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Assessing Theory Uncertainties in EFT Power Countings from Residual Cutoff Dependence

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I summarise a method to quantitatively assess the consistency of power-counting proposals in Effective Field Theories which are non-perturbative at leading order. It uses the fact that the Renormalisation Group evolution of an observable predicts the functional form of its residual cutoff dependence on the EFT breakdown scale, on the low-momentum scales, and on the order of the calculation. Passing this test is a necessary but not sufficient consistency criterion for a suggested power counting whose exact nature is disputed. For example, in χEFT with more than one nucleon, a lack of universally accepted analytic solutions obfuscates the relation between convergence pattern and numerical results, and led to proposals which predict different numbers of Low Energy Coefficients at the same chiral order. The method may provide an independent check whether an observable is properly renormalised at a given order, and allows one to estimate both the breakdown scale and the momentum-dependent order-by-order convergence pattern of an EFT. Conversely, it may help identify those LECs which produce renormalised observables at a given order. I also discuss its underlying assumptions and relation to the Wilsonian Renormalisation Group Equation; useful choices for observables and cutoffs; the momentum window in which the test provides best signals; its dependence on the values and forms of cutoffs as well as on the EFT parameters; the impact of fitting Low Energy Coefficients to data in different or the same channel; and caveats as well as limitations. Since the test is designed to minimise the use of data, it allows one to quantitatively falsify if the EFT has been renormalised consistently, rather than quantifying how an EFT compares to experiment. Its application in particular to the ^{3}P$_0$ and ^{3}P$_2$-^{3}F$_2$ partial waves of NN scattering in χEFT may elucidate persistent power-counting issues. Details and a better bibliography can be found in an upcoming publication [1].

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1. Motivation: Serious Theorists Have Error Bars

That our understanding of natural phenomena is based on concrete, falsifiable predictions is deeply ingrained in the scientific method. It is insufficient to compare numbers; one also must judge their reliability. And since we do not trust experiments without error bars, why should it be acceptable for a theorist to not assess uncertainties in a calculation independently of the data to be explained? Simply stating that this is "difficult" is certainly no sufficient excuse, especially after the recent surge of articles on theory errors; see e.g. [2, 3]. The prospect of a reproducible, objective, quantitative estimate of theoretical uncertainties lies thus at the heart not only of any Effective Field Theory (EFT). But EFTs claim to possess well-defined schemes to find just that. It is therefore befitting to explore how the validity of such prescriptions can be gauged.

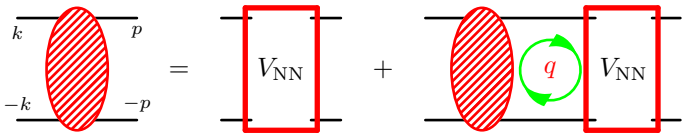
More than a decade ago, Lepage used a highly influential lecture to quantify convergence to data [4]. In contradistinction, the test presented here hopes to quantify the internal consistency of an EFT and takes *minimal resort* to experimental information. Ideally, theorists would perform "double-blind" calculations in which theory errors are assessed under the pretence that no or only very limited data is available. Such "post-dictions" are of course predictions when information is indeed experimentally unknown or hard to access, or when data consistency must be checked.

EFT uses scale-separation to expand interactions and observables in a dimension-less quantity

$$Q = \frac{\text{typical low momenta } k, p_{\text{typ}}}{\text{breakdown scale } \overline{\Lambda}_{\text{EFT}}} < 1 \ . \tag{1.1}$$

The numerator contains the relative momentum k between scattering particles and other intrinsic low scales, here summarily denoted by p_{typ}. At the breakdown scale $\overline{\Lambda}_{\text{EFT}}$, new dynamical degrees of freedom enter which are not explicitly accounted but whose effects at these short distances are simplified into Low-Energy Coefficients (LECs). In χEFT, p_{typ} includes the pion mass and the inverse scattering lengths of the NN system. $\overline{\Lambda}_\chi \approx [700 \ldots 1000]$ MeV is consistent with the masses of the ω and ρ as the next-lightest exchange mesons, and with the chiral symmetry breaking scale. The power counting (PC) is then determined by Naïve Dimensional Analysis [5, 6]. When all interactions are perturbative, as in the mesonic and one-baryon sectors, this amounts to not much more than counting powers of k and p_{typ}.

The situation is more complicated when some interactions must be treated non-perturbatively at leading order because of shallow real or virtual bound states with scales $p_{\text{typ}} \ll \overline{\Lambda}_{\text{EFT}}$ in the EFT's range of validity. In NN scattering, all terms in the LO Lippmann-Schwinger equation, including the potential, must be of the same order when all nucleons are close to their non-relativistic mass-shell. If not, one term could be treated as perturbation of the others and there would be no shallow bound-state. Picking the nucleon-pole in the energy-integration, $q_0 \sim \frac{q^2}{2M}$, leads to the consistency condition that both amplitude T_{NN} (ellipse) and potential V_{NN} (rectangle) must be of order Q^{-1}:



$$\begin{array}{ccccccc} T_{\text{NN}} & = & V_{\text{NN}} & + & T_{\text{NN}}\, G_{\text{NN}}\, V_{\text{NN}} & & \\ Q^m & \sim & Q^m & + & Q^{2m+3-2} & \Longrightarrow m \stackrel{!}{=} -1 \end{array} \tag{1.2}$$

In particular, the one-pion exchange appears to scale as $(\vec{\sigma}_1 \cdot \vec{q})(\vec{\sigma}_2 \cdot \vec{q})/(\vec{q}^2 + m_\pi^2) \sim Q^0$ when one counts only explicit low-momentum scales, but must be of order Q^{-1} if its iteration is to be mandated. In a straightforward extension, amplitude and interaction between n nucleons scale as

$$T_{n\mathrm{N}} \sim V_{n\mathrm{N}} \sim Q^{1-n} \tag{1.3}$$

if it is nonperturbative at LO. Since both the interactions and the LECs themselves carry inverse powers of Q, finding their importance by counting momenta is insufficient. This result only assumes the existence of unnaturally small scales, irrespective of the form of the interaction. It does not reveal *which* terms constitute the LO potential; only how those terms must be power-counted.

