Temperature dependence of shear viscosity in $SU(3)$–gluodynamics

V. Braguta[*]

Institute for Theoretical and Experimental Physics, Moscow, 117218 Russia
Institute for High Energy Physics NRC "Kurchatov Institute", Protvino, 142281 Russian
Federation
Far Eastern Federal University, School of Biomedicine, 690950 Vladivostok, Russia
National Research Nuclear University MEPhI (Moscow Engineering Physics Institute), 115409,
Russia, Moscow
E-mail: braguta@itep.ru

N. Astrakhantsev

Institute for Theoretical and Experimental Physics, Moscow, 117218 Russia
Moscow Institute of Physics and Technology, Dolgoprudny, 141700 Russia
E-mail: nikita.astrakhantsev@itep.ru

A. Kotov

Institute for Theoretical and Experimental Physics, Moscow, 117218 Russia
E-mail: kotov@itep.ru

This report is devoted to the study of temperature dependence of shear viscosity in $SU(3)$-gluodynamics. To calculate shear viscosity we measured the correlation function of the energy-momentum tensor $T_{12}T_{12}$ for a set of temperatures in the region $T/T_c \in (0.9, 1.5)$. The measurements were carried out using multilevel algorithm which considerably improves the accuracy of the data. For determination of the shear viscosity two methods were used: physically motivated ansatz and non-parametrical Backus-Gilbert method. Both methods give consistent results. The calculation allows to determine temperature dependence of the ratio of shear viscosity to the entropy density η/s.

---

[*]Speaker.

1. Introduction

One of the main purposes of heavy ion collision experiments at RHIC and LHC is the experimental study of Quark-gluon plasma (QGP). It was shown that experimental results can be described within hydrodynamical approach [1, 2]. In hydrodynamics transport coefficients such as shear viscosity, bulk viscosity, conductivity, etc. determine the dynamics of conserved charges. These transport coefficients can be determined from the experiment or calculated theoretically. For instance, the elliptic flow measurement [3, 4] allows to determine the shear viscosity of QGP. It turns out that the ratio of shear viscosity η to entropy density s lies in range $\eta/s = (1-3) \times 1/4\pi$ [5], which is close to $N = 4$ Super Yang Mills (SYM) theory prediction at strong coupling $\eta/s = 1/4\pi$ [6] and differs from a weak coupling result $\eta/s \sim c/(g^4 \log(1/g)) \sim 1$ [7, 8]. Thus a ratio η/s is very small and cannot be described within perturbation theory. It would be interesting to study the value of the shear viscosity within such nonperturbative approach as lattice simulations of QCD.

There were only few attempts to calculate shear viscosity of SU(3)–gluodynamics[9, 10, 11, 12, 13] and SU(2)–gluodynamics [14, 15]. In this paper we are going to study temperature dependence of shear viscosity of SU(3)–gluodynamics in the vicinity of confinement/deconfinement phase transition $T/T_c \in [0.9, 1.5]$.

2. Details of the calculation

Shear viscosity is related to the Euclidean correlation function of the energy-momentum tensor $T_{\mu\nu}$:

$$C(x_0) = T^{-5} \int d^3\mathbf{x} \langle T_{12}(0) T_{12}(x_0, \mathbf{x}) \rangle, \tag{2.1}$$

where T is the temperature of the system. The correlation function (2.1) can be written in terms of the spectral function $\rho(\omega)$ as follows

$$C(x_0) = T^{-5} \int_0^\infty \rho(\omega) \frac{\cosh \omega(\frac{1}{2T} - x_0)}{\sinh \frac{\omega}{2T}} d\omega. \tag{2.2}$$

In order to find shear viscosity from the spectral function one uses the Kubo formula [16] $\eta = \pi \lim_{\omega \to 0} \frac{\rho(\omega)}{\omega}$.

Lattice calculation of shear viscosity can be divided into two parts. The first part is the measurement of the correlation function $C(x_0)$ with sufficient accuracy. This part of the calculation requires large computational resources but for the gluodynamics the accuracy of the correlator can be dramatically improved with the help of the two-level algorithm [17]. The second part is the determination of the spectral function $\rho(\omega)$ from the correlation function $C(x_0)$. The last part of the calculation is probably the most complicated, since one should determine continuous spectral function $\rho(\omega)$ from integral equation (2.2) for the set of $O(10)$ values of the function $C(x_0)$ measured in lattice simulation.

