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Non-Local effective SU(2) Polyakov-loop models from inverse Monte-Carlo methods

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The strong-coupling expansion of the lattice gauge action leads to Polyakov-loop models that effectively describe gluodynamics at low temperatures, and together with the hopping expansion of the fermion determinant provides insight into the QCD phase diagram at finite density and low temperatures, although for rather heavy quarks. At higher temperatures the strong-coupling expansion breaks down and it is expected that the interactions between Polyakov loops become non-local. Here, we therefore test how well pure SU(2) gluodynamics can be mapped onto different non-local Polyakov models with inverse Monte-Carlo methods. We take into account Polyakov loops in higher representations and gradually add interaction terms at larger distances. We are particularly interested in extrapolating the range of non-local terms in sufficiently large volumes and higher representations. We study the characteristic fall-off in strength of the non-local couplings with the interaction distance, and its dependence on the gauge coupling in order to compare our results to existing proposals for non-local effective actions.

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1. Introduction

Due to the sign problem, QCD at finite density is still a challenge, a number of different methods is currently actively being explored. One way to get further insight into the QCD phase diagram is to use effective theories in which the sign problem is either absent or weak enough to be dealt with [1, 2, 3, 4, 5]. The deconfinement phase transition of a pure Yang-Mills theory in d dimensions is determined by the dynamics of Polyakov loops. From the arguments by Svetitsky and Yaffe [6] it shares its universal behavior with a spin model in $d-1$ dimensions, and Polyakov-loop models are excellent candidates of effective field theories to describe this behavior. They can be derived in strong-coupling expansions by integrating out the spatial links. Unfortunately, however, with increasing temperature the strong-coupling expansion eventually breaks down. In particular, local Polyakov-loop models typically fail to describe the full SU(3) Yang-Mills theory when the temperature approaches the phase transition. Therefore, one is left with non-local Polyakov-loop models whose effective couplings need to be mapped to the full theory in other ways.

Here we restrict to the simpler case of SU(2) gauge theory in which the fermion determinant at finite density remains real because of the pseudo-reality of its gauge group. In a first instance we investigate different types of non-local Polyakov-loop models for the pure gauge theory as motivated from the strong-coupling expansion with and without resummations of higher-order terms. Because an analytical derivation of all the resulting terms is in general not feasible, we will use the inverse Monte-Carlo method together with geometric Ward identities to fix the couplings of our effective theories [7]. Taking into account only Polyakov loops that wind around the temporal direction once, we can write the effective Polyakov-loop action as

$$S_{\text{lin}} = \sum_{p} \sum_{r^2 \geq 1} \sum_{<i,j>=r} \lambda_{p,r^2} \chi_{p,i} \chi_{p,j} \,, \tag{1.1}$$

with lattice indices of sites i and j at distances r, coupling constants λ_{p,r^2} and Polyakov loops $\chi_{p,i}$ in represenations p given by their Dynkin labels. This leads to the first class of effective theories which we refer to as the *linear* Polyakov-loop models. Alternatively one may rearrange classes of higher-order terms arising in the strong-coupling expansion and perform partial resummations to motivate an ansatz for an effective theory with logarithmic terms in the action [8],

$$S_{\text{log}} = -\sum_{p} \sum_{r^2 \geq 1} \sum_{<i,j>=r} \ln\left(1 + g_{p,r^2} \chi_{p,i} \chi_{p,j}\right) \,, \tag{1.2}$$

with coupling contants g_{p,r^2}, which we refer to as the *logarithmic* Polyakov-loop models.

2. Inverse Monte-Carlo Method, Geometric Ward-Identities and DSE's

Since we cannot derive the actions of the effective Polyakov-loop models (1.1) and (1.2) in closed form by analytically integrating out the spatial links, we determine their coupling constants via Inverse Monte-Carlo (IMC). In the IMC method we first generate configurations of the full theory via Monte-Carlo, calculate the corresponding configurations in terms of the degress of freedom of the effective theory, and then use the latter to determine the couplings of the effective theory in the IMC step (see Fig. 1). In principle the IMC step is done by taking an ansatz for an effective ac-

