PROCEEDINGS OF SCIENCE

Monte Carlo simulation of ϕ_2^4 and $O(N)\phi_3^4$ theories

Barbara De Palma^{*a} **and Marco Guagnelli** ^{b}

[a] *Università degli studi di Pavia and INFN*

[b] *INFN*

E-mail: barbara.depalma01@universitadipavia.it, marco.guagnelli@pv.infn.it

We report lattice simulations of ϕ_2^4 and $O(N)\,\phi^4$ models, performed by means of a Monte Carlo method based on the all-order strong coupling expansion (worm algorithm). The investigation of the non-perturbative features of the ϕ^4 continuum limit in two dimensions lead us to the result $g/\mu^2 = 11.15 \pm 0.06_{stat} \pm 0.03_{syst}$ for the critical coupling. Furthermore we present preliminary results for the three-dimensional $O(2)\phi^4$ model using the worm algorithm with the extention to $O(N)\phi^4$ in D dimensions.

34th annual International Symposium on Lattice Field Theory
24-30 July 2016
University of Southampton, UK

*Speaker.

PoS(LATTICE2016)277

1. Introduction

The ϕ^4 theory plays a pedagogical role in the study of quantum field theories. Despite the simplicity of the model there is still interest in the investigation of its main features by means of numerical and theoretical tools. As often happens, significant results in the theoretical study goes hand by hand with the efficiency of algorithms and computational power. From the point of view of simulations on the lattice it is fundamental to obtain precise results in the continuum limit, i.e. when the lattice spacing a goes to zero and the lattice is removed. In this limit the correlation length ξ diverges and the problem of *critical slowing down*, in which the autocorrelation time τ grows like a power of ξ, occurs. For real simulations on a lattice of linear size L, $\xi \sim L$ and $\tau \sim L^z$, where z is the dynamical critical exponent depending only on the dynamic behavior of the algorithm and on the observable under study. In order to control this problem, we chose to use the *worm algorithm* [1][2], a non local algorithm based on the high temperature expansion that drastically reduces z. Here we apply this method to the ϕ^4 theory with $O(N)$ symmetry, whose lagrangian has the general expression

$$\mathscr{L} = \frac{1}{2}\left(\partial_\mu \phi\right)^2 + \frac{\mu_0^2}{2}\left(\phi\cdot\phi\right) + \frac{g}{4}\left(\phi\cdot\phi\right)^2 \tag{1.1}$$

where ϕ is a N component field, μ_0 is the bare mass and g is the bare coupling. In particular, in the first part we describe the application of this method to the investigation of the critical features of ϕ^4 theory in two dimension with $N = 1$, showing the upgrades adopted with respect to our previous work [3]. Secondly, we present a new version of the worm algorithm, that allows to perform simulation of ϕ^4 theory with $O(N)$ symmetry in arbitrary dimension. In particular we will show results in the cases with $N = 2$, $D = 3$. This is a preliminary work that lays the foundation for the investigation of the phase transition of an ideal three-dimensional Bose-Einstein gas with fixed density. The computation of first correction to the shift of the critical temperature [4], with arbitrarily weak interactions, is still surrounded by theoretical uncertainties and a way to improve this estimation is to study it by means of an effective field theory. In particular the $O(2)\phi_3^4$ theory on the lattice suits well for probing the non-perturbative features of this physical system.

2. ϕ^4 theory in two dimensions

Now we specialize to the case of ϕ^4 theory in two dimensions with real fields ϕ with $N = 1$. From dimensional analysis arguments we know that $[\mu_0^2] = [g]$ and thus the only relevant dimensionless parameter is the ratio $f \equiv g/\mu^2$ defining a critical line when both g and μ^2 goes to zero, where μ^2 now is the renormalised squared mass in some given renormalization scheme. Despite the simplicity of the model there is still debate around this quantity and one of the last Monte Carlo estimation it is presented in our previous work [3], obtained by means of the *worm algorithm* tecnique [2][1].