First we recall very general properties: the positivity of the spectral function $\rho(\omega)/\omega \geq 0$ and oddness: $\rho(-\omega) = -\rho(\omega)$. At large frequencies one expects asymptotic freedom behaviour. For

this reason it is also important to write the expression for the spectral function in the leading-order approximation in strong coupling constant expansion [18]:

$$\rho^{LO}(\omega) = \frac{1}{10} \frac{d_A}{(4\pi)^2} \frac{\omega^4}{\tanh(\frac{\omega}{4T})} + \left(\frac{2\pi}{15}\right)^2 d_A T^4 \, \omega\delta(\omega), \tag{2.3}$$

where $d_A = N_c^2 - 1 = 8$ for the SU(3)-gluodynamics.

One also knows next-to-leading order expression for the spectral function at large ω [19]:

$$\lim_{\omega\to\infty} \rho^{NLO}(\omega) = \frac{1}{10} \frac{d_A}{(4\pi)^2} \omega^4 \left(1 - \frac{5\alpha_s N_c}{9\pi}\right) \tag{2.4}$$

It should be noted here that at large ω the spectral function scales as $\rho(\omega) \sim \omega^4$, what leads to a large perturbative contribution to the correlation function for all values of Euclidean time x_0. Calculation shows that even at the $x_0 = 1/(2T)$ the tree level contribution is $\sim 80 - 90\%$ of the total value of the correlation function. Note also that large ω behaviour of the spectral function leads to a fast decrease of the correlation function $C(x_0) \sim 1/x_0^5$ for small x_0. For this reason the signal/noise ratio for the $C(x_0)$ is small at $x_0 \gg a$ and lattice measurement of the correlation function at $x_0 \sim 1/(2T)$ becomes computationally very expensive.

In numerical simulation we use Wilson gauge action for the SU(3)–gluodynamics

$$S_g = \beta \sum_{x,\mu<\nu} \left(1 - \frac{1}{3} Re \text{Tr} U_{\mu,\nu}(x)\right), \tag{2.5}$$

where $U_{\mu,\nu}(x)$ is the product of the link variables along elementary rectangular (μ, ν), which starts at x.

For the tensor $F_{\mu\nu}$ we use the clover discretization scheme.

To calculate shear viscosity one should measure the correlation function (2.1) using two-level algorithm [17] to decrease computation time. Note also that instead of the correlation function $\langle T_{12}(x)T_{12}(y)\rangle$ we measure the correlation function $\frac{1}{2}(\langle T_{11}(x)T_{11}(y)\rangle - \langle T_{11}(x)T_{22}(y)\rangle)$. Both correlation functions are equal in the continuum limit[9].

It is conventional to present the value of shear viscosity as a the ratio viscosity-to-entropy density η/s. For homogeneous systems the entropy density s can be expressed as $s = \frac{\varepsilon+p}{T}$, where ε is the energy density and p is the pressure. These thermodynamic quantities were measured with the method described in [20].

For the correlation function considered in this paper the renormalization is multiplicative [21]. Renormalization factors depend on the discretization scheme. For instance, for the diagonal component of $T_{\mu\nu}$ (when $\mu = \nu$) and the plaquette-based discretization of $T_{\mu\nu}$ the renormalization factors are related to the anisotropy coefficients [22, 23]: $T_{\mu\nu}^{(ren)} = Z^{(plaq)} T_{\mu\nu}^{(plaq)}$, $Z^{(plaq)} = 1 - \frac{1}{2}g_0^2(c_\sigma - c_\tau)$, where c_σ and c_τ are defined in [20] and for $SU(2)$ are computed on the lattice in [24].

Using the renormalization factors for the plaquette-based discretization of T_{00}, we can find the renormalization factors for the clover discretization simply by fitting the vacuum expectation values of the renormalized T_{00}: $Z^{(plaq)} \langle T_{00}^{(plaq)}\rangle = Z^{(clov)} \langle T_{00}^{(clov)}\rangle$.

The spectral function from fitting procedure

We measured the correlation functions $C(x_0)$ for the temperatures $T/T_c = 0.9, 0.925, 0.95, 1.0, 1.1, 1.2, 1.35, 1.425, 1.5$. Application of two-level algorithm allowed us to get the uncertainty $\sim 2-3\%$ at the distance $Tx_0 = 1/2$ for all lattices. For the other points the accuracy is better.

