\frac{c}{2} & \frac{c}{2} \\ \frac{c\omega}{2} & \frac{c\omega}{2} & -\frac{c}{2} & 1-\frac{c}{2} \end{pmatrix} \begin{pmatrix} G(x_1,0_+) \\ G(x_1,0_-) \\ \partial_2 G(x_1,0_+) \\ \partial_2 G(x_1,0_-) \end{pmatrix} = \frac{e^{-\omega|x_1|}}{2\omega} \begin{pmatrix} 1 \\ 1 \\ \omega\,\mathrm{sgn}\,x_1 \\ \omega\,\mathrm{sgn}\,x_1 \end{pmatrix}, \tag{2.4}$$

where $\partial_2 G$ stands for the derivative of G with respect to its second argument, and we have used the fact that G and its derivatives are left (right) continuous on the left (right) of the wall.

2.3 Solution of Integral Equation

We find that the two sides of the wall only decouple for $c = \pm 1$, which crucially does not depend on the frequency ω. The solution of the integral equation for $c = -1$ is

$$\omega G(x_1,y_1) = \begin{cases} e^{\omega y_1}\cosh\omega x_1 & \text{for } y_1 \le x_1 \le 0 \\ e^{\omega x_1}\cosh\omega y_1 & \text{for } x_1 < y_1 \le 0 \\ 0 & \text{for } x_1 \le 0 < y_1 \\ 0 & \text{for } y_1 \le 0 < x_1 \\ e^{-\omega x_1}\sinh\omega y_1 & \text{for } 0 < y_1 \le x_1 \\ e^{-\omega y_1}\sinh\omega x_1 & \text{for } 0 < x_1 < y_1. \end{cases} \tag{2.5}$$

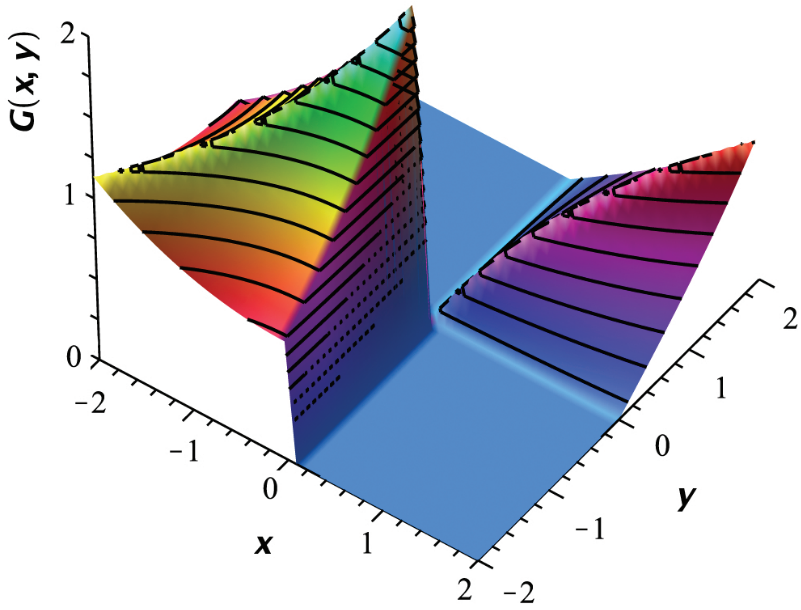

Figure 1: The Green's function $G(x_1, y_1)$ with $\omega = \frac{1}{2}$ and $c = -1$ for a scalar field with a wall interation K at $\sigma(x) = x_1 = 0$. This Green's function is not translationally invariant, so it is not just a function of $x_1 - y_1$. It satisfies Dirichlet boundary conditions on the right and Neumann boundary conditions on the left.

This solution for $\omega = \frac{1}{2}$ is shown in Figure 1. It satisifies Dirichlet boundary conditions on the right and Neumann boundary conditions on the left. Changing the sign of the wall interaction to $c = 1$ interchanges these.

3. Divergences and Renormalization

3.1 Feynman Rules

As well as the usual bulk divergences we have new divergences associated with wall vertices. For simplicity we consider the wall $\sigma(x) = x_1 - \ell$ which is orthogonal to the x_1 axis and intersects it at $x_1 = \ell$. The wall vertex is thus

$$K(x,y) = \int dz\, \delta(x - z_-)\delta(y - z_+)\delta'(z_1 - \ell); \tag{3.1}$$

in momentum space this is

$$\begin{aligned} \tilde{K}(q,q') &= \int \frac{dx\,dy}{(2\pi)^D} K(x,y) e^{-i(q\cdot x + q'\cdot y)} = \int \frac{dz}{(2\pi)^D} e^{-i(q\cdot z_- + q'\cdot z_+)} \delta'(z_1 - \ell) \\ &= \frac{i}{2\pi}(q+q')_1 e^{-i\ell(q+q')_1} e^{i\varepsilon(q-q')_1} \delta((q+q')_\perp) \\ &= \frac{i}{2\pi} \int dp\, p_1 e^{-i\ell p_1} \delta(q+q'-p)\delta(p_\perp) e^{i\varepsilon' \operatorname{sgn}(q-q')_1}. \end{aligned} \tag{3.2}$$

The location of the wall is specified by the phase $e^{-i\ell p_1}$, and its orientation by the dependence on the sign of $(q - q')_1$. We have associated an "external" momentum p with the wall source so that momentum is conserved at the wall vertex; this corresponds to the two-point vertex coupling the field the external source shown in Figure 2.

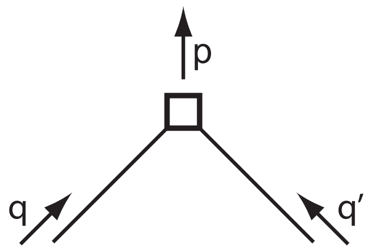

Figure 2: Two-point vertex coupling the field to the external source, which is localized on the wall.

