PROCEEDINGS OF SCIENCE

PoS(CD15)114

Surpassing Wigner's causality bound in relativistic scattering with zero-range interaction

Vladimir Pascalutsa*

Institut für Kernphysik, Johannes Gutenberg Universität Mainz, D-55099 Mainz, Germany

It is shown that the relativistic zero-range potential scattering surpasses Wigner's causality bound while being consistent with causality. The relativistic theory shows in addition a richer analytic structure such as a K-matrix pole necessarily accompanying the bound-state solution. Possible implications of these results for the effective-field theory of nuclear forces are briefly considered.

The 8th International Workshop on Chiral Dynamics, CD2015
29 June 2015 – 03 July 2015
Pisa, Italy

*Supported by the Deutsche Forschungsgemeinschaft (DFG) through Collaborative Research Center SFB 1044.

1. Introduction

Wigner's seminal work on causality bounds for the effective range of low-energy scattering [1] has been revisited quite recently in connection to the effective-field-theoretic (EFT) description of few-nucleon systems and cold atoms, see e.g. [2–5]. Zero-range forces play an important role in these considerations as they are expected to provide a leading-order description of any finite-range force, be it nuclear or Van der Waals. Indeed, the very low-energy (long-distance) probes of systems bound by finite-range forces cannot resolve the extent at which the forces act, and hence the zero-range approximation should naively be fine. For the nuclear force, however, it does not appear to be too fine. As first noted by Phillips and Cohen [2], in case of zero-range forces the Wigner's causality bound infers negative values for the s-wave effective-range parameters, in appreciable disagreement with what is observed in nucleon-nucleon (NN) scattering. This problem can be overcome by treating range corrections in perturbation theory, along with other interactions needed for renormalization-group invariance [6–8]. Further difficulties arise, however, when pions are included (perturbatively) in this framework, see e.g. [9] and references therein. A commonly accepted solution nowadays is to "promote" a finite-range (one-pion-exchange) force into the leading order, see Refs. [10–12] for reviews. Here, however, we would like to pursue a different route and demonstrate that a relativistic theory of zero-range forces can both be consistent with causality and yield positive effective-range parameters.

More specifically, introducing the s-wave scattering phase-shift $\delta(k)$, which is a function of the relative momentum k, the effective-range expansion is written as:

$$k\cot\delta(k) = -\frac{1}{a} + \frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n+1}\, r_n\, k^{2n}, \tag{1.1}$$

where a is the scattering length, r_1 is the effective range, and $r_{n\geq 2}$ are (up to an overall factor of 2) the effective-shape parameters. While Wigner's causality bound for scattering through a δ-function potential (zero-range force) yields [2]:

$$r_1 \leq 0 \quad \text{(Wigner's bound)}, \tag{1.2}$$

we establish here that the effective range is non-negative for causal scattering, together in fact with all the effective-shape parameters, i.e.:

$$r_n \geq 0 \quad \text{(present work)}. \tag{1.3}$$

This result is in near perfect disagreement with Wigner's bound, however, will be shown to reconcile with it in the non-relativistic limit where $r_1 = 0$. Away from non-relativistic limit this result may open up a venue for an EFT description of nuclear forces where the pion exchange is suppressed with respect to the zero-range interaction.

2. Light-by-light sum rule as causality criterion

Given the nearly perfect disparity of the two causality bounds quoted above, we start by noting that they are based on different interpretations of causality. Wigner's bound is based on positivity of time delay between the incoming and scattered wave, which translates into the following condition for the phase shift [13]:

$$\mathrm{d}\delta/\mathrm{d}k \geq (2k)^{-1}\sin 2\delta. \tag{2.1}$$

Taking here $k \to 0$ one arrives to Eq. (1.2). We, on the other hand, adopt a causality criterion based on dispersion theory. Namely, we follow up on the proposal [14] to exploit the analog of the Gerasimov-Drell-Hearn (GDH) sum rule for the light-light ($\gamma\gamma$) system [15–17]:

$$\int_0^\infty \mathrm{d}s\,\frac{\sigma_2(s)-\sigma_0(s)}{s} = 0, \tag{2.2}$$

where $\sigma_2(s)$ and $\sigma_0(s)$ are the cross sections of two-photon fusion process ($\gamma\gamma \to X$) with photons circularly polarised in the same or opposite directions, respectively. The total invariant energy squared is $s = (q_1+q_2)^2$, for q_1 and q_2 the colliding photon four-momenta.