In the calculation of viscosity we use ansatz motivated by QCD sum rules [25]. We join hydrodynamic behaviour at small frequencies with asymptotic freedom at large frequencies [1]

$$\rho_1(\omega) = BT^3 \, \omega \, \theta(\omega_0 - \omega) + A\rho_{lat}(\omega) \, \theta(\omega - \omega_0). \tag{2.6}$$

In the last formula $\rho_{lat}(\omega)$ is a tree level lattice expression for the spectral function calculated for the correlation function $\sim \frac{1}{2}(\langle T_{11}(x)T_{11}(y)\rangle - \langle T_{11}(x)T_{22}(y)\rangle)$ with clover discretization of the tensor $F_{\mu\nu}$ at lattice with fixed L_t and $L_s \to \infty$. The function $\rho_{lat}(\omega)$ was calculated in paper [15].

Low frequency part of the spectral function (2.6) is given by the first-order hydrodynamic expression $\sim \omega$. Comparison of the spectral functions of the energy-momentum tensor correlation functions obtained in $N = 4$ SYM [26] and the first-order hydrodynamic expressions allows us to expect that this approximation works well up to $\omega \le \pi T \simeq 1$ GeV[27].

On the other hand high frequency perturbative expression for the spectral function is rigidly fixed and works well up to $\omega \ge \omega_0 \sim 3$ GeV. The form of the spectral function in the region 1 GeV $\le \omega \le 3$ GeV is not clear. This is the main source of the uncertainty of the calculation based on the fitting procedure. In order to get rid of these uncertainties, one should apply non-parametrical estimation techniques.

3. The Backus-Gilbert method for the spectral function

In this section we are going to determine the ratio η/s using the Backus-Gilbert(BG) method [2]. This method does not require assumptions about the form of spectral function.

The method can be formulated as follows[3]. One needs to solve equation (2.2). To do this we divide the spectral function by $f(\omega)$ and multiply the kernel by $f(\omega)$.

Our aim is to determine the $\rho(\omega)$. In the BG method instead of the true function $\rho(\omega)$ we consider the estimator of this function $\bar{\rho}(\bar{\omega})$ which can be written as

$$\bar{\rho}(\bar{\omega}) = f(\bar{\omega}) \int_0^\infty d\omega \delta(\bar{\omega}, \omega) \frac{\rho(\omega)}{f(\omega)}, \tag{3.1}$$

where the function $\delta(\bar{\omega}, \omega)$ is called the resolution function. This function has a peak around the point $\bar{\omega}$ and it is normalized as $\int_0^\infty d\omega \delta(\bar{\omega}, \omega) = 1$. The sharper the peak, the closer $\bar{\rho}$ is to ρ. The function is expanded over the $K(x_i, \omega)$ as $\delta(\bar{\omega}, \omega) = \sum_i q_i(\bar{\omega})K(x_i, \omega)$. If a resolution functon is chosen this way, the estimator is a linear conbination of the values of the correlation function $\bar{\rho}(\bar{\omega}) = f(\bar{\omega}) \sum_i q_i(\bar{\omega})C(x_i)..$

---

[1]Note that the frequency ω is measured in physical units.

[2]In QCD this approach was recently applied in papers [28, 29]. The Backus-Gilbert was also recently applied in the lattice calculation of graphene conductivity [30].

[3]Here we follow the designations of [28, 29]

To get a better approximation for the $\rho(\omega)$ by the estimator $\bar{\rho}(\bar{\omega})$ one should sharpen the $\delta(\bar{\omega}, \omega)$ function. BG–method uses the second moment of the resolution function as the measure of its width: $\int_0^\infty d\omega \delta(\bar{\omega}, \omega)(\omega - \bar{\omega})^2$.

Minimization requirement gives the following values of the coefficients

$$q_i(\omega) = \frac{\sum_j W_{ij}^{-1}(\bar{\omega})R(x_j)}{\sum_{kj} R(x_k)W_{kj}^{-1}(\bar{\omega})R(x_j)}, \, W_{ij}(\bar{\omega}) = \int_0^\infty d\omega K(x_i, \omega)(\omega - \bar{\omega})^2 K(x_j, \omega), \, R(x_i) = \int_0^\infty d\omega K(x_i, \omega).$$

(3.2)

The matrix $W_{ij}(\bar{\omega})$ is poorly conditioned and there are large cancelations between different terms in equation (3). We regularize matrix $W_{ij}(\bar{\omega})$ by the covariance matrix of our data S_{ij}, $W_{ij} \rightarrow \lambda W_{ij} + (1 - \lambda)S_{ij}$.

To determine the spectral density at small frequencies $\omega \ll \Lambda_{QCD}$ it is reasonable to choose the function $f(\omega) = \omega$. In this case the ratio $\rho(\omega)/\omega|_{\omega \rightarrow 0} = \frac{\eta}{\pi}$.

To calculate shear viscosity, we choose $\lambda = 0.002$. For this value of the λ the uncertainty in the restored spectral density at zero frequency is less than 1% for all temperatures.

