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Isospin-breaking effects in K_{e4} decays of the charged kaon

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The very precise data on the decay channels $K^\pm \to \pi^+\pi^- e^\pm \overset{(-)}{\nu_e}$ and $K^\pm \to \pi^0\pi^0 e^\pm \overset{(-)}{\nu_e}$ obtained by the NA48/2 collaboration at the CERN SPS have made it necessary to consider effects due to isospin breaking. Three issues related to this aspect will be considered: isospin breaking due to the pion mass difference in the phases of the form factors for the $K^\pm \to \pi^+\pi^- e^\pm \overset{(-)}{\nu_e}$ decay amplitude, the cusp in the decay distribution of the $K^\pm \to \pi^0\pi^0 e^\pm \overset{(-)}{\nu_e}$ mode, and radiative corrections. This contribution provides a summary of the material and the more extensive discussions published in Refs. [1, 2, 3].

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1. Introduction

Low-energy $\pi\pi$ scattering remains a privileged place where we can test our understanding of the chiral structure of the QCD vacuum. So far, there are, however, only a handful of experimental processes that provide access to low-energy $\pi\pi$ scattering data with the required degree of precision. Among these, one may mention the lifetime and energy levels of pionic atoms [4, 5], the decay of the charged kaon into one charged and two neutral pions [6], and $K_{\ell 4}$ decays. This presentation is devoted to the latter case, where the experimental situation has witnessed an impressive evolution in recent years. For quite some time, the high-statistics Geneva-Saclay experiment [7], with its 30 000 events of the $K^+ \to \pi^+\pi^- e^+ \nu_e$ decay mode of the charged kaon, has remained unequalled. The situation improved notably with the $\sim 400\,000$ events collected by the Experiment 865 [8, 9] at the Brookhaven AGS about fifteen years ago. Finally, an even more spectacular number of more than 1 000 000 events, with comparable statistics in both charged modes, i.e. $K^+ \to \pi^+\pi^- e^+ \nu_e$ (roughly 2/3 of the total sample) and $K^- \to \pi^+\pi^- e^- \bar{\nu}_e$ (roughly 1/3 of the total sample), was collected and analysed by the NA48/2 Collaboration [10, 11, 12] at the CERN SPS. These decay modes will be referred to as K^{+-}_{e4}.

In addition, the NA48/2 Collaboration has also published [13] an analysis concerning a sample of $\sim 65\,000$ events in the mode with two neutral pions, i.e. $K^+ \to \pi^0\pi^0 e^+ \nu_e$ and $K^- \to \pi^0\pi^0 e^- \bar{\nu}_e$, referred to as K^{00}_{e4}. These modes offer an interesting cross-check with the K^{+-}_{e4} modes. Indeed, in the isospin limit, their amplitudes have one form factor in common. Measuring this form factor independently in each mode thus allows to test our understanding of isospin-breaking effects.

Let us now briefly present the three issues to be discussed in relation with these recent experimental achievements.

• A standard angular analysis of the K^{+-}_{e4} form factors [14, 15] provides information on low-energy $\pi\pi$ scattering (Watson's theorem) through the phase difference between the S and P waves, $[\delta_S(s) - \delta_P(s)]_{\text{exp}}$. Comparison with solutions of the Roy equations [16] for these phase shifts allows one to extract the values of the $\pi\pi$ S-wave scattering lengths a_0^0 and a_0^2 in the isospin channels $I = 0, 2$,

$$[\delta_S(s) - \delta_P(s)]_{\text{exp}} = f_{\text{Roy}}(s; a_0^0, a_0^2). \tag{1.1}$$

