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(Dimensional) twisted reduction in large N gauge theories

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We show that the spontaneous breaking of center symmetry can be avoided on a $L^2 \times 1^2$ lattice with the appropriate choice of twisted boundary conditions. In order for this to work it is crucial that the twisted boundary conditions are chosen in the reduced plane. This suggests that the choice of twist tensor can influence the directions in which color and space degrees of freedom become indistinguishable. We also present some preliminary quantitative data comparing the value of the plaquette for different forms of reduction.

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1. Introduction

$SU(N)$ Yang-Mills (YM) theories become volume independent in the large N limit [1]. In the context of lattice gauge theories it allows, in principle, to use a one-site lattice to obtain results in the large N limit and infinite volume. If implemented in the most naive way, this proposal was shown to fail [2], the reason being that the proof of volume independence relies on center symmetry, while for very small volumes (below the confinement scale), center symmetry is spontaneously broken. The literature contains several proposals to save the idea of volume independence [2, 3, 4, 5, 6, 7, 8, 9, 10]. In this work we will focus on the twisted reduction approach [3, 4, 11, 5]. The key idea is that the behavior of a field theory at small volumes depends crucially on the choice of boundary conditions, while the proof of volume reduction does not. In particular it has been shown recently that a judicious choice of twisted boundary conditions can avoid the spontaneous breaking of center symmetry. The choice of twisted boundary conditions are encoded in some phases $z_{\mu\nu} = \exp(2\pi\imath n_{\mu\nu}/N)$, where $n_{\mu\nu}$ is a tensor of integers modulo N.

The work [4] correctly argued that any choice of twist tensor $n_{\mu\nu}$ could do the job as long as it prevents the breaking of center symmetry. The study of this model at weak coupling suggests the particular choice of taking N a perfect square $N = \hat{L}^2$ and the symmetric twist tensor $n_{\mu\nu} = k\hat{L}$ where k and $\hat{L}$ are co-prime. The reason for the notation $\hat{L}$ is that the Feynman rules and propagators in the one-site model are identical to the Feynman rules on a lattice with $\hat{L}$ points in each dimension, except for some momentum-dependent phases that cancel in all non-planar diagrams. In this way the $\mathscr{O}(N^2)$ color degrees of freedom of the gluons are indistinguishable from the usual space degrees of freedom on a $\hat{L}^4$ lattice in the large N limit.

In this work we will explore the possibility of a non-symmetric twist. We will chose to twist only one plane (say the $x_1 - x_2$ plane), while our gauge fields will be strictly periodic in the other two directions x_0, x_3. We will argue that in this case the $\mathscr{O}(N^2)$ color degrees of freedom transform in space-time degrees of freedom only into the twisted plane $x_1 - x_2$. This will allow us to reproduce $SU(\infty)$ results in the infinite volume limit by exploring the large N limit of a theory with two short directions (x_1, x_2), and two infinite directions (x_0, x_3). We will call this the 2d TEK model. We will show that the pattern of spontaneous symmetry breaking will be consistent with this expectation: the strictly periodic directions x_0, x_3 must be kept "long" in order to avoid symmetry breaking.

2. Choice of twisted boundary conditions and simulation details

A comprehensive review of the twisted boundary conditions is beyond the scope of this work. We will stick to the basics and refer the reader to the existing literature [12, 13].

We are going to work in a four dimensional torus $\mathscr{T}^4$ of sides $L_0 \times L_1 \times L_2 \times L_3$. Twisted boundary conditions are imposed by requiring

$$A_\mu(x + L_\nu \hat{\nu}) = \Omega_\nu(x) A_\mu(x) \Omega_\nu^+(x) + \Omega_\nu(x) \partial_\mu \Omega_\nu^+(x)\,, \tag{2.1}$$

where matrices $\Omega_\mu(x)$ are called twist matrices, and they have to obey the consistency relation

$$\Omega_\mu(x + L_\nu \hat{\nu}) \Omega_\nu(x) = z_{\mu\nu} \Omega_\nu(x + L_\mu \hat{\mu}) \Omega_\mu(x) \tag{2.2}$$

N	24	36	40	56
k	7	13	11	23
$\bar{k}/N$	0.291666...	0.305555...	0.275	0.303571...

