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V_{us} from inclusive determinations based on hadronic tau data

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The conventional implementation of the inclusive hadronic τ decay data based, flavor-breaking (FB) finite-energy sum rule (FESR) determination of V_{us} is know to produce results $> 3\sigma$ low compared to kaon physics based results and 3-family-unitarity expectations. We revisit this implementation, showing that it fails a number of self-consistency tests, and that the problems originate from a breakdown of assumptions employed for treating higher dimension OPE contributions. A recently proposed alternate implementation, which cures these problems, and uses lattice data to more reliably quantify leading $D = 2$ OPE uncertainties, is then briefly reviewed. Employing this new implementation, using also preliminary BaBar results for the $\tau \to K^-\pi^0\nu_\tau$ exclusive branching fraction, yields a result, $V_{us} = 0.2228(23)_{exp}(6)_{th}$, in excellent agreement with that from $K_{\ell 3}$, and, within errors, with three-family-unitarity expectations. Limitations in the near-term possibilities for reducing the experimental error by the desired factor of ~ 2 reduction are then highlighted. These serve to motivate a new proposal for determining V_{us} via a dispersive analysis employing strange hadronic τ data and lattice data in place of the OPE.

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1. Introduction

Using $|V_{ud}| = 0.97417(21)$ from super-allowed $0^+ \to 0^+$ nuclear β decays [1] as input to the three-family-unitary relation leads to the expectation $|V_{us}| = 0.2258(9)$. This is compatible, within errors, with the results of direct determinations from $K_{\ell 3}$ and $\Gamma[K_{\mu 2}]/\Gamma[\pi_{\mu 2}]$, using the recent 2014 FlaviaNet experimental results, $f_+(0)|V_{us}| = 0.2165(4)$ and $|f_K V_{us}|/|f_\pi V_{ud}| = 0.2760(4)$ [2] and 2016 FLAG $n_f = 2+1+1$ lattice input, $f_+(0) = 0.9704(33)$ and $f_K/f_\pi = 1.193(3)$ [3], which yield $|V_{us}| = 0.2231(9)$ and $0.2253(7)$, respectively.

Much lower values are obtained from conventional implementations of FB FESR analyses of inclusive non-strange and strange hadronic τ decay distributions [4], the most recent update of this approach [5] producing, for example, a result

$$|V_{us}| = 0.2176(21)\,, \tag{1.1}$$

3.6σ lower than the three-family-unitarity expectations.

In the Standard Model (SM), with $R_{V/A;ij} \equiv \Gamma[\tau^- \to \nu_\tau \, \mathrm{hadrons}_{V/A;ij}(\gamma)]/\Gamma[\tau^- \to \nu_\tau e^- \bar{\nu}_e(\gamma)]$, the differential distributions, $dR_{V/A;ij}/ds$, for flavor $ij = ud, us$, vector (V) or axial vector (A) current mediated decays are related to $\rho^{(J)}_{V/A;ij}$, the spectral functions of the $J = 0,1$ scalar correlators, $\Pi^{(J)}_{V/A;ij}$, which characterize the flavor ij, V or A current-current two-point function, by [6]

$$\begin{aligned} \frac{dR_{V/A;ij}}{ds} &= \frac{12\pi^2 |V_{ij}|^2 S_{EW}}{m_\tau^2} \left[w_\tau(y_\tau)\rho^{(0+1)}_{V/A;ij}(s) - w_L(y_\tau)\rho^{(0)}_{V/A;ij}(s) \right] \,, \\ &\equiv \frac{12\pi^2 |V_{ij}|^2 S_{EW}}{m_\tau^2} \,(1-y_\tau)^2 \,\tilde{\rho}(s)\,, \end{aligned} \tag{1.2}$$

where $y_\tau = s/m_\tau^2$, $w_\tau(y) = (1-y)^2(1+2y)$, $w_L(y) = 2y(1-y)^2$, $\tilde{\rho}(s) = (1+2y_\tau)\rho^{(1)}_{V/A;ij}(s) + \rho^{(0)}_{V/A;ij}(s)$, S_{EW} is a known short-distance electroweak correction, and V_{ij} is the flavor ij CKM matrix element. The accurately known, non-chirally-suppressed π and K pole contributions dominate $\rho^{(0)}_{A;ij}(s)$, up to $O\left[(m_i \mp m_j)^2\right]$ continuum V and A corrections, which are negligible for $ij = ud$, and small for $ij = us$. With the latter estimated (in a mildly model-dependent manner) from associated $ij = us$ scalar and pseudoscalar sum rules [7, 8], the experimental $dR_{V/A;ij}/ds$ distributions provide direct determinations of $\rho^{(0+1)}_{V/A;ud,us}(s)$.

