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Progress in the exploratory calculation of the rare kaon decays $K \to \pi\ell^+\ell^-$

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The rare decays of a kaon into a pion and a lepton/antilepton pair proceed via a flavour changing neutral current and therefore first arise in the Standard Model only as a second order electroweak interaction. This natural suppression makes these decays sensitive to the effects of potential New Physics. However the rare decay channels $K^+ \to \pi^+\ell^+\ell^-$ are dominated by long-distance contributions, where the two electroweak processes are separated by distances over which non-perturbative QCD effects play a significant role. In this talk I provide an update on the progress of our exploratory calculations of the long-distance contributions to $K^+ \to \pi^+\ell^+\ell^-$ amplitudes, which make use of the Domain Wall Fermion ensembles of the RBC and UKQCD collaborations, and outline the prospects for further progress of the calculation.

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1. Introduction

The rare kaon decays $K \to \pi\ell^+\ell^-$ are flavour changing neutral current (FCNC) processes, which are naturally suppressed in the Standard Model as they first arise only as second-order electroweak processes. This suppression makes them ideal probes for potential New Physics effects, such as lepton flavour universality violation [1]. One significant difficulty in the theoretical understanding of second-order weak processes is that there may be significant contributions when the two electroweak vertices are separated by distances as large as $1/\Lambda_{\rm QCD}$. These long-distance effects contain non-perturbative contributions, hence a complete theoretical study of these processes can be achieved only by utilising non-perturbative methods such as lattice QCD.

The processes $K^+ \to \pi^+\ell^+\ell^-$ and $K_S \to \pi^0\ell^+\ell^-$ in particular are completely dominated by long-distance hadronic effects, and thus lattice QCD is a natural way to evaluate the corresponding matrix elements [2, 3]. The matrix elements for these decays are given by

$$\mathscr{A}_\mu^i\left(q^2\right) = \int d^4x \left\langle \pi^i\left(\mathbf{p}\right)\left|T\left[J_\mu\left(0\right)\mathscr{H}_W\left(x\right)\right]\right|K^i\left(\mathbf{k}\right)\right\rangle, \tag{1.1}$$

where $q = k - p$, $i = +, 0$, $\mathscr{H}_W$ is the $\Delta S = 1$ weak Hamiltonian and J_μ is the electromagnetic current. Conservation of the electromagnetic current implies that this matrix element can be written as

$$\mathscr{A}_\mu^i\left(q^2\right) \equiv -iG_F\frac{V^i\left(z\right)}{\left(4\pi\right)^2}\left(q^2\left(k+p\right)_\mu - \left(M_K^2 - M_\pi^2\right)q_\mu\right), \tag{1.2}$$

where non-perturbative QCD effects are contained in the form factor $V^i\left(z\right)$, $z = q^2/M_K^2$. Theoretical effort to understand these processes has been carried out using effective theories such as Chiral Perturbation Theory (ChPT) [4, 5, 6], culminating in a parametrisation of the $K \to \pi\ell^+\ell^-$ form factor of the form

$$V_i\left(z\right) = a_i + b_i z + V_i^{\pi\pi}\left(z\right), \tag{1.3}$$

where $z = q^2/M_K^2$, and $V_i^{\pi\pi}\left(z\right)$ $\left(i = +, 0\right)$ is introduced to account for $\pi\pi \to \gamma^*$ rescattering in $K \to \pi\pi\pi$ decays. This parametrisation has been used as a phenomenological ansatz for fits of experimental spectra to obtain the coefficients a_i and b_i. One opportunity of lattice QCD is to test this ansatz and determine the coefficients a_i and b_i from first principles.

In this paper we provide a summary of the results of lattice QCD studies of $K^+ \to \pi^+\ell^+\ell^-$ decays recently described in [7]. In section 2 we briefly review the operators involved in this calculation, and discuss our implementation. We give a short description of the analysis in section 3. In 4 we present our numerical results for the $K^+ \to \pi^+\gamma^*$ matrix elements, and in section 5 we discuss the results of preliminary simulations exploring the charm mass dependence of these matrix elements. In section 6 we make our conclusions and discuss the future prospects of this calculation.