This behaviour has been long-recognised in pionless EFT (EFT($\not\pi$)) and its variants (like Halo-EFT and EFT of point-like interactions), where the scaling of operators and the β functions of couplings between up to 3 nucleons are well-established [7, 8]. For example, analytic results in well-controlled limits show one momentum-independent $3N$ operator at LO. Likewise, non-relativistic QED and QCD count the Coulomb potential as Q^{-1} to allow its resummation.

The situation in χEFT for two and more nucleons is less obvious. Weinberg suggested to still count LECs as Q^0, and to apply the perturbative counting of momenta not to amplitudes but to the few-nucleon potential, which is then iterated to produce shallow bound states. How this translates into a PC of observables is under dispute. Further disagreement persists about the interpretation of approximate solutions (large off-shell momenta, semi-classical limit, etc.), and about unrelated numerical problems (deeply bound states etc.). In addition, a cutoff Λ becomes numerically necessary. It is conceptually quite different from the breakdown scale $\overline{\Lambda}_{\mathrm{EFT}}$, albeit the two symbols are similar. It cannot be much smaller than the breakdown scale in order not to "cut out" physical, low-resolution momenta in loops. But even how far Λ should be varied is under dispute: Is any value $\Lambda \gtrsim \overline{\Lambda}_{\mathrm{EFT}}$ legitimate, including $\Lambda \to \infty$; or should the range be constrained to $\Lambda \approx \overline{\Lambda}_{\mathrm{EFT}}$?

order	**Weinberg** (modified) [9]	**Birse** 2005 [10, 11]	**Pavon et al.** 2006 [12, 13, 14, 15]	**Long/Yang** 2012 [16, 17]
Q^{-1}	LO of ^{1}S$_0$, ^{3}S$_1$, OPE	LO of ^{1}S$_0$, ^{3}S$_1$, OPE, ^{3}D$_1$, ^{3}SD$_1$	LO of ^{1}S$_0$, ^{3}S$_1$, OPE, ^{3}P$_{0,2}$, ^{3}D$_2$	LO of ^{1}S$_0$, ^{3}S$_1$, OPE, ^{3}P$_{0,2}$
$Q^{-\frac{1}{2}}$	none	LO of ^{3}P$_{0,1,2}$, ^{3}PF$_2$, ^{3}F$_2$, ^{3}D$_2$	LO of ^{3}SD$_1$, ^{3}D$_1$, ^{3}PF$_2$, ^{3}F$_2$	none
Q^0	none	NLO of ^{1}S$_0$	NLO of ^{1}S$_0$	NLO of ^{1}S$_0$
$Q^{\frac{1}{2}}$	none	NLO of ^{3}S$_1$, ^{3}D$_1$, ^{3}SD$_1$	none	none
Q^1	LO of ^{3}SD$_1$,^{1}P$_1$, ^{3}P$_{0,1,2}$, TPE; NLO of ^{1}S$_0$, ^{3}S$_1$	none	none	LO of ^{3}SD$_1$,^{1}P$_1$, ^{3}P$_1$, ^{3}PF$_2$, TPE; NLO of ^{3}S$_1$, ^{3}P$_0$, ^{3}P$_2$; N^{2}LO of ^{1}S$_0$
$Q^{\frac{3}{2}}$	none	NLO of ^{3}D$_2$, ^{3}P$_{0,1,2}$, ^{3}PF$_2$, ^{3}F$_2$	none	none
Q^2	NLO of TPE	LO of TPE, ^{1}P$_1$; NLO of OPE; N^{2}LO of ^{1}S$_0$	LO of TPE, ^{1,3}P$_1$; NLO of ^{3}S$_1$, ^{3}D$_{1,2}$, ^{3}SD$_1$, ^{3}P$_{0,2}$, ^{3}PF$_2$; N^{2}LO of ^{1}S$_0$	NLO of TPE; N^{3}LO of ^{1}S$_0$
# at Q^{-1}	2	4	5	4
# at Q^0	+0	+7	+5	+1
# at Q^1	+7	+3	0	+8
total at Q^1	9	14	10	13

Table 1: Order Q^n at which some LECs and the One- as well as Two-Pion-Exchange (OPE, TPE) enter in partial waves, for proposed power-countings in NN χEFT [18]. LECs of mixing angles are denoted e.g. by ^{3}SD$_1$. The bottom part summarises the number of LECs at a given order. Not all schemes have contributions at an order, and some do not list all higher partial waves. While the information was collected with feedback from the respective authors, only I am to blame for errors. The results of Weinberg's PC have been shifted by -1 so that its potential starts at order Q^{-1}, as mandated by the general arguments of eq. (1.2).

It is thus no surprise that four active PC proposals emerged in χEFT, all with the same degrees of freedoms: nucleons and pions only [9, 10, 11, 12, 13, 14, 15, 16, 17]. Table 1 lists their predictions for the order at which a LEC enters in the lower NN partial waves. Each finds a different number of LECs at any given order – and each claims consistency. Not all can be right, though. Articles, panels and sessions at Chiral Dynamics and other conferences as well as dedicated workshops led to little consensus (see e.g. Daniel Phillips' even-handed account at the last Chiral Dynamics [19]); some additional notable contributions include Refs. [20, 21, 22, 23, 24, 25, 26].

This is not just stamp-collecting or a philosophical question which potentially exposes the soft underbelly of χEFT and the credibility of its error assessments, but which is "otherwise" of little practical consequence. A central EFT promise is that it encodes the unresolved short-distance information at a given accuracy into not just some, but the *smallest-possible* number of independent LECs. For example, the PC proposals of NN χEFT differ most for attractive triplet waves: the ^{3}P$_2$-^{3}F$_2$ system at order Q^0 has no LEC parameter [9] – or 3 of similar size [10, 11] – or 3, but with different weights [12, 13, 14, 15] – or 1 [16, 17]. To bring it to a boil: If all proposals are renormalised and fit NN data with the same χ^2, the one with the least number of parameters wins.

