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Extraction of low energy QCD parameters from $\eta \to 3\pi$ and beyond

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The $\eta \to 3\pi$ decays are a valuable source of information on low energy QCD. Yet they were not used for an extraction of the three flavor chiral symmetry breaking order parameters until now. We use a Bayesian approach in the framework of resummed chiral perturbation theory to extract information on the quark condensate and pseudoscalar decay constant in the chiral limit, as well as the mass difference of the light quarks. We compare our results with recent CHPT and lattice QCD fits and find some tension, as the $\eta \to 3\pi$ data seem to prefer a larger ratio of the chiral order parameters. The results also disfavor a very large value of the chiral decay constant, which was found by some recent works. In addition, we present preliminary results of a combined analysis including $\eta \to 3\pi$ decays and $\pi - \pi$ scattering.

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1. Introduction

Spontaneous breaking of chiral symmetry (SBχS) is a prominent feature of the QCD vacuum and thus its character has been under discussion for a long time [1, 2]. The principal order parameters are the quark condensate and the pseudoscalar decay constant in the chiral limit

$$\Sigma(N_f) = -\langle 0|\bar{q}q|0\rangle|_{m_q\to 0}\,, \tag{1.1}$$

$$F(N_f) = F_P^a|_{m_q\to 0}\,, \quad ip_\mu F_P^a \;=\; \langle 0|A_\mu^a|P\rangle, \tag{1.2}$$

where N_f is the number of quark flavors q considered light and m_q collectively denotes their masses. A_μ^a are the QCD axial vector currents, while F_P^a the decay constants of the light pseudoscalar mesons P.

Chiral perturbation theory (χPT) [3, 4, 5] is constructed as a general low energy parameterization of QCD based on its symmetries and the discussed order parameters appear at the lowest order of the chiral expansion as low energy constants (LECs). Interactions of the light pseudoscalar meson octet, the pseudo-Goldstone bosons of the broken symmetry, directly depend on the pattern of SBχS and thus can provide information about the values of these observables.

A convenient reparameterization of the order parameters, relating them to physical quantities connected with pion two point Green functions, can be introduced [2]

$$Z(N_f) = \frac{F(N_f)^2}{F_\pi^2}, \quad X(N_f) = \frac{2\hat{m}\Sigma(N_f)}{F_\pi^2 M_\pi^2}, \tag{1.3}$$

where $\hat{m}=(m_u+m_d)/2$. Defined in this way, $X(N_f)$ and $Z(N_f)$ are limited to the range $(0,1)$. $Z(N_f)=0$ would correspond to a restoration of chiral symmetry and $X(N_f)=0$ to a case with vanishing chiral condensate. Standard approach to chiral perturbation series tacitly assumes values of $X(N_f)$ and $Z(N_f)$ not much smaller than one, which means that the leading order terms should dominate the expansion.

Several recent results for the two and three flavor order parameters are listed in tables 1 and 2, respectively. As can be seen, while the two flavor case is quite settled, the values of $X(2)$ and $Z(2)$ indeed being not much smaller than one, the situation in the three flavor one is much less clear. Some analyses suggest a significant suppression of X(3) and/or Z(3) and thus a non-standard behavior of the spontaneously broken QCD vacuum.

	$Z(2)$	$X(2)$
$\pi\pi$ scattering [6]	0.89±0.03	0.81±0.07
RBC/UKQCD+ReχPT [7]	0.86±0.01	0.89±0.01
FLAG 2013 N_f=2 [8]	0.87±0.01	0.86±0.09
FLAG 2013 N_f=2+1 [8]	0.886±0.004	0.84±0.14

Table 1: Chosen results for the two flavor order parameters.

phenomenology	$Z(3)$	$X(3)$
NNLO χPT (main fit) [9]	0.59	0.63
NNLO χPT (free fit) [9]	0.51	0.48
NNLO χPT ("fit 10") [10]	0.89	0.66
ReχPT $\pi\pi$+πK [11]	>0.2	<0.8
lattice QCD	$Z(3)$	$X(3)$
RBC/UKQCD+ReχPT [12]	0.54±0.06	0.38±0.05
RBC/UKQCD+large N_c [13]	0.91±0.08	
MILC 09A [14]	0.72±0.06	0.62±0.07

Table 2: Chosen results for the three flavor order parameters.