The Roy equations follow from dispersion relations (that is, analyticity, unitarity, crossing, and the Froissard bound), data at energies $\sqrt{s} \geq 1$ GeV, and *isospin symmetry*. Solutions $f_{\text{Roy}}(s; a_0^0, a_0^2)$ to these equations have been constructed for (a_0^0, a_0^2) belonging to a restricted domain called the Universal Band, see Refs. [17, 18]. Once radiative corrections have been taken care of (see below), it is still necessary to take *isospin-breaking corrections* due to $M_\pi \neq M_{\pi^0}$ into account (M_π stands for the mass of the charged pion) before comparing with the data [19]. Such a calculation has been done at one loop in chiral perturbation theory [20], and as a result, Eq. (1.1) becomes

$$[\delta_S(s) - \delta_P(s)]_{\text{exp}} = f_{\text{Roy}}(s; a_0^0, a_0^2) + \delta f_{\text{IB}}(s; (a_0^0)^{\text{LO}}_{\text{ChPT}}, (a_0^2)^{\text{LO}}_{\text{ChPT}}). \tag{1.2}$$

However, the correction term $\delta f_{\text{IB}}(s; (a_0^0)^{\text{LO}}_{\text{ChPT}}, (a_0^2)^{\text{LO}}_{\text{ChPT}})$ is evaluated at fixed values of the scattering lengths, given by their lowest-order values in chiral perturbation theory, i.e. $(a_0^0)^{\text{LO}}_{\text{ChPT}} = 7M_\pi^2/(32\pi F_\pi^2) = 0.16$ and $(a_0^2)^{\text{LO}}_{\text{ChPT}} = -M_\pi^2/(16\pi F_\pi^2) = -0.045$ [21]. This limitation is shared by other studies [22, 23, 24] of isospin-breaking corrections at one loop in the low-energy expansion.

The crucial question is whether this situation induces a bias in the determination of the scattering lengths from the NA48/2 data, at the level of precision that has been reached today. In order to answer this question, one would like to obtain an expression of the form

$$[\delta_S(s) - \delta_P(s)]_{\rm exp} = f_{\rm Roy}(s; a_0^0, a_0^2) + \delta f_{\rm IB}(s; a_0^0, a_0^2), \tag{1.3}$$

where the scattering lengths appear as free parameters, to be determined from the data, both in the solution of the Roy equations $f_{\rm Roy}(s; a_0^0, a_0^2)$ *and* in the correction factor $\delta f_{\rm IB}(s; a_0^0, a_0^2)$. The first issue to be discussed shows how this question can be answered in a positive manner by constructing such a function $\delta f_{\rm IB}(s; a_0^0, a_0^2)$ that is in addition valid at two loops in the low-energy expansion.

• The distribution with respect to the invariant mass of the two neutral pions in the K^{00}_{e4} decay channels shows a unitarity cusp at $2M_\pi$, corresponding to the opening of the intermediate state with two charged pions [25]. Like in the case of the decay modes $K^\pm \to \pi^0\pi^0\pi^\pm$ [26, 27], this cusp contains information on the combination $a_0^0 - a_0^2$ of the $\pi\pi$ scattering lengths. The second issue is the question whether the phenomenological description of the cusp can impinge on the determination of the normalization of the form factor, an issue that is relevant for the next point, as well as the expected precision with which the information on $a_0^0 - a_0^2$ can be extracted from K^{00}_{e4} data.

• In the *isospin limit*, the matrix elements for K^{+-}_{e4} and K^{00}_{e4} have a form factor in common. This feature can be tested with the available data. Denoting by f_s this form factor, its experimental determinations from K^{+-}_{e4} [10] and from K^{00}_{e4} [13] give

$$\begin{aligned} |V_{us}|\, f_s[K^{+-}_{e4}] &= 1.285 \pm 0.001_{\rm stat} \pm 0.004_{\rm syst} \pm 0.005_{\rm ext} \\ (1+\delta_{EM})|V_{us}|\, f_s[K^{00}_{e4}] &= 1.369 \pm 0.003_{\rm stat} \pm 0.006_{\rm syst} \pm 0.009_{\rm ext}, \end{aligned} \tag{1.4}$$

respectively. This implies

$$(1+\delta_{EM}) \frac{f_s[K^{00}_{e4}]}{f_s[K^{+-}_{e4}]} = 1.065 \pm 0.010. \tag{1.5}$$

