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Renormalization of two-dimensional XQCD

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Recently, Kaplan proposed an interesting extension of QCD named Extended QCD or XQCD with bosonic auxiliary fields [1]. While its partition function is kept exactly the same as that of QCD, XQCD naturally contains properties of low-energy hadrons. We apply this extension to the two-dimensional QCD in the large N_c limit ('t Hooft model) [2]. In this solvable model, it is possible to directly examine the hadronic picture of the 2d XQCD and analyze its renormalization group flow to understand how the auxiliary degrees of freedom behave in the low energy region. We confirm that the additional scalar fields can become dynamical acquiring the kinetic term, and its parity-odd part becomes dominant in the low energy region. This renomalization of XQCD provides an "extension" of the renormalization scheme of QCD, inserting different field variables from those in the original theory, without any changes in physical observables.

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1. Introduction

In Ref. [1], Kaplan proposed an interesting reformulation of QCD named Extended QCD or XQCD. This new formulation contains additional auxiliary bosonic fields, keeping the partition function of QCD unchanged. The physics of XQCD is exactly the same as that of QCD, as long as the source operators of the ordinary quark and gluon fields are inserted. Kaplan has shown that XQCD can describe several low energy hadronic pictures such as the quark models, chiral perturbation theory and the bag models more naturally than QCD itself.

In this work, we study the Wilsonian renormalization group (RG) flow of the two-dimensional version of (X)QCD (we will simply denote QCD_2 or XQCD_2 in the following), in the large N_c limit, for which we have already published a paper [3]. This theory is known as the 't Hooft model [2] and the advantage of studying this model is that the theory is particularly simplified in the large N_c limit and solvable. We find that the auxiliary fields become dynamical when we take into account quantum corrections. In particular, the (pseudo)scalar auxiliary field should play a key role in the low energy effective action. It contains the degrees of freedom of pions, the lightest hadrons, as a consequence of the dynamical chiral symmetry breaking [4].

We also find that XQCD provides an interesting extension of the renormalization "scheme". Namely, we can insert an arbitrary number of new bosonic degrees of freedom at an arbitrary scale Λ_{cut} and the RG flow is extended to the space of their new interactions which are completely absent in the RG flow of QCD.

2. Extended QCD and its two-dimensional version

In this section, we review the original Extended QCD [1] in four dimensions and construct its two-dimensional version. We also summarize what is known in this two-dimensional large N_c QCD (the 't Hooft model).

2.1 XQCD in four dimensions

Let us consider QCD with N_f flavors of quarks and gauge group $SU(N_c)$ in four-dimensional Euclidean spacetime. XQCD is defined by introducing three types of auxiliary fields, the scalar field Φ, vector $\mathbf{v}_\mu$ and axial vector $\mathbf{a}_\mu$, with the action in a Gaussian form which keeps the original QCD partition function intact (up to a constant) :

$$Z_{\text{QCD}} = \int e^{-S_{\text{QCD}}[\psi,\bar{\psi},\mathbf{A}_\mu]} = \int e^{-S_{\text{QCD}}[\psi,\bar{\psi},\mathbf{A}_\mu]-S_{\text{aux}}[\Phi,\Phi^\dagger,\mathbf{v}_\mu,\mathbf{a}_\mu]} \equiv Z_{\text{XQCD}}, \tag{2.1}$$

where the above path integration is over all fields. Our new theory is given by

$$S_{\text{XQCD}} = N_c \int d^4x \left[\bar{\psi}(\mathscr{D}+m)\psi + \frac{1}{4g^2}\text{Tr }\mathbf{F}_{\mu\nu}\mathbf{F}_{\mu\nu} + \lambda^2\left(\text{Tr }\Phi^\dagger\Phi + \frac{1}{2}\text{Tr }[\mathbf{v}_\mu\mathbf{v}_\mu + \mathbf{a}_\mu\mathbf{a}_\mu]\right)\right], \tag{2.2}$$

where

$$\mathscr{D} \equiv \not{D} + \not{\mathbf{v}} + i\not{\mathbf{a}}\gamma_5 + 2(\Phi P_+ + \Phi^\dagger P_-). \tag{2.3}$$

