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Form factors in the $\mathrm{B_s} \to \mathrm{K}\ell\nu$ decays using HQET and the lattice

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We report on a recent computation of the form factors in semi-leptonic decays of the $\mathrm{B_s}$ using Heavy Quark Effective Theory (HQET) formalism applied on the lattice. The connection of the form factors with the 2-point and 3-point correlators on the lattice is explained, and the subsequent non-perturbative renormalization of HQET and it's matching to $N_f = 2$ QCD is outlined. The results of the (static) leading-order calculation in the continuum limit is presented.

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1. Motivations

Testing the consistency and the correctness of the Standard Model is a central goal of particle physics. The decays of the B-mesons and B-baryons can be used to improve the determination of several poorly known matrix elements of the Cabibo-Kobayashi-Maskawa (CKM) quark mixing matrix, and in particular $|V_{\mathrm{ub}}|$. The mean values of this fundamental parameter of the Standard Model extracted from inclusive decays agree with those extracted from the different exclusive decays (such as $\mathrm{B} \to \pi\ell\nu$ and $\mathrm{B} \to \tau\nu$) only when the quoted uncertainties are stretched by a factor three [1]. It needs to be resolved whether this arises due to systematic uncertainties inherent in different treatments, or due to beyond Standard Model (BSM) physics.

We report on the determination of the form factors in the $\mathrm{B_s} \to \mathrm{K}\ell\nu$ decay in $N_f = 2$ QCD, and the heavy quark effective theory (HQET) to account for the heavy quark on the lattice [2]. The formalism will be outlined here, especially the non-perturbative renormalization procedure and continuum limit extrapolation. The invariant mass of the leptons is kept fixed as the continuum limit is taken. The differential decay rates at a given q^2 is related to the renormalized form factors (at the same q^2) and $|V_{\mathrm{ub}}|$. A precise experimental determination of this differential decay rate would then allow for a reliable and accurate determination of $|V_{\mathrm{ub}}|$, when all the errors have been systematically accounted for. In our theoretical calculation we identify all the possible systematic sources of error, providing a robust estimate of the exclusive observable. The complementary proceeding [3] describes the details involved in the extraction of the bare form factors on the lattice.

On the lattice, the most challenging aspect for this reliable computation is the heavy quark itself [4]. Due it's mass $m_{\mathrm{b}} \sim 5\mathrm{GeV}$ being comparable to that of inverse lattice spacing of the finest lattice ensembles in use today, the discretization effects due to the heavy quark are particularly severe. A theoretically clean way to avoid this problem is the use of an effective theory for the heavy quark. However, the state-of-art computations [5, 6, 7, 8] use either a relativistic heavy quark action or a non-relativistic formulation of QCD on the lattice, which are affected by perturbative renormalization or uncontrolled discretion effects. A fully non-perturbative programme to renormalize the currents does not yet exist. Conventional discretization errors are estimated only by power-counting methods, while the actual extrapolation to the continuum limit with heavy quarks may involve a complicated dependence on the lattice spacing. Our computation seeks to address these issues and demonstrate a clean extrapolation to the continuum limit of the form factors.

2. Methodology

To make contact with phenomenology, we list the following "five-point" program:

(i) *reliable* computation of ground state matrix elements $\langle \mathrm{K}|V^{\mu}(0)|\mathrm{B_s}\rangle$,

(ii) *renormalization* of the matrix elements, either in full QCD, or if an effective theory is being used, then in the effective theory, which is then matched to QCD,

(iii) *extrapolation* of the renormalized quantity at finite lattice spacings *to the continuum limit*,

(iv) *extrapolation* of the (light) quark masses to their *physical values*,

(v) *mapping* out the q^2 dependence of the form factors, since experiments measure values of the differential cross-section.

We employ lattice discretized HQET to compute the form factors on the lattice. While the first challenge is the subject of [3], this proceeding is concerned with the second and third items. Work on the fourth point is in progress. At the moment, our computation focuses on a fixed value of q^2.