Up, down and strange quark masses are other parameters with strong influence on low energy QCD physics. A commonly used reparameterization can be used

$$\hat{m} = \frac{m_u + m_d}{2}, \qquad r = \frac{m_s}{\hat{m}}, \qquad R = \frac{m_s - \hat{m}}{m_d - m_u}. \tag{1.4}$$

The values for the light quark mass average and the strange to light quark mass ratio are well known from lattice QCD and QCD sum rules [8, 15]. On the other hand, as can be seen in table 3, the isospin breaking parameter R, directly related to the light quark mass difference, has not been determined with sufficient precision by these methods yet.

phenomenology	R
Dashen's theorem LO [16]	44
Dashen's theorem NNLO [16]	37
$\eta \to 3\pi$ NNLO χPT [16]	41.3
$\eta \to 3\pi$ dispersive [17]	37.8±3.3
lattice QCD	R
FLAG 2013 N_f=2 [8]	40.7±4.3
FLAG 2013 N_f=2+1 [8]	35.8±2.6

Table 3: Chosen results for the isospin breaking parameter R.

2. $\eta \to 3\pi$ decay

The $\eta \to 3\pi$ isospin breaking decays have not been exploited for an extraction of the chiral order parameters so far, yet we argue there is valuable information to be had. The theory seems to converge slowly for the decays. One loop corrections were found to be very sizable [18], the result for the decay width of the charged channel was 160±50 eV, compared to the current algebra prediction of 66 eV. However, the experimental value is still much larger, the current PDG value is [19]

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$$\Gamma^{+}_{\mathrm{exp}} = 300 \pm 12 \text{ eV}. \tag{2.1}$$

Only the two loop χPT calculation [16] has succeeded to obtain a reasonable result for the widths. The latest experimental value for the neutral decay width is [19]

$$\Gamma^{0}_{\mathrm{exp}} = 428 \pm 17 \text{ eV}. \tag{2.2}$$

As will be shown elsewhere [20], we argue that the four point Green functions relevant for the $\eta \to 3\pi$ amplitude (see (4.1) below) do not necessarily have large contributions beyond next-to-leading order and a reasonably small higher order remainder is not in contradiction with huge corrections to the decay widths. The widths do not seem to be sensitive to the details of the Dalitz plot distribution, but rather to the value of leading order parameters - the chiral decay constant, the chiral condensate and the difference of u and d quark masses, i.e. the magnitude of isospin breaking. Moreover, access to the values of these quantities is not screened by EM effects, as it was shown that the electromagnetic corrections up to NLO are very small [21, 22]. This is our motivation for our effort to extract information about the character of the QCD vacuum from this decay.

The Dalitz plot distributions are experimentally well known as well [23, 24, 25, 26, 27]. However, as we will discuss in detail in [20], we have not found the convergence of the theory in the case of the slopes reliable enough to include all the Dalitz plot parameters into the analysis. To stay on the conservative side, we used the lowest order parameter in the charged channel only [23]

$$a = -1.09 \pm 0.02. \tag{2.3}$$

3. Resummed χPT

Our calculation closely follows the procedure outlined in [28], results presented here are a significant update on our initial reports [29, 30]. We use an alternative approach to chiral perturbation theory, dubbed resummed χPT (ReχPT) [31], which was developed in order to accommodate the possibility of irregular convergence of the chiral expansion. The procedure can be very shortly summarized in the following way:

- standard χPT Lagrangian and power counting
- only expansions of quantities related linearly to Green functions of QCD currents trusted
- explicitly to NLO, higher orders implicit in remainders
- remainders retained, treated as sources of error
- manipulations in non-perturbative algebraic way

The hope for resummed χPT is that by carefully avoiding dangerous manipulations a better converging series can be obtained. The procedure also avoids the hard to control NLO and NNLO LECs by trading them for remainders with known chiral order.

