Absence of bilinear condensate in three-dimensional QED

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There are plausibility arguments that QED in three dimensions has a critical number of flavors of massless two-component fermions, below which scale invariance is broken by the presence of bilinear condensate. We present numerical evidences from our lattice simulations using dynamical overlap as well as Wilson-Dirac fermions for the absence of bilinear condensate for any even number of flavors of two-component fermions. Instead, we find evidences for the scale-invariant nature of three-dimensional QED.
1. Introduction

Parity-invariant QED\(_3\) with 2\(N_f\) flavors of massless two-component fermions coupled to three-dimensional non-compact Abelian gauge-fields has been studied in the past as a quantum field theory which can be tuned to be conformal or to have a mass-gap by changing \(N_f\). The question is the following – is there a critical number of flavors of two-component fermions 2\(N_f\) below which massless QED\(_3\) in a finite box of length \(\ell\) generates other low-energy length scales which are independent of \(\ell\) as \(\ell \to \infty\)? One such low-energy length scale that is of interest is the bilinear condensate \(\Sigma\) which, if non-zero, governs the following scaling of the low-lying eigenvalues \(\lambda_i\) of the massless Dirac operator:

\[
\lambda_i \approx z_i\frac{1}{\ell^{3}},
\]

where \(z_i\) are universal numbers depending only on the symmetries of the Dirac operator, and can be obtained from a random matrix model with the same symmetries (refer [1] for such a model corresponding to QED\(_3\)). In this talk, based on our publications [2, 3], we primarily address the existence of \(\Sigma\) for small \(N_f\) (= 1, 2, 3, 4) by asking if \(\lambda_i \sim \ell^{-(1+p)}\) with \(p = 2\). We summarize the status of the understanding of the critical \(N_f\) before our studies in Figure 1 (see [2] and references therein, for a complete literature survey). The analytical computations, each with their own limitations, suggested that the critical \(N_f\) lie between 0 and 4. The previous lattice studies suggested that it could be 1 or 2.

![Figure 1](image-url)

**Figure 1**: Few representative older calculations [4, 5, 6, 7, 8, 9] of the critical value of \(N_f\) below which bilinear condensate exists. The large-\(N_f\) computation points to an infra-red fixed point. Various perturbative calculations as well as approximate solutions to the gap equation have been carried out to investigate the stability of the infra-red fixed point. These calculations suggest the critical value might lie anywhere between 0 and 4. The previous non-perturbative lattice studies of QED\(_3\) suggest this critical value might be 1 or 2.

2. Lattice details

We regulated QED\(_3\) in a finite box of physical volume \(\ell^3\) using \(L^3\) lattices. The lattice coupling
appearing in the gauge action is $\beta = L/\ell$; the continuum limit at a fixed physical length $\ell$ is taken by extrapolating to $L \to \infty$. We regulated the two flavors of massless two-component fermions in a parity-invariant way using the Wilson-Dirac as well as overlap fermions. The fermion propagator $G$ for the parity-preserving Wilson-Dirac fermion is

$$ G^{-1} = \begin{bmatrix} 0 & X \\ -X^\dagger & 0 \end{bmatrix} ; \quad X = C_n + B - m_t. \quad (2.1) $$

$C_n$ is the two-component naive Dirac operator, $B$ is the Wilson term and $m_t$ is tuned such that the lowest eigenvalue $\lambda_1$ of $iG^{-1}$ is minimum. We further improved it by adding a Sheikholeslami-Wohlert term and by using HYP smeared links in the Dirac operator. The fermion propagator $G$ for the overlap fermion, which has the full $U(2N_f)$ symmetry even at finite lattice spacing, is given in terms of a unitary matrix $V = (X^\dagger X)^{-1}X$ as

$$ G^{-1} = \begin{bmatrix} 0 & \frac{1-V}{1+V} \\ \frac{1-V}{1+V} & 0 \end{bmatrix}. \quad (2.2) $$

We define the “eigenvalues of the Dirac operator” in either case to be the eigenvalues $\lambda_i$ of $iG^{-1}$ which are real. We used standard HMC for generating $\sim 500 - 1000$ independent gauge configurations at all the simulation points ($4 \leq \ell \leq 250$). Using Wilson-Dirac fermions we studied $N_f = 1, 2, 3$ and 4. With the overlap fermion, we studied $N_f = 1$. At each $\ell$, we used multiple $L^3$ lattices ($12 \leq L \leq 24$) in order to take the continuum limits.