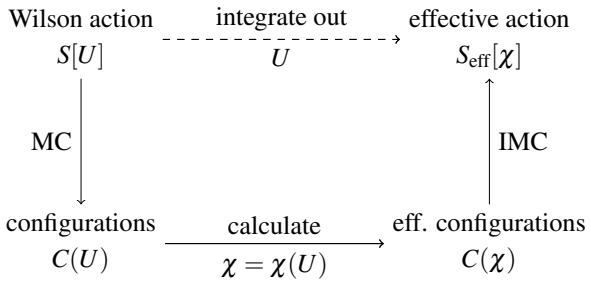

Figure 1: Inverse Monte-Carlo-Method

tion $S_{\text{eff}}(\lambda)$ with yet to determine coupling constants λ. As in the derivation of Dyson-Schwinger equations (DSEs) we use that expectation values of total derivatives with respect to the fields in the effective action must vanish and require that this remains true when replacing the effective theory with the full theory for the calculation of these expectation values via Monte-Carlo, i.e.

$$0 = \left\langle \frac{\delta S_{\text{eff}}}{\delta \varphi}(\lambda) \right\rangle_{\text{eff}} \stackrel{!}{=} \left\langle \frac{\delta S_{\text{eff}}}{\delta \varphi}(\lambda) \right\rangle_{\text{full}} . \tag{2.1}$$

This requirement implicitly determines the coupling constants of the effective theory. Since in lattice gauge theory we are dealing with link variables, which are elements of a gauge group G, in order to derive a DSE we need derivatives and integrations with respect to group elements. From the left invariance of the Haar measure, yielding for the left derivative L_a of a function f on G,

$$\int d\mu(g)(L_a f)(g) = 0 , \quad f \in L_2(G) , \tag{2.2}$$

one can derive a DSE from geometric Ward-identities [9]. Setting the function f to $\vec{L}\cdot(F\vec{L}\tilde{F})$, with class functions F and $\tilde{F}$, which then itself is a class function likewise, and using the fact that the fundamental characters χ_q, with $q \in \{1,\dots,r = \text{rank}(G)\}$, provide a basis for class functions, we can apply a character expansion

$$L_a F(\chi) = \sum_{q=1}^{r} \frac{\partial F(\chi)}{\partial \chi_q(g)} L_a \chi_q(g) . \tag{2.3}$$

Setting $\tilde{F} = \chi_p$, for some $p \in \{1,...,r\}$, one derives the master equation

$$0 = \int_G d\mu_{red} \left\{ \frac{1}{2} \sum_q K_{pq} \frac{\partial F(\chi)}{\partial \chi_q(g)} - c_p \chi_p(g) F \right\}, \quad K_{pq} := \left[(c_p + c_q)\chi_p \chi_q - \sum_\rho C^\rho_{pq} c_\rho \chi_\rho \right], \tag{2.4}$$

where C^ρ_{pq} are Clebsch-Gordon coefficients, c_ρ eigenvalues of corresponding Casimir operators, and the sum runs over all irreducible represenations ρ. One equation of the form (2.4) holds independently for every point on the $d-1$ dimensional lattice of our effective theory. Therefore, inserting unity in terms of $\Pi = \exp(-S_{\text{eff}})$ times its inverse, one can write the lattice average of these equations in the form of expectation values. In the last step we replace the measure for these expectation values with that of the full theory,

$$V^{-1} \sum_{i \in L} \left\langle \frac{1}{2} \sum_q K_{pq,i} \frac{\partial \vec{F}_i}{\partial \chi_{q,i}} \Pi(\vec{\lambda})^{-1} - c_p \chi_{p,i} \vec{F}_i \Pi(\vec{\lambda})^{-1} \right\rangle_{\text{full}} = \vec{0}. \tag{2.5}$$

Moreover, we have collected sets of as yet unspecified class functions per lattice site i in large vectors $\vec{F}_i$ because their number needs to match that of the couplings in the ansatz for the effective action, i.e. $\dim(\vec{F}_i) = \dim(\vec{\lambda})$, so that the resulting system of DSEs (2.5) can be solved to determine the couplings $\vec{\lambda}$ via IMC.

For $SU(2)$ there is only one fundamental represenation, with $p = q = 1$ and $c_1 = 3/2$, $c_3 = 4$, $C_{11}^3 = 1$ in Eq. (2.4). By setting the functions $\vec{F}_i \equiv \vec{f}_i \Pi$ with $\Pi = \Pi_{\text{lin}}$ for the linear model (1.1) and $\Pi = \Pi_{\log}$ for the logarithmic model (1.2), respectively, most of the factors Π^{-1} in (2.5) cancel. A convenient choice for the functions $\vec{f}_i$ in either case is then obtained from