As already noted, we expect that hydrodynamic approximation works well up to $\omega \sim \pi T$ [27]. At the same time from the width of the resolution function at $\lambda = 0.002$ is $\Delta\omega \sim 4T$. We conclude that the convolution of the spectral density with the resolution function at $\lambda = 0.002$ averages over reasonable region and gives good appoximation for the shear viscosity.

According to the results of previous section asymptotic freedom starts to work for frequencies $\omega/T \geq 7 - 8$. The resolution function is considerably suppressed in this region. However, the spectral function at large frequencies rises very quickly $\rho(\omega) \sim \omega^4$, so this convolution contributes to the answer. So, to get reliable estimate one should subtract the asymptotic freedom.

The simplest way to do this is to subtract the convolution

$$\rho_{uv}(0) = \int \delta(0, \bar{\omega})\rho_{uv}(\bar{\omega})d\bar{\omega},$$

where the $\rho_{uv}(\bar{\omega}) = A\theta(\bar{\omega} - \omega_0)\rho_{lat}(\bar{\omega})$.

The determination of A and ω_0 parameters can be done by many procedures, the simplest way is to fit colleration function $C(\tau)$ for low τ, where it is completely governed by the UV-contribution. Despite the fact that this method of subtraction makes assumptions about the UV spectral function form, it still leaves the intermediate and hydrodynamic regions untouched and the usage of BG method remains sensible.

On Fig. [1] we plot η/s temperature dependence for parametrical and non-parametrical (Backus-Gilbert) methods of estimation. We also show theoretical lower bound and experimentsl bound. Both results coincide within errors and agree with the experimental data and theory, which means that the ratio η/s is robust to the choise of middle-range spectral function form ansatz.

4. Conclusion

In this paper we studied the energy-momentum tensor correlation function $\langle T_{12}(0)T_{12}(x)\rangle$ for SU(3)-gluodynamics using lattice simulation of QCD. We carried out the measurements of this correlation function at temperatures $T/T_c = 0.9, 0.925, 0.95, 1.0, 1.1, 1.2, 1.35, 1.425, 1.5$ using two-level algorithm, which gave accuracy $\sim 2 - 3\%$ at $x_0 = 1/2T$. For the other points the accuracy is
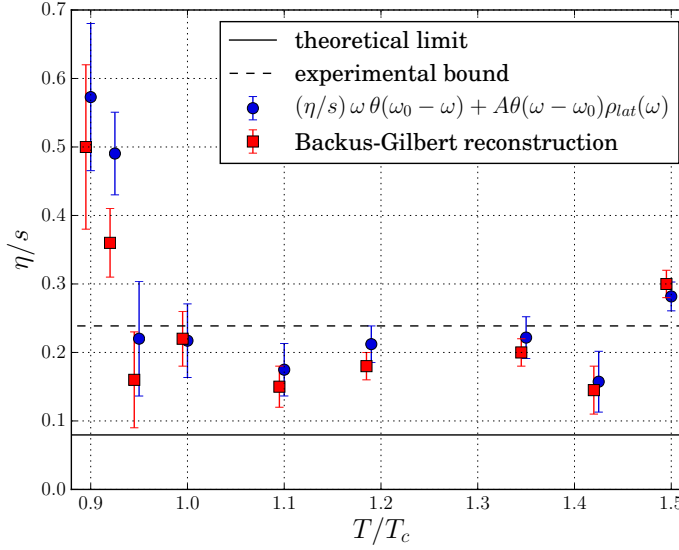
Figure 1: The ratio η/s temperature dependence for parametrical and BG procedures.

much better. Using lattice data for the correlation function we tried to study the spectral function. It was shown that physically motivated anzatz which joins the first-order hydrodynamical behaviour at small frequencies with asymptotic freedom at large frequencies fits our data very well for all temperatures. In addition, non-parametrical Backus-Gilbert method was applied in order to make model-independent estimation that would not make assumptions about low-frequencies spectral function shape. The results obtained be both methods coincide within errors and agree with the experiment and $N = 4$ SYM.

Acknowledgements

References

[1] P. F. Kolb and U. W. Heinz, In *Hwa, R.C. (ed.) et al.: Quark gluon plasma* 634-714.

[2] J. Y. Ollitrault, Eur. J. Phys. **29**, 275 (2008).

[3] K. H. Ackermann *et al.* [STAR Collaboration], Phys. Rev. Lett. **86**, 402 (2001).

[4] C. Adler *et al.* [STAR Collaboration], Phys. Rev. C **66**, 034904 (2002).