2. Lattice Implementation

The effective weak Hamiltonian relevant to the transition $s \to d\ell^+\ell^-$, renormalised at a scale $M_W \gg \mu > m_c$, is defined by [8]

$$\mathscr{H}_W = \frac{G_F}{\sqrt{2}}V_{us}^*V_{ud}\left(\sum_{j=1}^{2}C_j\left(Q_j^u - Q_j^c\right) + \sum_{j=3}^{8}C_jQ_j + \mathscr{O}\left(\frac{V_{ts}^*V_{td}}{V_{us}^*V_{ud}}\right)\right) = \frac{G_F}{\sqrt{2}}V_{us}^*V_{ud}\,H_W. \tag{2.1}$$

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In our implementation we consider only the operators $Q_1^{u,c}$ and $Q_2^{u,c}$; all others vanish at tree level and have much smaller Wilson coefficients than these two. The operators $Q_{1,2}^q$ are defined as [2, 3]

$$Q_1^q = \left(\bar{s}_i \gamma_\mu^L d_i\right)\left(\bar{q}_j \gamma^{L,\mu} q_j\right)\& \qquad Q_2^q = \left(\bar{s}_i \gamma_\mu^L d_j\right)\left(\bar{q}_j \gamma^{L,\mu} q_i\right), \tag{2.2}$$

where i, j are summed colour indices and $\gamma_\mu^L = \gamma_\mu(1-\gamma_5)$. The details of the non-perturbative renormalisation of this operator to match to the $\overline{\text{MS}}$ scheme in the continuum are given in Ref. [9]. The electromagnetic current J_μ is taken to be the standard flavour-diagonal conserved vector current

$$J_\mu = \frac{1}{3}\left(2V_\mu^u - V_\mu^d - V_\mu^s + 2V_\mu^c\right), \tag{2.3}$$

where V_μ^q is the current for the flavour q. The conserved current requires no renormalisation.

The diagram topologies resulting from the Wick contractions of Eq. 1.1 are shown in Fig. 1. The current may be inserted on any of the quark lines in each topology, or it may be inserted as a self-contracted loop to produce a disconnected diagram. Logarithmic divergences may arise as J_μ and H_W approach each other in the S and E diagram topologies; however by simulating with both valence up and charm quarks, these logarithmic divergences cancel via the GIM mechanism [2, 3].

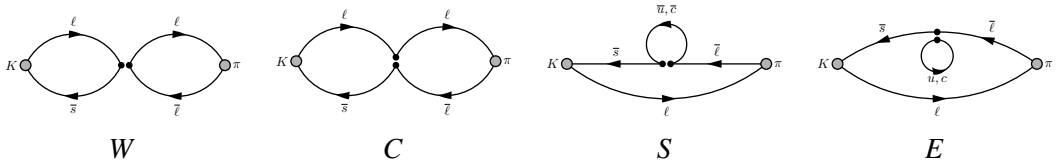

Figure 1: The four diagram topologies obtained after performing the Wick contractions for the H_W operator.

In our exploratory simulations we simulate with a kaon at rest decaying into a pion with non-zero momentum $\mathbf{p} = \frac{2\pi}{L}(1,0,0)$, $\frac{2\pi}{L}(1,1,0)$, $\frac{2\pi}{L}(1,1,1)$, where L is the spacial extent of the lattice. The full set of diagrams corresponding to the rare kaon decay are computed with 14 propagators [7]. Four are required to connect the kaon/pion sources to the H_W insertion: one strange and one light for the kaon; two light propagators with momenta $\mathbf{0}$ and $\mathbf{p}$ to produce a pion with momentum $\mathbf{p} \neq \mathbf{0}$ (this also allows us to make a pion with momentum $\mathbf{0}$). Two more propagators are needed for the loops in the S and E and disconnected diagrams (one light, one charm), and one more for the strange loop in disconnected diagrams. We omit disconnected diagrams from our present study, as we do not have the statistical precision to obtain a reasonable signal for them. The loops are evaluated using random volume source propagators [10]. We use each of these 7 propagators to calculate a sequential propagator to achieve the current insertion to bring us up to 14. In addition, to construct all the 2pt and 3pt functions required for our analysis, we compute another strange propagator with momentum $\mathbf{p}$ such that we can produce a kaon with momentum $\mathbf{p}$.

3. Analysis

To recover the amplitude of the decay we consider the time-integrated 4pt correlator,

$$I_\mu(T_a, T_b, \mathbf{k}, \mathbf{p}) = \int_{-T_a}^{T_b} dt_H \left\langle \pi(\mathbf{p}) \left| T\left[H_W(t_H) J^\mu(0)\right] \right| K(\mathbf{k}) \right\rangle, \tag{3.1}$$

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in the limit $T_a, T_b \to \infty$. The spectral decomposition for this integrated 4pt correlator is

$$I_\mu(T_a, T_b, \mathbf{k}, \mathbf{p}) = -\sum_n \frac{1}{2E_n} \frac{\langle \pi(\mathbf{p}) | J_\mu | n, \mathbf{k} \rangle \langle n, \mathbf{k} | H_W | K(\mathbf{k}) \rangle}{E_K(\mathbf{k}) - E_n} \left(1 - e^{(E_K(\mathbf{k}) - E_n)T_a}\right) + \sum_m \frac{1}{2E_m} \frac{\langle \pi(\mathbf{p}) | H_W | m, \mathbf{p} \rangle \langle m, \mathbf{p} | J_\mu | K(\mathbf{k}) \rangle}{E_m - E_\pi(\mathbf{p})} \left(1 - e^{-(E_m - E_\pi(\mathbf{p}))T_b}\right) \quad (3.2)$$

