PROCEEDINGS OF SCIENCE

Determining α_s by using the gradient flow in the quenched theory

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We present preliminary results to determine the strong coupling constant by using the gradient flow. Pure SU(3) gauge theory is studied. We carry out a direct analysis on very fine zero temperature lattices where the gradient flow is calculated in the continuum limit. As a second method we applied the finite-size scaling method to move towards the perturbative regime.

The 34th International Symposium on Lattice Field Theory
24 -30 July 2016
University of Southampton, UK

*Speaker.

PoS(LATTICE2016)196

1. Introduction

The determination of the Λ-parameter in QCD is of fundamental importance and the way of determining it is via the computation of the running strong coupling constant, α_s. For over 20 years this topic has gained the attention of many people in the particle physics community both experimentalists and theorists and especially in the lattice field community various methods have been used to determine α_s. However, there is still a lot of room for improvement in the calculations as there is a sparse deviation in the value of Λ-parameter from different working groups.

In the recent review by the Flag Working Group [1], all lattice works that estimated the Λ-parameter so far have been reviewed and classified according to a list of criteria that they fulfill. Most importantly one has to be careful so that a controlled continuum extrapolation should be able to be done, one has to ensure that the perturbative region of the expansion is reached correctly and discretization effects are not present, the perturbative behaviour is manifested for a significant range in the scale and the matching to the perturbative β-function must occur for small coupling values [2]. Finally, one has to make sure that finite-size effects are not effecting the result and also there is no effect from a possible topological freezing.

In this work, we present preliminary results of the determination of the Λ parameter in the quenched case by using the gradient flow. The strategy of our calculation is to use two different methods and compare the results. The first one is to estimate the strong coupling of QCD directly from the continuum extrapolated gradient flow using very large lattices at zero temperature and the second one is to use a fixed topology and do a finite size scaling method.

2. The gradient flow

Since 2010, when the uses of the gradient flow were explained in [3], the latter became very popular amongst the lattice community and found many applications in lattice field theory. For example it is used for setting the scale (similar to Sommer parameter), to renormalize composite operators, to define operator relations as a novel approach to measure observables like the energy-momentum tensor on the lattice and also to measure the topological susceptibility. The idea behind Yang-Mills gradient flow is to introduce an extra dimension, the so-called flow time, t, so that the gauge fields, $B_\mu(x,t)$ obey the flow equation

$$\frac{dB_\mu(x,t)}{dt} = D_\nu G_{\nu\mu}(x,t) \tag{2.1}$$

where $G_{\nu\mu}$ is the field strength tensor. It was clearly stated in [3] that as it is a monotonically decreasing function of t, the gauge fields are smeared out and automatically gauge invariant observables become renormalizable.

The simplest gauge invariant quantity that one can construct out of the gauge fields is the energy density

$$E(x,t) = \frac{1}{4} G^a_{\mu\nu}(x,t) G^a_{\mu\nu}(x,t). \tag{2.2}$$

Its expectation value has mainly a twofold use. First it serves as a non-perturbative definition of a reference scale. In [3], this was defined as t_0 when $t^2\langle E(t)\rangle\big|_{t=t_0} = 0.3$ and in [4] it was shown

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that its derivative provides a better scale setting, i.e. $t\frac{d}{dt}t^2\langle E(t)\rangle\big|_{t=w_0^2} = 0.3$, the so-called w_0 scale. Secondly, a renormalized coupling can be defined through the expectation value of the energy density, using the perturbative relation in $\overline{\text{MS}}$ scheme given by

$$t^2\langle E(t)\rangle = \frac{3\alpha_s}{4\pi}\left(1+\alpha_s k_1+\alpha_s^2 k_2+\mathscr{O}(\alpha_s^3)\right) \tag{2.3}$$

where k_1 and k_2 are finite coefficients calculated in [3] and [5] respectively.