3.2 Single Wall Vertex

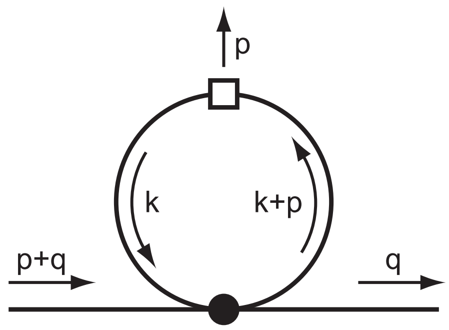

Figure 3: One-loop contribution to the two-point function that includes a single wall vertex.

Consider the one-loop diagram of Figure 3 contributing to the two-point function that includes a single wall vertex. This is logarithmically divergent in $D = 4$ dimensions, and therefore its divergent part is independent of q and is proportional to the wall vertex $\tilde{K}(p+q,q)$. This divergence may be absorbed into a renormalization of the coefficient of the wall vertex, $c \to c + \delta c$. We may impose the renormalization condition that the finite part of δc vanishes so as to maintain the decoupling of the two sides of the wall.

3.3 Multiple Wall Vertices

Consider the one-loop diagram of Figure 4 contributing to the two-point function that includes two wall vertices. This is logarithmically divergent in $D = 6$ dimensions (do not be distracted by the fact that ϕ^4 theory without walls is not renormalizable in six dimensions), and therefore its divergent part is independent of q, but it is *not* proportional to a single wall vertex. Therefore this divergence is *not* localized on the wall, and cannot be absorbed into a renormalization of the coefficient of the wall vertex.

In general, if more than one wall vertex appears in an overall divergent graph then the divergence is not localized on the wall.

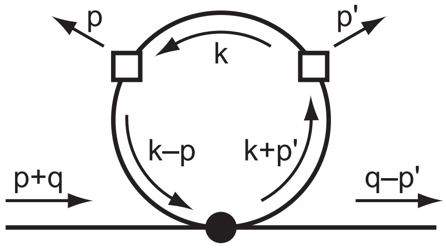

Figure 4: One-loop contribution to the two-point function that includes two wall vertices.

3.4 Power Counting

We can easily apply Dyson's power-counting threorem to wall vertices. In our example the wall interaction monomial in the action in D dimensions has dimension $D-2$, just like a mass term. Therefore for $D = 4$ the only overall divergent n-point functions with one wall vertex must have $n \leq 2$. If there are two or more wall vertices then $n \leq 0$. $n = 1$ is forbidden by $\phi \to -\phi$ symmetry, and $n = 0$ is uninteresting, so the only new counterterm required is proportional to the wall vertex and is therefore localized on the wall.

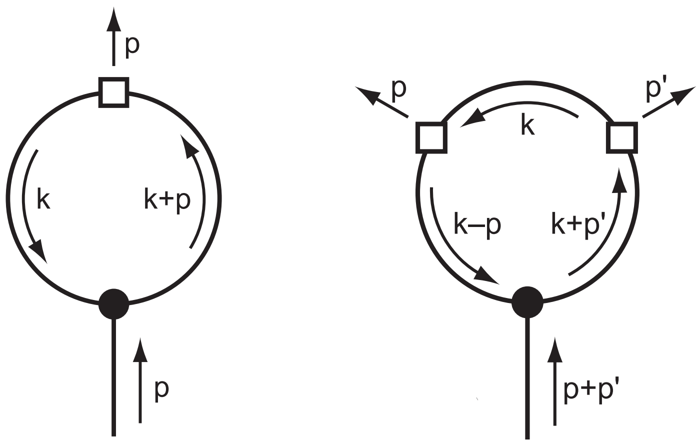

Figure 5: One-loop "tadpole" contributions to the one-point function.

The analysis for multi-loop diagrams is essentially the same. After removing all subdivergences using Bogoliubov's $\bar{R}$-operation there are only logarithmic divergences involving a single wall vertex for $D = 4$, which just gives a higher-order contribution to the counterterm δc in the loop expansion.

Observe that if we were to consider ϕ^3 theory instead of ϕ^4 theory then we would also need to consider divergent tadpoles such as the one-loop diagrams of Figure 5. These contribute to a non-uniform background source $J(x)$ for the field ϕ, but do not lead to any coupling of the opposite sides of the wall.

4. Conclusions

Momentum is not conserved at a wall vertex: this is not suprising, as the wall violates translational invariance. This corresponds to an "external" momentum p flowing into the wall; there is an integral of all possible values of p with a uniform distribution corresponding to the Fourier transform of the δ' function "shape" of the wall.

The wall interaction monomial in the action always has the same power-counting dimension as the mass term. Imposition of boundary conditions on the field by a local wall interaction induces counterterms that remain localized on the wall to all orders in perturbation theory provided that no more than one wall vertex can appear in any overall divergent two-point function.

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References

[1] H. B. G. Casimir, Proc. Kon. Nederland. Akad. Wetensch. **B51**, 793–795 (1948).

[2] K. Symanzik, Nucl. Phys. B **190**, 1–44 (1981).

[3] Martin Luscher, Rajamani Narayanan, Peter Weisz, and Ulli Wolff, Nucl. Phys. B **384** 168–228 (1992).

[4] S. Sint, Nucl. Phys. B **421**, 135 (1994) [hep-lat/9312079], Nucl. Phys. B **451**, 416 (1995) [hep-lat/9504005].

[5] J. H. Lowenstein and E. R. Speer, Commun. Math. Phys. **47**, 43–51 (1976).