For the sake of this note, I am agnostic about this dispute. Rather, I propose to test if a predicted convergence pattern is reflected in the answers, i.e. if a proposed power counting is consistent.

On a historical note, the origin of these remarks goes back to publications in 2003 and 2005 [27, 28], and to lectures at the 2008 US National Nuclear Physics Summer School [29]. When the issue was revisited at two more recent workshops [18, 30], its conclusions were generally perceived as not immediately straightforward or widely known. Input on some aspects was also provided for two recent publications [31, 32]. It seems therefore fitting to present the test in the form of an expanded Technical Note. This article summarises an upcoming publication [1].

2. The Test: Turning Cutoff Dependence into an Advantage

Assume we calculated an observable $\mathcal{O}$ up to and including order Q^n in an EFT:

$$\mathcal{O}(k,p_{\text{typ}};\Lambda;\overline{\Lambda}_{\text{EFT}}) = \sum_i^n \left(\frac{k,p_{\text{typ}}}{\overline{\Lambda}_{\text{EFT}}}\right)^i \mathcal{O}_i(k,p_{\text{typ}};\overline{\Lambda}_{\text{EFT}}) + \mathcal{C}_n(\Lambda;k,p_{\text{typ}},\overline{\Lambda}_{\text{EFT}}) \left(\frac{k,p_{\text{typ}}}{\overline{\Lambda}_{\text{EFT}}}\right)^{n+1} \quad (2.1)$$

[Non-integer n and non-integer steps from one order to the next will be discussed in Sect. 4.1.] The notation indicates that numerators may depend on both k and p_{typ}. In a properly renormalised result, effects attributed to a regulator Λ appear only at orders which are higher than the last order n which is known in full. The residual $\mathcal{C}_n(\Lambda;k,p_{\text{typ}},\overline{\Lambda}_{\text{EFT}})$ may still depend on $\overline{\Lambda}_{\text{EFT}}$, k and p_{typ}, but it should be of natural size for all $k, p_{\text{typ}} < \overline{\Lambda}_{\text{EFT}}$, so that its contribution is parametrically suppressed by $\left(\frac{k,p_{\text{typ}}}{\overline{\Lambda}_{\text{EFT}}}\right)^{n+1}$ relative to the known terms of the series. If not, cutoff variations produce corrections which are comparable in size to the regulator-independent terms $\mathcal{O}_i(k,p_{\text{typ}};\overline{\Lambda}_{\text{EFT}})$ and contradict the EFT assumption that higher-order corrections are parametrically small.

The relative difference of $\mathcal{O}(k,p_{\text{typ}};\Lambda)$ at any two cutoffs is then:

$$\frac{\mathcal{O}_n(k,p_{\text{typ}};\Lambda_1)-\mathcal{O}_n(k,p_{\text{typ}};\Lambda_2)}{\mathcal{O}_n(k,p_{\text{typ}};\Lambda_1)} = \left(\frac{k,p_{\text{typ}}}{\overline{\Lambda}_{\text{EFT}}}\right)^{n+1} \times \frac{\mathcal{C}_n(\Lambda_1;k,p_{\text{typ}},\overline{\Lambda}_{\text{EFT}})-\mathcal{C}_n(\Lambda_2;k,p_{\text{typ}},\overline{\Lambda}_{\text{EFT}})}{\mathcal{C}_n(\Lambda_1;k,p_{\text{typ}},\overline{\Lambda}_{\text{EFT}})} \,. \quad (2.2)$$

One can now vary k or p_{typ} to read off both the order n to which the calculation is complete and the breakdown scale $\overline{\Lambda}_{\text{EFT}}$ – if the cutoff behaviour cannot be eliminated in its entirety, i.e. $\mathscr{C}_n(\Lambda_1) \neq \mathscr{C}_n(\Lambda_2)$ for at least some cutoff pairs, and if the residuals $\mathscr{C}_n$ vary more slowly with k and p_{typ} than with Λ. The results of such a fit may certainly be inconclusive; see extended remarks in Sect. 4. But if higher orders are *not* parametrically suppressed and the slope comes out *smaller* than the PC prediction $n+1$, then the EFT is necessarily not properly renormalised. As will be discussed in Sect. 4.1, a slope $\geq n+1$ does not suffice to demonstrate consistency or establish failure.

Equation (2.2) is formulated in terms of renormalised quantities only and therefore holds for any regulator, but it is most useful for cutoffs: Answers in nonperturbative EFTs are usually found only numerically and for a *cutoff regulator*, i.e. for a regulator which explicitly suppresses high momenta $q \gtrsim \Lambda$ in loops. It is this case which we use to our advantage from now on.

Cutoffs are of course only sensible if all loop momenta are sampled which lie in the domain of validity of the EFT, i.e. if $\Lambda \gtrsim \overline{\Lambda}_{\text{EFT}}$. Only then can the coefficients $\mathscr{C}_n$ be expected to be of natural size relative to $\overline{\Lambda}_{\text{EFT}}$ (with the caveats mentioned around eq. (1.3)). Except for this, no particular assumption is necessary as to the size of Λ relative to $\overline{\Lambda}_{\text{EFT}}$ in eq. (2.2). In dimensional regularisation and other analytic schemes, on the other hand, renormalisation can be performed exactly and no cutoff Λ or residual regulator scale appears in observables at all. Equation (2.2) is then an exact zero, with no information about n and $\overline{\Lambda}_{\text{EFT}}$. But doubts about proper renormalisation of a calculation which is analytic at each step do not arise, so the test is moot anyway.