$$f_{l,d,i} = \frac{1}{\lambda_{l,d}} \frac{\partial \ln(\Pi_{l,d,i})}{\partial \chi_{1,i}}, \qquad \text{and} \qquad f_{l,d,i} = \frac{1}{g_{l,d}} \frac{\partial \Pi_{l,d,i}}{\partial \chi_{1,i}}, \tag{2.6}$$

with $(\vec{f}_i)_{d+(l-1)\cdot r_{max}^2} = f_{l,d,i}$ for the quadratic distance $d \in \{1, \ldots, r_{max}^2\}$ and the represenation $l \in \{1, \ldots, p_{max}\}$. It avoids any coupling constant from only occuring in terms that contain odd powers of some Polyakov loop $\chi_{l,i}$ which would otherwise lead to an independence of Eqs. (2.5) on that coupling after group integration.

3. Results

For the linear model all dependences on the coupling constants are linear and Eq. (2.5) reduces to a matrix equation of expectation values which can be solved by simple matrix inversion. In the logarithmic case we solve the then non-linear Eq. (2.5) by applying a secant method. Doing so we will investigate the linear and logarithmic model with differnt values of r_{max}^2 up to 81, and p_{max} up to Dynkin label 3, i.e. including the fundamental, the adjoint and the 4-dimensional representation, so that the maximum number of independent couplings is given by $|\{\lambda_{l,r^2} : \quad r^2 = 1, \ldots, r_{max}^2,\ l = 1, \ldots, p_{max}\}| = r_{max}^2 \cdot p_{max} = 243$.

3.1 Linear vs. Logarithmic Polyakov-loop Models

First we compare the linear and the logarithmic models with only nearest neighbour interactions. As shown in Fig. 2, the linear model significantly improves the expectaion value of the

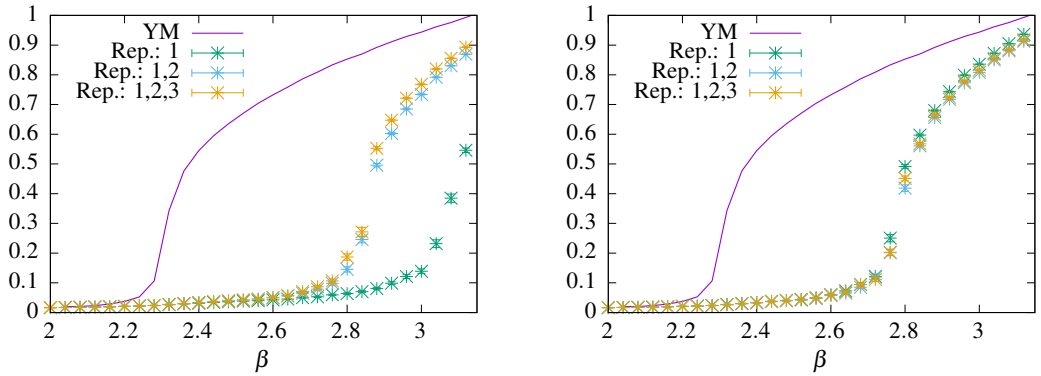

Figure 2: Polyakov-loop expecation values in local ($r_{\max} = 1$) linear (left) versus logarithmic (right) models compared to the $SU(2)$ gauge theory (YM) on a $16^3 \times 4$ lattice.

Polyakov loop around criticality if we add up to 3 representations, while the logarithmic resummation seems to work equally well with only the fundamental representation already. Adding higher

representations does not seem to improve the result much. Below the phase transition, where the strong coupling expansion is valid, the logarithmic resummation is expected to improve the results. For $\beta \sim \beta_c$ there is still a large discrepancy between the best local model and the full theory. We therefore gradually increase the non-locality up to $r_{\max} = 9$ to improve the logarithmic model. The

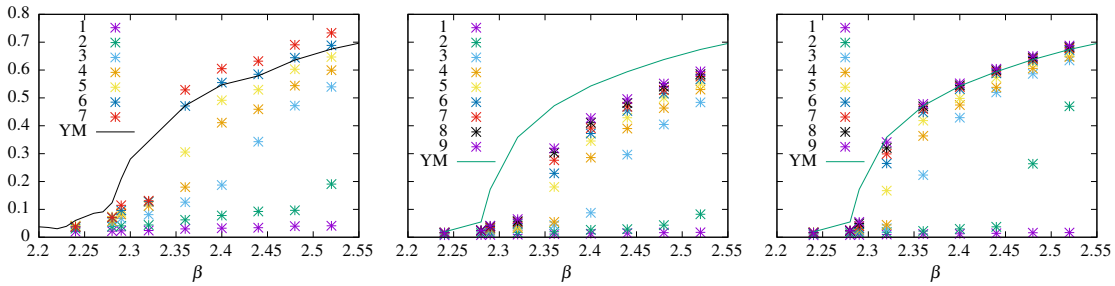

Figure 3: Logarithmic model with non-local couplings on a $16^3 \times 4$ lattice and fundamental representation only (left), and on a $32^3 \times 4$ lattice with one (middle) and two representations (right).