The validity of this sum rule relies on only general principles such as Lorentz and gauge symmetries, unitarity and analyticity. The latter requirement is associated with causality and is the less trivial to satisfy in a given modeling of these cross sections. This is why the sum rule verification is an indicator of causality above all the other aforementioned principles.

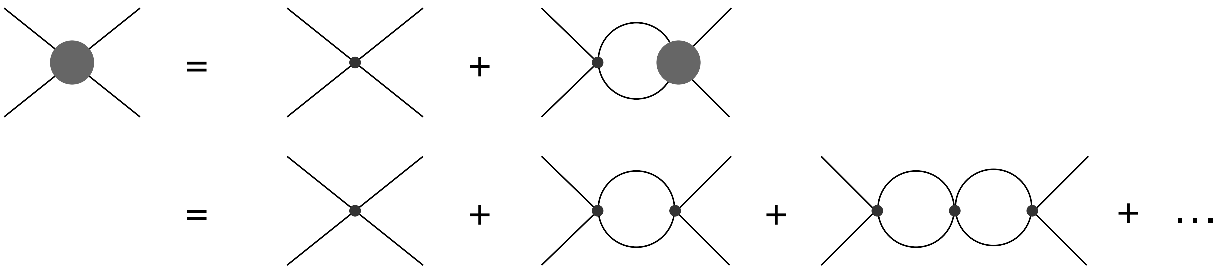

Figure 1: The Bethe-Salpeter equation and its iterative solution.

The sum-rule criterion is applicable to a relativistic scattering theory by constructing a particle-antiparticle scattering amplitude and then considering the $\gamma\gamma$ fusion into the pair. Assuming, for instance, that the scattering amplitude is found as the solution of the Bethe-Salpeter equation graphically represented in Fig. 1, the corresponding $\gamma\gamma$-fusion process is given by diagrams in Fig. 2.

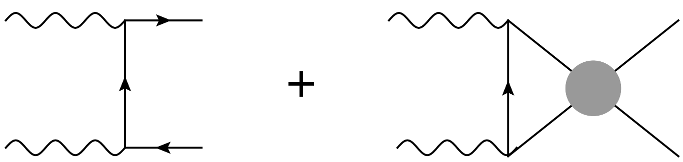

Figure 2: Photon-photon fusion with rescattering.

For the relevant case of the δ-function potential, given in momentum space by a constant $V = \lambda$, this criterion has first been employed by Pauk *et al.* [18], who showed that the sum rule is satisfied, unless $\lambda \in (-8\pi^2, 0)$.

We recall that the solution of the Bethe-Salpeter equation ($T = V + VGT$) is algebraic in this case and for the equal-mass situation reads:[1]

$$T(s) = \frac{1}{\lambda^{-1} - (4\pi)^{-2} B(s)}, \tag{2.3}$$

where $B(s)$ is a subtracted Passarino-Veltman one-loop integral B_0 [20]:

$$B(s) \equiv B_0(s, m^2, m^2) - B_0(4m^2, m^2, m^2) = -2v \,\text{arctanh}\, v^{-1}, \tag{2.4}$$

with m denoting the particle mass and $v = \sqrt{1 - 4m^2/s}$ their relative velocity. The subtraction is chosen such that at the threshold (zero velocity) the interaction strength is given by λ. Then, the scattering length is $a = -\lambda/(16\pi m)$ and hence the sign of the potential unambiguously implies that negative or positive a corresponds respectively to repulsive or attractive interaction.

In the center-of-mass frame, the two scatterers share the energy equally and hence their relative momentum is

$$k = \tfrac{1}{2} v s^{1/2} = \left(\tfrac{1}{4} s - m^2\right)^{1/2}. \tag{2.5}$$

In the following we use s, v, or k interchangeably as the energy variable. The amplitude is independent of scattering angle in this case, hence has no partial waves beyond the s-wave.

