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# $\theta$-dependence of the massive Schwinger model

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Understanding the role of the $\theta$ parameter in QCD and its connection with the strong CP problem and axion physics is one of the major challenges for high energy theorists. Due to the sign problem, at present only the QCD topological susceptibility at low temperatures is well known. Using an algorithmic approach that could potentially be extended to QCD, we study as a first step the $\theta$-dependence in the massive Schwinger model, and try to verify a conjecture of Coleman.

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## 1. Introduction

Our motivation is to understand the behavior of QCD with a $\theta$-term. A better knowledge of this system could provide some answers to open questions like the strong CP problem, or the dynamics of the axion field. Moreover, if a method capable of computing observables of interest in $\theta$-like systems with dynamical fermions is developed, one could think about possible extensions to other systems with a severe sign problem, like finite density QCD or other condensed-matter models.

As a first step in this program, we present in this proceeding our current work in a toy model of $\theta$-QCD: the massive Schwinger model with a $\theta$-term. We have analysed its $\theta$-dependence by means of a method developed in recent years [1], reviewed briefly in Section 2. Our intention has been to perform a proof of concept, that is, check if it is feasible to use the method of [1] in a toy model, keeping the rest of technical details as simple as possible, even if we had to sacrifice some computational performance. In Section 3 we describe the principal features of the Schwinger model, and present some preliminary results. Finally, we expose our conclusions and outline future work in Section 4.



## 2. Dealing with $\theta$-terms

The main difficulty that arises when we study a $\theta$-like system is the infamous Sign Problem. The $\theta$-term adds to the lagrangian a pure imaginary term, and makes impossible to perform standard Monte Carlo simulations, at least at real values of the $\theta$ parameter (which are the physical ones). In a few cases it is possible to reformulate the physical degrees of freedom in such a way that the new variables are completely free of it. However, this is not possible in the general case, and other methods must be explored. In the last few years, there has been an intense activity trying to overcome the sign problem: Complex Langevin Dynamics, Lefschetz Thimbles and Density of States methods, have been used on a wide variety of lattice gauge field theories.

In our case, we have been using two different methods [2, 1] that allow to reconstruct the $\theta$-dependence of observables like the topological charge. Both methods share the same input data: standard Monte Carlo simulations performed at imaginary (unphysical) values of the parameter $\theta$. This methods have been applied by us to a number of models, including the Ising model with an imaginary magnetic field [3], $CP^1$ [4], $CP^3$ [5], $CP^9$ [6] and the $O(3)$ nonlinear sigma model [7, 8], obtaining quite satisfactory results. It must be noted that the *first* method [2], fails in systems where the order parameter is not monotonous (i.e. when we have symmetry restoration at $\theta = \pi$). But in the cases where symmetry is spontaneously broken at $\theta = \pi$, both methods can be used as a check of consistency.

In the preliminary results presented in the next section, we will use the $\gamma_\lambda(y)$ exponent [1], whose limit for $\gamma(y \to 0)$[1], governs the behavior of the topological charge $q$ as $\theta \to \pi$ in the following way:

$$q(\theta) \propto (\pi - \theta)^{\gamma - 1} \qquad as \qquad \theta \to \pi. \tag{2.1}$$

If we are able to compute the previous exponent $\gamma_\lambda(y)$ for small values of $y$, which is possible via a standard (i. e. real action) Monte Carlo simulation, we can extrapolate to zero and find $\gamma$.

[1] Note that $y \to 0$ is the same limit that $\theta \to \pi$, provided that $y = q(\theta)/\tan(\theta/2)$ [1].

If it is equal to one, we have spontaneous symmetry breaking of the given model at $\theta = \pi$; on the contrary, if $\gamma = 2$ the symmetry is restored. And if $\gamma \in (1,2)$, we have a second order phase transition.

