PROCEEDINGS OF SCIENCE

Matrix Geometry and Coherent States

Goro Ishiki*
Graduate School of Pure and Applied Science, University of Tsukuba,
Tsukuba, Ibaraki 305-8571, Japan
E-mail: ishiki@het.ph.tsukuba.ac.jp

We propose a new description of the classical limit of the matrix geometry using the coherent states. We define the classical space as a set of the coherent states and also define geometric objects on the classical space in terms of the matrix elements which define the matrix geometry. Our method gives a new class of observables in matrix models, which contains geometric information of matrix configurations.

The 33rd International Symposium on Lattice Field Theory
14 -18 July 2015
*Kobe International Conference Center, Kobe, Japan**

*Speaker.

PoS(LATTICE 2015)295

1. Introduction

The matrix regularization plays an important role in formulating the superstring theory and M-theory [1]. This regularization preserves supersymmetries, so that it can be applied to the world-sheet or the worldvolume theory of a superstring or a supermembrane. The regularized theories are called the matrix models, which are defined by path integrals of some Hermitian matrices and are nonperturbatively well-defined. The matrix models are believed to contain not only the single string/membrane state but also multi-string/membrane states. Namely, they are expected to give the second quantized theories of the superstring or M-theory [2] [3].

A formal definition of the matrix regularization can be found in [4]. For a given symplectic manifold $\mathscr{M}$, its matrix regularization is defined as follows. Let $T_N : C^\infty(\mathscr{M}) \to M_N(C)$ be a linear map from smooth functions on $\mathscr{M}$ to $N\times N$ matrices and let I be a fixed index set made of a strictly increasing sequence of natural numbers. Then, if the sequence of the linear maps $\{T_N|N \in I\}$ satisfies the the four conditions shown in eq. (4.1)-(4.4) in [4], it is called the matrix regularization of $\mathscr{M}$. In particular, important properties of the matrix regularization are the following. For any $f,g \in C^\infty(\mathscr{M})$, it satisfies

$$\lim_{N\to\infty} ||T_N(f)T_N(g) - T_N(fg)|| = 0, \tag{1.1}$$

$$\lim_{N\to\infty} ||iN[T_N(f),T_N(g)] - T_N(\{f,g\})|| = 0, \tag{1.2}$$

where $||\cdot||$ is a matrix norm and $\{\ ,\ \}$ is the Poisson bracket on $\mathscr{M}$, respectively. Roughly speaking, the matrix regularization is an approximation for functions on $\mathscr{M}$ using $N\times N$ matrices, such that the algebraic structure is preserved, the Poisson tensor is realized as the commutator of matrices and the approximation becomes exact in the large-N limit.

Let us consider the situation relevant to the string and M-theories in which the manifold $\mathscr{M}$ is embedded in D-dimensional flat space. For a given embedding function $\{y^\mu\} : \mathscr{M} \to R^D (\mu = 1,2,\cdots,D)$, we denote by $X^\mu = T_N(y^\mu)$ the image under the matrix regularization. Here, an important condition for X^μ to be a regularization of y^μ is that

$$\lim_{N\to\infty} ||iN[X^\mu,X^\mu] - W^{\mu\nu}(X)|| = 0. \tag{1.3}$$

Here, $W^{\mu\nu}(X)$ is the image of the induced Poisson tensor, $W^{\mu\nu}(y) := \{y^\mu,y^\nu\}$[1]. Once X^μ with the desired property is obtained, the image of any function $f(y)$ can also be obtained by fixing an ordering rule, namely, the image is given by just replacing y^μ with X^μ under the ordering rule. So the problem of constructing the matrix regularization for the embedded space reduces to the problem of finding X^μ. For some symmetric space such as the sphere and torus, such coordinate matrices have been explicitly constructed. In the matrix model formulation of string theories, X^μ are treated as dynamical variables and their path integral is considered to realize various regularized configurations of strings or membranes.