4. Calculation

We start by expressing the charged decay amplitude in terms of 4-point Green functions G_{ijkl}, obtained from the generating functional of the QCD currents. The neutral decay amplitude can be straightforwardly obtained from the charged one. We compute at first order in isospin breaking, the amplitude then takes the form

$$F_\pi^3 F_\eta A(s,t,u) = G_{+-83} - \varepsilon_\pi G_{+-33} + \varepsilon_\eta G_{+-88} + \Delta_{G_D}^{(6)}, \tag{4.1}$$

where $\Delta_{G_D}^{(6)}$ is the direct higher order remainder to the 4-point Green functions. The physical mixing angles to all chiral orders and first order in isospin breaking can be expressed in terms of quadratic mixing terms of the generating functional to NLO and related indirect remainders

$$\varepsilon_{\pi,\eta} = -\frac{F_0^2}{F_{\pi^0,\eta}^2} \frac{(M_{38}^{(4)} + \Delta_{M_{38}}^{(6)}) - M_{\eta,\pi^0}^2 (Z_{38}^{(4)} + \Delta_{Z_{38}}^{(6)})}{M_\eta^2 - M_{\pi^0}^2}. \tag{4.2}$$

In accord with the method, $\mathscr{O}(p^2)$ parameters appear inside loops, while physical quantities in outer legs. Such a strictly derived amplitude has an incorrect analytical structure due to the leading order masses in loops, cuts and poles being in unphysical positions. To account for this, we exchange the LO masses in unitarity corrections and chiral logarithms for physical ones, as described in [28].

The next step is the treatment of the LECs. As discussed, the leading order ones, as well as quark masses, are expressed in terms of convenient parameters X, Z, r and R. At next-to-leading order, the LECs L_4-L_8 are algebraically reparametrized in terms of pseudoscalar masses, decay constants and the free parameters X, Z and r using chiral expansions of two point Green functions, similarly to [31]. Because expansions are formally not truncated, each generates an unknown higher order remainder.

We don't have a similar procedure ready for L_1-L_3 at this point, therefore we collect a set of standard χPT fits [9, 10, 32, 33] and by taking their mean and spread, while ignoring the much smaller reported error bars, we obtain an estimate of their influence

$$L_1^r(M_\rho) = (0.57 \pm 0.18) \cdot 10^{-3} \tag{4.3}$$

$$L_2^r(M_\rho) = (0.82 \pm 0.28) \cdot 10^{-3} \tag{4.4}$$

$$L_3^r(M_\rho) = (-2.95 \pm 0.38) \cdot 10^{-3} \tag{4.5}$$

As will be shown in [20], the results depend on these constants only very weakly.

The $O(p^6)$ and higher order LECs, notorious for their abundance, are implicit in a relatively smaller number of higher order remainders. We have eight indirect remainders - three generated by the expansions of the pseudoscalar masses, three by the decay constants and two by the mixing angles. We expand the direct remainder to the 4-point Green functions around the center of the Dalitz plot $s_0 = 1/3(M_\eta^2 + 2M_{\pi^+}^2 + M_{\pi^0}^2)$

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$$\Delta_{G_D}^{(6)} = \Delta_A + \Delta_B(s-s_0) + \Delta_C(s-s_0)^2 + \Delta_D[(t-s_0)^2 + (u-s_0)^2] \tag{4.6}$$

and thus get four derived direct remainders, two NLO and two NNLO ones. As the experimental curvature of the Dalitz plot is very small [23], we argue that for our purpose of calculating the decay widths and the lowest order Dalitz slope a the expansion to second order in the Mandelstam variables is sufficient.

5. $\pi\pi$ scattering

In addition to the $\eta \to 3\pi$ parameters discussed above, we employ $\pi\pi$ scattering in a very similar way to [31]. We use the two lowest order subthreshold parameters in the expansion of the polynomial part of the amplitude, $\alpha_{\pi\pi}$ and $\beta_{\pi\pi}$

$$\begin{aligned} A_{\pi\pi}(s,t,u) \;=\; & \frac{\alpha_{\pi\pi}}{F_\pi^2}\frac{M_\pi^2}{3} + \frac{\beta_{\pi\pi}}{F_\pi^2}\left(s - \frac{4M_\pi^2}{3}\right) + \frac{\lambda_1}{F_\pi^4}\left(s-2M_\pi^2\right)^2 + \frac{\lambda_2}{F_\pi^4}\left[(t-2M_\pi^2)^2 + (u-2M_\pi^2)^2\right] + \\ & + \frac{\lambda_3}{F_\pi^6}\left(s-2M_\pi^2\right)^3 + \frac{\lambda_4}{F_\pi^6}\left[(t-2M_\pi^2)^3 + (u-2M_\pi^2)^3\right] + U^{(2+4+6)}(s|t,u) + \mathscr{O}(p^8) \end{aligned} \tag{5.1}$$

We use the numerical values extracted in [6]

$$\alpha_{\pi\pi}^{\text{exp}} = 1.381 \pm 0.242, \quad \beta_{\pi\pi}^{\text{exp}} = 1.081 \pm 0.023. \tag{5.2}$$

It should be noted that we consider the combined analysis which includes both processes to be preliminary. Our aim at this point is to test whether $\pi\pi$ scattering could significantly constrain the parameters we are interested in.