3. An Application: Confirming the Hierarchy of $3N$ Interactions in EFT($\not\pi$)

Before continuing the discussion of the parameters of the test in Sect. 4, consider the first application (to my knowledge) of this test: the ${}^2\text{S}_{\frac{1}{2}}$ Nd wave in EFT($\not\pi$). It is well-known that its $3N$ interaction without derivatives does not follow simplistic PC rules ("just count momenta") which predict H_0 at N^{2}LO or $\mathscr{O}(Q^0)$ [7, 8]. Instead, it is needed at LO to stabilise the system (Thomas-collapse, Efimov effect); its scaling, $H_0 \sim Q^{-2}$, follows from eq. (1.3) for $n=3$. If the first momentum-dependent $3N$ interaction $k^2 H_2$ follows the simplistic argument and scales as Q^2, then new LECs need to be determined from $3N$ data only at N^{4}LO. Therefore, one could find $2N$ interaction strengths from few-N data with only one new $3N$ datum up to an accuracy of better than 1% at low momenta. This is crucial for example for hadronic flavour-conserving parity violation since it considerably extends the number of targets and observables [33].

Based on the asymptotic off-shell amplitudes, Refs. [27, 28] proposed that H_2 is only suppressed by Q^2 relative to LO, i.e. that calculations at N^{2}LO or on the 10%-level do already need one additional $3N$ datum as input. In Ref. [34], this was confirmed and extended to a general scheme to find the order at which any given $3N$ interaction starts contributing. The argument analyses perturbations to the asymptotic form of the LO integral equation. It is not immediately transparent, as witnessed by a subsequent claim that a k-dependent $3N$ interaction enters not earlier than N^{3}LO [35]. Upon closer inspection, it was later refuted [36].

Refs. [27, 28] also supplied numerical evidence from solutions of the Faddeev equations in momentum space with a step-function cutoff: a double-logarithmic plot of eq. (2.2) for the inverse K matrix, $\mathscr{O} = k\cot\delta$ at $\Lambda_1 = 900$ MeV and $\Lambda_2 = 200$ MeV, both well above the breakdown scale $\overline{\Lambda}_{\not\pi} \approx m_\pi$ of EFT($\not\pi$). A slight variant is reproduced here as Fig. 1. The cutoff dependence decreases

order-by-order as expected when the theory is perturbatively renormalised in the EFT sense. There is no decrease from NLO to N^{2}LO when $H_2 \equiv 0$. That by itself could be accidental – after all, would one not expect better convergence with one more parameter to tune?

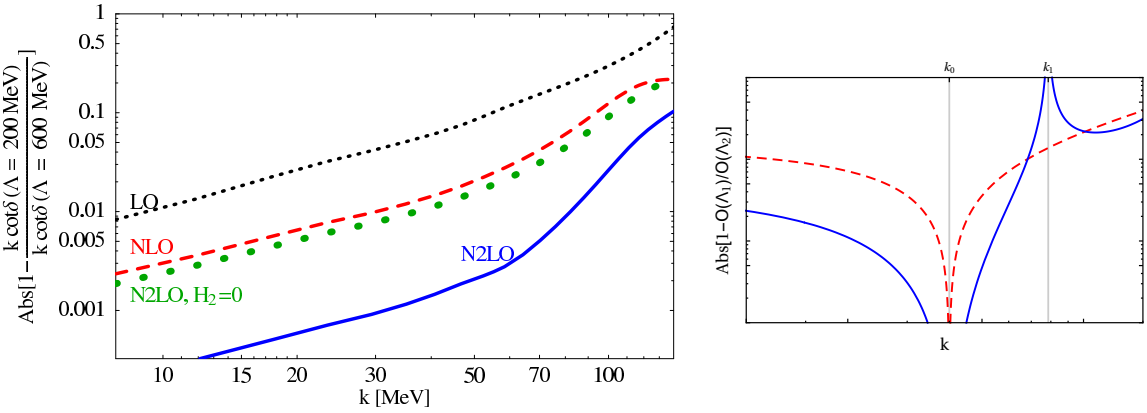


Figure 1: *Left*: Double-logarithmic error plot for the $^2S_{\frac{1}{2}}$ wave of *Nd* scattering in EFT($\not\pi$); cf. Refs. [27, 28]. *Right*: Qualitative example of the impact of zeroes in $\mathcal{O}(\Lambda_2) - \mathcal{O}(\Lambda_1)$ (exact reproduction of datum at k_0), and in $\mathcal{O}(\Lambda)$ ("accidental zero" of $\mathcal{O}(\Lambda_2)$ at k_1). Red dashed line: $n = 1$; blue solid: $n = 2$.

More informative is a look at the slopes. Lines at different orders are near-parallel for small k because there are additional natural low-energy scales p_{typ}, namely the binding momenta of the deuteron ($\gamma_t \approx 45$ MeV) and of the virtual singlet-S state ($\gamma_s \approx 8$ MeV). For $k \lesssim \gamma_{t,s}$, eq. (2.2) is not very sensitive to k, so all slopes should be small and near-identical. However, when $k \gg \gamma_{t,s}$ (but of course still $k \ll \overline{\Lambda}_{\not\pi}$, so that the EFT converges), they converge towards one region.

Indeed, the fits of n to the nearly straight lines in the momentum range between 70 and 100 to 130 MeV $< \overline{\Lambda}_{\not\pi}$ compare well to the PC prediction when H_2 is added at N^{2}LO [28]:

	LO	NLO	N^{2}LO	N^{2}LO without H_2
$n+1$ fitted	~ 1.9	2.9	4.8 [*sic!*]	3.1
$n+1$ predicted	2	3	4	not renormalised

(3.1)

Without H_2 at N^{2}LO, the slope does not improve from NLO. This is a clear signal that the PC is inconsistent without a momentum-dependent $3N$ interaction at N^{2}LO: Its assumptions do not bear out in the functional behaviour of this observable on k. On the other hand, when H_2 is included, the slope is markedly steeper than at NLO. The general agreement between predicted and fitted slope is astounding, and actually quite stable against variation of the fit range or of the two cutoffs Λ_1 and Λ_2. Only the LO numbers are somewhat sensitive, and only to the upper limit [28].