The states m have the quantum numbers of a kaon, hence for this time ordering we will always have $E_m > E_\pi(\mathbf{p})$. This part of the integral will converge as $T_b \to \infty$. However the states n have the quantum numbers of a pion. For physical pion and kaon masses there are three permitted on-shell intermediate states (namely one, two and three pion intermediate states), which satisfy $E_n < E_K(\mathbf{k})$ and will cause the integral to diverge with increasing T_a. These exponentially growing terms do not contribute to the overall decay amplitude and thus should be removed.

There are two main options for the removal of the single pion exponentially growing term. Method 1 involves measuring the matrix elements $\langle \pi(\mathbf{p}) | J_\mu | \pi(\mathbf{k}) \rangle$ and $\langle \pi(\mathbf{k}) | H_W | K(\mathbf{k}) \rangle$, such that the exponentially growing term can be constructed and subtracted explicitly. Method 2 involves an unphysical shift of the weak Hamiltonian by a density $\bar{s}d$ [11] such that

$$\langle \pi(\mathbf{k}) | H'_W | K(\mathbf{k}) \rangle = \langle \pi(\mathbf{k}) | H_W | K(\mathbf{k}) \rangle - c_s(\mathbf{k}) \langle \pi(\mathbf{k}) | \bar{s}d | K(\mathbf{k}) \rangle = 0. \quad (3.3)$$

By replacing H_W by H'_W in Eq. (3.2) it is straightforward to see that the divergent single pion contribution vanishes.

The two- and three-pion intermediate states do not contribute in our exploratory simulations with $M_\pi \simeq 430\,\text{MeV}$ and $M_K \simeq 600\,\text{MeV}$; however the $\pi\pi$ intermediate state does not arise in this decay because of parity conservation, and $\pi\pi\pi$ intermediate states are expected to be heavily phase space suppressed; we thus expect their contribution to be negligibly small within the target statistical precision of our calculation.

4. Numerical Results

Our exploratory studies [7] have been performed using a $24^3 \times 64$ lattice with an inverse lattice spacing of $1/a = 1.78\,\text{GeV}$, employing Shamir domain wall fermions [12] with Iwasaki gauge action [13], a pion mass of $\sim 430\,\text{MeV}$ and a kaon mass of $\sim 625\,\text{MeV}$ [14, 15]. We use a sample of 128 configurations, each separated by 20 molecular dynamics time units. In order to cancel divergences with the GIM mechanism we include a charm quark with a bare mass of $m = 0.2$. Using the mass renormalisation factor $Z_m^{\overline{MS}}(2\,\text{GeV}) = 1.498$ for this lattice [14], this corresponds to an unphysical charm quark of mass $m_c^{\overline{MS}}(2\,\text{GeV}) = 533\,\text{MeV}$.

The main difficulty of this analysis is the removal of the exponential term that grows with increasing T_a; however in practice we find that it is necessary to also consider a term that falls exponentially with T_b, which does not converge in the available time extent (i.e. we are outside the $T_b \to \infty$ limit). This term arises due to a single kaon intermediate state, i.e. when we see $E_m = E_K(\mathbf{p})$ in Eq. 3.2. The slow decay can be attributed to the fact that here the kaon-pion mass difference is rather small. In practice therefore it is necessary also to remove this decaying contribution in a manner similar to the exponentially growing term by reconstructing the state

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using results of fits to 2pt and 3pt correlators. However with Method 2 the removal of the single pion growing exponential also happens to remove this exponentially falling contribution within our available statistical precision: this can be attributed to the fact that the parameter $c_s(\mathbf{k})$ is very weakly dependent upon momentum. We remark that the matrix element $\langle \pi(\mathbf{p}) | H_W | K(\mathbf{p}) \rangle$ is difficult to determine with a high level of statistical precision when we have $\mathbf{p} \neq \mathbf{0}$; the statistical error on this quantity thus drowns out the signal for the matrix element. For this reason method 2 gives us a more precise result.