In the case of the K^{+-}_{e4} modes, radiative corrections were taken into account. In the K^{00}_{e4} case, no radiative corrections were applied, hence the presence of the factor δ_{EM}. This correction factor is not available from the existing literature (the discussion in Ref. [28] is not very explicit, and hence not useful). Besides radiative corrections, there are also isospin-breaking corrections due to the difference between the *up* and *down* quark masses m_u and m_d, conveniently described by the parameter R, with $1/R = (m_d - m_u)/(m_s - m_{ud})$, where m_s is the mass of the *strange* quark, whereas m_{ud} denotes the average mass of the *up* and *down* quarks, $m_{ud} = (m_u + m_d)/2$. For instance, at lowest order in the chiral expansion, one has [22, 29]

$$\frac{f_s[K^{00}_{e4}]}{f_s[K^{+-}_{e4}]} = \left(1 + \frac{3}{2R}\right). \tag{1.6}$$

Barring contributions of higher-order corrections, values of R as small as [30] $R = 35.8(1.9)(1.8)$ can account for about two thirds of the effect in Eq. (1.5). The third issue concerns thus the evaluation of δ_{EM}. In order to make the interpretation of Eq. (1.5) meaningful, the evaluation of δ_{EM} should be carried out within the *same* framework as used in the analysis of the K^{+-}_{e4} data.

2. Isospin breaking in the phases of the two-loop K_{e4} form factors

The goal here is to obtain a representation for the K_{e4} form factors: i) that is valid at two loops in the low-energy expansion, ii) where the $\pi\pi$ scattering lengths occur as free parameters, and iii) with isospin-breaking effects due to $M_\pi \neq M_{\pi^0}$ included. This has been done in Ref. [2] by adapting the approach ("reconstruction theorem") first introduced and described in Ref. [31] for the $\pi\pi$ scattering amplitude, and implemented explicitly in Ref. [32]. This method rests on very general principles, relativistic invariance, analyticity, unitarity, crossing, and chiral counting. Isospin symmetry itself is not required. An iterative two-step construction then yields a two-loop representation for meson scattering amplitudes and K_{e4} form factors. As an outcome of this construct, the phases of the S- and P-wave projections of the form factors can be expressed as

$$\delta_S(s,s_\ell) = \sum_{\{a',b'\}} \frac{1}{\mathscr{S}_{a'b'}} \frac{\lambda^{\frac{1}{2}}_{a'b'}(s)}{s} \left[\varphi_0^{a'b';+-}(s)\, \frac{F^{a'b'}_{S[0]} + F^{a'b'}_{S[2]}(s,s_\ell)}{F_{S[0]} + F_{S[2]}(s,s_\ell)} + \psi_0^{a'b';+-}(s)\, \frac{F^{a'b'}_{S[0]}}{F_{S[0]}} \right] \theta(s - s_{a'b'}) \\ + \mathscr{O}(E^6), \tag{2.1}$$

and

$$\delta_P(s,s_\ell) = \sum_{\{a',b'\}} \frac{\lambda^{\frac{1}{2}}_{a'b'}(s)}{s} \frac{\lambda^{\frac{1}{2}}_{a'b'}(s)}{\lambda^{\frac{1}{2}}_{ab}(s)} \left[\varphi_1^{a'b';+-}(s)\, \frac{G^{a'b'}_{P[0]} + G^{a'b'}_{P[2]}(s,s_\ell)}{G_{P[0]} + G_{P[2]}(s,s_\ell)} + \psi_1^{a'b';+-}(s)\, \frac{G^{a'b'}_{P[0]}}{G_{P[0]}} \right] \theta(s - s_{a'b'}) \\ + \mathscr{O}(E^6). \tag{2.2}$$