The simplest example of the embedded matrix geometry is the fuzzy sphere. The ordinary unit sphere embedded in R^3 is described by the embedding functions $y^\mu (\mu = 1,2,3)$ satisfying

[1] The condition (1.1) says that any polynomial of the form $y^\mu y^\nu \ldots$, whose degree does not depend on N, is mapped to the matrix polynomial $X^\mu X^\nu \ldots$. Also the condition (1.2) implies that X^μ become mutually commuting. So if one ignores terms of $\mathscr{O}(1/N)$, $W^{\mu\nu}(X)$ can be defined as $W^{\mu\nu}(y)$ with y^μ replaced with X^μ under some fixed ordering rule.

$y^\mu y^\mu = 1$. The fuzzy sphere is defined by their images $X^\mu = T_N(y^\mu)$ given by

$$X^\mu = \frac{1}{\sqrt{\Lambda(\Lambda+1)}} L^\mu. \ (\mu = 1,2,3) \tag{1.4}$$

Here, L^μ is the representation matrix of the $SU(2)$ generators in the spin Λ representation and it satisfies $[L^\mu, L^\nu] = i\varepsilon^{\mu\nu\rho} L_\rho$ and $L^\mu L^\mu = \Lambda(\Lambda+1)\mathbf{1}$. The normalization constant in (1.4) is chosen so that $X^\mu X^\mu = \mathbf{1}$. From the $SU(2)$ Lie algebra and the fact that $W^{\mu\nu}(y) = \{y^\mu, y^\nu\} = \varepsilon^{\mu\nu\rho} y_\rho$ is satisfied for the standard Poisson bracket on S^2, one can easily check that X^μ satisfies (1.3). One can also construct the mapping for general functions on sphere. In general, a smooth function f on S^2 can be expanded by the spherical harmonics as

$$f(\Omega) = \sum_{J=0}^{\infty} \sum_{m=-J}^{J} f_{Jm} Y_{Jm}(\Omega), \tag{1.5}$$

where the harmonics are given by symmetric polynomials of y^μ as

$$Y_{Jm}(\Omega) = \sum_{l=0}^{J} c^{lm}_{\mu_1 \dots \mu_l} y^{\mu_1} \dots y^{\mu_l}, \tag{1.6}$$

with $c^{lm}_{\mu_1 \cdots \mu_l}$ constants. Let the image of (1.6) be

$$\hat{Y}_{Jm} = \sum_{l=0}^{J} c^{lm}_{\mu_1 \dots \mu_l} X^{\mu_1} \dots X^{\mu_l}. \tag{1.7}$$

Then, one can check that the equations (1.1) and (1.2) are satisfied for this mapping rule. What is important here is that any function is replaced by a matrix with the matrix size $N = 2\Lambda + 1$ and only a finite number (N^2) of $\hat{Y}_{Jm}$ is needed to expand matrices with $N = 2\Lambda + 1$. In fact, $\{\hat{Y}_{Jm} | J = 1, 2, \dots, N-1, m = -J, -J+1, \dots, J\}$ forms a complete basis of $N \times N$ matrices. Hence, a natural UV cutoff is introduced to the angular momentum by the matrix size and this is the reason why this procedure is called regularization. In contrast to the usual naive momentum cutoff scheme, the cutoff given by the matrix size is compatible with symmetries in field theories such as the gauge symmetry or the supersymmetry. This is because the regularized functions are $N \times N$ matrices and they form a ring (i.e. closed under the matrix multiplication), while the functions with the naive momentum cutoff do not. This is a great advantage of the matrix regularization.

A shortcoming of the matrix regularization is that once the functions are replaced by the matrices, the original geometric information of $\mathscr{M}$ becomes invisible at first sight. In the matrix models, the theory is defined by the path integration over all possible configurations of the Hermitian matrices X^μ. In order to understand those configurations as some states of strings or membranes, one needs a method of translating the matrix configurations into the usual differential geometric language. The main purpose of our study is to find a new quantities in matrix models, by which we can read off differential geometric information such as the shape of string worldsheet and the geometric objects defined on it.

This problem can be seen as the inverse problem of the construction of the matrix regularization. Namely, in our problem, a large-N sequence of the Hermitian matrices $\{(X^1, \dots, X^D) | N \in I\}$ are given and we consider how to find the associated classical space $\mathscr{M}$ and how to construct the geometric objects from the matrices. We assume that the given matrices X^μ are norm-bounded. This condition is needed to define some geometric objects [5].