Note that S_{XQCD} is manifestly renormalizable. Here, the color singlet Φ transforms as a bifundamental representation under the $SU(N_f)_L \times SU(N_f)_R$ chiral symmetry, and the flavor singlets $\mathbf{v}_\mu$

and $\mathbf{a}_\mu$ are $N_c \times N_c$ matrices (the singlet plus adjoint representations of the $SU(N_c)$ gauge group). Notice that when we integrate out all auxiliary fields from the theory, the resulting four-quark interactions are canceled by the Fierz identity:

$$(P_+)_{mn}(P_-)_{m'n'} + (P_-)_{mn}(P_+)_{m'n'} = \tfrac{1}{4}[(\gamma_\mu)_{mn'}(\gamma_\mu)_{m'n} - (\gamma_\mu\gamma_5)_{mn'}(\gamma_\mu\gamma_5)_{m'n}]. \tag{2.4}$$

Since the integration over auxiliary fields is just a constant, the expectation value of any operator involving gluon and quark fields only, is equivalent to that of QCD.

2.2 Application to the 't Hooft model

In this work, we consider the large N_c limit of QCD in two-dimensional Lorentzian spacetime, which is the so-called 't Hooft model [2]. The advantage of studying this model is that the theory is particularly simplified in the large N_c limit and solvable. In this subsection, we briefly review this model and construct its extended version.

It is most convenient to take the light-cone gauge: $\mathbf{A}_- = \mathbf{A}^+ = \frac{1}{\sqrt{2}}(\mathbf{A}^0 + \mathbf{A}^1) = 0$ to analyze this model. This gauge greatly simplifies the Feynman rule in the large N_c limit and enables us to non-perturbatively compute the quark self-energy. For example, the self-energy $\Sigma(p)$ of quarks satisfies a self-consistent equation (see Fig. 1) whose exact solution gives the square of the constituent quark mass M^2 as $M^2 = m^2 - g^2/\pi$.

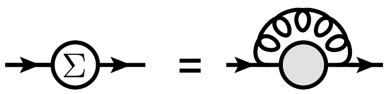

Figure 1: A diagrammatic expression of the self-consistent equation for the self-energy Σ(p).

To "extend" the 't Hooft model is almost straightforward as the original XQCD in four-dimensions with use of the Fierz identity of two-dimensional theories. The total action of XQCD_2 is given by

$$S_{\mathrm{XQCD}} = \int d^2x \left[\bar{\psi}\mathscr{D}'\psi + \frac{1}{2}\mathrm{Tr}\,(\partial_-\mathbf{A}_+)^2 - \lambda^2\left(\mathrm{Tr}\,\Phi^\dagger\Phi - \frac{1}{2}\mathrm{Tr}\,\mathbf{v}_\mu\mathbf{v}^\mu\right)\right], \tag{2.5}$$

where

$$\mathscr{D}' \equiv i\partial\!\!\!/ - \frac{g}{\sqrt{N_c}}\mathbf{A}_+\gamma^+ + \frac{i\alpha\lambda}{\sqrt{N_c}}\mathbf{v}\!\!\!/ - \frac{\sqrt{2}\alpha\lambda}{\sqrt{N_c}}(\Phi P_+ + \Phi^\dagger P_-), \tag{2.6}$$

and λ and α are arbitrary real parameters. The mass dimensions of auxiliary fields and parameters are given by $[\Phi] = [\mathbf{v}_\mu] = 0,\ [\alpha] = 0,\ [\lambda] = 1$.

3. Extended renormalization scheme

As explained above, although QCD and XQCD are exactly equivalent, their low energy expressions are expected to be different. To understand this more clearly, we perform the Wilsonian RG transformation on both theories and compare their low energy effective actions.

We would like to address two possible features of XQCD. One is how the mesonic degrees of freedom become dynamical. As Φ is expected to play a role of the NG boson at low energy, the RG flow should develop its kinetic term at low energy, keeping its mass almost zero. Another issue is to see what happens on the original quark and gluon sectors along the RG flow. As hadrons play more important roles at low energy, the original quarks and gluons should decrease their relevance, and can eventually be decoupled from the effective action, near the scale of their (constituent) masses.

