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$B \to K^*$ decays in a finite volume

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We study the extraction of the $B \to K^*$ transition form factors on the lattice by using the non-relativistic effective field theory in a finite volume. The determination of the matrix element at the resonance pole is considered in the case of coupled-channels. It is shown that, in the limit of the infinitely narrow width, the expected result for the matrix element is reproduced. The results, contained in this contribution, are discussed at length in Ref. [1].

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1. Introduction

Rare B decay modes provide one of the best opportunities in the search for physics beyond the Standard Model (BSM). Among them, $B \to K^* l^+ l^-$ is regarded as one of the most important channels, as the polarization of the K^* allows a precise angular reconstruction resulting in many observables which can be tested in the Standard Model (SM) and its extensions [2]. In order to perform this beautiful test of the Standard model, however, one should have the corresponding hadronic matrix elements, which affect the branching ratios involved in the fit to the experimental data, under full control. Lattice QCD provides a framework to calculate these matrix elements in the low-recoil region [3]. Here, the calculations can be carried out from first principles and are devoid of any model-dependent assumptions. A potential source of the systematic uncertainty is, however, the fact that K^* is a resonance and not a stable particle, whereas the technique, which is used to extract this matrix element on the lattice, is designed for stable particles. The resonance matrix elements are defined through the analytic continuation into the complex energy plane to the resonance pole, and the purpose of the present work is to demonstrate explicitly, how this procedure can be performed with the lattice input. Moreover, we describe this procedure for the general case, where multiple channels are present. In the present context, this issue is more of academic interest (for the physical quark masses, the ηK channel lies significantly higher than K^* resonance and couples very weakly to it), but the general formalism might be useful for the analysis of other systems (e.g., the electromagnetic form factors of the $\Lambda(1405)$ resonance).

It should be pointed out that the computation of the matrix elements, which involve strongly interacting particles in the in- or out- states, has already been addressed in the past. In their seminal paper [4], Lellouch and Lüscher have shown, how the finite volume artifacts due to the final state interactions can be removed in the $K \to 2\pi$ weak decay amplitude. The subsequent work (see, e.g., Refs. [5]) was mainly focused on the generalization of the above result to the case of the multi-channel scattering, non-rest frames, etc. The analytic continuation to the resonance pole was considered in our previous papers [6, 7]. In the present work, we complete the job by performing the continuation in the multi-channel case, as well as by clarifying the issue with the photon virtuality at the resonance pole.

We use the non-relativistic EFT framework in a finite volume to achieve the goals outlined above. With the same non-relativistic Lagrangian, one calculates the matrix elements in question twice – in the infinite and in a finite volume – and establishes a connection between these two quantities. In the final results, any notion of the non-relativistic approach disappears, as it should. We find this approach algebraically simpler than the one based on the Bethe-Salpeter equation (see, e.g., Refs. [8]). In the end, both methods have the same range of applicability and one arrives at the same results.

2. Matrix elements on the lattice

In the infinite volume, Euclidean space, rest-frame of the K^* meson, the seven semileptonic $B \to K^*$ form factors $V, A_0, A_1, A_{12}, T_1, T_2, T_{23}$ are defined in a standard manner (see, e.g., Ref. [1]). These expressions are written, however, under the assumption that the K^* meson is stable and should be modified accordingly in case of a resonance. Namely, in this case one measures – at a

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given lattice volume – the matrix elements of the currents between the one-B-meson state $|B(\mathbf{p})\rangle$ and the eigenstate $|n\rangle$ of the total Hamiltonian H in a finite volume, which corresponds to the discrete eigenvalue E_n and strangeness $S=+1$. One may denote these matrix elements as F^M, $M=1,\cdots,7$ (all other indices are suppressed). The volume-dependence of F^M is irregular, and the infinite-volume limit cannot be performed straightforwardly. The question now is, how are the quantities F^M related to the seven semileptonic form factors $V,A_0,A_1,A_{12},T_1,T_2,T_{23}$ and how the latter are defined in the case when the K^* meson is unstable.