3. Determination of α_s by using fine zero temperature lattices

The first determination of α_s is attempted on very fine lattices at zero temperature, at which we simulated the tree-level Symanzik action. In all of our simulations we kept the physical volume constant by keeping $LT_c \simeq 2$. The value of the lattice coupling was varied between 5.357 to 6.360, at which we had our biggest lattice of $N = 160$. We used periodic boundary conditions and we restricted the simulations to the $Q = 0$ sector. As $w_0^{Q=0} \neq w_0^{all\ Q}$, the ratio between the two is a volume dependent quantity. This can be calculated for $LT_c \simeq 2$ using the coarse lattices, where we do not experience topological freezing. For setting the scale we used a version of the scale w_0 expalined in Section 2, that we call w_1 and it is defined as

$$t\frac{d}{dt}t^2\langle E(t)\rangle\big|_{t=w_1^2} = 0.03 \tag{3.1}$$

Our scale was expressed in r_0 units using the ratios of w_1/w_0 and w_0/r_0 [6] and thus finding $w_1/r_0 = 0.115(2)$.

In the strategy that we followed, first we apply a tree-level improvement to the flow as it is known that for small t/a^2 the discretization effects are sizable. This is done by applying a discretization correction [7] given by

$$C(a^2/t) = 1+\sum_{m=1}^{\infty} C_{2m}\frac{a^{2m}}{t^m} \tag{3.2}$$

and a finite volume correction [8] given by

$$\delta(t/L^2) = 1-\frac{64t^2\pi}{3L^2}+8\mathrm{e}^{-L^2/8t}+24\mathrm{e}^{-L^4/4t}+\ldots \tag{3.3}$$

It is interesting that even though the improved and unimproved flow have different values, when taking the continuum limit they tend to meet at the same point as shown in Fig.1.

Using the improved flow, we solve directly Eq. (2.3) to find α_s. Then using the well-known 4-loop β-function in the $\overline{\text{MS}}$ scheme we run α_s to a high scale and from that we determine the Λ-parameter of QCD using its non-perturbative relation to the coupling constant. This was our first approach. However, it was noticeable that a NNNLO term in Eq. (2.3) could be of significant importance and thus we tried to eliminate its contribution to our calculation by determining a combination of the flow and its derivative instead of the flow itself. Mainly we calculated the

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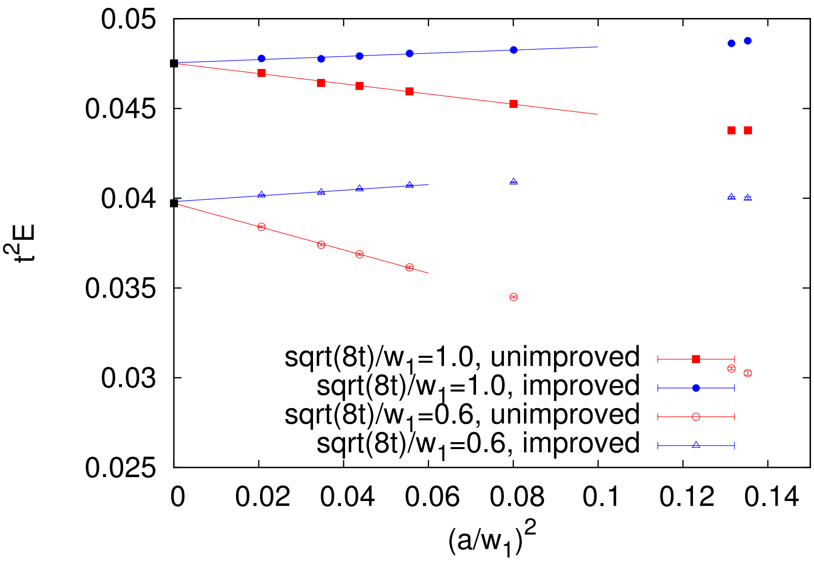

Figure 1: The continuum extrapolation of the flow $t^2\langle E\rangle$ for two different normalized times $\sqrt{8t}/w_1 = 0.6$ and $\sqrt{8t}/w_1 = 1.0$. The red points show the flow when no corrections were done and the blue points show the flow after the discretization and volume corrections were applied (Eq.(3.2) and Eq.(3.3) respectively).