3. HQET on the lattice and non-perturbative renormalization

A $\mathrm{B_s}$ state with a momentum $p_{\mathrm{B_s}}$ decays into a Kaon K with momentum p_{K} mediated by the vector current, $V^\mu(x) = \bar{\psi}_{\mathrm{u}}(x)\gamma_\mu \psi_{\mathrm{b}}(x)$. In QCD, we parameterize the matrix element into two equivalent form-factor decompositions:

$$\begin{aligned} \langle \mathrm{K}(p_{\mathrm{K}})|V^\mu(0)|B_s(p_{\mathrm{B_s}})\rangle &= \left(p_{\mathrm{B_s}} + p_{\mathrm{K}} - \frac{m^2_{\mathrm{B_s}} - m^2_{\mathrm{K}}}{q^2} q \right)^\mu \cdot f_+(q^2) + \frac{m^2_{\mathrm{B_s}} - m^2_{\mathrm{K}}}{q^2} q^\mu \cdot f_0(q^2) \\ &= \sqrt{2m_{\mathrm{B_s}}} \left[v^\mu \cdot h_\parallel(p_{\mathrm{K}} \cdot v) + p^\mu_\perp \cdot h_\perp(p_{\mathrm{K}} \cdot v) \right] \end{aligned} \tag{3.1}$$

The last line can be seen as a definition of the form factors $h_\parallel$ and $h_\perp$ which we compute. The velocity, momentum and squared momentum transfer (q^2) variables are related as: $v^\mu = p^\mu_{\mathrm{B_s}}/m_{\mathrm{B_s}}$, $p^\mu_\perp = p^\mu_{\mathrm{K}} - (v \cdot p_{\mathrm{K}})v^\mu$, $q^\mu = p^\mu_{\mathrm{B_s}} - p^\mu_{\mathrm{K}}$, $p_{\mathrm{K}} \cdot v = \frac{m^2_{\mathrm{B_s}} + m^2_{\mathrm{K}} - q^2}{2m_{\mathrm{B_s}}}$. Using non-relativistic state normalization in HQET removes the mass dependence present in the relativistic normalization: $\langle \mathrm{B_s}(p')|\mathrm{B_s}(p)\rangle = 2E(\mathbf{p})(2\pi)^3\delta(\mathbf{p} - \mathbf{p}')$. Consequently, $h_\perp, h_\parallel$ are independent of the heavy quark mass, modulo a logarithmic dependence coming from the matching function between QCD and HQET. Note that in the latter, the effective heavy quark fields are mass independent, and hence no additional dependence originates from the vector current.

The lattice computation is performed in the rest frame of the $\mathrm{B_s}$ meson: $v^\mu = (1,0,0,0)$, where the QCD matrix elements are simply related to the form factors via:

$$(2m_{\mathrm{B_s}})^{-1/2} \langle \mathrm{K}(p_{\mathrm{K}})|V^0(0)|\mathrm{B_s}(p_{\mathrm{B_s}})\rangle = h_\parallel(E_{\mathrm{K}}) \tag{3.2}$$

$$(2m_{\mathrm{B_s}})^{-1/2} \langle \mathrm{K}(p_{\mathrm{K}})|V^k(0)|\mathrm{B_s}(p_{\mathrm{B_s}})\rangle = p^k_{\mathrm{K}} h_\perp(E_{\mathrm{K}}) \tag{3.3}$$

The only kinematic variable is $E_{\mathrm{K}} = p_{\mathrm{K}} \cdot v$, the energy of the final state Kaon. Neglecting terms of $\mathscr{O}(m^2_\ell/m^2_{\mathrm{B_s}}, m^2_\ell/q^2)$, the differential decay rate can be directly related to the form factor:

$$\frac{\mathrm{d}\Gamma(\mathrm{B_s} \to \mathrm{K}\ell\nu)}{\mathrm{d}q^2} = \frac{G^2_{\mathrm{F}}}{24\pi^3} |V_{\mathrm{ub}}|^2 |\mathbf{p}_K|^3 [f_+(q^2)]^2 \tag{3.4}$$