In these expressions, the sums run over all possible mesonic two-particle intermediate states $\{a',b'\}$ that can contribute when s, the square of the invariant mass of the $\pi^+\pi^-$ pair, exceeds the threshold value $s_{a'b'}$. Their complete list, together with the corresponding expressions for the leading-order (in the low-energy expansion) form factors $F^{a'b'}_{S[0]}$ and $G^{a'b'}_{P[0]}$ (note that $F_{S[0,2]} \equiv F^{\pi^+\pi^-}_{S[0,2]}$, $G_{P[0,2]} \equiv G^{\pi^+\pi^-}_{P[0,2]}$) are given in Table 2 of Ref. [2]. The phase-space factors are expressed in terms of the appropriate triangle or Källen functions, $\lambda_{ab}(s) = s^2 - 2s(M_a^2 + M_b^2) + (M_a^2 - M_b^2)^2$. In the case of the P-wave phase $\delta_P(s,s_\ell)$, there can be no contribution from states with two identical particles due to Bose symmetry, making the symmetry factor $1/\mathscr{S}_{a'b'}$ in $\delta_P(s,s_\ell)$ superfluous. Furthermore, $\varphi_0^{a'b';+-}(s)$ and $\varphi_1^{a'b';+-}(s)$ denote the partial-wave projections of the lowest-order scattering amplitudes for the processes $a'b' \to \pi^+\pi^-$. These are the only quantities that contribute to the phases of the one-loop form factors. The phases of the two-loop form factors receive corrections at the next order in the low-energy expansion. These corrections materialize as corrections $F^{a'b'}_{S[2]}$ and $G^{a'b'}_{P[2]}$ to the form factors, and as corrections $\psi_0^{a'b';+-}(s)$ and $\psi_1^{a'b';+-}(s)$ to the partial-wave projections. Through these corrections, the phases $\delta_S(s,s_\ell)$ and $\delta_P(s,s_\ell)$ depend also on s_ℓ, the square of the invariant mass of the lepton pair, as soon as a second intermediate state $a'b' \neq +-$ is involved. For the description of the K^{+-}_{e4} processes, s ranges from $4M_\pi^2$ to M_K^2, so that only two-pion intermediate states are relevant, i.e. $\{a',b'\} = \{\pi^+,\pi^-\}, \{\pi^0,\pi^0\}$. Due to Bose symmetry, the second possibility does not occur in the P wave, so that the dependence on s_ℓ occurs only in the S wave. In other words, while Watson's theorem does not apply to the case of the phase of the S-wave projection of the form factors, it is still operative for the P wave in the range of s allowed by the phase space of the K^{+-}_{e4} decay mode. It appears that the available statistics has not allowed the NA48/2 experiment to identify a dependence of the phases on s_ℓ [11, 12]. We haver checked that, from the theoretical side, the dependence on s_ℓ is indeed sufficiently small, as compared to other sources of error. We have therefore taken $s_\ell = 0$ in our formulas. Let us stress that the dependence on s_ℓ is also not present in $\delta_S(s,s_\ell)$ at lowest order (i.e. the case considered in Ref. [20]), where the expression for δ_S reduces to

$$\delta_S(s) = \sum_{\{a',b'\}} \frac{1}{\mathscr{S}_{a'b'}} \frac{\lambda^{\frac{1}{2}}_{a'b'}(s)}{s} \varphi_0^{a'b';+-}(s) \frac{F^{a'b'}_{S[0]}}{F_{S[0]}} \theta(s - s_{a'b'}) + \mathscr{O}(E^4). \tag{2.3}$$

In the isospin limit, the dependence on s_ℓ also drops out from $\delta_S(s,s_\ell)$, and Watson's theorem is recovered, i.e. the phases tend towards

$$\delta_S(s,s_\ell) \to \delta_0(s),\ \delta_P(s) \to \delta_1(s) \tag{2.4}$$

where $\delta_0(s)$ and $\delta_1(s)$ denote the $\pi\pi$ phases in the $l=0$, $I=0$ and $l=1$, $I=1$ channels, respectively. The quantity that is determined from experiment is the difference $\delta_S(s,s_\ell) - \delta_P(s)$ and our aim is to compute its deviation from the difference $\delta_0(s) - \delta_1(s)$.