function A(t) given by

$$\begin{aligned} A(t) &\equiv (t^2\langle E\rangle)^2 + C\Big(t\frac{dt^2\langle E\rangle}{dt}\Big) \\ &= \alpha_s^2\Big(\frac{9}{(4\pi)^2}+\frac{3\beta_0 C}{(4\pi)^2}\Big)+\alpha_s^3\left(\frac{18k_1}{(4\pi)^2}+C\Big(\frac{3\beta_1}{(4\pi)^3}+\frac{6k_1\beta_0}{(4\pi)^2}\Big)\right) \\ &+\alpha_s^4\left(\frac{9(k_1^2+2k_2)}{(4\pi)^2}+C\Big(\frac{3\beta_2}{(4\pi)^4}+\frac{6k_1\beta_1}{(4\pi)^3}+\frac{9k_2\beta_0}{(4\pi)^2}\Big)\right) \\ &+\alpha_s^5\left(\frac{9(2k_1k_2+2k_3)}{(4\pi)^2}+C\Big(\frac{3\beta_3}{(4\pi)^5}+\frac{6k_1\beta_2}{(4\pi)^4}+\frac{9k_2\beta_1}{(4\pi)^3}+\frac{12k_3\beta_0}{(4\pi)^2}\Big)\right)+\mathscr{O}(\alpha_s^6) \end{aligned} \tag{3.4}$$

and we notice that for $C = -\frac{3}{2\beta_0} = -0.13636364$ the k_3 contribution was eliminated to $\mathscr{O}(\alpha_s^5)$. The strong coupling α_s was then found using the exact same procedure described above and consequently the Λ-parameter was determined as shown in Fig. 2. The value of $r_0\Lambda$ shows a good plateau for the range of $\alpha = 0.1708$ to $\alpha = 0.2757$, covering a quite large range of α values to make it compatible with the criterium concerning the perturbative regime. We consider the value of $r_0\Lambda$ to be 0.664(14), the first value of the plateau that we find. Even though the value has a large error the determination can be considered successful as it also has a controlled continuum extrapolation and the renormalization scale criteria are met.

4. Determination of α_s by using step-scaling

The second method used to determine α_s is by using the finite size scaling method [9]. In order

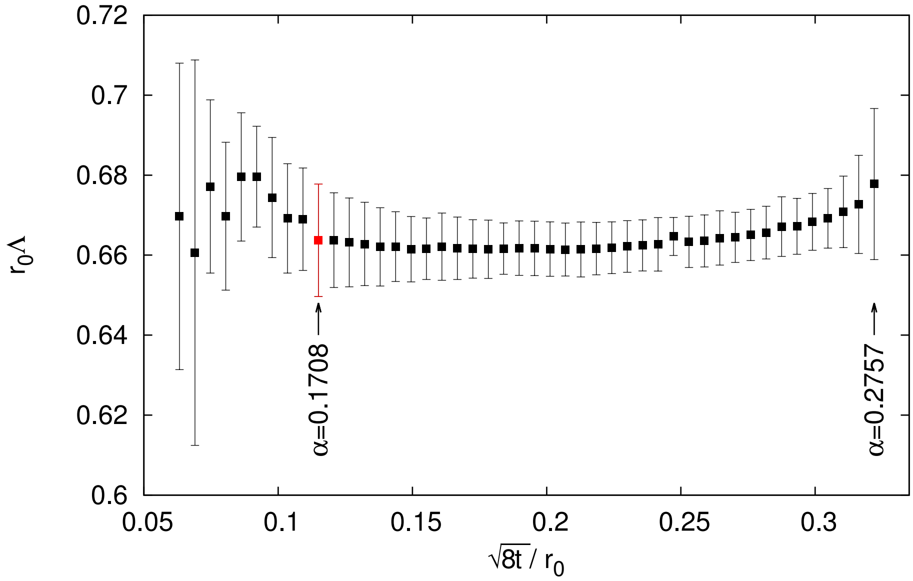

Figure 2: The value of Λ-parameter in r_0 units as a function of the normalized time as found using the function $A(t)$. The value of Λ-parameter was taken to be the first value of the plateau, i.e. $r_0\Lambda = 0.664(14)$.

to do that we used lattices of fixed topology, that is lattices of sizes $N = 14, 16, 20, 24, 28, 32, 40, 48$ and various values of the lattice coupling β. The step-scaling function, called D, was defined to be the difference of the inverse of the quantity $t^2\langle E\rangle$ at scales 2μ and μ, where $\mu = 1/\sqrt{8t}$, that is

$$D(t^2\langle E\rangle\big|_{\mu}) = \frac{1}{t^2\langle E\rangle}\Big|_{2\mu} - \frac{1}{t^2\langle E\rangle}\Big|_{\mu}. \tag{4.1}$$