## 3. The massive Schwinger model with a $\theta$ term

The one-flavor massive Schwinger model is QED in $1+1$ dimensions. Adding a $\theta$-term, it serves as a toy model for $\theta$-QCD: it is a model with fermions, confining, that has a non-trivial topology and shows explicitly the $U_A(1)$ axial anomaly. Its euclidean continuum action is given by

$$S = \int d^2x \{\bar{\psi}\gamma_\mu \left(\partial_\mu + iA_\mu\right) \psi + m\bar{\psi}\psi + \frac{1}{4e^2}F_{\mu\nu}^2 + \frac{i\theta}{4\pi}\varepsilon_{\mu\nu}F_{\mu\nu}\}. \tag{3.1}$$

The behavior of the topological charge $q$ in this model depends on the fermion mass. At large $m$, it tends to pure gauge two-dimensional electrodynamics, which is exactly solvable, and presents spontaneous symmetry breaking at $\theta = \pi$. However, at small $m$, it is possible to expand the topological charge in powers of the mass [9]. Provided that the expansion converges, we have

$$\langle q \rangle = m\Sigma \sin\theta + O(m^2), \tag{3.2}$$

that is, we recover the symmetry. Separating the small and large fermion masses regimes we expect a critical point, as conjectured by Coleman [10], and supported by strong numerical evidence working in the Hamiltonian formalism [11, 12] and also in recent years with the Grassman tensor renormalization group [13][2].

In order to perform the Monte Carlo simulation, we formulate the model in the lattice following the standard conventions. We use the compact formulation for the link variables

$$U_{n\mu} \equiv U_\mu(n) = e^{i\varphi_{n\mu}}; \quad \varphi \in [-\pi, \pi], \tag{3.3}$$

and the usual Wilson gauge action with staggered fermions:

$$\begin{aligned} S = &\frac{1}{2}\sum_{n,\mu} \eta_\mu(n)\bar{\chi}(n)\{U_\mu(n)\chi(n+\mu) - U_\mu^\dagger(n-\mu)\chi(n-\mu)\} \\ &+ m\sum_n \bar{\chi}(n)\chi(n) - \beta\sum_n \Re(U_{\Box n}) - i\theta\sum_n q(n), \end{aligned} \tag{3.4}$$

where $q(n)$ is the local topological charge, defined as the sum of the phases $\varphi_{n\mu}$ of the plaquette modulo $2\pi$. We also take the square root of the fermionic determinant in order to stay in the one-flavor case.

Taking the above action, which is real-valued for imaginary values of $\theta$, we can perform our MC simulation. As this work is a proof of concept, we have chosen a standard Metropolis algorithm. At each sweep of the algorithm, we try to update each link sequentially, re-evaluating the fermionic determinant at each attempt. We have performed simulations at three different values of the coupling $\beta$ and at several fermion masses. The preliminary results we present below are

[2]These numerical methods are free of the sign problem , but they are limited to two-dimensional systems.

computed in a $16^2$ lattice, and are grouped by its value of $\beta$ or the mass. Each point in the figures is the result of two simulations with $\sim 10^6$ measurements.

Our preliminary results are presented in figures 1, 2 and 3, and the same data grouped in a different way is displayed in figures 4 and 5. In all the plots we show the exponent $\gamma_\lambda(y)$, computed from Monte Carlo simulations at imaginary values of $\theta$, together with the pure gauge $(m \to \infty)$ analytical solution. In figure 1 we can see how the $m = 0.50$ data is lying, within the error bars, over the $m \to \infty$ curve, thus being compatible with an extrapolation to $\gamma = 1$: the symmetry then would be broken at $\theta = \pi$, as expected for masses over the critical point. The other two masses show a different behavior: the data suggest that their extrapolations to zero go beyond $\gamma = 1$, thus restoring the symmetry at $\theta = \pi$, although it is unclear if they can arrive to $\gamma = 2$ or somewhere in between. Figures 2 and 3 show a qualitatively similar behavior, but with bigger error bars. Nevertheless, increasing the coupling $\beta$ also allows us to reach lower values of $y$, making the extrapolation easier. This can be seen in figures 4 and 5, which display the same data as the previous ones, but for fixed mass.

With the previous considerations in mind, we can say that the preliminary results are at least compatible with previous work, showing the existence of two different mass phases: one phase of low mass that recovers the symmetry at $\theta = \pi$, and another of high mass where the symmetry at $\theta = \pi$ is broken. Determining other properties, such as the precise position of the critical point, would require a great optimization of the MC simulations, and is out of the scope of this work.