Let us now come to the main point, namely the dependence on the scattering lengths a_0^0 and a_0^2. Along with the form factors describing the K_{e4} form factors, one also needs to construct the various amplitudes for $\pi\pi$ scattering. This can be done within the framework provided by the "reconstruction theorem" of Ref. [31], even when isospin is broken [1]. In doing so, one can parameterise these amplitudes, and thus the partial-wave projections that appear in the phases $\delta_S(s,s_\ell)$ and $\delta_P(s)$, directly in terms of the scattering lengths. The same can be done for the phases $\delta_0(s)$ and $\delta_1(s)$ in the isospin limit. Doing this for the one-loop form factors, one obtains this way the expression

$$\begin{aligned}\delta f_{\mathrm{IB}}(s;a_0^0,a_0^2) \;=\; & \frac{1}{2}\sigma(s)\left[-\frac{2}{3}a_0^0 + \frac{2}{3}a_0^2 - 4a_0^2\frac{\Delta_\pi}{M_\pi^2} - \frac{1}{12}(2a_0^0 - 5a_0^2)\frac{s-4M_\pi^2}{M_\pi^2}\right] \\ & -\frac{1}{2}\sigma_0(s)\left(1+\frac{3}{2R}\right)\left[-\frac{2}{3}a_0^0 + \frac{2}{3}a_0^2 + a_0^2\frac{\Delta_\pi}{M_\pi^2} - \frac{1}{12}(2a_0^0 - 5a_0^2)\frac{s-4M_\pi^2}{M_\pi^2}\right] + \mathscr{O}(E^4),\end{aligned} \tag{2.5}$$

where $\sigma(s) = \sqrt{1-4M_\pi^2/s}$, $\sigma_0(s) = \sqrt{1-4M_{\pi^0}^2/s}$, and $\Delta_\pi = M_\pi^2 - M_{\pi^0}^2$. If one replaces the scattering lengths by their lowest-order values $(a_0^0)^{\mathrm{LO}}_{\mathrm{ChPT}}$ and $(a_0^2)^{\mathrm{LO}}_{\mathrm{ChPT}}$ given previously, one recovers the result of Ref. [20]. In contrast, in the expression (2.5), the scattering lengths appear as free parameters. It is also possible to work out [2] the $\mathscr{O}(E^4)$ corrections to the above expression of $\delta f_{\mathrm{IB}}(s;a_0^0,a_0^2)$, thus obtaining an expression whose dependence on the scattering lengths a_0^0 and a_0^2 is correct up to corrections of the order $\mathscr{O}(E^8)$ in the low-energy expansion. Numerically, we observe that $\delta f_{\mathrm{IB}}(s;a_0^0,a_0^2)$ shows significant variations with respect to the scattering lengths a_0^0 and a_0^2, as these are varied away from the lowest-order chiral prediction, see the Figures in Section 6 of Ref. [2].

We have redone the fit to the NA48/2 data using our determination of the correction factor $\delta f_{\mathrm{IB}}(s;a_0^0,a_0^2)$, obtaining

$$a_0^0 = 0.221(18) \qquad a_0^2 = -0.0453(106). \tag{2.6}$$

This result compares well with the values $a_0^0 = 0.222(14)$ and $a_0^2 = -0.0432(97)$ obtained in Ref. [12] with the correction factor of Ref. [20], but with slightly larger errors once the dependence of the isospin-breaking corrections on the scattering lengths is taken into account. Actually, the interference between the S and P waves from the K_{e4}^{+-} angular analysis shows a strong correlation between a_0^0 and a_0^2. In order to circumvent this problem, one may supplement the NA48/2 data with additional information. Two options have been considered: using data in the $I = 2$ S wave [18], or using a theoretical constraint on the scalar radius of the pion [33]. With the first option, our fit gives the result

$$a_0^0 = 0.232(9) \qquad a_0^2 = -0.0383(40), \tag{2.7}$$

while with the second option we obtain

$$a_0^0 = 0.226(7) \qquad a_0^2 = -0.0431(19). \tag{2.8}$$

We have estimated higher-order corrections (in the low-energy expansion) to $\delta f_{\mathrm{IB}}(s; a_0^0, a_0^2)$ in various manners, and have found that they affect the results of our fits in a marginal way. For more quantitative statements concerning this issue we refer the interested reader to Section 7 of Ref. [2].