The formulation of the ϕ^4 theory on the lattice lead us to the euclidean action

$$\mathscr{S} = \sum_x \left\{ -\sum_\nu \phi_x\phi_{x+\nu} + \frac{1}{2}\left(\mu_0^2+4\right)\phi_x^2 + \frac{g}{4}\phi_x^4 \right\}, \tag{2.1}$$

[1]For a summary of the last determinations of f see the Table IV in [3], with [7] as the most recent estimation.

where $\phi_{x\pm\hat{\nu}}$ are fields at neighbor sites in the $\pm\nu$ directions and μ_0 and g are expressed in lattice units. For the computation of $f \equiv g/\mu^2$ the general idea is to fix a value of g and search for a value of μ_0^2 such that we get, in the infinite volume limit, a second order phase transition point in the plane (g, μ_0^2). In order to securely get the continuum limit it is necessary to deal with an additive renormalization of the mass parameter, since μ_0^2 diverges like $\log(a)$ in this limit. In the end we extrapolate the quantity g/μ^2 to the limit $g \to 0$ and finally obtain the critical value in the continuum limit.
For our scope is useful to introduce another parametrization of the action:

$$\begin{aligned}\mathscr{S}_E &= -\beta \sum_x \sum_\nu \varphi_x \varphi_{x+\hat{\nu}} + \sum_x \left[\varphi_x^2 + \lambda(\varphi_x^2 - 1)^2\right] \\ &= \mathscr{S}_I + \mathscr{S}_{Site},\end{aligned} \tag{2.2}$$

where $\mathscr{S}_I$ is the interaction term between neighbor sites with a coupling constant of strength β and $\mathscr{S}_{Site}$ is the term related to a single site. The relations between $(\mu_0^2,\ g)$ and $(\beta,\ \lambda)$ are:

$$\phi_x = \sqrt{\beta}\varphi, \qquad \mu_0^2 = 2\frac{1-2\lambda}{\beta} - 4, \qquad g = \frac{4\lambda}{\beta^2}. \tag{2.3}$$

In the next section we briefly describe our computational strategy for the first estimation and for the improved one.

2.1 Strategy for the computation of f

For the computation of f we use the *worm algorithm*[2] and consider the lattice action (2.2). In order to compute the critical point in the new representation, we fix a value of λ and search for the corresponding critical value β. The physical condition we impose for such scope is

$$mL = \frac{L}{\xi} = \text{const} = z \tag{2.4}$$

which implies that ξ grows linearly with L and when $L/a \to \infty$ we arrive at the critical point. m is determined by the implicit formula

$$\frac{G(p^*)}{G(0)} = \frac{m}{p^{*2} + m^2} \tag{2.5}$$

where $G(p)$ is the two-point function in momentum space, and p^* is the smallest momentum on a lattice of linear size L. Basically we simulate several lattices with different values of $N \equiv L/a$; for each couple (λ, N) we obtain a value of $\beta(\lambda, N)$ such that $mL = z$. After this step we extrapolate our results to $a/L \to 0$ in order to compute $\beta(\lambda)$. Now, using relations in (2.3) we derive $g(\lambda, \beta)$ and $\mu_0^2(\lambda, \beta)$. Using renormalisation condition specified in [3] we finally pin down $\mu^2(g)$ and hence the ratio g/μ^2. We repeat all this procedure for several values of λ, and hence of g; in the end we extrapolate our results to $g \to 0$, in order to get f. Our final value is $g/\mu^2 = 11.15 \pm 0.06_{stat} \pm 0.03_{syst}$ obtained imposing $z = 4$ in (2.4).