PoS(LATTICE 2015)295

2. Classical limit of matrix geometry

2.1 Classical space

A reasonable way to associate a classical space with the given sequence $\{(X^1,...,X^D)|N \in I\}$ is provided by the coherent states. Let us recall the noncommutative plane (quantum mechanics), where the coordinate operators satisfy the canonical commutation relation, $[X^1,X^2] = i\theta$. In this case the classical space, which should be a plane, can be defined as a set of all canonical coherent states. The coherent states are those which have minimal wave packets and their wave packets shrink to points in the classical limit $\theta \to \infty$. So they are the most natural analogue of the points on the classical space. It is easy to imagine that such minimal wave packet can exist everywhere on the classical space and in fact the coherent states are labelled by points on the classical phase space $(p,q) \in R^2$. The classical space can then be reconstructed as a set of all coherent states.

It is also possible to find the similar construction for the fuzzy sphere. The so-called Bloch coherent states, which minimize the squared sum of the standard deviations of the coordinate matrices, are in one-to-one correspondence with the points on the sphere.

In this way, at least in those specific examples, the classical spaces can be reconstructed from the given matrices as a set of the coherent states. In order to generalize this construction to more general cases, we first extract some common features of the canonical and Bloch coherent states [5]. Firstly, we note that the both coherent states are the ground states of the Hamiltonians of the form,

$$H(y) = \frac{1}{2}(X^\mu - y^\mu \mathbf{1})^2. \tag{2.1}$$

where $y^\mu (\mu = 1,2,...,D)$ are real parameters and $D = 2,3$ for the noncommutative plane and the fuzzy sphere, respectively.

Secondly, the classical spaces are given by the loci of zeros of the ground state energy $E_0(y)$ of $H(y)$ in the classical limit. One can understand this as follows. The ground state energy of the Hamiltonian can be written as

$$E_0(y) = \frac{1}{2}(\Delta X^\mu)^2 + \frac{1}{2}(\langle X^\mu \rangle - y^\mu)^2, \tag{2.2}$$

where ΔX^μ is the standard deviation of the coordinate operator and the $\langle X^\mu \rangle$ is the expectation value for the ground state. When $E_0(y)$ is vanishing, both ΔX^μ and $\langle X^\mu \rangle - y^\mu$ are vanishing. From this, it follows that the ground state energy vanishes if and only if there exists a wave packet which shrinks to the point.

This description can be generalized to more general cases [5]. Namely, for a given matrices, $\{(X^1,...,X^D)|N \in I\}$, we first define the Hamiltonian by (2.1) and then define the associated classical space by

$$\mathscr{M} = \{y \in R^D | f(y) = 0\}, \tag{2.3}$$

where the function $f(y)$ is the large-N limit of the ground state energy,

$$f(y) = \lim_{N\to\infty} E_0(y). \tag{2.4}$$

Since the function $f(y)$ is in principle computable from the given matrices, the definition (2.3) gives a practical method of finding the classical space form the matrices.

For generic matrices, the classical space defined in (2.3) would be a non-manifold or an empty set. However, an interesting situation relevant to the string or M-theory is the case in which the classical space forms a smooth manifold. So in the following, we assume that when N is sufficiently large the ground state energy is differentiable near $\mathscr{M}$, so that $\mathscr{M}$ is a smooth manifold.

In the string theory, one can find the origin of the Hamiltonian (2.1) in some systems with D-branes. The hamiltonian is just the Laplacian for the massless bosonic modes of the open strings connecting a probe brane with the target D-branes [6] [7] or it can also be seen as the tachyon potential for non-BPS D-brane systems [8].