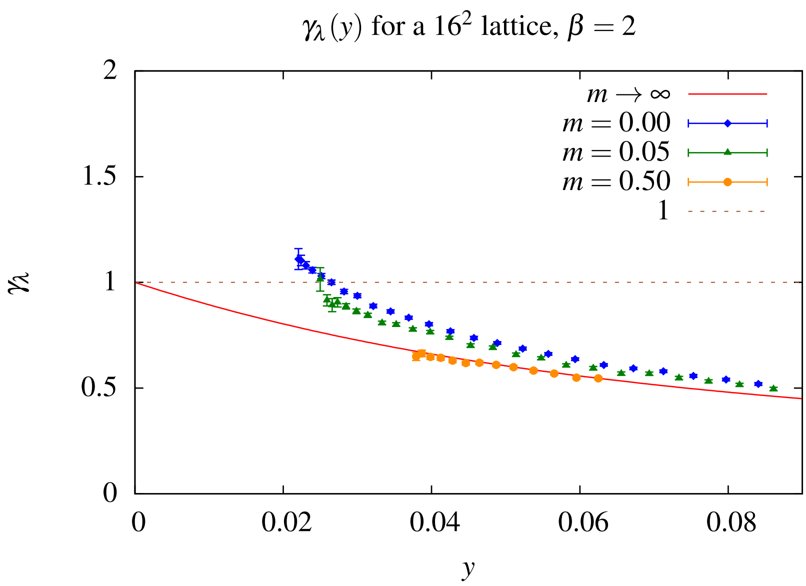


**Figure 1:** The results of the exponent $\gamma_\lambda(y)$, computed from Monte Carlo simulations at imaginary $\theta$, together with the analytic solution of the infinite mass case (continuous curve). Different masses at $\beta = 2$.