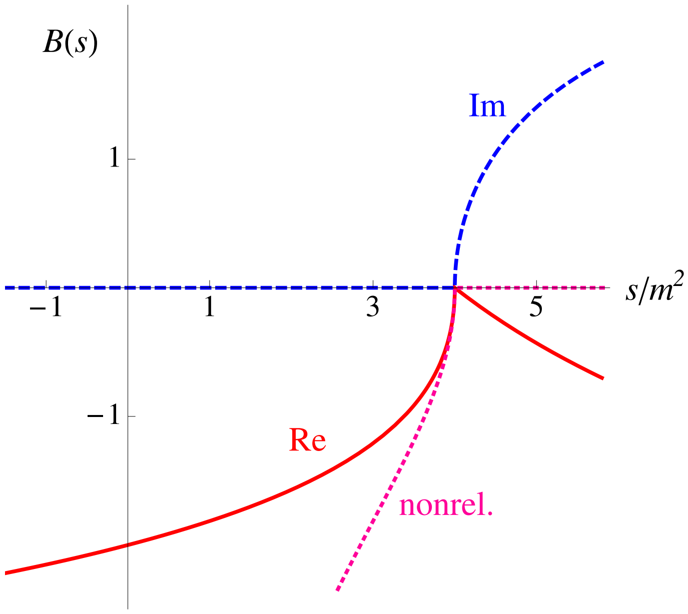

Figure 3: The real (solid red) and imaginary (blue dashed) parts of the loop function $B(s)$. The "nonrel." (dotted magenta) curve shows the non-relativistic approximation to the real part.

The analytic properties of the amplitude T are determined by the loop function B plotted in Fig. 3. For negative λ, the amplitude develops a pole at the position where

$$(4\pi)^2 \lambda^{-1} = B(s). \tag{2.6}$$

[1]Although this solution may seem arbitrary from field-theoretic point of view, it emerges in the $O(N)$ models as an exact solution in the large-N limit, see e.g. [19].

Solving this equation for s one finds the mass squared M^2 of the corresponding bound state solution. Since $B(s)$ is negative, there is no solution for positive λ. Furthermore, above the threshold the loop function develops an imaginary part,

$$\mathrm{Im}\,B(s) = \pi v\,\theta(v^2) = \frac{\pi}{(1+m^2/k^2)^{1/2}}\,\theta(k^2), \tag{2.7}$$

and since λ is real, there is only a solution below the threshold: a bound state with $M^2 < 4m^2$. There are no poles for complex s as is demonstrated in the Appendix.

Importantly, since the loop function extends to negative s, for $(4\pi)^2\lambda^{-1} < -2$ one finds $M^2 < 0$, i.e. the tachyon. The appearance of a tachyon solution is in apparent conflict with causality, and indeed the light-by-light scattering sum rule cannot be satisfied in this case [18]. In the bound-state case (i.e., $M^2 \geq 0$) the sum rule is satisfied provided the bound state is treated as an asymptotic state, and hence the channel of $\gamma\gamma$ fusion into the bound state is included.

To summarize, while the helicity-difference sum rule given in Eq. (2.2) is easily verified for the repulsive ($\lambda > 0$) δ-function potential [18], for attractive interaction ($\lambda < 0$) there is a causal (bound-state) and acausal (tachyon) regimes. We thus distinguish the following two domains:

$$\text{causal:} \quad -\infty < \lambda \leq -8\pi^2 \cup \lambda \geq 0, \tag{2.8a}$$

$$\text{acausal:} \quad -8\pi^2 < \lambda < 0. \tag{2.8b}$$

We next consider how these domains project onto the effective-range parameters.