With the experimental measurement of the differential decay rate and a reliable estimate of $f_+(q^2)$, $|V_{\mathrm{ub}}|$ can be accurately measured. We now discuss the renormalized form factors $h^{\mathrm{stat,RGI}}_{\parallel,\perp}$, related to the QCD form factors $h_{\parallel,\perp}$ via the matching function(s):

$$h_{\parallel,\perp}(E_{\mathrm{K}}) = C_{\mathrm{V_0,V_k}}(M_{\mathrm{b}}/\Lambda_{\overline{\mathrm{MS}}}) h^{\mathrm{stat,RGI}}_{\parallel,\perp}(E_{\mathrm{K}}) \cdot [1 + \mathscr{O}(1/m_{\mathrm{b}})] \tag{3.5}$$

Our computations use the $N_f = 2$ QCD ensembles. A parallel project is non-perturbatively matching the currents $V_{0,k}$ between QCD and HQET, and will allow for the $1/m$ corrections to be computed [12]. For now, we remain with the static order, where most of the non-perturbative results are available. The bare currents $V^{\mathrm{stat}}_{0,k}$ are:

$$\mathrm{V}^{\mathrm{stat}}_0 = \bar{\psi}_{\mathrm{u}}\gamma_0\psi_{\mathrm{h}} + ac_{\mathrm{V_0}}(g_0)\bar{\psi}_{\mathrm{u}} \sum_l \overleftarrow{\nabla}^S \gamma_l \psi_{\mathrm{h}}; \quad \mathrm{V}^{\mathrm{stat}}_{\mathrm{k}} = \bar{\psi}_{\mathrm{u}}\gamma_k\psi_{\mathrm{h}} - ac_{\mathrm{V_k}}(g_0)\bar{\psi}_{\mathrm{u}} \sum_l \overleftarrow{\nabla}^S \gamma_l\gamma_k \psi_{\mathrm{h}}. \tag{3.6}$$

The leading term is the same as in QCD, except that the heavy quark fields are denoted by $\bar{\psi}_{\mathrm{h}}$, and the HYP1 and HYP2 heavy quark actions are considered [9], which have an exponentially better signal-to-noise ratio as compared to the classic Eichten-Hill action for the heavy quarks. In addition, there is the $\mathscr{O}(a)$ improvement term, whose coefficients are currently known to 1-loop order: $c_x = c_x^{(1)} g_0^2 + O(g_0^4)$. They are relatively small [10]: for HYP1, $c_{\mathrm{V_0}} = 0.0223(6), \;\; c_{\mathrm{V_k}} = 0.0029(2)$ while for HYP2, $c_{\mathrm{V_0}} = 0.0518(2), \;\; c_{\mathrm{V_k}} = 0.0380(6)$.

In HQET, the currents get renormalized multiplicatively. At the static order, the spin symmetry of HQET results in the same renormalization of the vector current as the axial current: $Z^{\mathrm{stat,RGI}}_{\mathrm{V_k}} = Z^{\mathrm{stat,RGI}}_{\mathrm{A_0}}$, but an extra factor $Z^{\mathrm{stat}}_{\mathrm{V/A}}(g_0)$ for $\mathrm{V_0}$, as Wilson fermions break chiral symmetry:

$$\mathrm{V}_0^{\mathrm{stat,RGI}} = Z^{\mathrm{stat,RGI}}_{\mathrm{A_0}}(g_0) Z^{\mathrm{stat}}_{\mathrm{V/A}}(g_0) \mathrm{V}_0^{\mathrm{stat}}; \quad \mathrm{V}_{\mathrm{k}}^{\mathrm{stat,RGI}} = Z^{\mathrm{stat,RGI}}_{\mathrm{A_0}}(g_0) \mathrm{V}_{\mathrm{k}}^{\mathrm{stat}}. \tag{3.7}$$

The crucial aspect of our non-perturbative renormalization setup proceeds via the calculation of the RGI (renormalization group invariant) quantities Φ^{RGI}, which are both scheme and scale independent. For a renormalized static heavy-light current, the the differential equation for the physical quantity at different scales μ, can be integrated in perturbation theory:

$$(A^{\mathrm{RGI}})_0 = \lim_{\mu \to \infty} [2b_0 \bar{g}^2(\mu)]^{-\gamma_0/2b_0} (A_{\mathrm{R}}^{\mathrm{stat}})_0(\mu) \tag{3.8}$$