$|V_{us}|$ can be determined from inclusive FB τ decay data using FESRs involving the FB polarization difference, $\Delta\Pi_\tau \equiv \Pi^{(0+1)}_{V+A;ud} - \Pi^{(0+1)}_{V+A;us}$, and associated spectral function, $\Delta\rho_\tau \equiv \rho^{(0+1)}_{V+A;ud} - \rho^{(0+1)}_{V+A;us}$ [4]. Generally, for any $s_0 > 0$ and any $w(s)$ analytic in the region $|s| \le s_0$,

$$\int_0^{s_0} w(s)\Delta\rho_\tau(s)\, ds = -\frac{1}{2\pi i}\oint_{|s|=s_0} w(s)\Delta\Pi_\tau(s)\, ds\,. \tag{1.3}$$

Experimental data is to be used on the LHS and, for large enough s_0, the OPE on the RHS. For general w, one must first construct the $J = 0+1$ analogue, $dR^{(0+1)}_{V/A;ij}/ds$, of $dR_{V/A;ij}/ds$ by subtracting $J = 0$ contributions, and then form the re-weighted integrals

$$R^w_{V+A;ij}(s_0) \equiv \int_0^{s_0} ds\, \frac{w(s)}{w_\tau(s)} \frac{dR^{(0+1)}_{V+A;ij}(s)}{ds}\,. \tag{1.4}$$

Then, using the OPE representation of the FB difference

$$\delta R^w_{V+A}(s_0) \equiv \frac{R^w_{V+A;ud}(s_0)}{|V_{ud}|^2} - \frac{R^w_{V+A;us}(s_0)}{|V_{us}|^2}, \tag{1.5}$$

given by the LHS of Eq. (1.3), one finds, solving for $|V_{us}|$ [4],

$$|V_{us}| = \sqrt{R^w_{V+A;us}(s_0) / \left[\frac{R^w_{V+A;ud}(s_0)}{|V_{ud}|^2} - \delta R^{w,OPE}_{V+A}(s_0)\right]}, \tag{1.6}$$

where the resulting $|V_{us}|$ will be independent of s_0 and w if all experimental and OPE input is reliable. Checking for stability as s_0 and w are varied thus allows one to expose problems and/or test for self-consistency.

The low values of $|V_{us}|$ noted above result from a conventional implementation of Eq. (1.6) [4] employing the single s_0 value, $s_0 = m_\tau^2$ and single weight $w = w_\tau$. With these choices, the associated spectral integrals are fixed by the inclusive non-strange and strange branching fractions. Self-consistency tests using variable s_0 and w are then no longer possible. With w_τ having degree 3, unsuppressed OPE contributions up to dimension $D = 8$ are present in $\delta R^{w_\tau,OPE}_{V+A}(s_0)$. $D = 2$ and 4 contributions are known [9], being fixed by α_s, $m_{u,d}$, m_s, $\langle \bar{u}u\rangle$ and $\langle \bar{s}s\rangle$ [3, 10, 11]. $D = 6$ and 8 condensates, however, are not known experimentally. $D = 6$ contributions have typically been estimated using the very crude vacuum saturation approximation (VSA) and $D = 8$ contributions neglected [4, 12]. The lack of self-consistency tests in the standard implementation makes these "approximations" potentially dangerous, especially in view of the known crudeness of the VSA in the *ud* sector [13] and the very strong (factor of ~ 20) cancellation in the FB $D = 6$ VSA estimate.