3. Two-body scattering in a finite volume

To keep things completely general, we derive the Lüscher-Lellouch formula in the two-channel case (adding more two-particle channels is straightforward). The infinite-volume T-matrix, corresponding to the final-state interactions, is parameterized in terms of two scattering phases $\delta_1(p_1)$, $\delta_2(p_2)$ and mixing parameter $\varepsilon(E)$

$$T=8\pi\sqrt{s}\begin{pmatrix}\frac{1}{p_1}(c_\varepsilon^2e^{i\delta_1}\sin\delta_1+s_\varepsilon^2e^{i\delta_2}\sin\delta_2) & \frac{1}{\sqrt{p_1p_2}}c_\varepsilon s_\varepsilon(e^{i\delta_1}\sin\delta_1-e^{i\delta_2}\sin\delta_2)\\ \frac{1}{\sqrt{p_1p_2}}c_\varepsilon s_\varepsilon(e^{i\delta_1}\sin\delta_1-e^{i\delta_2}\sin\delta_2) & \frac{1}{p_2}(c_\varepsilon^2e^{i\delta_2}\sin\delta_2+s_\varepsilon^2e^{i\delta_1}\sin\delta_1)\end{pmatrix}, \tag{3.1}$$

where $s_\varepsilon\equiv\sin\varepsilon(E)$, $c_\varepsilon\equiv\cos\varepsilon(E)$. Here, p_1 and p_2 denote the relative 3-momenta in the πK and ηK channels, respectively. The T-matrix obeys the Lippmann-Schwinger equation $T=V+VGT$, where all quantities are taken on shell [7]. Here, V denotes a potential and $G(s)$ is a loop function matrix given by

$$G=\text{diag}\left(\frac{ip_1}{8\pi\sqrt{s}},\frac{ip_2}{8\pi\sqrt{s}}\right). \tag{3.2}$$

The parameterization of the potential V in terms of the parameters $\delta_1(p_1)$, $\delta_2(p_2)$ and $\varepsilon(E)$ reads:

$$V=8\pi\sqrt{s}\begin{pmatrix}\frac{1}{p_1}(t_1+s_\varepsilon^2t) & -\frac{1}{\sqrt{p_1p_2}}c_\varepsilon s_\varepsilon t\\ -\frac{1}{\sqrt{p_1p_2}}c_\varepsilon s_\varepsilon t & \frac{1}{p_2}(t_2-s_\varepsilon^2t)\end{pmatrix}, \tag{3.3}$$

where $t_i\equiv\tan\delta_i(p_i)$ and $t=t_2-t_1$. Clearly, the potential matrix V is real and symmetric. The finite-volume counterpart of the loop function matrix Eq. (3.2), which we denote by G_L, reads

$$G_L=\text{diag}\left(-\frac{p_1}{8\pi\sqrt{s}}\cot\phi(p_1),-\frac{p_2}{8\pi\sqrt{s}}\cot\phi(p_2)\right). \tag{3.4}$$

Here, $\phi(p_\alpha)$ are the volume-dependent functions that are related to the Lüscher zeta-function [1]. Further, the T_L-matrix is a scattering amplitude in a finite volume that is defined formally also through a Lippmann-Schwinger equation with the same potential V and G replaced by G_L. It takes the form:

$$T_L=\frac{8\pi\sqrt{s}}{f(E)}\begin{pmatrix}\frac{1}{p_1}[t_1\tau_1(t_2+\tau_2)+s_\varepsilon^2\tau_1\tau_2t] & -\frac{1}{\sqrt{p_1p_2}}c_\varepsilon s_\varepsilon\tau_1\tau_2t\\ -\frac{1}{\sqrt{p_1p_2}}c_\varepsilon s_\varepsilon\tau_1\tau_2t & \frac{1}{p_2}[t_2\tau_2(t_1+\tau_1)-s_\varepsilon^2\tau_1\tau_2t]\end{pmatrix}, \tag{3.5}$$