In Fig. 1 a direct comparison with the Monte Carlo result obtained with *cluster algorithm* [6] is shown: blue round circles are our determinations while red triangular points are the results obtained in [6]. In the region of the minimum of the curves the two determinations are in almost

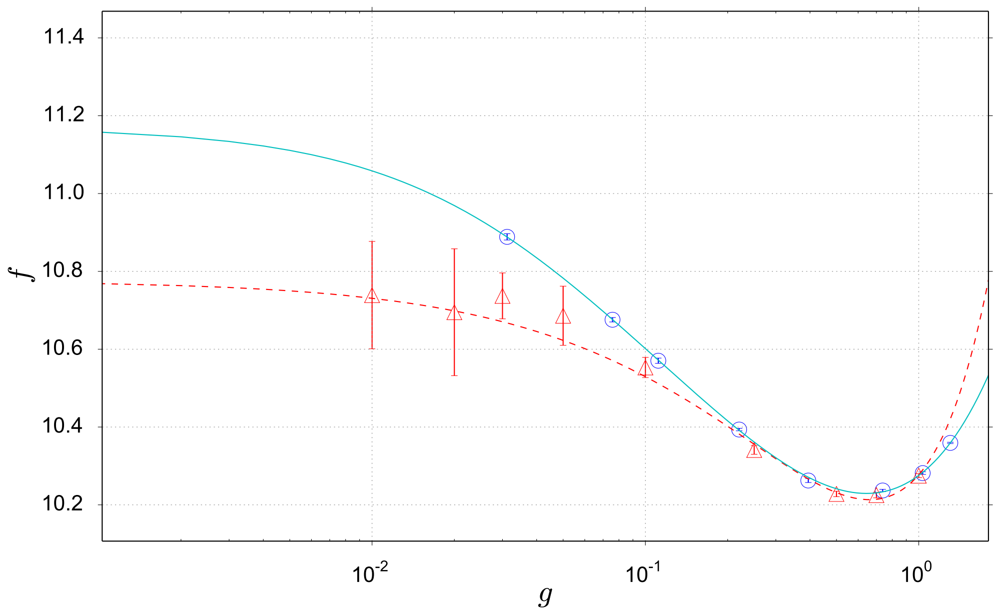

Figure 1: Final results for f(g) in logarithmic scale. The extrapolations are carried out with different methods and with different fit functions.

perfect agreement, despite the fact that the infinite volume limit extrapolations are obtained with completely different strategy. At lowest values of g our points seem to be a little bit higher but, even if we have more accurate results, we were not able to assert something definitely.

In order to improve our estimation we have started a new set of simulations with a slightly different strategy. For several values of $N \equiv L/a$ we perform two independent simulations, following the procedure described above, in which we consider two different values of $z = 2, 4$. The final extrapolation is obtained by means of a combined fit for the two different choices of z. In this way we do not completely solve the technical problem to probe low g-region, but we hope to get more accurate and precise results in order to identify the right behavior of the critical coupling in the continuum limit.

3. Extended worm algorithm

Now we show the extension of the worm algorithm for a ϕ^4 theory with N component fields ϕ and $O(N)$ symmetry. The starting point is the work of U.Wolff [5] in which the application of the *all-order strong coupling expansion* to the $O(N)$ sigma model is described.

Consider the partition function with two field insertions, using the action (2.2):

$$Z(u,v) = \int \left[\prod_z d\phi(z)\, e^{-S_0} \right] \left(\prod_{l=\langle xy \rangle} e^{\beta\phi(x)\cdot\phi(y)} \right) \phi(u)\cdot\phi(v) \tag{3.1}$$

We want that the term in the squared brackets becomes the new functional measure

$$\int \prod_x d\phi(x) e^{-|\phi|^2-\lambda(|\phi|^2-1)^2} = \int \prod_x d\mu[\phi(x)] \tag{3.2}$$

where the integrations employ the normalized $O(N)$ invariant measure on the sphere, which acts on a test function $f(\phi)$ as follows

$$\int d\mu(\phi) f(\phi) = C_N \int dr\, d\theta \frac{d\Omega}{2} r^{N-1} (\sin\theta)^{N-2} f(r,\theta,\Omega). \tag{3.3}$$