Inclusion of the auxiliary fields extends the (relevant) parameter space of the theory. The new terms of the effective Lagrangian we should consider are

$$\mathrm{Tr}\partial_\mu\Phi^\dagger\partial^\mu\Phi,\ \mathrm{Tr}\partial_\nu\mathbf{v}_\mu\partial^\nu\mathbf{v}^\mu,\ \mathrm{Tr}(\partial_\mu\mathbf{v}^\mu)^2,\ \mathrm{Tr}\,\Phi^\dagger\Phi,\ \mathrm{Tr}\,\mathbf{v}_\mu\mathbf{v}^\mu,\ \bar{\psi}(\Phi P_+ + \Phi^\dagger P_-)\psi,\cdots \tag{3.1}$$

However, as the original theory has only two parameters g and m, the new interactions are not independent, but essentially controlled by these two parameters. Namely, the RG flows are restricted on a two-dimensional surface in the extended parameter space.

Which two-dimensional surface we take is determined by the choice of the regularization we use, and the re-definition of the coupling constants (by giving counterterms). Therefore, the choice of the surface corresponds to nothing but the choice of the renormalization scheme. Thus, XQCD can be regarded as the extension of the renormalization scheme to the extended theory space. The physics remains to be unchanged as the observables do not depend on the renormalization scheme. The extended renormalization scheme, allowing the new field contents, provides us a wider choice of the effective actions sharing the same physics.

4. RG flow of QCD$_2$ in the large N_c limit

In this subsection, we analyze the RG flow of QCD$_2$ itself, without introducing any auxiliary fields. We find a non-perturbative "solution" (in a truncated theory space), which reasonably interpolates the theory in the continuum limit and that at the constituent quark mass.

Our goal is to integrate out the high energy modes of the quark and gluon fields in QCD$_2$ and obtain an effective action S_Λ at a finite cut-off Λ. If we could employ a gauge-invariant regularization, we expect that S_Λ has a similar form to the bare action :

$$S_\Lambda^{\mathrm{QCD}} = \int d^2x\,\left[-\frac{1}{2}\mathrm{Tr}\,(\mathbf{A}_+)_R\partial_-^2(\mathbf{A}_+)_R + \bar{\psi}_R(i\partial\!\!\!/ - m_R(\Lambda))\psi_R - \frac{g_R(\Lambda)}{\sqrt{N_c}}\bar{\psi}_R\mathbf{A}_+\gamma^+\psi_R + \cdots\right], \tag{4.1}$$

where $(\mathbf{A}_+)_R$ and ψ_R denote the renormalized fields, and $m_R(\Lambda)$ and $g_R(\Lambda)$ are the renormalized mass and coupling constant. In this work, we truncate the higher order terms and neglect irrelevant contributions at $O(1/\Lambda^4)$.

The renormalized mass and coupling constant are computed in Ref. [3]. Here we just show the results,

$$m_R^2(\Lambda) = m^2\left(1+\frac{\frac{2g^2}{\pi\Lambda^2}\log\left|\frac{\Lambda^2}{M^2}\right|}{1-\frac{g^2}{\pi\Lambda^2}\log\left|\frac{\Lambda^2}{M^2}\right|}\right),\quad g_R^2(\Lambda) = \frac{1-\frac{g^2}{\pi\Lambda^2}}{\left(1-\frac{g^2}{\pi\Lambda^2}\log\left|\frac{\Lambda^2}{M^2}\right|\right)^2}g^2. \tag{4.2}$$

This is the non-perturbative result at the large N_c limit. The running of them are shown in Fig. 2. The solid curves are non-perturbative solutions, while the dashed ones are the one-loop results.

The flow shows that the coupling and mass do not monotonically increase but return to near the original bare values at $\Lambda \sim M$. Here, we make all quantities dimensionless using an arbitrarily chosen parameter Λ_0, and use $\bar{\Lambda} = \Lambda/\Lambda_0$ for the horizontal axis. The bare parameters are set to $g/\Lambda_0 = 1$ and $m/\Lambda_0 = 0.1$.

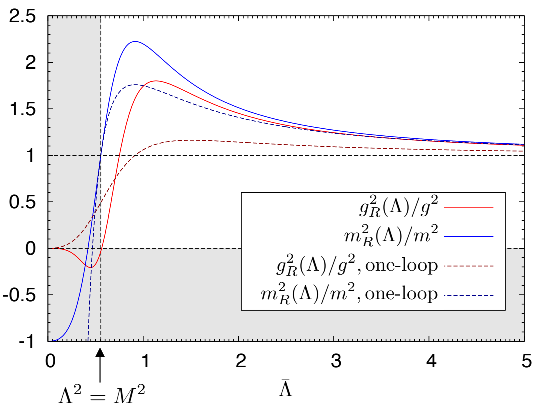

Figure 2: The RG running of the mass and coupling of QCD_2.