3. Causality bound in effective-range expansion

Our suitably normalized elastic scattering amplitude is given by:

$$\frac{v}{16\pi}\,T(s) \equiv f(k) = \left(\frac{16\pi}{\lambda\sqrt{1+m^2/k^2}} + \frac{2}{\pi}\,\mathrm{arctanh}\,\frac{1}{\sqrt{1+m^2/k^2}} - i\right)^{-1}, \tag{3.1}$$

and is related to the phase shift and the K-matrix as:

$$f(k) = e^{i\delta(k)}\sin\delta(k) = \left[K^{-1}(k) - i\right]^{-1}. \tag{3.2}$$

The effective-range expansion proceeds then as follows:[2]

$$\begin{aligned} k\cot\delta(k) &= 16\pi\lambda^{-1}\sqrt{m^2+k^2} + (2k/\pi)\,\mathrm{arccoth}\sqrt{1+m^2/k^2} \\ &= 16\pi\lambda^{-1}m + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}\Gamma(n-1/2)}{(n-1)!(2n-1)\pi^{3/2}}\left(1+\frac{2n-1}{4n}(4\pi)^2\lambda^{-1}\right)\frac{k^{2n}}{m^{2n-1}}. \end{aligned} \tag{3.3}$$

[2]In doing the expansion one uses (for $x \geq 0$):

$$\mathrm{arccoth}\sqrt{1+1/x^2} = \mathrm{arccosh}\sqrt{1+x^2} = \mathrm{arcsinh}\,x = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}\Gamma(n-1/2)}{(n-1)!(2n-1)\sqrt{\pi}}\,x^{2n-1}.$$

Comparing to Eq. (1.1) we identify the scattering length, the effective-range and shape parameters:

$$a = -\frac{\lambda}{16\pi m}, \tag{3.4a}$$

$$r_1 = \frac{4}{\pi m}\left(1+4\pi^2\lambda^{-1}\right), \tag{3.4b}$$

$$r_n = \frac{2\Gamma(n-1/2)}{n!\,\pi^{3/2}m^{2n-1}}\left(\frac{n}{2n-1}+4\pi^2\lambda^{-1}\right). \tag{3.4c}$$

It is obvious that r_n can only turn negative provided λ satisfies:

$$-4\pi^2\frac{2n-1}{n} < \lambda < 0. \tag{3.5}$$

This domain, however, is well within the acausal region [Eq. (2.8b)], at least for any integer n. Hence, as long as λ is within the allowed causal range [Eq. (2.8a)], we obtain the central result of this work:

$$r_n \geq \frac{\Gamma(n-1/2)}{n!\,(2n-1)}\frac{1}{\pi^{3/2}m^{2n-1}} \geq 0, \tag{3.6}$$

for any integer n. In particular, for the effective range we obtain:

$$r_1 \geq \frac{2}{\pi m}. \tag{3.7}$$

As noted above, this is in near perfect disagreement with the corresponding Wigner's bound: $r_1 \leq 0$. In the following section we point out a possible origin of this disagreement and further discuss the analytical properties of the new solution.

4. K-matrix pole as satellite of the bound-state pole

The Wigner's bound arises in non-relativistic scattering theory [13]. Our causality criterion is based on relativistic dispersion theory. The difference between the bounds [Eqs. (1.2) vs. (1.3)] should therefore be pinned on "relativistic effects". Indeed, by taking the non-relativistic limit ($k/m \to 0$) in our example one obtains $r_1 = 0$, which honors the Wigner's bound, albeit quite trivially.

The non-relativistic limit on the other hand ruins the analyticity in s as can be seen for the "nonrel." curve in Fig. 3 which displays the real part of the loop function in the non-relativistic limit; the imaginary part remains unchanged. One sees that, while in the threshold region ($s \approx 4m^2$) the non-relativistic limit may serve as a good approximation, it is missing important features away from the threshold. One such feature is the K-matrix pole which appears in relativistic theory at $s = s_K > 4m^2$ such that

$$\mathrm{Re}\,B(s_K) = (4\pi)^2\lambda^{-1}. \tag{4.1}$$

This pole disappears in the non-relativistic limit, since then $\mathrm{Re}\,B(s) = 0$, for $s \geq 4m^2$.

In the full theory, however, the bound-state pole appearing at $s = M^2 < 4m^2$ is always accompanied by a K-matrix pole.[3] The closer is the bound state to the threshold (the "shallower" it is),

[3]The K-matrix pole is sometimes indicative of a resonance, however not in this case. The present solution for the amplitude T, and hence for the S-matrix, has no poles for complex s. A simple proof of this statement is given in the Appendix.

the closer is the K-matrix pole. The phase shift, of course, crosses 90 degrees at the K-matrix pole position, as $K(k) = \tan\delta(k)$ by definition. Hence, for a shallow bound state such the deuteron, the corresponding phase-shift (i.e., ${}^3S_1$ in case of NN scattering) starts at $\delta(0) = \pi$ (due to Levinson's theorem) and then quickly goes down to cross $\pi/2$ at a fairly low k. This is how in fact the empirical ${}^3S_1$ phase shift behaves. In the non-relativistic description with zero-range potential the phase shift never crosses $\pi/2$. We therefore expect a more effective description of the deuteron phase-shift within the relativistic theory.