C_N is the normalization coefficient, r is the radial integration variable and θ, Ω constitute the total solid angle for a N-sphere. Rewriting (3.1) as integral over this new functional measure, we have to work out integration over spherical coordinates in N dimension. In order to obtain the loop representation is useful to introduce the single generating function for a general source j

$$G_N(j) \equiv \int \prod_x d\mu[\phi(x)] e^{j\cdot\phi} = \sum_{k=0}^{\infty} c[k;N] (j\cdot j)^k. \tag{3.4}$$

The coefficients of the series expansion are given by the resolution of the integral by means of the modified Bessel function $I_{N/2-1}$, using the measure defined in (3.2):

$$c[k;N] = \frac{\rho(N+k-1)\Gamma(N/2)}{\rho(N-1)2^{2k}k!\Gamma(N/2+k)}. \tag{3.5}$$

With ρ we indicate the solution of the radial part of (3.4) that can be solved only numerically, while the other terms in (3.5) come from the resolution of the solid angular part. The (3.5) is the key for computing the observables in a $O(N)\phi^4$ model.

3.1 Algorithm

The technique *high temperature expansion* allows to translate the original system in a new representation where the new fields are located at links connecting each pair of neighboring sites and have discrete values. Configurations in this formulation have a graphical representation as collections of paths that are called *loops* and the worm algorithm samples them by local moves. In particular there are several update steps that may constitute one local move. Here we only mention those moves that have a different acceptance probabilities with respect to [5], implying that the other ones remain the same. For this purpose we need to introduce some definitions: the active loop is the loop participant to the update process, u indicates the head and v the tail of the active loop, and an active loop is called trivial if it contains no 2-vertex and $u = v$.

Now we write down the ratio q that controls the acceptance probability $P = min(1,q)$ in the cases mentioned before.

- Extension: we try to move the head u to one of the nearest neighbor $\tilde{u}$.

$$q_1 = \frac{\rho(N+d(\tilde{u})/2)}{\rho(N+d(\tilde{u})/2-1)} \frac{\beta}{N+d(\tilde{u})} \tag{3.6}$$

- Retraction: we try to retract the head u by on link along the active loop and $\tilde{u}$ is the new head.

$$q_2 = \frac{\rho(N+d(u)/2-2)}{\rho(N+d(\tilde{u})/2-1)} \frac{N+d(\tilde{u})-1}{\beta} \tag{3.7}$$

PoS(LATTICE2016)277

- Kick: if the loop is trivial, we randomly pick a site x and try to move the trivial loop in that site.

$$q_3 = \frac{\rho(N+d(x)/2)\rho(N+d(u)/2-2)}{\rho(N+d(x)/2-1)\rho(N+d(u)/2-1)} \frac{N+d(u)-2}{N+d(x)} \tag{3.8}$$

4. Application

We perform several simulations to verify our algorithm. In particular we well reproduce the results in the case of the non-linear sigma model for the A-series in [5], .

We apply our method to the study of the phase transition of the three-dimensional $O(2)\phi^4$, basing on [8]. In Fig 2 we show the extrapolation of β_c to the infinite volume limit for $\lambda = 0.05$. The behavior of the function used for the fit is suggested by finite size scaling arguments and has