The integration constant in the left hand side is completely independent of scale of the running coupling $\bar{g}^2(\mu)$, the renormalized current and the scheme used to compute them. γ_0 and b_0 are the universal 1-loop coefficients of the running mass and the couplings. The goal therefore is to determine the integration constant. The non-perturbative analog is expressed as:

$$Z^{\mathrm{stat}}_{\mathrm{A,RGI}}(g_0) = \frac{\Phi^{\mathrm{RGI}}}{\Phi(\mu)} \times Z^{\mathrm{stat}}_{\mathrm{A}}(g_0, a\mu)\big|_{\mu = \frac{1}{2L_{\max}}}. \tag{3.9}$$

The first universal factor relates the renormalization of A_0^{stat} at a scale $\mu_0 = 1/L_{\max}$ calculated in a scheme to the RGI operator, while the second factor knows about the lattice discretization. The computation of both factors was done in [11], where the Schrödinger functional scheme was used to compute the RGI quantity, and yielded $\Phi^{\mathrm{RGI}}/\Phi(\mu) = 0.880(7)$. Finally, for our computation, we used a a generous range $[Z^{\mathrm{stat}}_{V/A}(g_0)]^{-1} = 0.97(3)$, motivated by comparing the value for the quenched approximation, noting the absence of N_f dependence at the 1-loop order. This uncertainly only affects the $1/m_{\mathrm{h}}$ terms, and will be eliminated in the non-perturbative matching program [12].

4. Matching to QCD

The renormalized form factors need to be matched to that of QCD via the conversion functions C_x. These can be obtained by calculating a matrix element upto a given order in both theories and then matching the expressions. As a pedagogical example, consider the matrix element $\mathscr{M}$ of the renormalized axial current in both 1-loop QCD and HQET in the leading order (stat). To make the theories agree, we match the expressions [4] obtained for QCD, with that of HQET:

$$\mathscr{M}_{\mathrm{QCD}}(L, m_{\mathrm{h}}) \; = \; [1 + g^2(-\gamma_0 \ln(m_{\mathrm{h}} L) + B_{\mathrm{QCD}})] \mathscr{M}^{(0)} + \mathscr{O}(g^4) + \mathscr{O}(\frac{1}{m_{\mathrm{h}} L}), \tag{4.1}$$

$$\mathscr{M}_{\mathrm{stat}}(\mu L) \; = \; [1 + g^2(-\gamma_0 \ln(\mu L) + B_{\mathrm{lat}})] \mathscr{M}^{(0)} + \mathscr{O}(g^4). \tag{4.2}$$

Expressed in terms of the running coupling g^2, and renormalized mass, equating the leading order gives the multiplicative matching function between the two theories: $C_{\mathrm{match}} = 1 + g^2 \gamma_0 \ln(\mu/m_{\mathrm{h}}) + (B_{\mathrm{QCD}} - B_{\mathrm{lat}}) + \mathscr{O}(g^4)$, motivating the the logarithmic dependence on the heavy quark mass.

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A non-perturbative matching of the currents is underway by the ALPHA collaboration [12]. Meanwhile, for our purposes, we use the best available perturbative results. A major advantage in our procedure of using the RGI operators is that we can directly use the results from renormalized continuum perturbation theory [13], as opposed to other results in the literature which use bare perturbation theory. The limitation here is on the knowledge of the running quark mass, which have uncertainties of $\mathcal{O}(\alpha_s^3)$, and translates to the same level of uncertainties on our estimate of the total $Z_x = C_x \times Z_x^{\mathrm{stat,RGI}}$ [4]. This is already better by one more order in α_s than the other approaches used in the literature which have uncertainties of at least $\mathcal{O}(\alpha_s^2)$.

5. The continuum limit

We now outline our results for the continuum limits of the form factors. Computations of the bare form factors are repeated for different lattice ensembles with decreasing lattice spacings. For this, the $N_f = 2$ ensembles by the Coordinated Lattice Simulations (CLS) effort [14] were used. The chosen ensembles (with labels A5, F6 and N6) all have roughly the same pion mass (330, 310, 340 MeV respectively), and the lattice spacing decreases like 0.0749(8), 0.0652(6) and 0.0483(4) fm respectively. The spatial dimension of the lattices satisfy $m_\pi L > 4$ and an aspect ratio of 2. The ensembles have degenerate light quarks. The strange quark mass is fixed by fixing the Kaon mass in units of the Kaon decay constant to it's physical value at our light quark masses.