Table 1: Parameters of the runs

with $z_{\mu\nu}$ being elements of the center of $SU(N)$. They are gauge invariant, and therefore encode the physical part of the twisted boundary conditions. We are going to use a particular setup of this general scheme: we twist the plane x_1, x_2, while the gauge potential will be periodic in the directions x_0, x_3. This amounts to the choice

$$z_{\mu\nu} = z^*_{\nu\mu} = \begin{cases} \exp(2\pi\imath k/N) & \mu = 1 \text{ and } \nu = 2 \\ 1 & \text{otherwise} \end{cases} . \tag{2.3}$$

Finally if we choose the integer k to be co-prime with N, it is guaranteed that there is a unique field configuration that minimizes the action modulo gauge transformations (i.e. the only zero-modes of the action are the gauge degrees of freedom). This is crucial for our purposes, since constant configurations that are minima of the action ("torons") are the source of spontaneous center symmetry breaking.

It can be proved [12] that any gauge connection compatible with these particular boundary conditions can be written as

$$A^a_\mu(x)T^a = \frac{1}{\prod_\mu L_\mu} \sum_{\tilde{p}\neq 0} \tilde{A}_\mu(x,\tilde{p}) e^{\imath \tilde{p} x} \hat{\Gamma}(\tilde{p}), \tag{2.4}$$

where $\tilde{A}_\mu(x,\tilde{p})$ are complex functions (*not* matrices) periodic in x, and $\hat{\Gamma}(\tilde{p})$ are some well defined matrices (see [12] for more details). Finally the color-momentum $\tilde{p}_\mu$ is defined as

$$\tilde{p}_\mu = \frac{2\pi\tilde{n}_\mu}{NL_\mu}, \quad \tilde{n}_\mu = \begin{cases} 0 & \mu = 0,3 \\ 0,\ldots,N-1 & \mu = 1,2 \end{cases} . \tag{2.5}$$

Note that the function $\tilde{A}_\mu(x,\tilde{p})$, being periodic in the torus, has naturally momentum modes quantized in units of $2\pi/L_\mu$, but Eq. (2.4) suggests that actually in the directions x_1, x_2 the unit of quantization of the momentum is $2\pi/NL_\mu$ (i.e. the gauge field lives in an apparently larger lattice of size NL in the x_1, x_2 directions).

We still have to choose k. The experience with the TEK model and symmetric twist suggests that keeping $k = 1$ would result in the spontaneous breaking of center symmetry [11]. We will see that the same holds here, but the advice of [5] (i.e. $k/N > 1/9$) will also avoid the spontaneous breaking of center symmetry in our case. Moreover, based on the experience with the symmetric twist it seems reasonable to keep $\bar{k}/N$ as constant as possible, where $\bar{k}$ is defined by the condition $k\bar{k} = 1 \mod N$. The parameters that we will use in our runs are specified in Table 1. All simulations are done using standard over-relaxation techniques for $SU(N)$ after introducing auxiliary variables to linearize the action [14, 15].

3. Pattern of spontaneous symmetry breaking

Pushing the idea of volume independence to the limit, we should be able to obtain continuum results of $SU(\infty)$ in $\mathbb{R}^4$ by computing any observable on a $L_0/a \times 1 \times 1 \times L_3/a$ lattice. The $N \to \infty$

limit has to be taken at fixed bare coupling, and only later the continuum limit ($b \to \infty$ keeping a line of constant physics). It is easy to understand why center symmetry must be preserved: on an infinite lattice, open paths are protected from getting an expectation value by gauge invariance. On the other hand on our reduced lattice, the quantities $\mathrm{Tr}U_i(x)$ for $i = 1,2$ (i.e. the reduced directions in which the lattice has only one point) are gauge invariant, although not center invariant. The preservation of center symmetry in the reduced model can therefore be related with the preservation of gauge invariance on the infinite lattice. This naturally suggests to use as order parameters to study the breaking of center symmetry the quantities [11]

$$P_n = \frac{1}{N}\langle|\mathrm{Tr}U_i^n(x)|\rangle; \quad Q_{n,m} = \frac{1}{N}\langle|\mathrm{Tr}U_i^n(x)U_j^m(x)|\rangle\,. \tag{3.1}$$

These quantities have to be zero[1] for all $0 < n,m < N$. The most stringent constraint comes in fact from P_1, and this is the case that we will study in more detail. Figure 1 shows the comparison of the cases in which center symmetry is spontaneously broken, and cases in which this is not the case. The points are results of the simulations with the parameters of table 1 and the choice $L_{0,3}/a = N$. As the reader can observe, this choice of k (following [5]) avoids the breaking of center symmetry. On the other hand the same plot shows that the value of k plays a crucial role in avoiding the breaking of center symmetry: simulations with $k = 1$ shows this breaking. Also the choice of which directions are large and which ones are short cannot be arbitrary: changing the twist from the plane $x_1 - x_2$ to the plane $x_0 - x_3$ but keeping the same geometry (i.e. $L_{1,2}/a = 1$ and $L_{0,3}/a = N$) shows again signs of spontaneous symmetry breaking.