It is somewhat surprising that the slope increases by two units from NLO to N^{2}LO when one includes H_2. One would have expected the change from each order to the next to be by only one unit. This may stem from the "partially resummed formalism" used at that time. Since that resums some higher-order contributions, it may be worth revisiting this issue with J. Vanasse's method to determine higher-order corrections in "strict perturbation" [37]. But we will see in the notes on "Assumptions of the Expansion" in Sect. 4 that a fitted slope which is larger than predicted does not invalidate the power counting – the converse does.

Finally, one reads off a rough value of $\overline{\Lambda}_{\not\pi} \approx [120 \ldots 150]$ MeV as the region where the fitted lines coalesce. This is not in disagreement with the breakdown scale expected of EFT($\not\pi$).

4. Notes of Note

With this example in mind, let us consider assumptions, strengths, extensions, features, caveats and limitations of such an analysis to assess the consistency of a PC proposal.

4.1 Matters of Principle

Renormalisation Group Evolution Multiply eq. (2.2) by $(\Lambda_1-\Lambda_2)/\Lambda_1$ and take $\Lambda_2\to\Lambda_1$:

$$\frac{\Lambda}{\mathscr{O}}\,\frac{\mathrm{d}\mathscr{O}}{\mathrm{d}\Lambda} = \left(\frac{k,p_{\text{typ}}}{\overline{\Lambda}_{\text{EFT}}}\right)^{n+1}\,\frac{\mathrm{d}\ln\mathscr{C}_n(\Lambda)}{\mathrm{d}\ln\Lambda}\ . \tag{4.1}$$

This is Wilson's Renormalisation Group Equation for the observable $\mathscr{O}$. Note that eq. (4.1) features a *total* derivative: LECs in $\mathscr{O}$ are readjusted as Λ changes.

In practise, an EFT at finite order n and with finite cutoff tolerates cutoff artefacts which are parametrically small, i.e. at least of order $n+1$. This also limits the rate of change in the residual $\mathscr{C}_n$: I call an observable "perturbatively renormalised" when the right-hand side of eq. (2.2) is smaller than any term on the left-hand side. To some, this condition implies Λ can only be varied in a range around $\overline{\Lambda}_{\text{EFT}}$; the functional dependence on k and n is then still a quantitative prediction. A double-logarithmic plot reveals quantitative aspects of the Renormalisation Group evolution and can be utilised to falsify claims of consistency in an EFT.

Extending the Expansion The order n is not counted relative to LO. It is not even necessarily an integer, as Table 1 shows, and the first omitted order is not always Q^{n+1}, but more generally $Q^{n+\alpha}$, $\text{Re}[\alpha]>0$. To replace $n+1\to n+\alpha$ in eqs. (2.1), (2.2), (4.1) – and indeed throughout – is straightforward. In EFT($\not\pi$), the slope-fit in eq. (3.1) endorses that the $3N$ PC proceeds in integer steps. Including non-analytic dependencies of the residuals on k or p_{typ} is also straightforward. For the remainder of the presentation, all such replacements are implied, but we stick to the integer case for convenience.

Assumptions of the Expansion The assumptions on the residual $\mathscr{C}_n$ are endorsed if order n and breakdown scale $\overline{\Lambda}_{\text{EFT}}$ follow indeed the functional form of eq. (2.2) or its variant (4.1). Naïve Dimensional Analysis (NDA) sets the magnitude of $\mathscr{C}_n$ to the scale of its running [5, 6]. Its cutoff-dependence and other effects are eventually absorbed into higher-order LECs (see also below).

We can actually be somewhat more specific about the condition that the variation of the residual $\mathscr{C}_n$ variation with respect to Λ should be larger than that for other parameters. Since $k,p_{\text{typ}}\ll\Lambda_1,\Lambda_2$, the dimensionless ratio on the left-hand side of eq. (2.2) can be expanded as

$$\frac{\mathscr{C}_n(\Lambda_1;k,p_{\text{typ}},\overline{\Lambda}_{\text{EFT}})-\mathscr{C}_n(\Lambda_2;k,p_{\text{typ}},\overline{\Lambda}_{\text{EFT}})}{\mathscr{C}_n(\Lambda_1;k,p_{\text{typ}},\overline{\Lambda}_{\text{EFT}})} = c_0(\Lambda_1,\Lambda_2;\overline{\Lambda}_{\text{EFT}})+c_1(\Lambda_1,\Lambda_2;\overline{\Lambda}_{\text{EFT}})\,\frac{k,p_{\text{typ}}}{\Lambda_1,\Lambda_2}+\ldots \tag{4.2}$$

If the first term dominates, then the dependence of eq. (2.2) on k and p_{typ} is indeed indicative of the order Q^{n+1}. If subsequent terms dominate, the slope may be larger than $n+1$ – but never smaller.

Necessary but Not Sufficient This shows that a slope smaller than $n+1$ conclusively demonstrates failure of the PC to be consistent. However, the criterion is necessary rather than sufficient:

Slopes $\geq n+1$ are proof neither of failure, nor of success. Indeed, a PC may be inconsistent but the coefficient of the terms with slope $< n+1$ may be anomalously small, leading to a "false positive".

Estimating the Expansion Parameter When the cutoffs Λ_1 and Λ_2 are both varied over a wide range[1], the analysis also gives a practical way to find the size of the expansion parameter as a function of k. Ratios between different orders estimate $Q(k, p_{\text{typ}})$, and hence residual theoretical uncertainties as function of k. This is of course only one way to assess $Q(k)$; within reason, the least optimistic and hence most conservative of several methods should be picked. For example, Ref. [38] combined this with the convergence pattern of the EFT series; see also [39].

Choice of Expansion Parameter In Sect. 3, k is varied while the other scales p_{typ} are fixed, but any combination of the low-energy scales may serve as variable(s). For example, scanning in the pion mass at fixed $k \ll m_\pi$ may elucidate the m_π-dependence of some couplings, with particular relevance to extrapolating lattice computations at non-physical pion masses. Here, I will continue to concentrate on variations with k, but most issues transfer straightforwardly to other variations.