2.2 Tangent space

Most theories of modern differential geometry are based on the notion of the tangent space. In order to develop differential geometry on $\mathscr{M}$, here we define the tangent space of $\mathscr{M}$ in terms of the given matrices. At each point $y \in \mathscr{M}$, we can consider D-dimensional vectors $A^\mu(y)$ in R^D. There must exist a projection operator from those vectors to the tangent vectors of $\mathscr{M}$. We find that the projection operator is given by [5]

$$g^\mu{}_\nu = \delta^\mu_\nu - \partial^\mu \partial_\nu f(y), \tag{2.5}$$

where we raise and lower the D-dimensional indices by the Kronecker delta (the flat metric on the target space), so that $\partial^\mu = \partial_\mu$. The tangent vectors are characterized by the equation,

$$g^\mu{}_\nu A^\nu = A^\mu. \tag{2.6}$$

Note that in principle the derivatives of $f(y)$ (and hence $g_{\mu\nu}$) can be computed from the given matrices by using the perturbation theory. For a sufficiently small vector ε^μ, the Hamiltonian varies by $H(y+\varepsilon) = H(y) + \varepsilon \cdot (y - X) + \frac{1}{2}\varepsilon^2$. The deviation of the ground state energy can be computed by treating the terms with ε^μ as the perturbative correction. The large-N limits of those correction terms for the ground state energy give the derivatives of $f(y)$. More specifically, if we introduce the eigenstates of $H(y)$ by

$$H(y)|n,y\rangle = E_n(y)|n,y\rangle, \tag{2.7}$$

the projection operator can also be written as

$$g^\mu{}_\nu(y) = 2 \lim_{N\to\infty} \sum_{n\neq 0} \mathrm{Re} \frac{\langle 0,y|X^\mu|n,y\rangle \langle n,y|X_\nu|0,y\rangle}{E_n(y) - E_0(y)}. \tag{2.8}$$

This relation directly connects the matrix elements to the geometric object $\{g^\mu{}_\nu\}$ through the formula in the perturbation theory of quantum mechanics.

3. Geometric objects

In this section, we propose a description of the metric, connection, curvature and Poisson tensor on $\mathscr{M}$ in terms of the given matrices.

Metric

The metric is a positive, symmetric and non-degenerate tensor on $\mathscr{M}$. Since (2.8) satisfies these properties on the tangent vectors[2], it gives the metric on $\mathscr{M}$. This is the induced metric on $\mathscr{M}$.

For the case of fuzzy sphere (1.4) with the unit radius, it is given by

$$g^{\mu}{}_{\nu}(y) = \frac{1}{|y|}\left(\delta^{\mu}_{\nu} - \frac{y^{\mu}y_{\nu}}{|y|^2}\right). \tag{3.1}$$

This is indeed a projection to the tangent vectors on the unit sphere (i.e. for $|y| = 1$) and hence gives a metric on S^2. If one use the standard polar coordinates on S^2, which solve $(y^{\mu})^2 = 1$, one finds that it gives the standard metric on S^2, $ds^2 = g_{\mu\nu}dy^{\mu}dy^{\nu} = d\theta^2 + \sin^2\theta d\phi^2$.

Levi-Civita connection

Let $A^{\mu}(y)$ and $B^{\mu}(y)$ be two tangent vectors at y satisfying (2.8). Then, we define the following linear covariant derivative:

$$(\nabla_B A)^{\mu} = B^{\nu}(\partial_{\nu}A^{\mu} + \Gamma^{\mu}_{\nu\rho}A^{\rho}). \tag{3.2}$$

We choose the connection $\Gamma^{\mu}_{\nu\rho}$ in such a way that the image of ∇_B is again a tangent vector, namely, $g^{\mu}{}_{\nu}(\nabla_B A)^{\nu} = (\nabla_B A)^{\mu}$. We find a solution as

$$\Gamma^{\mu}_{\nu\rho} = (\partial^{\mu}\partial^{\lambda} f)(\partial_{\nu}\partial_{\rho}\partial_{\lambda} f). \tag{3.3}$$

We also find that this choice preserves the metric, so it defines the Levi-Civita connection associated with $g^{\mu}{}_{\nu}$. The connection (3.3) can also be directly computed from the given matrices, since the second and the third derivatives of f are computable using the perturbation theory.