3. The cusp in the K_{e4}^{00} decay distribution

Two questions related to the presence of the unitarity cusp in the decay distribution of the K_{e4}^{00} decay modes were addressed in Ref. [3]. The first one aims at determining to which extent the phenomenological parameterisations of the cusp considered in the data analysis could influence the outcome, in particular as far as the value of the form factor $f_s[K_{e4}^{00}]$ is concerned. The second one is to determine the statistics that would be necessary in order to extract the information on the $\pi\pi$ scattering lengths with a certain level of accuracy from K_{e4}^{00} data.

We have addressed both issues in the somewhat simpler situation of the scalar form factors of the pions, for which two-loop expressions were obtained in Ref. [1]. These expressions again retain the full dependence on the scattering lengths and on isospin-breaking effects, and thus provide a theoretical description of the cusp (in the scalar form factor of the neutral pion) that is accurate at that level in the low-energy expansion.

We have used these two-loop representations in order to generate pseudo data, which have then been analysed with various phenomenological parameterisations of the form factor, inspired by those in use for the analyses of the K_{e4}^{+-} and K_{e4}^{00} experimental data, and which do not fully agree with the general properties that can be inferred from the exact expressions of the form factors. The outcome of this study is that the determination of the normalization of the form factor is actually not sensitive to the parameterisations used, and can be determined accurately (at the percent level with a statistical sample of the size of the one collected by NA48/2). Consequently, the fit procedure adopted in Ref. [13] does not bias the determination of $f_s[K_{e4}^{00}]$, and thus cannot explain even part of the surprisingly higher value obtained for it by the NA48/2 collaboration as compared to the value for $f_s[K_{e4}^{+-}]$ determined from the K_{e4}^{+-} channel, see Eq (1.5). Although our study was carried out for the scalar form factor of the neutral pion, we expect that the conclusion also holds for the K_{e4}^{00} form factor. This expectation rests on the fact that the scalar and K_{e4}^{00} form factors have similar shapes, in particular as far as the cusp is concerned.

As far as the extraction of the combination $a_0^0 - a_0^2$ is concerned, the presence of a cusp similar to the one observed in the three-body $K^+ \to \pi^+\pi^0\pi^0$ decay [34] suggests that it should, in principle, be possible to extract information on the scattering lengths from an accurate measurement of the K^{00}_{e4} differential decay rate. We have found that, unfortunately, with the sample of events presently available, the statistical uncertainties remain large and the scattering lengths are only weakly constrained. A substantial increase of the statistical sample would be required in order to reach a precision that would become close to the precision obtained by the Dirac experiment [35].

4. Radiative corrections to the K^{00}_{e4} decay rate

Because of the smallness of the electron mass and of the limited experimental precision, the decay of the charged kaon into two neutral pions, $K^\pm \to \pi^0\pi^0 e^\pm \overset{(-)}{\nu}_e$ can be described in terms of a single form factor. This form factor also occurs in the description of the decay into two charged pions, $K^\pm \to \pi^+\pi^- e^\pm \overset{(-)}{\nu}_e$, and up to isospin-breaking contributions, the two determinations should agree. Having eliminated possible biases due to the parameterisations used in the data analysis, the significant difference displayed in Eq. (1.5) should therefore be ascribed to radiative corrections, i.e. to the factor δ_{EM}.

In the data analyses, radiative corrections were addressed differently in the K^{+-}_{e4} and K^{00}_{e4} cases. In the latter case, no radiative corrections were applied to the measured decay rate [13], and the factor δ_{EM} was left unspecified. In the K^{+-}_{e4} case, two types of radiative corrections were implemented. Virtual photon exchanges between all possible pairs of charged external lines were considered, and the corresponding Sommerfeld-Gamow-Sakharov factors applied. The corrections induced by emission of real photons were treated with PHOTOS [36, 37, 38, 39]. The latter also implements wave-function renormalisation on the external charged legs. The couplings of photons to mesons are treated as point-like interactions, given by scalar QED. The result is then free from infrared singularities. Furthermore, contributions that vanish when the electron mass goes to zero, which is a sensible limit to consider for the K_{e4} decay channels, are neglected.