Window of Opportunity One can read off slopes most easily in the range $p_{\text{typ}} < k < \overline{\Lambda}_{\text{EFT}}$. In EFT($\not\pi$), that window is narrow but suffices: $\overline{\Lambda}_{\not\pi}/(p_{\text{typ}} \sim \gamma_{t,s}) \lesssim 3$. In χEFT with dynamical $\Delta(1232)$ degrees of freedom, we expect a wider range: $\overline{\Lambda}_{\chi}/(p_{\text{typ}} \sim m_\pi) \gtrsim 4$. One may of course also fit the variables n and $\overline{\Lambda}_{\text{EFT}}$ in eq. (2.2) to the numerical results outside that window, but then one needs to specify p_{typ} and determine its contribution relative to k.

Choice of Regulator Residual cutoff dependence comes naturally in numerical computations. This tests uses it as a tool to check consistency. The example used a "hard" cutoff, but $\overline{\Lambda}_{\text{EFT}}$ and n do not depend on a specific regulator. If the theory can be renormalised exactly, all residual regulator dependence disappears by dimensional transmutation; cf. (4.1).

Choice of Cutoffs The functional dependencies of eqs. (2.2) and (4.1) on n and $\overline{\Lambda}_{\text{EFT}}$ do not depend on Λ_1 and Λ_2. While any two cutoffs $\Lambda_1, \Lambda_2 \gtrsim \overline{\Lambda}_{\text{EFT}}$ will do in principle, small leverage may lead to numerical artefacts. The larger $\Lambda_2 - \Lambda_1$, the clearer the signal should be. For our example, Fig. 2 shows that an upper cutoff of 900 MeV instead of 600 MeV leads to different curves but very similar slopes. Infinities, zeroes and oscillations of $\mathscr{O}$ with k for any pair Λ_1, Λ_2 can lead to problems (see "*Observables: Accidental Zeroes and Infinities*" below) which are readily avoided by choosing a cutoff pair such that $\mathscr{O}(\Lambda_1) - \mathscr{O}(\Lambda_2) > 0$ for all k. Even when one does not choose to take one of the cutoffs to infinity[2], a reasonable range of allowed cutoffs exists. If $\Lambda_1 \approx \Lambda_2$, one may of course directly consider the numerical derivative of eq. (4.1) – over a range of cutoffs. [To reiterate: exact cutoff independence $\mathscr{O}(\Lambda_1) \equiv \mathscr{O}(\Lambda_2)$ for any cutoff pair is not considered.]

Decreasing Cutoff Dependence Equation (2.2) is a variant of the Renormalisation Group evolution of $\mathscr{O}$, eq. (4.1), which in turn quantifies the fundamental EFT tenet that observables must become order-by-order less sensitive to loop contributions beyond $\overline{\Lambda}_{\text{EFT}}$, the range of applicability. Cutoff dependence in observables should therefore generically decrease from order to order, irrespective whether or not LECs are fitted. This does not apply to the $\mathscr{C}_n$ themselves, but to the entire left-hand side of eq. (2.2). Refitting LECs may of course help to absorb some cutoff dependence. Indeed, no new LECs enter at NLO in the example above (H_0 is just refitted), and the cutoff

[1] Some claim that renormalisability requires that $\mathscr{O}$ has a unique limit as $\Lambda \to \infty$.

[2] One could adhere to the philosophy that cutoffs and breakdown scales should be similar.

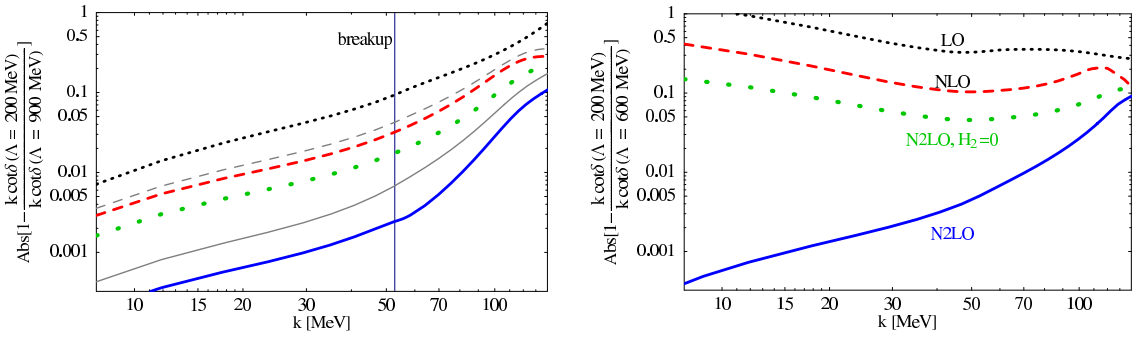


Figure 2: *Left*: Thick coloured lines: Z-parametrisation of the NN amplitude as in Fig. 1, but for $\Lambda_1 = 900$ MeV, not 600 MeV; thin gray lines: Bethe's Effective Range Parametrisation; from Ref. [28]. *Right*: Test when the leading $3N$ interaction is determined not by the Nd scattering length as in Fig. 1, but by the position of the triton pole. The N^{2}LO fit uses again the Nd scattering length and triton binding energy.

dependence decreases from LO to NLO. While it is conceivable that the residual $\mathscr{C}_n$ is sometimes somewhat larger than NDA predicts, NDA should apply "most of the time", statistically speaking.

Still, a specific regulator form may produce a very small residual cutoff dependence at one order but a significantly larger one at a subsequent order, $\mathscr{C}_n(\Lambda_1) - \mathscr{C}_n(\Lambda_2) < \mathscr{C}_{n+1}(\Lambda_1) - \mathscr{C}_{n+1}(\Lambda_2)$. This may for example occur if the regulator produces only corrections with even powers of Λ and the numerics preserves this symmetry at least approximately (e.g. because $\Lambda_1 \approx \Lambda_2$, allowing for a perturbative expansion). If this overwhelms the expansion in Q, $\mathscr{O}$ may indeed systematically become more dependent on Λ between some orders, but not between all. Nonetheless, one should not just see some qualitatively improved cutoff dependence with increasing order, but one must see the quantitatively predicted slopes emerge for many orders: they must be $\geq n+1$; see eq. (4.2).