Curvature

Let A^{μ}, B^{μ} and C^{μ} be tangent vectors on $\mathscr{M}$. The curvature tensor is defined as usual by

$$R(A,B)C = [\nabla_A, \nabla_B]C - \nabla_{[A,B]}C, \tag{3.4}$$

where $[A,B]$ is the Lie bracket of the vector fields, $[A,B]^{\mu} = A^{\nu}\partial_{\nu}B^{\mu} - B^{\nu}\partial_{\nu}A^{\mu}$. Note that this is also an tangent vector satisfying (2.8). If we write $(R(A,B)C)^{\mu} = A^{\nu}B^{\rho}C^{\sigma}R^{\mu}{}_{\sigma\nu\rho}$, then each component of the curvature is given by

$$(R(A,B)C)^{\mu} = A^{\nu}B^{\rho}C^{\sigma}\left\{(\partial_{\mu}\partial_{\nu}\partial_{\lambda} f)(\partial_{\rho}\partial_{\sigma}\partial_{\lambda} f) - (\partial_{\mu}\partial_{\rho}\partial_{\lambda} f)(\partial_{\nu}\partial_{\sigma}\partial_{\lambda} f)\right\}. \tag{3.5}$$

Again, the formulae in perturbation theory directly relate the curvature with the given matrix elements.

[2]More precisely, $g_{\mu\nu} = \delta_{\mu\rho}g^{\rho}{}_{\nu}$ satisfies the properties.

Poisson tensor

Let us consider the $D \times D$ real antisymmetric matrix defined for each $y \in \mathscr{M}$ as

$$W^{\mu\nu}(y) = i \lim_{N\to\infty} c_N \langle 0,y|[X^\mu, X^\nu]|0,y\rangle, \tag{3.6}$$

where c_N is a positive constant depending on N, which makes $W^{\mu\nu}$ nonvanishing. One can prove that (3.6) gives a Poisson tensor on $\mathscr{M}$ [5]. Namely, it is a tangent bivector on $\mathscr{M}$ satisfying $g^\mu{}_\nu W^{\nu\rho} = W^{\mu\rho}$ and the Poisson bracket defined by $\{f,g\} = W^{\mu\nu}(\partial_\mu f)(\partial_\nu g)$ satisfies the Jacobi identity.

4. Summary and discussions

We proposed a new construction of the classical limit of the matrix geometry described by a sequence of D Hermitian matrices, $\{(X_1^{(N)}, \cdots, X_D^{(N)})|N \in I\}$. We used the Hamiltonian (2.1) to describe its classical limit. The classical space $\mathscr{M}$ is given by the loci of zeros of the ground state energy in the large-N limit as (2.3). When $\mathscr{M}$ is a smooth manifold (more precisely, when the ground state energy is smooth around the loci of its zeros in a sufficiently large-N region), we also found an appropriate description of the tangent space of $\mathscr{M}$. The tensor (2.8) gives a projection from the vectors in R^D to the tangent vectors on $\mathscr{M}$. We also proposed the description of some geometric objects on $\mathscr{M}$ such as the metric, Levi-Civita connection, curvature and Poisson tensor.

Our definitions of the geometric objects as well as $\mathscr{M}$ itself are invariant under the $U(N)$ transformation given by $X^\mu \to U^\dagger X^\mu U$, which is the gauge transformation in the context of the matrix models. So they provide a new class of observables in the matrix models, which enables us to extract geometric information of the matrix-regularized strings or membranes.

These observables can also be defined in higher dimensional gauge theories. We consider it very interesting if the emergent geometry in the gauge/gravity correspondence are indeed visualized by using those observables.

Acknowledgments

This work was supported, in part, by Program to Disseminate Tenure Tracking System, MEXT, Japan.

References

[1] B. de Wit, J. Hoppe and H. Nicolai, Nucl. Phys. B **305**, 545 (1988).

[2] T. Banks, W. Fischler, S. H. Shenker and L. Susskind, Phys. Rev. D **55**, 5112 (1997).

[3] N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, Nucl. Phys. B **498**, 467 (1997).

[4] J. Arnlind, J. Hoppe and G. Huisken, J. Diff. Geom. **91**, 1 (2012).

[5] G. Ishiki, Phys. Rev. D **92**, no. 4, 046009 (2015).

[6] A. Chatzistavrakidis, H. Steinacker and G. Zoupanos, JHEP **1109**, 115 (2011).

[7] D. Berenstein and E. Dzienkowski, Phys. Rev. D **86**, 086001 (2012).

[8] T. Asakawa, S. Sugimoto and S. Terashima, JHEP **0203**, 034 (2002).