### 3. Evidence from $\ell$-scaling of the low-lying eigenvalues of Dirac operator

![Figure 2](image.png)

**Figure 2:** On the left panel, the $\ell$ dependence of the six low-lying, continuum extrapolated, eigenvalues of the overlap operator is shown. The Padé approximations to their $\ell$ dependence with $p = 1$ are shown as the solid curves. On the right panel, the likelihood of different values of the exponent $p$, measured using the $\chi^2$/DOF for the best fit of the Padé approximation with various values of $p$ to the finite $\ell$ data, is shown.

1The Wilson mass $m_t = 1$ in overlap simulations
In a finite physical box, the spectrum of the Dirac operator is discrete. Thus, one can talk about the \( \ell \)-dependence of the individual low-lying eigenvalues. As we noted in the introduction, the \( i \)-th low-lying eigenvalue \( \lambda_i \) will scale as \( \ell^{-3} \) when there is a condensate \( \Sigma \). If \( \ell^{-3} \) scaling is not found, we can conclude that a bilinear condensate is absent and instead we can obtain the mass anomalous dimension of the scale-invariant theory; since \( \lambda \) has an engineering dimension of mass, the mass anomalous dimension \( \gamma_m \) is \( p \) if \( \lambda \sim \ell^{-p-1} \) and \( p < 1 \).

In the left panel of Figure 2, we show the dependence of the continuum extrapolated values of \( \lambda_i \ell \) as a function of \( 1/\ell \) for the six low-lying eigenvalues of the overlap operator in a log-log plot. At any finite \( \ell \) that we studied, the slope \( \frac{d \log(\lambda_i \ell)}{d \log(1/\ell)} \) is less than 2, the value that is expected if \( \Sigma \neq 0 \). In fact, it is less than 1. We estimate the exponent of the power-law that would be seen as \( \ell \to \infty \) by describing the \( \ell \)-dependence of our data by

\[
\lambda_i \ell = \ell^{-p} F(1/\ell),
\]

with an unknown scaling correction \( F \). We approximate \( F \) by a \([1/1]\) Padé approximant. We find it numerically stable to write the Padé approximant in terms of \( \tanh(1/\ell) \). The best fits of the above ansatz with \( p = 1 \) to the data are shown by the solid curves in the left panel of Figure 2. In the right panel, we show the \( \chi^2/\text{DOF} \) for such fits to the six low-lying eigenvalues as a function of the exponent \( p \). The value \( p = 2 \) is clearly ruled out, which implies the absence of a condensate. Assuming the theory does not generate other length scales as well, we can estimate the mass anomalous dimension \( \gamma_m = p \) of the theory to be 1.0(2) from the same plot. Further, we support the correctness of our result by comparing the \( \ell \)-dependence of the continuum extrapolated low-lying eigenvalues of the two different lattice Dirac operators in Figure 3. A perfect agreement between the Wilson-Dirac and the overlap formalisms is seen. Due to such an agreement, we study the \( N_f = 2, 3, 4 \) cases using only the Wilson-Dirac fermion.

**Figure 3:** The plot compares the \( \ell \)-dependence of the first three low-lying eigenvalues, after taking the continuum limit, using Wilson fermions (open symbols) and overlap fermions (filled symbols) for the \( N_f = 1 \) case.

In Figure 4, we show the \( \ell \)-dependence of the continuum extrapolated smallest eigenvalue for different number of flavors \( N_f = 1, 2, 3 \) and 4. The eigenvalues scale with a smaller exponent \( p \) as \( N_f \) increases, consistent with the expectation that if \( N_f = 1 \) does not have a bilinear condensate, the \( N_f = 2, 3, 4 \) also would not. Thus QED\(_3\) does not have a bilinear condensate for all non-zero \( N_f \). Again, assuming this means that QED\(_3\) is scale-invariant for all \( N_f \), we estimate the mass
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Figure 4: The $\ell$-dependence of the smallest eigenvalue of the Wilson-Dirac operator for $N_f = 1, 2, 3$ and 4. The expected scaling when a bilinear condensate is present, $\lambda \ell \sim \ell^{-2}$, is shown by the black straight line in this log-log plot. The exponent $p$ for the asymptotic $\ell$-scaling seems to decrease as $1/N_f$.

The anomalous dimension to be $\gamma_m = 1.0(2), 0.6(2), 0.37(6)$ and $0.28(6)$ for $N_f = 1, 2, 3, 4$ respectively. Surprisingly, this agrees with an analytical calculation [10] of $\gamma_m$ to $O(1/N_f^2)$ where no assumption about bilinear condensate is made; the analytical values are $\gamma_m = 1.19, 0.56, 0.37$ and $0.28$ for $N_f = 1, 2, 3, 4$ respectively.