expectation value of the Polyakov loop thereby indeed moves closer to that of the full theory near $\beta_c \approx 2.29$ at first, but evetually overshoots the gauge-theory result at larger β, as seen in the left panel of Fig. 3. Larger lattices appear to fix this problem at first, but adding higher representations makes the result worse again (the more non-local couplings, the more the higher represenations start to matter again) as can be seen in Fig. 3 (middle and right). The model does not seem to converge in any clear way towards the full theory. This might not be too suprising since the logarithmic resummation relies on the strong-coupling expansion which breaks down for larger β. Since the linear ansatz (1.1) is more general, one might expect it to yield better results at large β. Again we now investigate the non-local linear Polyakov model by gradually increasing $r_{\max}$ up to 9. As seen in Fig. 4 for the $16^3 \times 4$ lattice the linear model is impoved by increasing $r_{\max}$ without

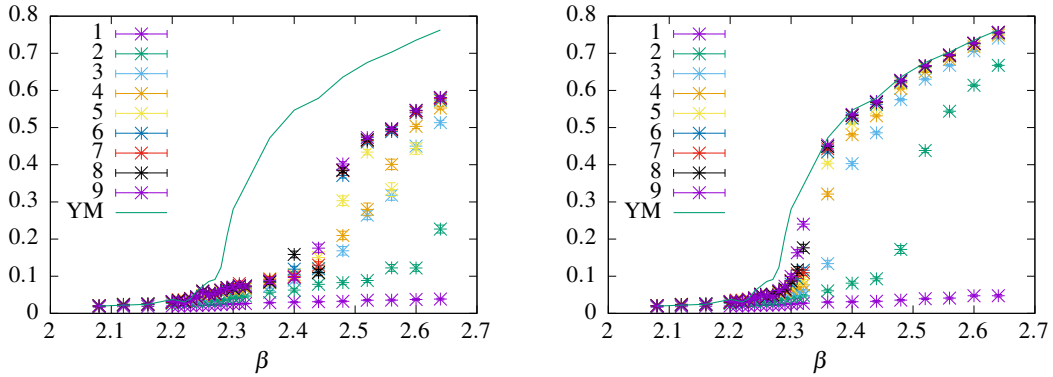

Figure 4: Non-local linear model with one (left) and two representations (right) on a $16^3 \times 4$ lattice.

the overshooting at large β. Moreover, adding the adjoint representation to the ansatz improves the result significantly and we match the full theory quite well in a wide range of β around β_c. We have checked that adding higher representations does not change these results anymore. The linear model is more stable than the logarithmic one when improving the truncation and provides the better approximation to the full theory at larger β. Nevertheless the model still approaches the full theory very slowly near β_c indicating that one might need to increase $r_{\max}$ much further in

this region (at the expense of larger spatial volumes) which might reflect the diverging correlation length of the theory at criticality. We have verified these results on the $32^3 \times 4$ lattice as well.

3.2 Long-Distance Behaviour of Non-Local Couplings

In this section we look at the long-distance behaviour of the couplings and compare it to a semi-analytical model proposed by Greensite and Langfeld [10], given by the effective action

$$S_{GL} = -\frac{1}{8} c_1 \sum_x \chi_x^2 + \frac{1}{2} c_2 \sum_{x,y} \chi_x Q(x-y) \chi_y \,, \tag{3.1}$$

where $Q(x-y)$ is the square root of the negative Laplacian, i.e.

$$Q(x-y) = \begin{cases} (\sqrt{-\Delta_L})_{xy} & |x-y| \leq r_{max} \equiv 3 \\ 0 & |x-y| > r_{max} \equiv 3 \end{cases} . \tag{3.2}$$