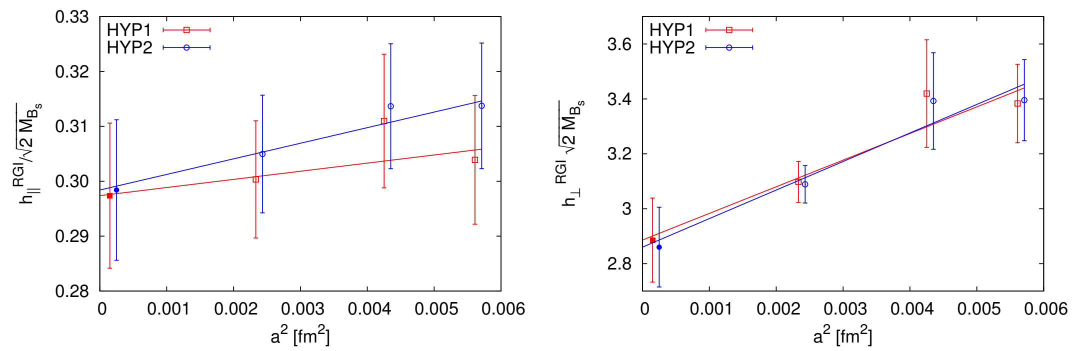

Figure 1: The linear extrapolation in a^2 to the continuum limit of the 1-loop $\mathcal{O}(a)$ improved RGI form factors $h_{\parallel}^{\mathrm{stat,RGI}}$ (left) and $h_{\perp}^{\mathrm{stat,RGI}}$ (right). The different discretizations HYP1/2 are shifted to ensure visibility.

Since the computations keep the physical momentum $\mathbf{p}_\mathrm{K} = 0.535$ GeV fixed, flavor twisted boundary conditions are necessary to impart the same (three-) momentum to the Kaon as the lattice spacing varies: $\psi_\mathrm{s}(x + L\hat{1}) = e^{i\theta}\psi_\mathrm{s}(x)$. The aforementioned Kaon momentum is obtained with a lattice momentum of $(1,0,0)$ on the finest lattice (N6). On the F6 and A5, therefore, we choose flavor twisted boundary conditions, such that, $\mathbf{p}_\mathrm{K} = (1,0,0)(2\pi + \theta)/L$. The F6 gets a larger $(\theta/(2\pi) = 0.350)$ twist as compared to A5 $(\theta/(2\pi) = 0.034)$ due to the respective lattice size. The $\mathrm{B_s}$ meson is always kept at rest. For our value of $\mathbf{p}_\mathrm{K}$, the momentum transferred is $q^2 = 21.22$ GeV^2 on all lattices, with an error of $0.03 - 0.05$ GeV^2 due to the lattice spacing.

While the extraction of the bare form factors is explained in [3], we show how the continuum limit can be extracted with the RGI form factors. In the error analysis, the statistical correlations and autocorrelations are all taken into account [15, 16]. The results of the continuum extrapolation is shown in Fig 1. A list of the values can be found in [2]. The continuum limit is obtained with an extrapolation linear in a^2, with the $c_x = c_x^{(1)} g_0^2$. While the $h_\parallel$ gives a compatible estimate by fitting it to a constant (and naturally with smaller error bars), we keep the a^2 extrapolations because there is no reason why these should disappear. Also, the $\mathscr{O}(a)$ corrections from perturbation theory do not matter at the precision of our data. This gives us: $h_\parallel^{\mathrm{stat,RGI}} = 0.976(41)\mathrm{GeV}^{1/2}$ and $h_\perp^{\mathrm{stat,RGI}} = 0.876(43)\mathrm{GeV}^{-1/2}$.