Constructing a PC by Trial-and-Error If the cutoff dependence of a given observable does not decrease consistently between subsequent orders, caution may be advisable. For example, Λ-dependence may increase from one order to the next, but then decrease markedly when another full order with a new LEC is included. This could signal that this LEC cures cutoff dependence already at a lower order – and hence that the PC is inconsistent. One should then study the convergence pattern as the LEC is promoted to a lower order such that the cutoff dependence decreases always between subsequent orders. This may help to construct a consistent PC by trial-and-error and iteration. Remember also that after a LEC starts contributing at a certain order, it is re-adjusted at each subsequent order to absorb both cutoff effects and still match its determining datum.

Calculating Higher Orders Traditionally, observables beyond LO have been found by "partially resumming" contributions, i.e. the power-counted potential is iterated like in Weinberg's original suggestion. Since corrections to the LO potential are defined as parametrically small, they can be included in "strict perturbation", avoiding potential problems with spurious deeply bound state which can be generated by iteration [37]. This may also provide clearer signals for the PC test.

4.2 Picking Observables

Isolating Dynamical Effects While any observable could be chosen, those which are free from kinematic or other constraints (e.g. from symmetries) are preferred. Consider the scattering ampli-

tude $\mathscr{A}_l$ in the lth partial wave (for simplicity, assume no mixing). Since it is complex, one could choose $\mathscr{O} = |\mathscr{A}_l|$. However, unitarity relates $\mathscr{A}_l = 1/(k\cot\delta_l - \mathrm{i}k)$ to the phase shift δ_l. This constraint dominates when δ_l is between about $\pi/4$ and $3\pi/4$ – which affects much of the NN S-wave phase shifts. Even outside this interval, the additional contribution to eq. (2.2) is not sensitive to dynamics. In addition, analyticity dictates that phase shifts approach zero like k^{2l+1} for $k \to 0$ in the lth partial wave. Since both numerator and denominator in eq. (2.2) are then zero, $\mathscr{O} = \delta_l$ is dominated by numerical uncertainties as $k \to 0$. This may not be a problem if the region in which the slopes are determined is far away, but only a closer inspection could tell if that holds. Likewise, one eliminates phase-space factors in decay constants, production cross sections, etc.

A sensible choice for single-channel scattering appears thus to be $\mathscr{O} = k^{2l+1}\cot\delta_l$: It is only constrained to be real below the first inelasticity, and imaginary parts are usually small above it. Indeed, the S-wave example above kept track of the imaginary part by plotting

$$\left| 1 - \frac{k\cot\delta_0(\Lambda_2)}{k\cot\delta_0(\Lambda_1)} \right| \quad . \tag{4.3}$$

While factors of k formally cancel, one computes $\mathscr{A}$ $(k\cot\delta_0)$, so that numerics is more benign.

Partial-Wave Mixing In the NN system, two partial waves with total angular momentum J mix. The corresponding unconstrained observables in the Stapp-Ypsilanti-Metropolis (SYM or "nuclear-bar") parametrisation are

$$k^{2\pm1-2J}\bar{\delta}_{J\pm1} \quad \text{and} \quad k^{-(2J+1)}\bar{\varepsilon}_J \quad . \tag{4.4}$$

In the Blatt-Biedenharn parametrisation, the same rules apply for the eigenphases, but $k^{-2}\varepsilon_J$ is the unconstrained variable for the mixing angle; see e.g. [40]. These choices do not suffer from unitarity constraints (except for being real below the first inelasticity) and can be used directly.

Dependence on Parameter Input Let us first consider processes in which $\mathscr{O}(k)$ is a parameter-free prediction, i.e. its LECs are all known from some other process(es). To what extent does the procedure depend on that choice? In the example, the two-nucleon interactions were determined to match the Z-parametrisation of NN-scattering (fit to pole position and residue of the scattering amplitude) [41]. Fig. 2 shows that results with Bethe's Effective-Range parametrisation have a markedly different rate of convergence, but the extracted slopes and $\overline{\Lambda}_{\not\pi}$ agree very well [28].

Accidental Zeroes and Infinities Some observables may show additional structures which should be avoided. For example, the $^3\mathrm{P}_0$ phase shift in NN scattering is zero at a lab energy of about 150 MeV, so that the relative deviation of $\mathscr{O} = \delta_l$ in eq. (2.2) diverges. Likewise, $\mathscr{O} = k^{2l+1}\cot\delta_l$ diverges (approaches zero) at $\delta_l = 0$ $(\pi/2)$, e.g. in the $^1\mathrm{S}_0$ wave at $k \approx 370$ MeV and $^3\mathrm{S}_1$ wave at $k \approx 90$ MeV and 400 MeV [32]. As the qualitative plot in Fig. 1 shows, the corresponding spikes may make it more difficult to determine slopes.

Fitting to a Point A "zero" in eq. (2.2) is induced intentionally when the observable contains a LEC that is determined in the channel in which one tests the PC. If the observable is tuned to exactly reproduce a certain value at some point (k_0, p_{typ}), then $\mathscr{O}(k_0;\Lambda_1) - \mathscr{O}(k_0;\Lambda_2) = 0$ – with all the problems mentioned just now. Obviously, one should choose the fit point to be outside the slope-region. In the example of Sect. 3, the strength of the $3N$ interaction H_0 without derivatives

was fixed at each order to the Nd scattering length, i.e. using $k = 0$ as fit point. That is far away from the slope-region. At N^{2}LO, the momentum-dependent $3N$ interaction H_2 was in addition determined from the triton binding energy $B_3 = 8.48$ MeV, i.e. the pole in the amplitude is fixed to $k_0 = \sqrt{-4MB_3/3} \approx 100\,\mathrm{i}$ MeV. If one chooses this fit point for H_0 at LO and NLO, instead of $k_0 = 0$, the pattern of the slopes is wiped out; see Fig. 2. It appears that fitting only at k_0 introduces a new low-energy scale p_{typ} and leaves no window $\overline{\Lambda}_{\not\pi} \gg k \gg |k_0| \approx 100$ MeV, while the N^{2}LO fit at both $k = 0$ and k_0 does not suffer this limitation.