[5] D. A. Teaney, arXiv:0905.2433 [nucl-th].

[6] G. Policastro, D. T. Son and A. O. Starinets, Phys. Rev. Lett. **87**, 081601 (2001) doi:10.1103/PhysRevLett.87.081601 [hep-th/0104066].

[7] P. B. Arnold, G. D. Moore and L. G. Yaffe, JHEP **0011**, 001 (2000).

[8] P. B. Arnold, G. D. Moore and L. G. Yaffe, JHEP **0305**, 051 (2003).

[9] F. Karsch and H. W. Wyld, Phys. Rev. D **35**, 2518 (1987). doi:10.1103/PhysRevD.35.2518

[10] A. Nakamura and S. Sakai, Phys. Rev. Lett. **94**, 072305 (2005) doi:10.1103/PhysRevLett.94.072305 [hep-lat/0406009].

[11] H. B. Meyer, Phys. Rev. D **76**, 101701 (2007) doi:10.1103/PhysRevD.76.101701 [arXiv:0704.1801 [hep-lat]].

[12] H. B. Meyer, Nucl. Phys. A **830**, 641C (2009) doi:10.1016/j.nuclphysa.2009.09.053 [arXiv:0907.4095 [hep-lat]].

[13] S. W. Mages, S. Borsáanyi, Z. Fodor, A. Schäd'fer and K. Szabás, PoS LATTICE **2014**, 232 (2015).

[14] V. V. Braguta, A. Y. Kotov, A. A. Nikolaev and S. N. Valgushev, JETP Lett. **101**, no. 11, 732 (2015) [Pisma Zh. Eksp. Teor. Fiz. **101**, no. 11, 827 (2015)]. doi:10.1134/S0021364015110053

[15] N. Y. Astrakhantsev, V. V. Braguta and A. Y. Kotov, JHEP **1509**, 082 (2015).

[16] R. Kubo, J. Phys. Soc. Jap. **12**, 570 (1957). doi:10.1143/JPSJ.12.570

[17] H. B. Meyer, JHEP **0301**, 048 (2003) doi:10.1088/1126-6708/2003/01/048 [hep-lat/0209145].

[18] H. B. Meyer, JHEP **0808**, 031 (2008) doi:10.1088/1126-6708/2008/08/031 [arXiv:0806.3914 [hep-lat]].

[19] A. L. Kataev, N. V. Krasnikov and A. A. Pivovarov, Nucl. Phys. B **198**, 508 (1982) Erratum: [Nucl. Phys. B **490**, 505 (1997)] doi:10.1016/0550-3213(82)90338-8, 10.1016/S0550-3213(97)00101-6 [hep-ph/9612326].

[20] J. Engels, F. Karsch and T. Scheideler, Nucl. Phys. B **564**, 303 (2000) doi:10.1016/S0550-3213(99)00522-2 [hep-lat/9905002].

[21] H. B. Meyer, Eur. Phys. J. A **47**, 86 (2011) doi:10.1140/epja/i2011-11086-3 [arXiv:1104.3708 [hep-lat]].

[22] J. Engels, F. Karsch, H. Satz and I. Montvay, Nucl. Phys. B **205**, 545 (1982). doi:10.1016/0550-3213(82)90077-3

[23] H. B. Meyer, Nucl. Phys. B **795**, 230 (2008) doi:10.1016/j.nuclphysb.2007.11.017 [arXiv:0711.0738 [hep-lat]].

[24] J. Engels, F. Karsch and K. Redlich, Nucl. Phys. B **435**, 295 (1995) doi:10.1016/0550-3213(94)00491-V [hep-lat/9408009].

[25] M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Nucl. Phys. B **147**, 385 (1979). doi:10.1016/0550-3213(79)90022-1

[26] P. Kovtun and A. Starinets, Phys. Rev. Lett. **96**, 131601 (2006) doi:10.1103/PhysRevLett.96.131601 [hep-th/0602059].

[27] H. B. Meyer, PoS LATTICE **2008**, 017 (2008) [arXiv:0809.5202 [hep-lat]].

[28] B. B. Brandt, A. Francis, H. B. Meyer and D. Robaina, Phys. Rev. D **92**, no. 9, 094510 (2015).

[29] B. B. Brandt, A. Francis, B. Jäd'ger and H. B. Meyer, Phys. Rev. D **93**, no. 5, 054510 (2016).

[30] D. L. Boyda, V. V. Braguta, M. I. Katsnelson and M. V. Ulybyshev, Phys. Rev. B **94**, 085421 (2016).