where $\tau_\alpha\equiv\tan\phi(p_\alpha)$ and $f(E)\equiv(t_1+\tau_1)(t_2+\tau_2)+s_\varepsilon^2(t_2-t_1)(\tau_2-\tau_1)$. The two-channel Lüscher equation that determines the discrete spectrum E_n reads $f(E_n)=0$. The quantity T_L, as a function of E, has simple poles on the real axis at $E=E_n$ with factorizing residues

$$T_L^{\alpha\beta}=\frac{f_\alpha f_\beta}{E_n+iP_0}+\cdots. \tag{3.6}$$

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Here, the quantities f_1, f_2 can be brought to the following form by applying the Lüscher equation:

$$f_1^2 = \frac{8\pi\sqrt{s}}{p_1}\,\frac{\tau_1^2(t_2+\tau_2-s_\varepsilon^2 t)}{f'(E)}\bigg|_{E=E_n}, \quad f_2^2 = \frac{8\pi\sqrt{s}}{p_2}\,\frac{\tau_2^2(t_1+\tau_1+s_\varepsilon^2 t)}{f'(E)}\bigg|_{E=E_n}, \tag{3.7}$$

where $f'(E) \equiv df(E)/dE$.

4. Derivation of the Lellouch-Lüscher equation

Let $O(x)$ be a local operator with quantum numbers of the K^* that transforms according to a given irrep, and let $\mathscr{O}(t) = \sum_{\mathbf{x}} O(\mathbf{x},t)$. Consider the following Euclidean two-point function

$$D(x_0-y_0) = \langle 0|\mathscr{O}(x_0)\mathscr{O}^\dagger(y_0)|0\rangle\,. \tag{4.1}$$

The spectral representation of this two-point function takes the form

$$D(x_0-y_0) = \sum_n e^{-E_n(x_0-y_0)}|\langle 0|\mathscr{O}(0)|E_n\rangle|^2\,. \tag{4.2}$$

On the other hand, this two-point function can be calculated within the non-relativistic EFT in a finite volume, where it is given by a sum of bubble graphs. Let the quantities X_α, $\alpha=1,2$ denote the couplings of the operator $\mathscr{O}$ to the respective channels. Since the corresponding Lagrangian contains terms with an arbitrary number of spatial derivatives, one has $X_\alpha = A_\alpha + B_\alpha \mathbf{p}_\alpha^2 + \cdots$, where $A_\alpha, B_\alpha, \ldots$ contain only short-range physics. Summing up all bubbles yields the two-channel T-matrix. Using Eq. (3.6), it can be straightforwardly demonstrated that the two-point function obeys the spectral representation (4.2). The expression of the matrix element in this spectral representation can be directly read off:

$$|\langle 0|\mathscr{O}(0)|E_n\rangle| = \frac{\mathscr{V}^{1/2}}{8\pi E_n}\left|\sum_{\alpha=1}^{2} X_\alpha p_\alpha(E_n)\tau_\alpha^{-1}(E_n) f_\alpha(E_n)\right|. \tag{4.3}$$

Next, we consider the three-point function

$$\Gamma^M(x_0,p) = \langle 0|\mathscr{O}(x_0)J^M(0)|B(p)\rangle, \quad M=1,\ldots 7. \tag{4.4}$$

Here, the $J^M(0)$ denote the current operators, see Ref. [1]. Inserting a complete set of states, we get the spectral representation of $\Gamma^M(x_0,p)$

$$\Gamma^M(x_0,p) = \sum_n e^{-E_n x_0}\langle 0|\mathscr{O}(0)|E_n\rangle F^M(E_n,|\mathbf{q}|)\,, \tag{4.5}$$

where $|\mathbf{q}| = p$ denotes the magnitude of the photon 3-momentum in this frame.