6. Statistical analysis

For the statistical analysis, we use an approach based on Bayes' theorem [31]

$$P(X_i|\text{data}) = \frac{P(\text{data}|X_i)P(X_i)}{\int \mathrm{d}X_i\, P(\text{data}|X_i)P(X_i)}\,, \tag{6.1}$$

where $P(X_i|\text{data})$ is the probability density of the parameters and remainders, denoted as X_i, having a specific value given the observed experimental data. $P(\text{data}|X_i)$ is the known probability density of obtaining the observed values of the included observables O_k in a set of independent experiments with uncertainties σ_k under the assumption that the true values of X_i are known

$$P(\text{data}|X_i) = \prod_k \frac{1}{\sigma_k\sqrt{2\pi}} \exp\left[-\frac{(\mathrm{O}_k^{\text{exp}} - \mathrm{O}_k(X_i))^2}{\sigma_k}\right]. \tag{6.2}$$

Our observables, treated as independent, are the charged and neutral decay widths and the Dalitz slope a of the $\eta \to 3\pi$ decays and the $\pi\pi$ scattering subthreshold parameters $\alpha_{\pi\pi}$ and $\beta_{\pi\pi}$.

$P(X_i)$ are the prior probability distributions of X_i. We use them to implement the theoretical uncertainties connected with our parameters and remainders. In such a way we keep the theoretical assumptions explicit and under control. This also allows us to test various assumptions and formulate if-then statements as well as implement additional constraints (see below).

We resort to Monte Carlo sampling in order to perform the numerical integration in (6.1). We have used 10000 samples per grid element, the total number of samples being $4 - 7.5 \cdot 10^6$.

7. Assumptions

The following list summarizes the higher order remainders we need to deal with:

- $\eta \to 3\pi$ direct ones: Δ_A, Δ_B, Δ_C, Δ_D
- $\eta \to 3\pi$ indirect ones: Δ_{M_π}, Δ_{F_π}, Δ_{M_K}, Δ_{F_K}, Δ_{M_η}, Δ_{F_η}, Δ_{M38}, Δ_{Z38}
- $\pi\pi$ scattering: $\Delta_{\alpha_{\pi\pi}}$, $\Delta_{\beta_{\pi\pi}}$

We use an estimate based on general arguments about the convergence of the chiral series [31]

$$\Delta_G^{(4)} \approx \pm 0.3G, \quad \Delta_G^{(6)} \approx \pm 0.1G, \tag{7.1}$$

where G stands for any of our 2-point or 4-point Green functions, which generate the remainders. We implement (7.1) by using a normal distribution with $\mu = 0$ and $\sigma = 0.3G$ or $\sigma = 0.1G$ for the NLO or NNLO remainders, respectively. The remainders are thus limited only statistically, not by any upper bound.

We assume the strange to light quark ratio r to be known and use the lattice QCD average [8]

$$r = 27.5 \pm 0.4. \tag{7.2}$$

At last, we are left with three free parameters: X, Z and R. These control the scenario of spontaneous breaking of chiral symmetry and isospin breaking in our results. In the case of X and Z, we use the constraint from the so-called paramagnetic inequality [2] and assume these parameters to be in the range

$$0 < X < X(2), \quad 0 < Z < Z(2). \tag{7.3}$$

For the two flavor order parameters, we use the lattice QCD values [7]. In addition, we implement a constraint following from $X(2), Z(2) > 0$, similarly to [31].