2. Problems with the conventional implementation and an alternate strategy

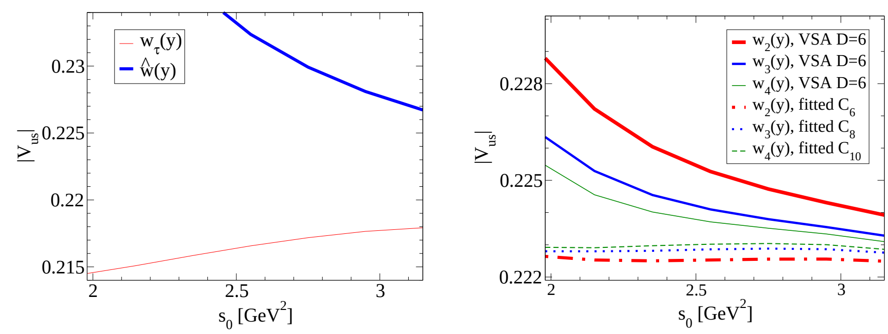

Figure 1: Left panel: $|V_{us}|$ from the w_τ and $\hat{w}$ FESRs with conventional OPE input (including contour improved perturbation theory for the $D = 2$ series). Right panel: Comparison of conventional implementation results (solid lines) with those obtained using central fitted $C_{6,8,10}$ and the fixed order perturbation theory $D = 2$ prescription favored by lattice results, for the weights $w_{2,3,4}$ (dashed lines).

Figure 1 shows the results for $|V_{us}|$ as a function of s_0 obtained using a range of w and the conventional implementation assumptions for $D = 6$ and 8 OPE contributions. Very significant s_0- and w-dependence is observed. Particularly illuminating is a comparison of the results from the

$w_\tau(y) = 1-3y^2+2y^2$ and $\hat{w}(y) = 1-3y+3y^2-y^3$ ($y = s/s_0$) FESRs, whose integrated $D=6$ OPE contributions are equal in magnitude but opposite in sign. In the conventional implementation, $D=6$ contributions are small and $D=8$ contributions negligible for the w_τ FESR. If these approximations are reasonable for w_τ, they should be similarly reasonable for $\hat{w}$, and the $|V_{us}|$ results obtained from the two FESRs should agree well and both display good s_0 stability. If not, the two FESRs should display s_0-instabilities of opposite signs. Moreover, since integrated $D=6$ and 8 contributions scale as $1/s_0^2$ and $1/s_0^3$, the difference between the the output $|V_{us}|$ from the two FESRs should decrease with increasing s_0. Obviously it is the second scenario which is realized. The break down of the conventional implementation assumptions suggested by these results is further confirmed by the s_0-instabilities of the solid lines of the right panel of Figure 1, which show the conventional implementation results for $|V_{us}|$ obtained from the $w_N(y)$ FESRs, $N = 2,3,4$, where

$$w_N(y) = 1-\frac{N}{N-1}y+\frac{1}{N-1}y^N . \tag{2.1}$$

These results suggest an alternate implementation in which assumptions about $D>4$ contributions are avoided and the effective $D>4$ condensates, $C_{D>4}$, are instead obtained from fits to data, taking advantage of the differing s_0-dependence of different D OPE contributions. The dashed lines show the much improved stability obtained when such fitted $C_{D>4}$ are employed as input to the w_N FESRs.

Another issue for the FB FESR approach is the slow convergence of the relevant $D=2$ OPE series. To four loops, one has, neglecting $O(m_{u,d}^2/m_s^2)$ corrections [9]

$$\left[\Delta\Pi_\tau(Q^2)\right]^{OPE}_{D=2} = \frac{3}{2\pi^2}\frac{m_s(Q^2)}{Q^2}\left[1+\frac{7}{3}\bar{a}+19.93\bar{a}^2+208.75\bar{a}^3+\cdots\right], \tag{2.2}$$

with $\bar{a} = \alpha_s(Q^2)/\pi$, and $\alpha_s(Q^2)$ and $m_s(Q^2)$ the running $\overline{MS}$ coupling and strange quark mass. With $\bar{a}(m_\tau^2) \simeq 0.1$, the $O(\bar{a}^3)$ term exceeds the $O(\bar{a}^2)$ term at the spacelike point on the contour $|s| = s_0$ for all kinematically accessible s_0, complicating the task of choosing an appropriate truncation order and estimating the associated truncation uncertainty. This issue was investigated by comparing OPE expectations to $n_f = 2+1$ RBC/UKQCD lattice data [14] for $\Delta\Pi_\tau(Q^2)$ over a range of Euclidean Q^2 [15]. An excellent match of the $D=2+4$ OPE sum to the lattice data was obtained over an interval stretching from $Q^2 \sim 10\ GeV^2$ down to $\sim 4\ GeV^2$ using the 3-loop-truncated version of the $D=2$ series with a fixed- rather than local-scale treatment of logarithmic contributions [15].[1] The high-Q^2 comparison also demonstrates that conventional $D=2+4$ OPE error estimates are extremely conservative [15]. Deviations of the $D=2+4$ OPE sum from the lattice data below $Q^2 \sim 4\ GeV^2$ [15] are also clearly incompatible with the conventional implementation assumptions regarding the effective $D>4$ OPE condensates.