Figure 5: (Left) The zero spatial momentum scalar correlator $G(t) = \langle \Sigma(0) \Sigma(t) \rangle$ as a function of temporal separation $t$. The different lines are tangents to the correlator, with slope $k(t)$, at various $t$ on the log-log plot. (Right) The mass anomalous dimension given by $\gamma_m(t) = 1 - k(t)/2$ is plotted as a function of the scale $t$.

The other way to obtain the mass anomalous dimension is to study the scalar correlator $G(t) = \langle \Sigma(0) \Sigma(t) \rangle$ projected to zero spatial momentum. The correlator is shown as a function of the temporal separation $t$ in the left panel of Figure 5. The first thing to notice is the concave-up nature of the correlator. This indicates the absence of a mass-gap, thereby ruling out the presence of another length scale in addition to a bilinear condensate. The slope on the log-log plot, $k(t) = \frac{d\log(G(t))}{d\log(t)}$, is related to a scale dependent mass anomalous dimension $\gamma_m(t)$ as $\gamma_m(t) = 1 - k(t)/2$. This is shown as a function of $1/t$ in the right panel of Figure 5. The mass anomalous dimension
at the IR fixed point to which QED$_3$ with $N_f = 1$ flows to, is $\gamma' = \lim_{t \to \infty} \gamma_m(t)$. We estimate by an extrapolation that $\gamma' = 0.8(1)$. This is consistent with the estimate 1.0(2) from the eigenvalues described above. The agreement between two different approaches to $\gamma'$ serves as a cross-check.

4. Evidence from Inverse Participation Ratio and number variance

The Inverse Participation Ratio (IPR) is defined as

\[ I_2 \equiv \left\langle \int (\psi_\lambda^*(x)\psi_\lambda(x))^2 \, d^3x \right\rangle, \quad (4.1) \]

where $\psi_\lambda$ is the normalized eigenvector corresponding to the eigenvalue $\lambda$. In random matrix models, which are ergodic, $I_2 \sim \ell^{-3}$. Thus, if the theory has a condensate, the low-lying eigensystem of the Dirac operator would be described by a random matrix model. Thus the IPR corresponding to the low-lying eigenvalues should show a $\ell^{-3}$ scaling. This is another test for the presence of $\Sigma$.

Instead, if the theory is scale-invariant, the finite size scaling of IPR would be $I_2 \sim \ell^{-3+\eta}$, where $\eta$ is a critical exponent. The exponent $\eta$ is related to a quantity called number variance $\Sigma_2$ which measures correlations between the eigenvalues. The number variance $\Sigma_2(n)$ is defined as the variance of the number of eigenvalues below a value $\lambda$ which on the average contains $n$ eigenvalues. In ergodic random matrix models, $\Sigma_2(n) \sim \log(n)$. For a critical theory, $\Sigma_2(n) \sim (\eta/6)n$, where $\eta$ is the critical exponent from the IPR [11].

In the left panel of Figure 6, we have shown the $\ell$-scaling of IPR for $N_f = 1$. For large $\ell$, the onset of scaling is clearly seen. The scaling is $I_2 \sim \ell^{-2.62(1)}$. Firstly, this rules out the ergodic $\ell^{-3}$ scaling. The theory has a non-zero critical exponent $\eta = 0.38(1)$. As explained above, in a critical theory, $\eta$ should satisfy a critical relation to the slope of number variance. In the right panel of Figure 6, we have shown $\Sigma_2(n)$ as a function of $n$. Again, clearly there is a disagreement with the
expectation from the nonchiral random matrix theory thereby ruling out condensate in another way. We see a linear rise in $\Sigma_2(n)$ indicating a critical behavior. As $\ell$ is increased, the slope of the linear rise seems to approach $\eta/6$, as shown by the black line in the figure. Thus, both the IPR and $\Sigma_2$ show critical behavior, and also they satisfy the critical relation between the two.

5. Conclusions

In this talk, we presented convincing numerical evidences for the absence of a bilinear condensate for all $N_f$. Instead, we found evidences for QED$_3$ to be scale-invariant, and we estimated the mass anomalous dimension at the infra-red fixed point at various $N_f$. In another work [12], we established the presence of a condensate in the 't Hooft limit using the same methods we described here. This suggests an interesting phase diagram in the $(N_f, N_c)$ plane whose one side is conformal while the other side has a mass-gap, providing a powerful system to understand the generation of mass in QFTs. We aim to present results on this in a future Lattice meeting.

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References