5. RG flow of $\mathbf{XQCD_2}$ in the large N_c limit

Now let us investigate the RG flow of XQCD_2 in the large N_c limit. As in the previous section, we truncate our theory space to neglect $O(1/\Lambda^4)$ terms. The large N_c limit also helps to reduce some redundancy of the extended theory space. For example, the kinetic term of $\mathbf{v}_\mu$ is never developed. With this simplification, the most general form of the effective action is

$$S_\Lambda^{\mathrm{XQCD}} = S_\Lambda^{\mathrm{QCD}} + \int d^2p \left[Z_\Phi(\Lambda)\mathrm{Tr}\, \Phi^\dagger p^2 \Phi - m_\Phi^2(\Lambda)\mathrm{Tr}\, \Phi^\dagger \Phi - \frac{\sqrt{2}y(\Lambda)}{\sqrt{N_c}} \bar{\psi}_R(\Phi P_+ + \Phi^\dagger P_-)\psi_R \right.$$
$$\left. + \frac{1}{2}\lambda^2 \mathrm{Tr}\, \mathbf{v}_\mu \mathbf{v}^\mu + i\frac{\alpha\lambda}{\sqrt{N_c}} \frac{Z_\psi(\Lambda)}{Z_\psi(\Lambda_{\mathrm{cut}})} \bar{\psi}_R \not{\mathbf{v}} \psi_R \right]. \tag{5.1}$$

Neglecting the overall normalization of the fields, our theory space is extended from 2 (with m_R and g_R) to 5 dimensions (since α and λ do not run).

As discussed in Sec. 3, we can define a number of new RG schemes in this extended theory space, by choosing a two-dimensional surface in it. The simplest (and trivial) scheme is to take the three constraints :

$$Z_\Phi(\Lambda) = 0, \quad m_\Phi^2(\Lambda) = \lambda^2, \quad y(\Lambda) = \alpha\lambda, \quad (\text{at any } \Lambda), \tag{5.2}$$

along the RG flow. With this scheme, one can always integrate Φ and $\mathbf{v}_\mu$ out and go back to original QCD_2 at any scale Λ. Since this scheme is exactly equivalent to the scheme in QCD_2, let us call it the "QCD scheme".

We are interested in more non-trivial schemes, where the hadronic degrees of freedom become relevant (let us denote it the "hadronization scheme"). Let us require the same form of the constraints as Eq. (5.2) but only at a point $\Lambda = \Lambda_{\mathrm{cut}}$:

$$Z_\Phi(\Lambda_{\mathrm{cut}}) = 0, \quad m_\Phi^2(\Lambda_{\mathrm{cut}}) = \lambda^2, \quad y(\Lambda_{\mathrm{cut}}) = \alpha\lambda. \tag{5.3}$$

Then, the RG flows can go inside the bulk of the extended five-dimensional space. In the following, we compute the RG flow of XQCD in this hadronization scheme and compare it with the QCD scheme.

5.1 One-loop analysis

Let us start with the computation at the one-loop. The three relevant diagrams in the large N_c limit are the quark self-energy, the Φ's self energy and Yukawa interaction (Fig. 3).

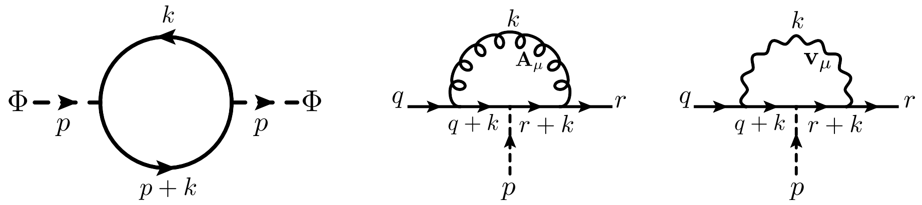

Figure 3: Φ's self energy and Yukawa interaction.