At the next step, we again calculate the three-point function in the non-relativistic EFT. Denoting the sum of all *two-particle irreducible* diagrams in the respective channel by $\bar F_\alpha^M(E,|\mathbf{q}|)$, $\alpha = 1,2$, we finally get

$$\begin{aligned}\Gamma^M(x_0,p) \;=\; & \frac{\mathscr{V}^{-1/2}}{64\pi^2 E_n^2}\sum_n e^{-E_n x_0}\sum_{\alpha,\beta=1}^{2}[X_\alpha p_\alpha(E_n)\tau_\alpha^{-1}(E_n) f_\alpha(E_n)] \\ & \times\; [p_\beta(E_n)\tau_\beta^{-1}(E_n) f_\beta(E_n)\bar F_\beta^M(E_n,|\mathbf{q}|)].\end{aligned} \tag{4.6}$$

From this expression, one may directly read off the expression for the finite-volume matrix element

$$|F^M(E_n,|\mathbf{q}|)| = \frac{\mathscr{V}^{-1}}{8\pi E}\big|p_1\tau_1^{-1}f_1\bar{F}_1^M + p_2\tau_2^{-1}f_2\bar{F}_2^M\big|\Big|_{E=E_n}. \tag{4.7}$$

The last step that needs to be done is to relate the above defined quantities $\bar{F}_1^M$, $\bar{F}_2^M$ to the (infinite-volume) decay amplitudes $\mathscr{A}_1^M(B\to\pi K l^+l^-)$ and $\mathscr{A}_2^M(B\to\eta K l^+l^-)$ through the two-channel Watson theorem. After summing up the two-particle reducible diagrams in the infinite volume, one gets

$$\mathscr{A}_1^M = \frac{1}{\sqrt{p_1}}(u_1^M c_\varepsilon e^{i\delta_1} - u_2^M s_\varepsilon e^{i\delta_2}), \quad \mathscr{A}_2^M = \frac{1}{\sqrt{p_2}}(u_2^M c_\varepsilon e^{i\delta_2} + u_1^M s_\varepsilon e^{i\delta_1}), \tag{4.8}$$

where

$$u_1^M = (\sqrt{p_1}c_\varepsilon\bar{F}_1^M + \sqrt{p_2}s_\varepsilon\bar{F}_2^M)\cos\delta_1, \quad u_2^M = (\sqrt{p_2}c_\varepsilon\bar{F}_2^M - \sqrt{p_1}s_\varepsilon\bar{F}_1^M)\cos\delta_2. \tag{4.9}$$

This is the generalization of the Lellouch-Lüscher formula for the two-channel case. Note that the key to the derivation was the fact that the two-particle irreducible diagrams, which describe the short-range dynamics, are the same in a finite and in the infinite volume (this statement holds up to the exponentially suppressed contributions). Note also that the unknown couplings X_α have been canceled in the final formula which does not bear any reference to the non-relativistic EFT framework either.

5. Form factors at the K* resonance pole

The current matrix elements involving resonances have the proper field-theoretical meaning only if they are analytically continued to the resonance pole position. The advantage of such a definition is that it is process-independent. In this section, we only briefly sketch the procedure which should be used for the continuation. The key to the procedure lies in the fact that the quantities $\tilde{u}_\alpha^M = u_\alpha^M/\sin\delta_\alpha,\ \alpha=1,2$, where u_α^M were introduced in Eq. (4.9), are low-energy polynomials in the vicinity of a narrow resonance (i.e., their expansion in Taylor series does not contain a small scale related to the resonance width). Owing to this property, it is possible to approximate these quantities with polynomials and determine the coefficients of these polynomials from fit to the lattice data. Recalling that the measured matrix elements are functions of two kinematic variables: the CM energy in the final state interactions and the photon momentum $\mathbf{q}$, it is clear that, in order to perform a fit, one should "scan" the CM energy range near the resonance mass, leaving the other variable $\mathbf{q}$ fixed. This can be done by performing simulations in asymmetric boxes or using (partially) twisted boundary conditions [1, 6]. Once the fit is done, the analytic continuation reduces to the evaluation of the polynomial at the (complex) resonance pole. The final result for the resonance form factors at the pole is given by