Transposing this discussion to the K^{00}_{e4} case, one notices that Sommerfeld-Gamow-Sakharov factors are not relevant, so that only the second type of corrections effectively contributes to δ_{EM}. In order to make the comparison with the K^{+-}_{e4} case meaningful, the evaluation of δ_{EM} should be done within the same framework as used there. This requires to analyse the content of PHOTOS in some greater detail, in order to identify which corrections are included, and in which manner they are implemented. We have done this study, and have evaluated the corresponding radiative corrections in the K^{00}_{e4} case. The result we obtain is

$$\delta_{EM} = 0.018. \tag{4.1}$$

This correction term has the expected size. Moreover, being positive, it indeed reduces the discrepancy in Eq. (1.5), from 6.5% to 4.6%, i.e.

$$\frac{f_s[K^{00}_{e4}]}{f_s[K^{+-}_{e4}]} = 1.046 \pm 0.010. \tag{4.2}$$

The remaining difference can then be accounted for by the isospin breaking in the quark masses, i.e. the value of R. Actually, one may even take an inverted point of view and, by combining Eqs. (4.2) and (1.6), extract $R = 32^{+9}_{-6}$. Of course, a more reliable statement would require one to evaluate the corrections to the lowest-order relation (1.6), as well as a more systematic treatment of radiative corrections in both K_{e4} channels.

5. Summary and conclusions

The high level of precision reached by the determination of $\delta_S(s) - \delta_P(s)$ from the data collected by the NA48/2 experiment requires to consider isospin-breaking corrections. Since the ultimate goal is to extract the $\pi\pi$ scattering lengths in the isospin limit, a_0^0 and a_0^2, the corrections due to the mass difference $M_\pi \neq M_{\pi^0}$ should not be computed at fixed values of the scattering lengths, but should be parameterised in terms of them.

General properties (analyticity, unitarity, crossing, chiral counting) provide the necessary tools, through the reconstruction theorem, to do this in a model-independent way. The corrections can be obtained in the form shown in Eq. (1.3), where in both terms on the right-hand side the scattering lengths appear as free parameters, to be fitted to the data. Moreover, the correction term $\delta f_{\rm IB}(s; a_0^0, a_0^2)$ has been worked out at next-to-leading order. Using this construction, we have redone the fit to NA48/2 data. Our results are compatible with those published by NA48/2 within errors.

We have also looked for possible sources of biases that could provide explanations for the discrepancy observed between the measurements of the form factor f_s in the K^{+-}_{e4} and K^{00}_{e4} channels. A possible bias due to the use, in the data analysis, of simplified phenomenological parameterisations of the cusp in the K^{00}_{e4} form factor, does actually not influence the determination of its normalization at the level of precision achieved with the NA48/2 data.

We have next evaluated radiative corrections to the K^{00}_{e4} decay rate, being careful to perform this evaluation in the same framework as used for the treatment of radiative corrections in the K^{+-}_{e4} decay, in order to make a comparison between the two meaningful. The resulting correction reduces the discrepancy in Eq. (1.5) from 6.5% to 4.5%. The remaining discrepancy can then be ascribed to the difference $m_u - m_d$ between quark masses, given the typical values of the quark-mass ratio R obtained by recent simulations of QCD on the lattice. A more quantitative statement would required a more involved treatment of radiative corrections.

This brings us to our final remark. While the treatment of radiative corrections in the data analysis might give reliable results as far as the decay rates are concerned, it might not quite do justice to the high quality of the data that have become available for the decay distributions. The issue of the dependence on the scattering lengths thus arises quite legitimately also in the context of radiative corrections. Existing studies do not take this aspect into account. We leave this as an interesting open problem for future work.

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