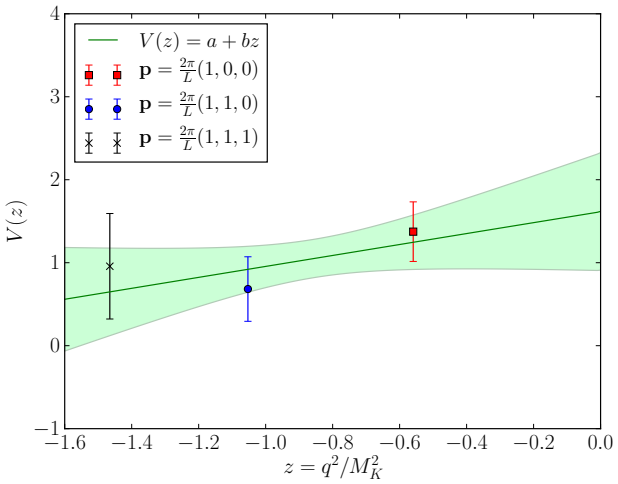

Figure 2: Dependence of the form factor for the decay $K^+ \to \pi^+\ell^+\ell^-$ upon $z = q^2/M_K^2$. Our lattice data is fit to a linear ansatz to obtain $a = 1.6(7)$ and $b = 0.7(8)$.

In Fig. 2 we display the dependence of the form factor extracted from lattice data upon $z = q^2/M_K^2$. Since we have only three data points at rather large space-like momenta, we are not able to fully explore the anastz in Eq. 1.3. Here we simply use a linear fit, which does provide a reasonable description of our data with a $\chi^2/\text{dof} = 0.74$. The parameters we obtain, $a_+^{lat} = 1.6(7)$ and $b_+^{lat} = 0.7(8)$, are different from the parameters obtained from phenomenological fits to experimental data, $a_+^{exp} = -0.578(16)$ and $b_+^{exp} = -0.779(66)$. However such a comparison must be taken with care given the unphysical masses used in our simulation.

5. Charm Mass Dependence

A proper treatment of the long-distance contributions to $K \to \pi\ell^+\ell^-$ involves a lattice calculation with 4 active flavours at their physical masses. Simulating with a physical up quark requires a large lattice volume to avoid errors from finite volume effects; conversely simulating with a physical charm requires simulating with a small lattice spacing to avoid discretisation errors. This prospect presents a huge computational challenge, and it is unclear whether this is feasible with our currently available computational resources. Our priority therefore is to first push to physical quark masses, and consider a calculation with a physical charm quark at a later stage.

To obtain an idea of the charm mass dependence, we have calculated our matrix element on the $24^3 \times 64$ lattice with three values of the bare charm quark mass: $m_c = 0.2, 0.25, 0.3$. In Fig. 3

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we show the dependence of the S and E diagram contributions to the matrix element on the mass of the charm quark. At this stage we effectively have no signal for the charm quark mass dependence, indicating that the systematic error from using an unphysically heavy charm quark mass is currently much smaller than our current statistical error.

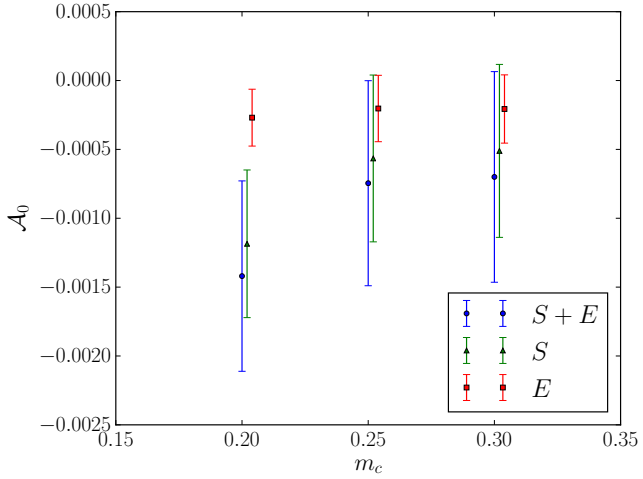

Figure 3: The dependence of the S and E diagrams on the mass of the charm quark.

6. Conclusions

Our exploratory lattice simulations of the rare kaon decay $K \to \pi\ell^+\ell^-$ demonstrate that we are able to extract the matrix elements with a statistical error that is one to two orders of magnitude larger than the statistical precision currently achieved by experiments. In the future we foresee much greater expenditure of computational resources in order to reduce the statistical errors on our matrix elements.

Because the simulation of an unphysically heavy charm quark requires a very fine lattice, it is an intriguing prospect to consider our calculation in a 3-flavour theory, where the charm has been integrated out. While this incurs a systematic error from the fact that perturbation theory cannot be applied reliably at the charm scale, it allows us to save much simulation time by simulating on coarser lattices. Our results in the 3-flavour theory may be compared to results obtained with unphysically light charm quarks; such a calculation would be an important intermediate step before a full physical point calculation is feasible.

We are currently in the process of finalising the procedures required to simulate in the 3-flavour theory, in particular removing the logarithmic divergence in the S and E diagrams without simulating the charm directly. The procedure required to remove this divergence is very similar to the renormalisation required in $K \to \pi\nu\bar{\nu}$ decays [16, 17]. We therefore aim to begin simulating this decay with physical light quark masses within a short timescale.

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