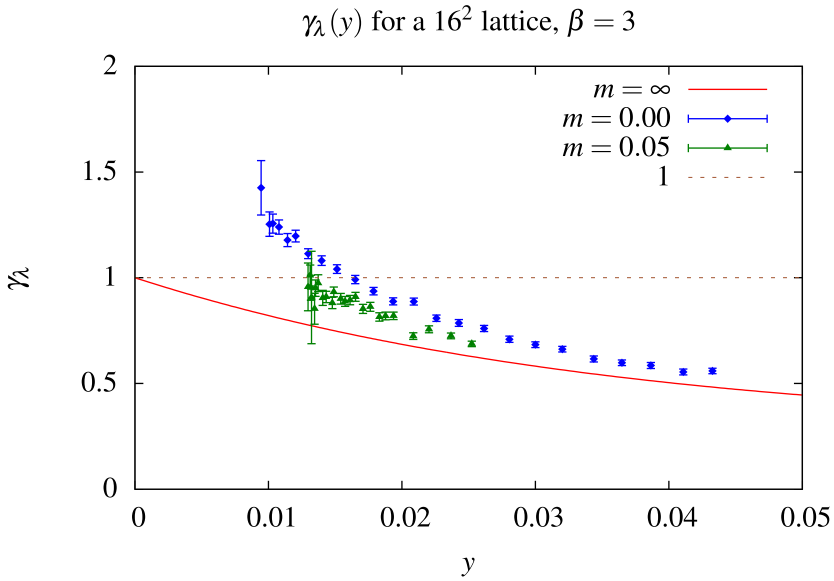


**Figure 2:** As in the figure above, $\gamma_\lambda(y)$ from MC simulations at $\beta = 3$.

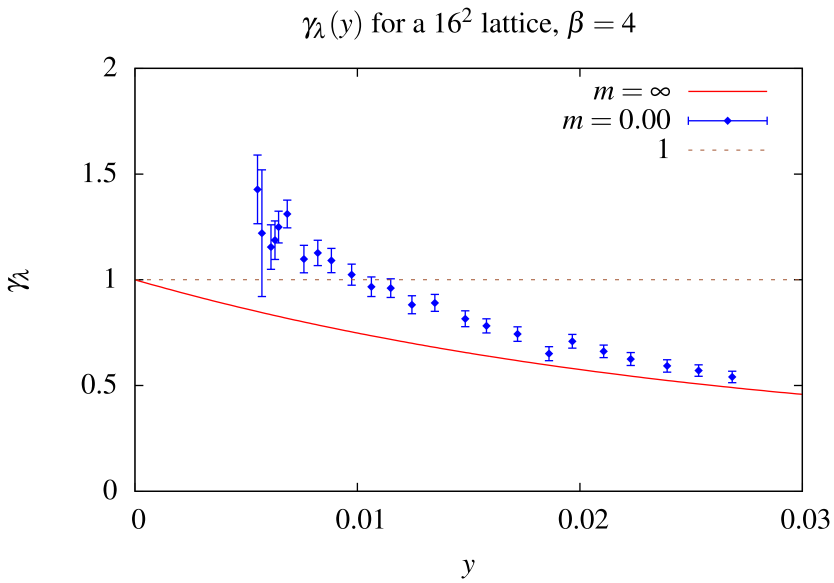


**Figure 3:** $\gamma_\lambda(y)$ for $\beta = 4$, results only for $m = 0$.

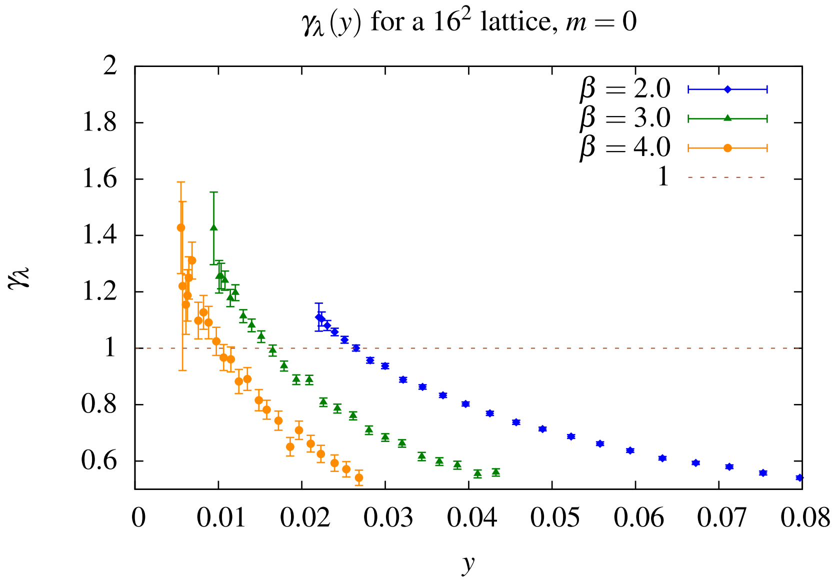


**Figure 4:** Results for the exponent $\gamma_\lambda(y)$ at different $\beta$ for $m = 0$.

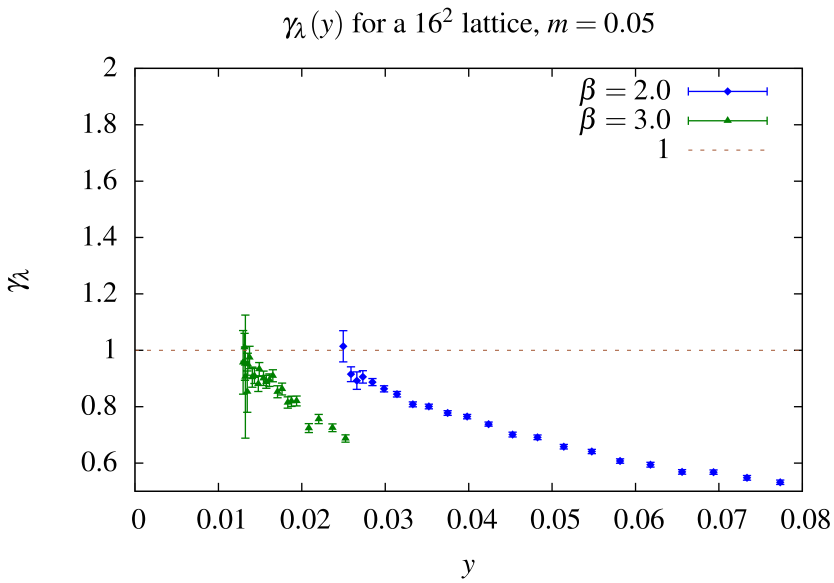


**Figure 5:** Results for the exponent $\gamma_\lambda(y)$ at different $\beta$ for $m = 0.05$.

## 4. Conclusions and Outlook

There are methods that can treat systems with a $\theta$-like term, which have been tested in a wide variety of models, including the Ising model, $CP^3$ or $CP^9$. We have applied them to a toy model of QCD with dynamical fermions, the massive Schwinger model with a $\theta$-term, obtaining results compatible with previous work. In principle, the methods described here should be applicable to QCD with a $\theta$-term. We are starting simulations, first in quenched QCD, to test concrete implementations of both the dynamics and the topological charge operator.

## 5. Acknowledgements

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