On a $16^3 \times 4$ lattice with $\beta = 2.22$ the constants were given as $c_1 \approx 4.417(4)$ and $c_2 \approx 0.498(1)$. Comparing Eqs. (3.1) and (1.1), we see that the effective theories are very similar if we use $p_{max} = 1$, add a quadratic Polyakov loop potential with coupling constant $\lambda_{1,0}$ (which yields an extended linear model), and apply the mapping

$$\lambda_{1,0} = -\frac{1}{8} c_1 + \frac{1}{2} c_2 Q(0) \quad \text{and} \quad \lambda_{1,r^2} = c_2 Q(r) \quad \text{with} \quad Q(r) \approx \frac{1}{|\{x : |x| = r\}|} \sum_{|x|=r} Q(x). \tag{3.3}$$

A fit to our couplings obtained from IMC then leads to $c_1 = 3.6(7)$ and $c_2 = 0.42(7)$, consistent with the values given by Greensite and Langfeld. In Fig. 5 (left) we compare the expectation value of the Polyakov loop in the extended linear model to our best results ($r_{\max} = 9$) with one and three representations in the linear model (1.1). We observe that the quadratic (adjoint) potential has almost the same effect as adding non-local interactions in the adjoint and higher representations. Close to the phase transition we have to take into account larger distances in the effective action.

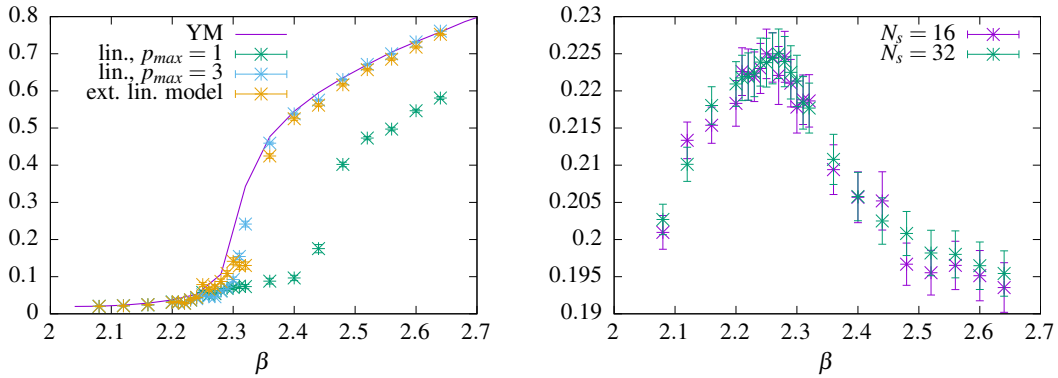

Figure 5: Polyakov loop in the non-local linear model with $p_{max} = 1$ and 3 compared to that in the extended linear model with potential term (left); and characteristic length $\xi(\beta)$ for the exponential fall-off of the couplings λ_{1,r^2} with r as a function of β for the $16^3 \times 4$ and $32^3 \times 4$ lattices (right).

In order to quantify this behaviour we define a characteristic length $\xi(\beta)$, via

$$\lambda(r) \propto a(\beta) \cdot \exp(-\frac{r}{\xi(\beta)}) \,, \tag{3.4}$$

and check for critical scaling. Fitting this exponential fall-off to the first couple of IMC couplings at small r for various β-values yields the results shown in Fig. 5 (right). While this characteristic fall-off does peak at around $\beta_c \approx 2.29$, the height of this peak does not grow with the volume as one would expect for a correlation length near criticality. On the other hand, this might indicate that the theory becomes local again in the thermodynamic limit. To confirm this we have to investigate larger N_t and extrapolate towards the contiuum limit, however.

4. Conclusion and Outlook

In this contribution we have presented the IMC method for different non-local pure gauge SU(2) Polyakov-loop models and saw that the methods works quite well. We found that the logarithmic resummation is only applicable for $\beta < \beta_c$, whereas for $\beta > \beta_c$ we can approximate the full theory, in a way that shows clear signs of convergence, only using the non-local linear Polyakov-loop models. We also compared our non-local extended linear ansatz to the model proposed by Greensite and Langfeld and found quite good agreement. As expected, around the phase transition both classes of models (linear and logarithmic) approximate the full theory only relatively poorly, however, and we would need to include even more non-local terms for a better description. On the other hand, we also found indications that the linear model might become local in the contiuum limit again. We have also computed more complicated observables, like local Polyakov-loop distributions and correlation functions which we will present in an upcoming full article. We will also investigate the behavior of our models in the contiuum limit in the future, investigate higher rank gauge groups such as G_2 and $SU(3)$, and add fermionic interactions with finite chemical potential as relevant for the phase diagram of QCD.

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