To find the continuum step scaling function as given in Eq. (4.1) the physical volume, the ratio $c = \sqrt{t}/N$ and the lattice spacing are kept the same and then the doubling of μ is achieved by doubling N. Afterwards, we did a combined fit to the data which can be seen in Fig. 3. Perturbative physics is accessible from small $t^2\langle E\rangle$, where data is rather inaccurate. However, in this range, the function D can be estimated analytically using the perturbative β-function without the knowledge of Λ. If we neglect the zero-modes, it is polynomial of $t^2\langle E\rangle$ as given in Eq. (4.2) and terms beyond the first two terms can be fitted together with $1/N^2$.

$$\frac{1}{t^2\langle E\rangle}\Big|_{2\mu} - \frac{1}{t^2\langle E\rangle}\Big|_{\mu} = \frac{4\pi}{3}\left[\frac{2\beta_0}{\pi}\ln 2 + \frac{2\beta_1}{\pi^2}\ln 2\frac{4\pi}{3}\mathrm{t}^2\langle \mathrm{E}\rangle\Big|_{\mu}\right] + \mathrm{a}(\mathrm{t}^2\langle \mathrm{E}\rangle|_{\mu})^2 + \mathrm{b}(\mathrm{t}^2\langle \mathrm{E}\rangle|_{\mu})^3 + \mathrm{c}(\mathrm{t}^2\langle \mathrm{E}\rangle|_{\mu})^4 \tag{4.2}$$

After determining the continuum function D, we applied the finite size scaling procedure for two different c-values, $c = 0.1, 0.12$ starting from a fixed w_1/L value. As can be seen in Fig. 4, most of our results agree within errors and they all agree with the value found using the brute-force elements as described in Section 3.

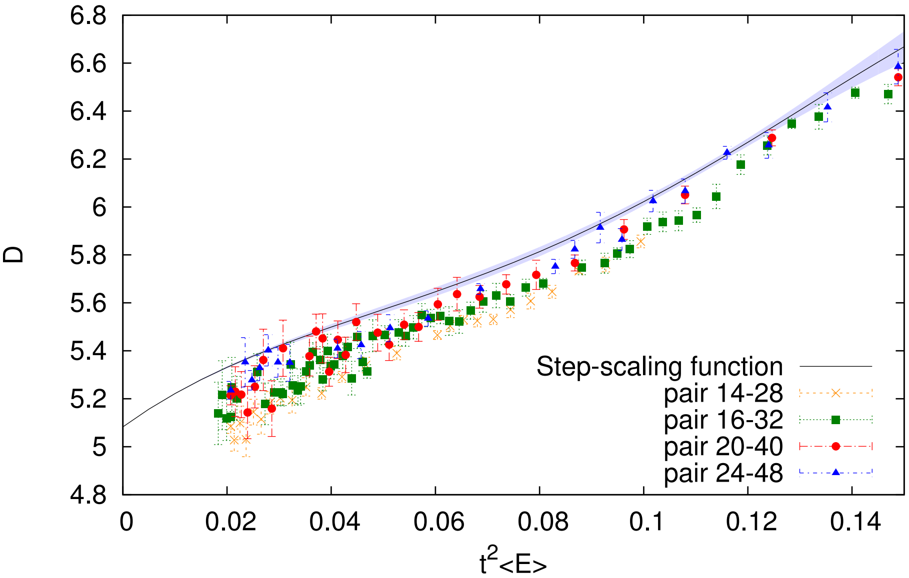

Figure 3: The solid line shows the continuum step-scaling function D as defined in Eq. (4.1). This was found by taking the continuum limit of the four pairs of lattices used (shown with points) and at the same time fitting to a polynomial.