An alternate implementation of the FB FESR approach, predicated on the observations above, was presented in Ref. [15]. The theory side employs the 3-loop-truncated, FOPT version of $D=2$ OPE contributions favored by the lattice, and fits $|V_{us}|$ and the relevant effective $D>4$ OPE condensates using the s_0-dependent w_N-weighted spectral integrals. Spectral integrals are evaluated using

[1] The fixed-/local-scale treatment of $[\Delta\Pi_\tau]^{OPE}_{D=2}$ is the analogue of the "fixed-order" (FOPT)/"contour-improved" (CIPT) treatment of the $D=2$ contribution on the OPE side of the FSER relation.

$\pi_{\mu 2}$, $K_{\mu 2}$ and SM expectations for the π and K pole contributions, ALEPH continuum *ud* V+A data [16], Belle [17] and BaBar [18, 19] results for the $\bar{K}^0\pi^-$ and $K^-\pi^0$ distributions, BaBar [20] and Belle [21] results for the $K^-\pi^+\pi^-$ and $\bar{K}^0\pi^-\pi^0$ distributions, and 1999 ALEPH results [22] for the sum of the distributions of those exclusive strange modes not remeasured by the B-factory experiments. Two different versions exist for the $K^-\pi^0\nu_\tau$ branching fraction, which normalizes the corresponding exclusive distribution: $0.00433(15)$ from the 2014 HFAG summer fit [23], and the preliminary BaBar thesis update $0.00500(14)$ [19]. The latter is favored by BaBar, whose earlier result dominates the 2014 HFAG average. Central results below correspond to the latter choice.

The w_N FESRs have the advantage that they involve, in addition to the known $D=2$ and 4 terms, only a single unknown $D=2N+2$ OPE contribution. Fits to the w_N FESR ($N=2,3,4$) thus yield $|V_{us}|$ and C_{2N+2}. The $|V_{us}|$ from the different w_N FESRs are in excellent agreement [15]. With the C_{2N+2} obtained from these fits as input to the conventional implementation of the $w_{2,3,4}$ FESRs yield the results shown by the dashed lines in Figure 1, which display excellent s_0- and w-stability. The excellent consistency allows a final result for $|V_{us}|$ to be obtained using a combined 3-weight fit. Normalizing the exclusive $K^-\pi^0$ distribution using the preliminary BaBar update for the $\tau^- \to K^-\pi^0\nu_\tau$ branching, one finds [15]

$$|V_{us}| = 0.2228(5)_{th}(23)_{exp} . \tag{2.3}$$

The theory error is dominated by the uncertainty in $\langle m_s\bar{s}s\rangle$, the experimental error by the the strange exclusive distribution errors [15]. This result agrees well with that from $K_{\ell 3}$, and, within errors, with 3-family unitarity expectations.[2] Compared to the conventional implementation results, roughly half of the improved agreement results from the data-based treatment of higher D OPE contributions, and half from the new preliminary BaBar $K^-\pi^0\nu_\tau$ branching fraction normalization.

Table 1: Relative contributions to the w_N-weighted *us* spectral integrals in the s_0 fit window employed in the alternate FB FESR implementation. $K\pi$ column entries are the sum of the $K^-\pi^0$ and $\bar{K}^0\pi^-$ contributions, $K\pi\pi$ (B factory) column entries the sum of the $K^-\pi^+\pi^-$ and $\bar{K}^0\pi^-\pi^0$ contributions, and *Residual* column entries the contributions of the residual part of the 1999 ALEPH distribution.