We can now estimate the form factor f_+ using different systematics for $1/m_\mathrm{h}$ suppressed terms. Two such estimates are $f_{+,1}$ and $f_{+,2}$. In the first one, all the known terms and kinematic factors are controlled, expect in $h_{\perp,\parallel}^{\mathrm{stat,RGI}}$ (which have suppressed uncontrolled $1/m_\mathrm{h}$ contributions), while in the second one $f_{+,2}$ all the $1/m_\mathrm{h}$ terms are systematically dropped:

$$f_{+,1} = \sqrt{\frac{m_{\mathrm{B_s}}}{2}} \left((1 - \frac{E_\mathrm{K}}{m_{\mathrm{B_s}}}) C_{\mathrm{V_k}} h_\perp^{\mathrm{stat,RGI}}(E_\mathrm{K}) + \frac{1}{m_{\mathrm{B_s}}} C_{\mathrm{V_0}} h_\parallel^{\mathrm{stat,RGI}}(E_\mathrm{K}) \right) \tag{5.1}$$

$$f_{+,2} = \sqrt{\frac{m_{\mathrm{B_s}}}{2}} C_{\mathrm{V_k}} h_\perp^{\mathrm{stat,RGI}}(E_\mathrm{K}). \tag{5.2}$$

Numerically, the values are: $f_{+,1} = 1.77(7)(7)$ and $f_{+,2} = 1.63(8)(6)$. Comparing these estimates, we see that the $\mathscr{O}(1/m_\mathrm{h})$ terms contribute like $(1 - \frac{E_\mathrm{K}}{m_{\mathrm{B_s}}})$, and we have a systematic $\sim 15\%$ ambiguity/uncertainty due to the dropping of these terms. Thus we have a preliminary estimate: $f_+(21.22\mathrm{GeV}^2) = f_{+,2} \pm 0.15 f_{+,2} = 1.63(8)(6) \pm 0.24$. The second error bars are the $\mathscr{O}(\alpha_s^3)$ perturbative uncertainties in the matching functions. We note that this systematic error is expected to drop to $\sim 1-2\%$ when all the $1/m_\mathrm{h}$ will be systematically included.

The quantity f_0 is also often quoted in the literature, which is defined as:

$$f_0 = \frac{\sqrt{2/m_{\mathrm{B_s}}}}{1 - m_\mathrm{K}^2/m_{\mathrm{B_s}}^2} \left((1 - \frac{E_\mathrm{K}}{m_{\mathrm{B_s}}}) C_{\mathrm{V_0}} h_\parallel^{\mathrm{stat,RGI}}(E_\mathrm{K}) + \frac{\mathbf{p}_\mathrm{K}^2}{m_{\mathrm{B_s}}} C_{\mathrm{V_k}} h_\perp^{\mathrm{stat,RGI}}(E_\mathrm{K}) \right) \tag{5.3}$$

For this number, we estimate $f_0 = 0.663(3)(1)$. The results for $f_{+,0}$ agree well with those existing in the literature: For Flynn et al. [6], the form factors extracted at our values of q^2 are $f_+ \simeq 1.65(10)$ and $f_0 \simeq 0.62(5)$, while Bouchard et al. [7] report $f_+ \simeq 1.80(20)$ and $f_0 \simeq 0.66(5)$. Given that our calculation has a very different source of systematic uncertainty, this agreement is very necessary for phenomenological applications.

6. Conclusion and Outlook

We report the first study of the continuum limit of non-perturbatively renormalized form factors. Since the stress is on the continuum limit we have kept the momentum transfer fixed at $q^2 = 21.22\mathrm{GeV}^2$. In the static limit, we have precise non-perturbative determinations of the RGI form factors. The matching to QCD is perturbative, and only has uncertainties of order $\mathscr{O}(\alpha_s^3)$. The discretization effects are not very large, allowing a smooth linear a^2 extrapolation to the continuum. On the other hand $1/m_\mathrm{h}$ effects need to incorporated.

The agreement of the various methods in the extraction of the form factors increase our confidence in their extraction using lattice techniques. Once the $1/m_\mathrm{h}$ are included they will be of

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direct phenomenological interest. This is envisaged as a direct follow-up of the project. Preliminary investigations are already in progress. For connection to phenomenology, the extrapolation to the physical light quarks will also be considered, as well as the computation at another value of the momentum. The basic results for the continuum extrapolation presented here inspire the confidence that precise lattice results, can be used together with experimental measurements for a reliable extraction of V_{ub} in the near future.

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