We use two approaches to deal with R. In the first one we assume it to be a known quantity. We use the value

$$R = 37.8 \pm 3.3, \tag{7.4}$$

obtained from a dispersive analysis of $\eta \to 3\pi$ [17]. However, one should be aware that this estimate is based on an assumption that NNLO standard χPT [16] converges well at a specific kinematic point found in unphysical region. Alternatively, we leave R free, or more precisely, assume it to be in a wide range $R \in (0, 60)$.

8. Results and conclusion

The obtained probability density distributions for the case of value of R fixed can be found in figure 1. When considering $\eta \to 3\pi$ observables only, we get

$$X = 0.57 \pm 0.21, \quad Z = 0.40 \pm 0.17, \quad Y = X/Z = 1.45 \pm 0.25. \tag{8.1}$$

As can be seen, when assuming $R = 37.8 \pm 3.3$, there is some tension with available results (table 2). The $\eta \to 3\pi$ data seem to prefer a larger value for the ratio of the chiral order parameters $Y = X/Z = 2\hat{m}B_0/M_\pi^2$ than recent χPT and lattice QCD fits. In addition, very large values of the chiral decay constant are excluded at 2σ C.L. and a relatively small value is favored. The uncertainties, however, are quite large.

The picture does not change appreciably when including the $\pi\pi$ scattering subthreshold parameters

$$X = 0.59 \pm 0.21, \quad Z = 0.46 \pm 0.17, \quad Y = X/Z = 1.30 \pm 0.25. \tag{8.2}$$

Though a bit disappointing, this outcome is not unexpected considering the weak constraints obtained in [31] and [11].

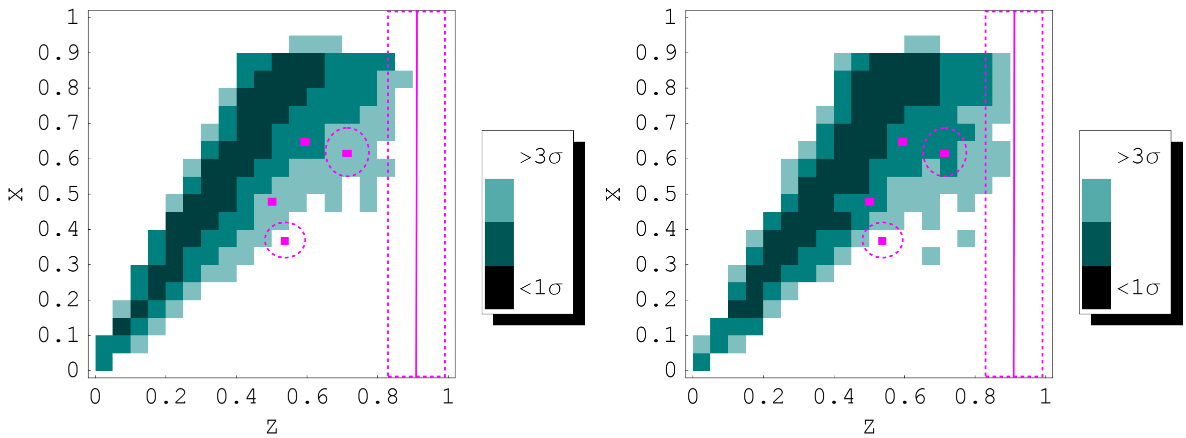

Figure 1: Probability density $P(X, Z|\text{data})$ for $R = 37.8 \pm 3.3$. Purple: results listed in table 2. Left: $\eta \to 3\pi$ only. Right: $\eta \to 3\pi$ and $\pi\pi$ scattering.