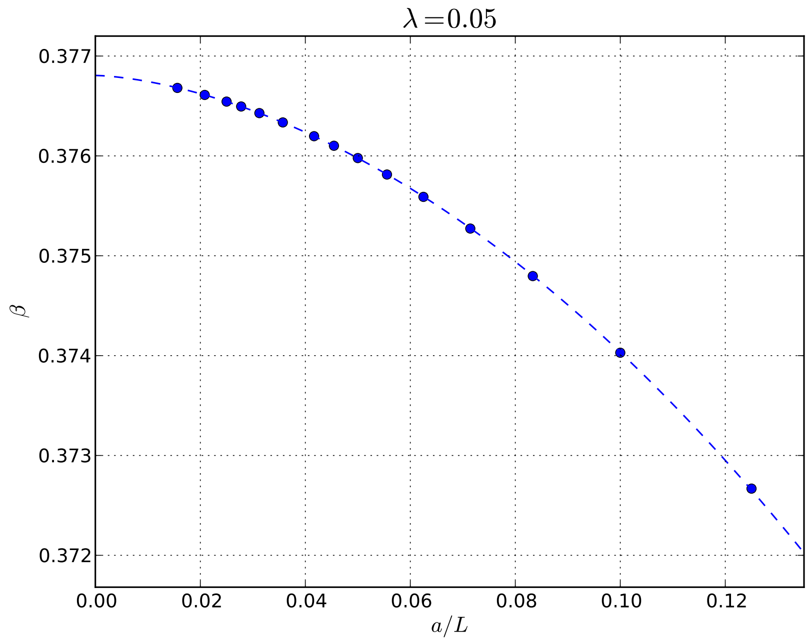

Figure 2: Example of extrapolation of β_c in the infinite volume lmit for $\lambda = 0.05$.

the following expression

$$\beta_\lambda(L) = \beta_c + b_1 L^{-1/\nu} + b_2 L^{-1/\nu+\omega}. \tag{4.1}$$

where the critical exponent ν, ω are taken from [8]. This is only a preliminary work and here we only want to show the quality of the fit. The next step is the computation of the difference between the value of $\langle\phi^2\rangle$ at the critical point for the case of *i*) small g and *ii*) $g = 0$, defined as

$$\Delta\langle\phi^2\rangle_c \equiv [\langle\phi^2\rangle_c]_g - [\langle\phi^2\rangle_c]_0. \tag{4.2}$$

This is obtained with the same technique adopted in [3]: at fixed λ we compute the quantity $\langle\phi^2\rangle$ for several lattice and then we extrapolate in the infinite volume limit. Finally we perform the continuum limit and estimate (4.2). We do not go further into details since the simulations are ongoing and the results will be presented in a dedicated article.

PoS(LATTICE2016)277

5. Conclusion

In the first part of this paper we summarize the plan we have for improving the estimation of the critical coupling g/μ^2 in ϕ^4. Since the simulations are still running, we have not yet a final result. Secondly, we briefly introduce the extension of the worm algorithm for the case of $O(N)\phi^4$ in arbitrary dimensions. In particular we present the application for the case of $D = 3$ and $N = 2$, summarizing the strategy we adopt for the estimation of (4.2). This quantity is related to the first correction to the shift of the critical temperature of an ideal Bose-Einstein gas and we hope to give soon a new estimation of it.

References

[1] N.V Prokof'ev, B.V Svistunov, I.S Tupitsyn, *Worm algorithm in quantum Monte Carlo simulations* Physics Letters A **238**, (1998) 253.

[2] T. Korzec, I. Vierhaus, U. Wolff, *Tomasz Korzec and Ingmar Vierhaus and Ulli Wolff*Computer Physics Communications **182**, (2011) 1477.

[3] P. Bosetti, B. De Palma, M. Guagnelli, *Monte Carlo determination of the critical coupling in ϕ_2^4 theory* Phys. Rev. D **92** (2015) 034509 [hep-lat/1506.08587].

[4] P. Arnold, G. Moore, *Transition temperature of a dilute homogeneous imperfect Bose gas*, Phys. Rev. Lett. **87** (2001) 120401 [cond-mat/0103228]

[5] U. Wolff, *Simulating the All-Order Strong Coupling Expansion III: O(N) sigma/loop models*, Nuclear Physics B **824** (2010) 254 [hep-lat/0908.0284].

[6] D. Schaich, W. Loinaz, Phys. Rev. D **79** (2009) 056008, [hep-lat/0902.0045].

[7] M. Burkardt, S. S. Chabysheva, J. R Hiller*Two-dimensional light-front ϕ^4 theory in a symmetric polynomial basis*, Phys. Rev. D **94** (2016) 065006 [hep-th/1607.00026].

[8] M. Campostrini, M. Hasenbusch, A. Pelissetto, P. Rossi, E. Vicari, *Critical behavior of the three-dimensional XY universality class*, Phys. Rev. B **63** (2001) 21 [cond-mat/0010360].