Weight	s_0 [GeV^2]	K	$K\pi$	$K\pi\pi$ (B-factory)	Residual
w_2	2.15	0.496	0.426	0.062	0.017
	3.15	0.360	0.414	0.162	0.065
w_3	2.15	0.461	0.446	0.073	0.019
	3.15	0.331	0.415	0.182	0.074
w_4	2.15	0.441	0.456	0.082	0.021
	3.15	0.314	0.411	0.194	0.081

Improvements to the low-multiplicity strange exclusive branching fractions would allow for significant reductions in the error on $|V_{us}|$ obtained from the new implementation of the FB FESR

[2] Normalizing the $K^-\pi^0$ distribution with the HFAG 2014 $\tau^- \to K^-\pi^0\nu_\tau$ branching fraction yields instead $|V_{us}| = 0.2200(5)_{th}(23)_{exp}$, 0.0024 higher than obtained from the conventional implementation using the same input. Further work on the branching fraction of this mode is desirable.

approach. Uncertainties in the combined, higher-multiplicity 1999 ALEPH "residual mode" distribution, however, are likely to prove an important limiting factor in the near future. The errors on the weighted spectral integrals over this residual distribution are $\sim 25\%$. A competitive determination of $|V_{us}|$ requires sub-0.5% precision, which requires sub-% precision on the weighted inclusive *us* spectral integrals. The relative contributions of the lower-multiplicity exclusive modes, as well as that of the residual mode sum, to the inclusive weighted *us* spectral integrals for the w_2, w_3 and w_4 FESRs are shown in Table 1 at the lowest and highest s_0 of the analysis fit window.

The $\sim 25\%$ residual mode errors correspond to $\sim 2\%$ errors on the inclusive *us* spectral integrals at the lower end of the s_0 fit window, indicating that a factor of ~ 2 or more improvement would be required in the normalization of the residual mode sum to make the FB FESR approach fully competitive with kaon-physics-based determinations.

A way around this current limitation is to switch to a dispersive analysis based on the inclusive *us* data alone in which weights are chosen that allow lattice data to be used in place of the OPE [15] on the theory side of the dispersion relations. This works as follows. From Eq. (1.2), the experimental $dR_{us;V+A}/ds$ distribution provides a direct determination of $|V_{us}|^2\,\tilde{\rho}(s)$, with no even mildly model-dependent continuum $J=0$ subtraction required. The combination $\tilde{\rho}(s)$ is the spectral function of the kinematic-singularity-free $J=0$ and 1 *us* V+A polarization combination,

$$\tilde{\Pi}_{us;V+A}(Q^2) \equiv \left(1 - 2\frac{Q^2}{m_\tau^2}\right) \Pi^{(J=1)}_{us;V+A}(Q^2) + \Pi^{(J=0)}_{us;V+A}(Q^2)\,, \tag{2.4}$$

where $Q^2 = -s$. Choosing weights, $W_N(s)$,

$$W_N(s) \equiv \frac{1}{\prod_{k=1}^{N}(s+Q_k^2)}\,, \tag{2.5}$$

which have poles at the N distinct Euclidean locations $Q^2 = Q_1^2, \cdots Q_N^2$, $Q_k^2 > 0$, one has, for $N \geq 3$, the convergent, unsubtracted dispersion relation

$$\int_{th}^{\infty} ds\, W_N(s)\,\tilde{\rho}_{us;V+A}(s) = \sum_{k=1}^{N} \frac{\tilde{\Pi}_{us;V+A}(Q_k^2)}{\prod_{j\neq k}\left(Q_j^2 - Q_k^2\right)}\,. \tag{2.6}$$

Lattice data is to be used to evaluate the $\tilde{\Pi}_{us;V+A}(Q_k^2)$ on the RHS of this relation. This can be done with good accuracy if all Q_k^2 are kept to a few to several tenths of a GeV^2. The $s \leq m_\tau^2$ contribution to the LHS is determinable, up to the unknown factor $|V_{us}|^2$, from experimental $dR_{us;V+A}/ds$ data. To control errors on the LHS, the number, N, and locations, $Q_1^2, \cdots, Q_N^2$, of the poles, are to be chosen such that contributions from both the region where *us* data errors are large and the region $s > m_\tau^2$ (where data do not exist and pQCD is used for $\tilde{\rho}(s)$) are very small. This goal can be accomplished by keeping all Q_k^2 below $\sim 1\ GeV^2$ and choosing N large enough. Increasing N, however, increases the errors on the lattice side of Eq. (2.6) (the level of cancellation in the sum of residues appearing there grows with increasing N). The error on $|V_{us}|$ extracted using Eq. (2.6) is minimized by optimizing the choice of N and pole locations, subject to these two competing constraints. A preliminary implementation of this approach is described in the write-up of H. Ohki's presentation, in these proceedings.

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