For a very shallow bound state ($\lambda < 0$, $|\lambda| \gg 8\pi^2$), the transcendental equations for the bound-state and K-matrix pole positions can be solved to yield, respectively:

$$M^2 \simeq \frac{4m^2}{1+(16\pi)^2\lambda^{-2}}, \tag{4.2a}$$

$$s_K \simeq \frac{4m^2}{1+8\pi^2\lambda^{-1}}. \tag{4.2b}$$

We thus can establish an approximate relation between the binding energy, $B = 2m - M$, and the momentum at which the corresponding phase shift crosses 90 degrees, $k_{\pi/2} = (1/2)\sqrt{s_K - 4m^2}$:

$$k_{\pi/2} \approx B^{1/4}m^{3/4}. \tag{4.3}$$

For the kinetic energy $k^2_{\pi/2}/m$, we simply have $\sqrt{mB}$, which shows that the position of the K-matrix pole is directly related with the soft scale emerging in the presence of the bound state. This scale arises here naturally, rather than as a result of fine-tuning the subleading contributions as in the non-relativistic theory (see e.g., [10]).

5. Conclusion

The zero-range forces should be playing the leading role in a low-energy EFT description of any short-range interaction such as nuclear or inter-atomic. However, at least in a non-relativistic formulation, a zero-range force is bound to yield non-positive effective-range parameters [2], and hence is bound not to be adequate empirically, unless a physical cut-off is introduced. We have shown that in relativistic theory the zero-range force yields only positive effective-range parameters, provided causality is respected. This appears to be in complete disagreement with Wigner's causality bound. The precise origin of this paradox has not been entirely understood here, however we certainly favor here the relativistic approach to causality.

A question of consistency of the bubble-chain approximation arises, as from field-theoretic point of view it presents a dramatic truncation of the full theory. Similar concerns may arise in developing a power counting in the EFT framework, as relativistic effects appear merely as effects of "higher order". The truncation considered in this work is consistent at least with respect to the agreement with the sum rule, hence has the correct analytic structure.

An interesting prediction of relativistic theory of zero-range interactions is the fact that a bound state is accompanied by a K-matrix pole. The latter shows up in the pertinent phase-shift crossing of 90 degrees. In the case of a shallow bound state, its binding energy determines the position of the 90 degree crossing according to Eq. (4.3). The K-matrix pole does not correspond to a resonance in this case.

It remains to be seen whether these findings will help to reorganize the EFT of nuclear forces so as to defer the finite-range considerations (e.g., the pion exchange) and 3-nucleon forces where the naive dimensional analysis places them — subleading orders. As result, the idea of 'perturbative pions' [21], which fails in the strictly nonrelativistic description, may be revived in the relativistic framework.

Acknowledgements

It is a pleasure to thank Mike Birse, Evgeny Epelbaum, Ubirajara van Kolck, Dean Lee, Daniel Phillips, and Marc Vanderhaeghen for valuable remarks on the manuscript. The work was partially supported by the Deutsche Forschungsgemeinschaft through Collaborative Research Center "The Low-Energy Frontier of the Standard Model" (SFB 1044) and the Cluster of Excellence "Precision Physics, Fundamental Interactions and Structure of Matter" (PRISMA).