Already at this moment, we can answer to our first question about the RG flow of the quark and gluon fields in XQCD$_2$. The three diagrams show that the scalar (and pseudo-scalar) Φ field receives quantum corrections from ψ and $\mathbf{v}_\mu$, but never gives a feedback to them. Namely, the RG flow of the quark and gluon sector is unchanged. This result is not what we originally expected : weakening of the quark and gluon interactions. It seems that the two-dimension, the light-cone gauge, and the large N_c limit simplify the theory too much. We still expect a non-trivial difference in the case of four-dimensional QCD with $N_c = 3$.

Although there is no essential change in the RG flow of the quark mass and gauge coupling, the Feynman diagrams are quite different from those in original QCD. The essential change is in inclusion of the Yukawa interaction, which makes the mesonic degrees of freedom more relevant, as will be discussed below.

In Ref. [3], the RG flows of the renormalized quantities at the one-loop level are computed as

$$
\begin{aligned}
&Z_\Phi(\Lambda) = \frac{5y^2(\Lambda)}{6\pi}\left(\frac{1}{\Lambda^2} - \frac{1}{\Lambda_{\text{cut}}^2}\right) + O(\Lambda^{-4}), \;\; m_\Phi^2(\Lambda) = \lambda^2 - \frac{y^2(\Lambda)}{\pi}\log\left(\frac{\Lambda_{\text{cut}}}{\Lambda}\right) + O(\Lambda^{-2}), \\
&y(\Lambda) = \frac{\alpha\lambda}{1+\frac{\alpha_R^2(\Lambda)}{\pi}\log\left(\frac{\Lambda_{\text{cut}}}{\Lambda}\right)} + O(\Lambda^{-2}),
\end{aligned}
\tag{5.4}
$$

where we have used the initial conditions Eq. (5.3). As expected, the Φ field becomes a dynamical variable, developing its kinetic term.

5.2 Non-perturbative analysis

Here we assume that the chiral symmetry breaking in this model. Namely the VEV of Φ, which is related to the VEV of $\bar{\psi}\psi$, takes a non-zero value. Thus we may re-parametrize Φ as

$$
\Phi = \langle\Phi\rangle e^{\frac{\sigma+i\pi}{\sqrt{2}}}, \quad \langle\Phi\rangle = -\frac{1}{\sqrt{2N_c}}\frac{\alpha}{\lambda}\langle\bar{\psi}\psi\rangle + O(m). \tag{5.5}
$$

where σ and π are $N_f \times N_f$ hermitian matrices. With this parametrization, the masses of σ and π are given by

$$m_\sigma^2 = \frac{\alpha^2}{N_c} \langle \bar{\psi}\psi \rangle^2 + O(m) \ , \ \ m_\pi^2 = \tfrac{1}{2} m \langle \bar{\psi}\psi \rangle + O(m^2). \tag{5.6}$$

Since the mass of π is proportional to the quark mass, it vanishes in the chiral limit $m \to 0$. This GMOR relation [5] is kept along the renormalization flow as long as our renormalization scheme preserves the chiral symmetry. For σ, their mass is proportional to Λ^2 since the mass $Z_\Phi^{-1}(\Lambda) m_\Phi^2(\Lambda)$ is proportional to Λ^2. For the quarks, its mass is proportional to Λ since the Yukawa coupling $Z_\Phi^{-1/2}(\Lambda) y(\Lambda)$ is proportional to Λ. As we continue to integrate out high momentum modes, σ and quarks would decouple from the low energy dynamics at some scale, while π continues to contribute to the low energy dynamics. Eventually the theory is expected to go to the chiral effective theory described by the π field only and this confirms the low energy hadronic picture. We never reach this picture from the RG flow without auxiliary fields. In this way, the extension of the RG scheme introducing auxiliary fields gives a different aspect of the theory.

6. Summary

We have studied the RG flow of QCD_2 in the large N_c limit and its extension to $XQCD_2$.

For the RG flow of XQCD, the parameter space of the theory is extended and the choice of the surface corresponds to that of the (extended) renormalization scheme. In “hadronization scheme”, where the scalar auxiliary field Φ becomes dynamical while the vector auxiliary field $\mathbf{v}_\mu$ still remains to be an auxiliary, we confirm the hadronic picture of QCD. Due to the chiral symmetry breaking in QCD_2 at the large N_c limit, the only pions remain almost massless and relevant in the low energy region. Since the RG flow of QCD does not show this picture, we emphasize that we can never realize such a picture without taking into account the RG flow with auxiliary field.

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