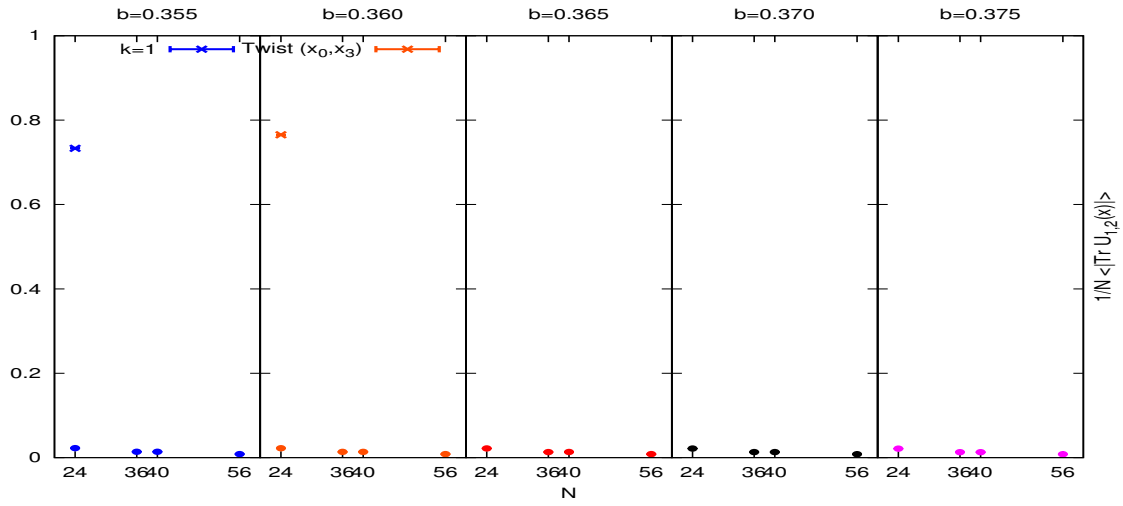


Figure 1: $\frac{1}{N}\langle|\mathrm{Tr}U_{1,2}(x)|\rangle$ for different values of b and N in a simulation of $SU(N)$ on a $N \times 1 \times 1 \times N$ lattice. Circles represent results obtained with our run parameters (Table 1), and crosses some test runs: One with $k = 1$ and another using twisted boundary conditions in the (x_0, x_3) plane.

4. Comparison with the TEK model with symmetric twist

Following the above discussion, results of the twisted reduced model with different choices

[1] More precisely, they have to go to zero as $N \to \infty$.

N	$b = 0.355$	$b = 0.360$	$b = 0.365$	$b = 0.370$
24	0.546681(11)	0.5592981(82)	0.5702507(67)	0.5801170(77)
36	0.5458441(63)	0.5585279(46)	0.5695444(41)	0.5794575(39)
40	0.5458078(47)	0.5584742(45)	0.5694683(31)	0.5793679(31)
56	0.5455784(56)	0.5582325(52)	0.5692438(46)	0.5791585(47)
∞ (d=2)	0.545326(40)	0.557988(35)	0.569018(17)	0.5789434(64)
∞ (d=4*)	0.545417(63)	0.558012(12)	0.569021(41)	0.578978(17)
∞ (d=0*)	0.545336(11)	0.558019(11)	0.569018(4)	0.578959(5)