Fitting in a Region The issue is less transparent when the LEC is not determined by exactly reproducing some data, but by least-χ^2 fitting over a whole region in k. That is the typical case in NN scattering; see e.g. Ref. [32]. The deviation of the fitted result from data is more regular at any given cutoff Λ than when it is exactly zero at k_0. A pronounced spike is therefore replaced by a more uniform behaviour inside the fit region. The comparison between two cutoffs in eq. (2.2) is therefore also more uniform as a function of k. Since cutoff variations can now be balanced by adjusting LECs, the coefficients $\mathscr{C}_n$ are artificially small in that régime. One still expects the cutoff dependence to decrease order-by-order, but the characteristic slopes are harder to see since the observable is constrained by the fit. Just like in the neighbourhood of a fit point, an observable will first have to shed the fit constraints outside the fit region for pronounced slopes.

Such a fit region must of course be inside the applicability range of the EFT. Traditional fits do not take into account that the systematic uncertainties of an EFT increase with k but assign a k-independent uncertainty weight. Eq. (2.1) suggests that this is justified for $k \lesssim p_{\text{typ}}$ since the error varies only mildly. In that case, one can speculate that the impact on the slopes at higher k is not too big. This limits a reasonable fit region to $k \lesssim \gamma_{t,s}$ in EFT($\not\pi$); and to $k \lesssim m_\pi$ in χEFT. In addition, one expects clearer signals if the same fit region is used at each order. It is difficult to see how slopes can clearly be identified when the fit region extends far towards $\overline{\Lambda}_{\text{EFT}}$. Practical considerations, like insufficient or low-quality data at low momenta may well override this choice.

Fitting to Pseudo-Data As a recourse and in order to assess the impact of a fit region on the slopes, one may create an artificial, "exact datum" $\mathscr{O}_0(k_0)$ at very low $k \to 0$ which agrees with low-energy data (e.g. a scattering length, effective range, etc); and then assess the dependence of the slope on reasonable variations of $\mathscr{O}_0(k_0)$. The goal is then not to find good agreement with actual data at higher energies, but to test the convergence pattern.

Summary: Choice of Observable Ideal candidates for $\mathscr{O}$ are positive-definite observables which are not subject to unitarity and other constraints, and which are nonzero and finite over a wide range in k and Λ, including the régime $k \gtrsim p_{\text{typ}}$ where one hopes to determine the slope. EFT parameters/LECs should be determined at very low k. A good signal may need some creativity. The choices $\mathscr{O} = k^{2l+1} \cot\delta_l$, $k^{2\pm1-2J}\bar{\delta}_{J\pm1}$ and $k^{-(2J+1)}\bar{\varepsilon}_J$, with effective-range parameters determining unknowns, appear suitable in most scattering cases.

4.3 Miscellaneous Notes

Consistency Assessment vs. "Lepage Plots" Double-logarithmic convergence plots are not unfamiliar. Lepage compared to data in order to quantify how accurately the EFT reproduces experimental information [4]. This triggered a series of influential studies of differences between

approximations and "exact results" in toy-models, see e.g. [42, 43, 44]. Recently, Birse perused similar techniques, after removing the strong influence of long-range Physics (One- and Two-Pion Exchange) from empirical phase shifts in a modified effective range expansion, allowing for a more detailed study of the residual short-distance interactions [45, 46]. Such investigations assume that the correct PC is known and quantitative comparison to data is needed.

The test advocated here aims to answer different questions: Does the output match the assumptions? Is the theory consistent? Recall that an EFT may converge by itself, but not to data, if some dynamical degrees of freedom are incorrect or missing. For example, a χEFT without dynamical $\Delta(1232)$ at $k \approx 300$ MeV cannot reproduce Delta resonance properties – but it may well be consistent. In other words, an EFT may be consistent, but not consistent with Nature.

Insensitivity to Some LECs This procedure can only help determine if a LEC is correctly accounted for when it is needed to absorb residual cutoff dependence. Eq. (4.1) then determines its running, and its initial condition is fixed by some input, for example data or results of a more fundamental theory. Some LECs do however start contributing just because of their natural size, and not to renormalise that order. For example, the magnetic moment of the nucleon enters the one-baryon Lagrangean of χEFT at NLO, albeit it is not needed to renormalise loops. Similarly, the contribution of a LEC to a particular observable may be unnaturally small (or even zero).

Numerics The analysis can be numerically indecisive. We would trust results only if n and $\overline{\Lambda}_{\text{EFT}}$ can be determined quite robustly in a reasonably wide range to cutoffs (and, possibly, cutoff forms), parameter sets and fit-windows. None of this provides, however, sufficient excuse not report results.

Sampling Tests Finding that the slope at each order Q^n is not smaller than $n+1$ is necessary but not sufficient for a consistent PC. We saw that fine-tuning, particular choices of regulator forms and observables, and anomalously small coefficients are some reasons which may hide signals of slopes $< n+1$ which violate the PC assumptions. If slopes are always $\geq n+1$ for a variety of independent observables, regulators etc., that may increase confidence in PC consistency – but cannot prove it.

4.4 Outlook

The χEFT power-counting proposals differ most starkly in the attractive triplet partial waves of NN scattering since they reflect different philosophies on how to treat the non-selfadjoint, attractive $1/r^3$ potential at short distances which appears at leading order; see Table 1. It would therefore be interesting to see this test applied to the ^{3}P$_0$ wave and to the ^{3}P$_2$-^{3}F$_2$ system. The test proposed here is not necessarily a silver bullet to endorse or reject a particular counting since its results may in the worst case be inconclusive. But that implies it is still worth a try.

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