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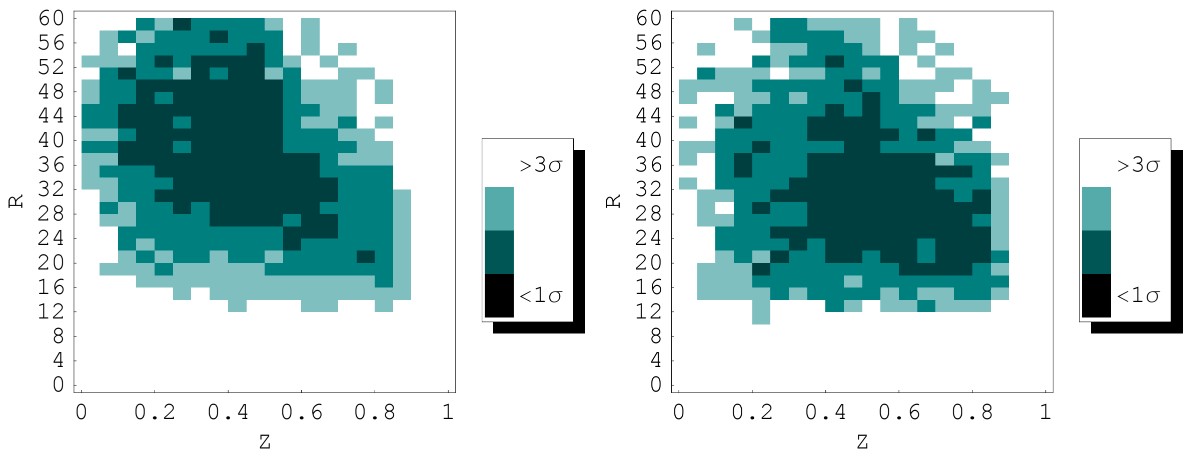

Figure 2: Probability density $P(R,Z|\text{data})$ for R free, X integrated out.
Left: $\eta \to 3\pi$ only. Right: $\eta \to 3\pi$ and $\pi\pi$ scattering.

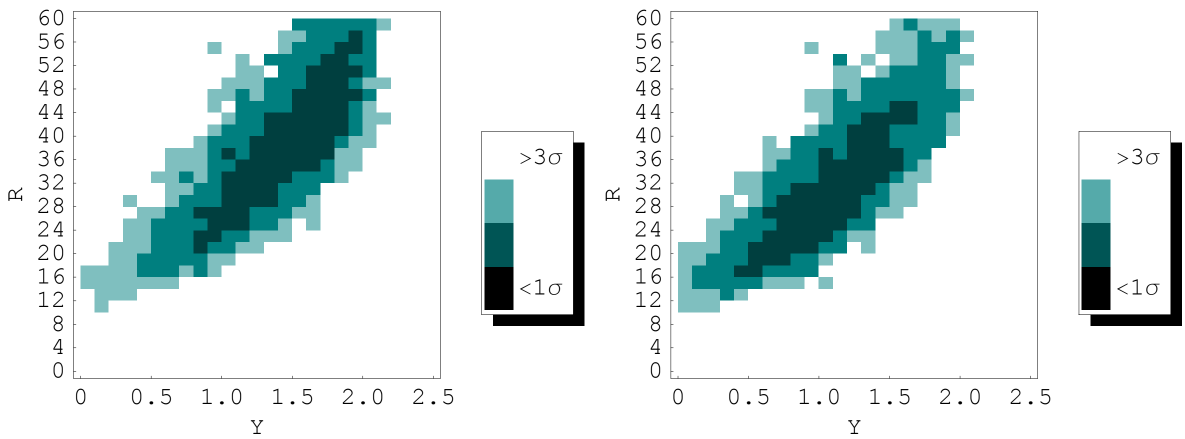

Figure 3: Probability density $P(R,Y|\text{data})$ for R free, Z integrated out.
Left: $\eta \to 3\pi$ only. Right: $\eta \to 3\pi$ and $\pi\pi$ scattering.

The results with R left as a free parameter are shown in figure 2. Not surprisingly, it's hard to constrain R without information on X and Z. Even in this case a part of the parameter space can be excluded at 2σ C.L. though. Integrating out one of the remaining parameters for the $\eta \to 3\pi$ case leads to

$$R = 38 \pm 10, \quad Z = 0.42 \pm 0.18, \tag{8.3}$$

while the combined $\pi\pi$ and $\eta \to 3\pi$ analysis gives us

$$R = 32 \pm 10, \quad Z = 0.50 \pm 0.18. \tag{8.4}$$

The obtained value for R is compatible with available results (table 3). The result for the chiral decay constant is quite low, similarly to the case with R fixed.

Figure 3 shows the dependence of R and the ratio $Y = X/Z$. As can be seen, they seem to be quite strongly correlated, which also explains the large value of Y obtained when fixing $R = 37.8 \pm 3.3$. The uncertainties when extracting Y with R free are thus larger: $Y = 1.41 \pm 0.37$ ($\eta \to 3\pi$) and $Y = 1.10 \pm 0.38$ ($\pi\pi + \eta \to 3\pi$).

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