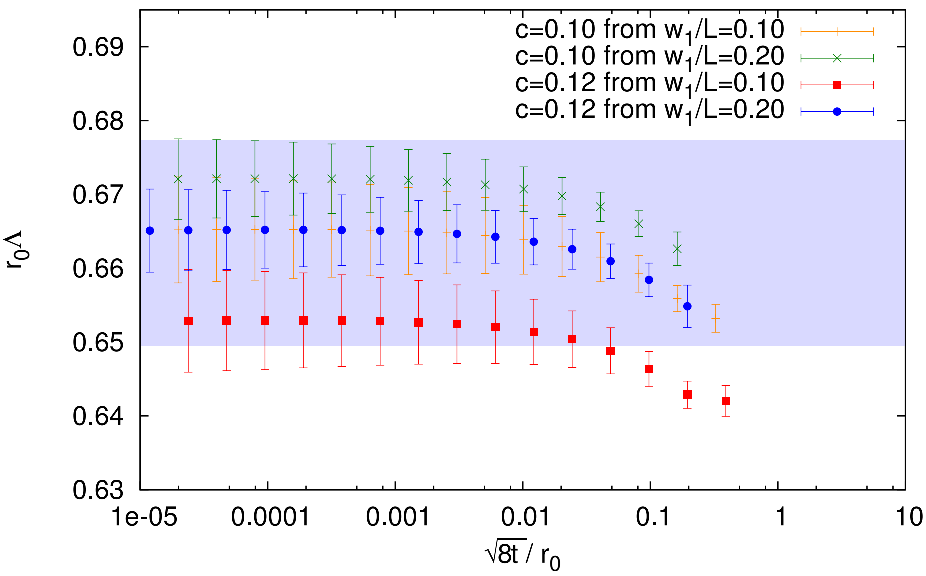

Figure 4: The value of $r_0\Lambda$ found using the finite size scaling procedure for two different values of $c = \sqrt{8t}/w_1$ and keeping fix w_1/L to 0.1 and 0.2. The blue band is the value of $r_0\Lambda$ with its error as estimated using the fine, zero-temperature lattices. We see that the results of the second method lie in this band.

5. Conclusions and Future work

In this work we present preliminary results of the determination of the Λ-parameter by using directly the gradient flow. The results agree within errors using the two different approaches implemented here. The first one is by using the gradient flow at very large lattices at zero temperature, where one can safely assume that is in the infinite volume regime. There we see that the perturbative behaviour is satisfied over a satisfactory range of α_s which was the most challenging issue that we had to address. Then a finite size scaling method was used on lattices of a fixed topology. There, the big challenge is to ensure that, as we are in a finite volume regime, volume corrections are applied correctly so that we can treat it as infinite volume. So far, we have neglected the NLO contribution to Eq. (3.3) that would lead to a contribution of $\sqrt{t^2\langle E\rangle}$ in Eq. (4.2), which we intend to include in future work. Nevertheless, the current results seem to agree within error with the value estimated from the brute-force determination of α_s. Overall, the agreement of the two methods encourages us to believe that the use of the gradient flow directly to estimate the Λ-parameter in the $\overline{\mathrm{MS}}$ scheme might provide a new way of its estimation using lattice simulations not only in the quenched case but also in the dynamical case.

Acknowledgments

Many thanks to Szabolcs Borsanyi and Zoltan Fodor for initiating the project and for the useful discussions. EL was supported by the DFG grant SFB/TR55.

References

[1] S. Aoki *et al.*, arXiv:1607.00299 [hep-lat].

[2] M. Dalla Brida, P. Fritzsch, T. Korzec, A. Ramos, S. Sint and R. Sommer, arXiv:1604.06193 [hep-ph].

[3] M. Lüscher, JHEP **1008** (2010) 071 Erratum: [JHEP **1403** (2014) 092] [arXiv:1006.4518 [hep-lat]].

[4] S. Borsanyi *et al.*, JHEP **1209** (2012) 010 [arXiv:1203.4469 [hep-lat]].

[5] R. V. Harlander and T. Neumann, JHEP **1606** (2016) 161 [arXiv:1606.03756 [hep-ph]].

[6] R. Sommer, PoS LATTICE **2013** (2014) 015 [arXiv:1401.3270 [hep-lat]].

[7] Z. Fodor, K. Holland, J. Kuti, S. Mondal, D. Nogradi and C. H. Wong, JHEP **1409** (2014) 018 [arXiv:1406.0827 [hep-lat]].

[8] Z. Fodor, K. Holland, J. Kuti, D. Nogradi and C. H. Wong, JHEP **1211** (2012) 007 [arXiv:1208.1051 [hep-lat]].

[9] M. Luscher, P. Weisz and U. Wolff, Nucl. Phys. B **359** (1991) 221.