Appendix: No poles for complex s

To show that the amplitude $T(s)$ given by Eq. (2.3) has no poles for complex s we need to show that $\lambda^{-1} = (4\pi)^{-2}B(s)$ has no solution for $s = s_\mathrm{r} + is_\mathrm{i}$, with $s_\mathrm{r}, s_\mathrm{i} \in \mathbb{R}$ and $s_\mathrm{i} \neq 0$. As due to hermiticity λ is real, we only need to show that $\operatorname{Im} B(s) \neq 0$, for $s_\mathrm{i} \neq 0$. For this we use the dispersion relation for the subtracted loop integral:

$$B(s) = \frac{1}{\pi}\int_{4m^2}^{\infty} ds' \, \frac{\operatorname{Im} B(s')}{s'-s}\left(\frac{s-4m^2}{s'-4m^2}\right), \tag{1}$$

with $\operatorname{Im} B(s)$ for real s given in Eq. (2.7). We then proceed to write

$$B(s) = \frac{s-4m^2}{\pi}\int_{4m^2}^{\infty} ds' \, \frac{\operatorname{Im} B(s')}{|s'-s|^2}\,\frac{s'-s^*}{s'-4m^2}. \tag{2}$$

Hence, the real and imaginary parts of B are given respectively as:

$$\operatorname{Re} B(s) = \frac{1}{\pi}\int_{4m^2}^{\infty} ds' \, \frac{\operatorname{Im} B(s')}{|s'-s|^2}\left(s_\mathrm{r} - 4m^2 + \frac{|s-4m^2|^2}{s'-4m^2}\right), \tag{3}$$

$$\operatorname{Im} B(s) = \frac{s_\mathrm{i}}{\pi}\int_{4m^2}^{\infty} ds' \, \frac{\operatorname{Im} B(s')}{|s'-s|^2}. \tag{4}$$

The integrand in the latter expression is positive definite, hence the integral is not zero, and hence for $s_\mathrm{i} \neq 0$, we indeed have $\operatorname{Im} B(s) \neq 0$. Therefore, $T(s)$ has no poles away from the real axis.

References

[1] E. P. Wigner, Phys. Rev. **98** (1955) 145.

[2] D. R. Phillips and T. D. Cohen, Phys. Lett. B **390** (1997) 7.

[3] M. Pavon Valderrama and E. Ruiz Arriola, Phys. Rev. C **74** (2006) 054001.

[4] H. -W. Hammer and D. Lee, Phys. Lett. B **681** (2009) 500; Annals Phys. **325** (2010) 2212.

[5] S. Elhatisari and D. Lee, Eur. Phys. J. A **48** (2012) 110.

[6] U. van Kolck, Nucl. Phys. A **645**, 273 (1999).

[7] M. C. Birse, J. A. McGovern and K. G. Richardson, Phys. Lett. B **464**, 169 (1999).

[8] J. Gegelia, Phys. Lett. B **429**, 227 (1998).

[9] T. D. Cohen and J. M. Hansen, Phys. Rev. C **59**, 3047 (1999).

[10] U. van Kolck, Prog. Part. Nucl. Phys. **43**, 337 (1999).

[11] S. K. Bogner, T. T. S. Kuo and A. Schwenk, Phys. Rept. **386**, 1 (2003).

[12] E. Epelbaum, Prog. Part. Nucl. Phys. **57**, 654 (2006).

[13] H. M. Nussenzveig, *Causality and Dispersion Relations* (New York: Academic Press, 1972).

[14] V. Pascalutsa, Few Body Syst. **54**, 53 (2013) [arXiv:1110.5792 [nucl-th]].

[15] S. B. Gerasimov and J. Moulin, Nucl. Phys. B **98** (1975) 349 [Yad. Fiz. **23** (1976) 142].

[16] S. J. Brodsky and I. Schmidt, Phys. Lett. B **351** (1995) 344.

[17] V. Pascalutsa and M. Vanderhaeghen, Phys. Rev. Lett. **105**, 201603 (2010).

[18] V. Pauk, V. Pascalutsa and M. Vanderhaeghen, Phys. Lett. B **725**, 504 (2013).

[19] H. J. Schnitzer, Phys. Rev. D **10**, 1800 (1974).

[20] G. Passarino and M. J. G. Veltman, Nucl. Phys. B **160** (1979) 151; T. Hahn and M. Perez-Victoria, Comput. Phys. Commun. **118** (1999) 153.

[21] D. B. Kaplan, M. J. Savage and M. B. Wise, Phys. Lett. B **424**, 390 (1998); Nucl. Phys. B **534**, 329 (1998).

PoS(CD15)114