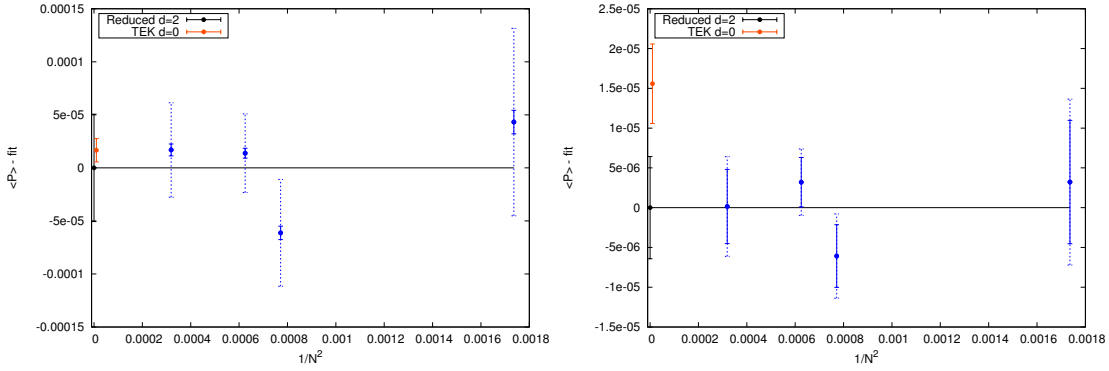
Table 2: Values of the average plaquette. Values with $N = \infty$ corresponds to extrapolated values. Values marked with an * are taken from [16]. See text for more details.

of twist should agree in the large N limit. In this section we will compare the usual one-site TEK model (which we will call 0d TEK) with symmetric twist with the choice of twist described above that we called 2d TEK model.

For this preliminary study we will choose as observable to perform the comparison the plaquette. Figure 3a shows the extrapolation of our data for the case of $b = 0.355$. In general the finite N corrections to any quantity will depend not only on N, but also on the choice of parameters L/a and k. How to take into account the variations of k, N and L/a in the $N \to \infty$ extrapolation requires a deeper understanding of the general structure of the corrections. For the case of the $0d$ TEK model, the corrections are parametrized by the quantity $\bar{k}/\sqrt{N}$ [13]. This suggests to keep constant the ratio $\bar{k}/N$ for the case of the $2d$ twisted reduction. We have chosen our parameters (Table 1) according to this criteria, but still we can observe some corrections to a perfect $1/N^2$ scaling. For example, Figures 2 show the difference between our data points and a linear $1/N^2$ fit. The solid lines are the statistical errors of the data points. The dotted lines represent an error bar the includes an estimate of the uncertainty due to our choices of k, N and L/a. This estimate is computed by using the χ^2/dof to weight the errors of the points in such a way that the fit becomes good (i.e. we multiply all error bars by a factor such that the χ^2/dof becomes 1). An important observation is that this effect is less important closer to the continuum.

Table 2 summarizes the values of the plaquette. The values at $N = \infty$ corresponds to extrapolations using different setups. The values quoted as $d = 4$ correspond to the naive extrapolation to $N \to \infty$ on a 4d lattice with linear size $L/a = 16$. The values quoted as $d = 0$ corresponds to results of the TEK model with symmetric twist. Both these values are taken from the existing literature [16]. Finally the values at finite N and the extrapolation quoted $d = 2$ corresponds to our setup with two large directions and two reduced ones. As the reader can see the quantitative agreement of the three scenarios is very good.

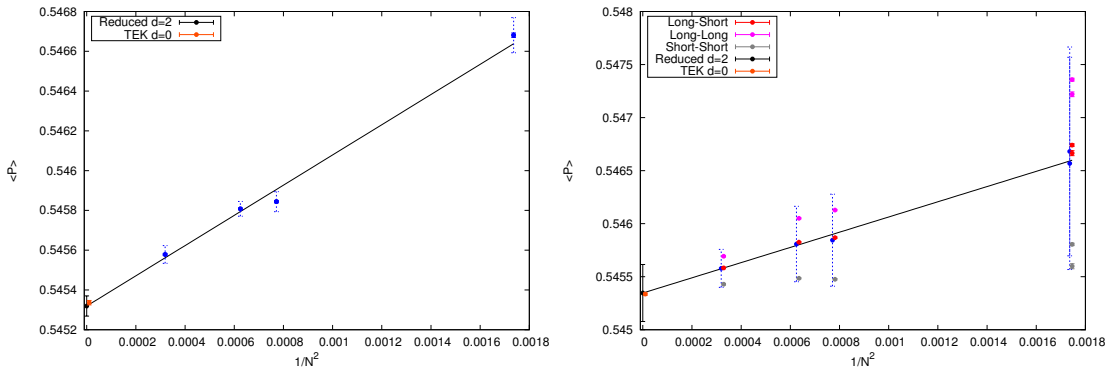
Fig. 3b shows the average plaquette values for each plane. In our simulations we have 3 types of planes, one made by short directions $(x_1 - x_2)$, one made by long directions $(x_0 - x_3)$, and four planes that mix the long and the short directions. The final plaquette values quoted in Table 2 are the average over all the planes, but since in the infinite volume and infinite N limit all planes are equivalent, the spread between these values can be used to estimate how far a certain simulation is from this limit. Figure 3b shows that indeed this is the case: the spread between different planes gets reduced when N increases. This is a sign that the $O(4)$ invariance of the infinite volume and