$$F_R^M(E_R,|\mathbf{q}|) = -\frac{i}{8\pi E}\big(p_1h_1\bar{F}_1^M - p_2h_2\bar{F}_2^M\big)\Big|_{E=E_R}, \tag{5.1}$$

where the $\bar{F}^M_\alpha$ are obtained from $\tilde{u}^M_\alpha$ evaluated at the pole, and the quantities h_α determine the residue of the infinite-volume scattering matrix at the pole on the second Riemann sheet

$$T_{II}^{\alpha\beta}(s) = \frac{h_\alpha h_\beta}{s_R - P^2} + \cdots\,, \tag{5.2}$$

$$h_1^2 = -\frac{8\pi\sqrt{s}}{p_1}\frac{2E(t_2+i-s_\varepsilon^2 t)}{h'(E)}\bigg|_{E=E_R}\,,\quad h_2^2 = -\frac{8\pi\sqrt{s}}{p_2}\frac{2E(t_1-i+s_\varepsilon^2 t)}{h'(E)}\bigg|_{E=E_R}\,, \tag{5.3}$$

where $h'(E) \equiv dh(E)/dE$ and $h(E) = (t_1-i)(t_2+i) - 2is_\varepsilon^2(t_2-t_1)$.

Finally, it can be explicitly checked that, in the limit of the infinitely narrow resonance,

$$F^M(E_n,|\mathbf{q}|) \to \frac{\mathscr{V}^{-1}}{2E_n} F_R^M(E_R,|\mathbf{q}|)\,,\quad E_n \to E_R\,, \tag{5.4}$$

where $\mathscr{V}$ denotes the lattice volume. In other words, in this limit, the form factor, extracted at the pole, up to a normalization factor, coincides with the one measured on the lattice. Note that, for the one-channel case, this has been already shown in Ref. [6].

6. Photon virtuality

The analytic continuation to the resonance pole yields the quantity $F_R^M(E_R,|\mathbf{q}|)$. Below, we would like to briefly discuss one conceptual issue, related to the interpretation of this quantity. Namely, we wish to know, what is the photon virtuality q^2 for the resonance form factor, extracted at the pole. In the literature, different statements have been made on this issue so far. We think that a clarification is needed at this point.

According to the procedure, which is proposed in the present paper (see also Ref. [6]), the finite-volume matrix element is measured at different two-particle energies $E_n(L)$ and a fixed value of $|\mathbf{q}|$. After that, an analytic continuation is performed to the complex resonance pole, keeping $|\mathbf{q}|$ fixed. Further, the photon virtuality becomes complex at the pole (see Fig. 1)

$$q^2 = \left(E_R - \sqrt{m_B^2 + \mathbf{q}^2}\right)^2 - \mathbf{q}^2\,. \tag{6.1}$$

From this pure kinematical consideration it becomes clear that it is not possible to keep the quantity q^2 real at the pole, unless one considers the B-meson of shell (which would be a rather unattractive option). On the other hand, there is no alternative to the analytic continuation, because there is always a non-vanishing background on the real axis and the form factors become process-dependent.

7. Conclusions

In this work, we have studied the extraction of the $B \to K^*$ transition form factors in the low recoil region on the lattice.

- We have applied the non-relativistic effective field theory in a finite volume and reproduced the two-channel analogue of the Lellouch-Lüscher formula.

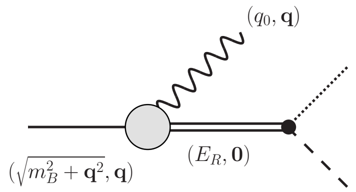

Figure 1: The factorization of the amplitudes at the resonance pole. The photon virtuality q^2 is complex.

- We have studied the extrapolation of the matrix elements to the complex resonance pole.
- We have shown that the photon virtuality is a complex quantity for the matrix elements extracted at the pole.
- We have shown that, in the limit of the infinitely small width, up to a kinematical normalization factor, the extracted form factors coincide with those directly measured on the lattice.

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