(a) $b = 0.355$ (b) $b = 0.370$

Figure 2: Oscillations around the $1/N^2$ scaling for $b = 0.355$ and $b = 0.370$. Solid error bars are the statistical errors. The difference between the dotted and the solid error bars are an estimation of the systematic uncertainty due to the variation of k, N and L/a. Note that this difference is smaller closer to the continuum.

infinite N limit is restored despite the fact that our simulations all have $L_1/a = L_2/a = 1$.



(a) $N \to \infty$ extrapolation (data of Table 2). The statistical error in the data is approximately of the size of the points. The dotted error bars are estimated with the fit quality (see text for more details).

(b) Average plaquette values for each plane. The spread between planes can be used to estimate how far observables at finite N are from the $N = \infty$ limit.

Figure 3: Average plaquette values for $b = 0.355$

5. Conclusions

In the context of twisted reduction [3, 4] we have studied the possibility of using a non-symmetric twist. We have shown that in order to avoid the spontaneous breaking of center symmetry (a necessary condition for reduction to hold), we have to keep the two strictly periodic directions large, while the directions on the twisted plane can be made arbitrarily small with a proper choice of twist tensor. In this sense the choice of twist tensor allows to make color degrees of freedom indistinguishable from space degrees of freedom in certain directions.

Beyond the theoretical interest in this form of twisted reduction, this choice of twist tensor might have some advantages. Among them one of the long directions can be naturally interpreted

as the time direction, allowing to compute masses by looking at the large euclidean time behavior of correlators (see also [17]). The reader should also note that the cost of matrix multiplications grows like $\mathcal{O}(N^3)$, while the computations on a lattice grows with the volume (therefore like $(L/a)^2$ for the non-reduced directions), making this choice of twisted reduction cheaper from a numerical point of view. Also the lattice structure provides a natural way to perform distributed computations. The main drawback is that, having to keep the space degrees of freedom in the simulations one can reach only smaller values of N, and therefore an extrapolation to $N \to \infty$ is still required. A deeper understanding of how the choices of k, N and L/a affect the $N \to \infty$ limit would be desirable.

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References

[1] T. Eguchi and H. Kawai, *Phys.Rev.Lett.* **48** (1982) 1063.

[2] G. Bhanot at al. *Phys.Lett.* **B113** (1982) 47.

[3] A. Gonzalez-Arroyo and M. Okawa, *Phys.Lett.* **B120** (1983) 174.

[4] A. Gonzalez-Arroyo and M. Okawa, *Phys.Rev.* **D27** (1983) 2397.

[5] A. Gonzalez-Arroyo and M. Okawa, *JHEP* **1007** (2010) 043, [arXiv:1005.1981].

[6] J. Kiskis et al. *Phys.Lett.* **B574** (2003) 65–74, [hep-lat/0308033].

[7] P. Kovtun et al. *JHEP* **0706** (2007) 019, [hep-th/0702021].

[8] G. Basar et al. *Phys.Rev.Lett.* **111** (2013), no. 12 121601, [arXiv:1306.2960].

[9] M. Unsal and L. G. Yaffe, *Phys.Rev.* **D78** (2008) 065035, [arXiv:0803.0344].

[10] T. Azeyanagi et al. *Phys. Rev.* **D82** (2010) 125013, [arXiv:1006.0717].

[11] M. Teper and H. Vairinhos, *Phys. Lett.* **B652** (2007) 359–369, [hep-th/0612097].

[12] A. González-Arroyo, *World Scientific. Proceedings of the Peñiscola 1997 advanced school on non-perturbative quantum field physics* (1998) Singapore, [hep-th/9807108].

[13] M. G. Perez et al. arXiv:1406.5655.

[14] H. Vairinhos, arXiv:1010.1253.

[15] M. G. Pérez et al. *JHEP* **06** (2015) 193, [arXiv:1505.0578].

[16] A. Gonzalez-Arroyo and M. Okawa, *JHEP* **1412** (2014) 106, [arXiv:1410.6405].

[17] A. González-Arroyo and M. Okawa, arXiv:1510.0542.

[18] A. Fanfarillo et al. *Proceedings of the 8th International Conference on Partitioned Global Address Space Programming Models. ACM* (2014).